

CLOSED MAGNETIC GEODESICS ON CLOSED HYPERBOLIC RIEMANN SURFACES

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ABSTRACT. We prove the existence of Alexandrov embedded closed magnetic geodesics on closed hyperbolic surfaces. Closed magnetic geodesics correspond to closed curves with prescribed geodesic curvature.

1. INTRODUCTION

Let (M, g) be a compact, two dimensional, oriented manifold equipped with a smooth metric g and $k : M \rightarrow \mathbb{R}$ a smooth positive function. We consider the following two equations for curves γ on M :

$$D_{t,g}\dot{\gamma} = k(\gamma)J_g(\gamma)\dot{\gamma}, \quad (1.1)$$

and

$$D_{t,g}\dot{\gamma} = |\dot{\gamma}|_g k(\gamma)J_g(\gamma)\dot{\gamma}, \quad (1.2)$$

where $D_{t,g}$ is the covariant derivative with respect to g , and $J_g(x)$ is the rotation by $\pi/2$ in T_xM measured with g and the given orientation.

Equation (1.1) describes the motion of a charge in a magnetic field corresponding to the magnetic form kdV_g and solutions to (1.1) will be called *(k-)magnetic geodesics* (see [1,11]). Equation (1.2) corresponds to the problem of *prescribing geodesic curvature*, as its solutions γ are constant speed curves with geodesic curvature $k_g(\gamma, t)$ given by $k(\gamma(t))$ (see [15]).

It is easy to see that a nonconstant magnetic geodesic γ lies in a fixed energy level E_c , i.e. there is $c > 0$, such that

$$(\gamma, \dot{\gamma}) \in E_c := \{(x, V) \in TM : |V|_g = c\}.$$

For fixed k and $c > 0$ the equations (1.1) and (1.2) are equivalent in the following sense: If γ is a nonconstant solution of (1.2) with k replaced by k/c , then the curve $\gamma_c(t) := \gamma(ct/|\dot{\gamma}|_g)$ is a k -magnetic geodesic in E_c , and a k -magnetic geodesic in E_c solves (1.2) with k replaced by k/c . We emphasise that k -magnetic geodesics on different energy levels are not reparameterizations of each other.

We study the existence of closed curves with prescribed geodesic curvature or equivalently the existence of periodic magnetic geodesics on prescribed

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energy levels E_c .

There is a vast literature on the existence of closed magnetic geodesics. We limit ourselves to quote [12, 18] for the approach via Morse-Novikov theory for (possibly multi-valued) variational functionals, [1, 5] for the application of the theory of dynamical systems and symplectic geometry, [4] concerning Aubry-Mather's theory, and [15], where the theory of vector fields on infinite dimensional manifolds is applied to (1.2). We refer to [4, 6, 7, 19] for a survey and additional references.

From the example of the horocycle flow below, closed magnetic geodesics need not exist on a fixed energy level in general. However, from [6, 17, 18], there are always closed magnetic geodesics in high and low energy levels, i.e. in E_c with $c \geq c_0$ and $c \leq (c_0)^{-1}$, where $c_0 > 0$ depends on (M, g) and k (in case of a flat torus and high energy levels k is assumed not to vanish). If the magnetic form is exact, i.e. $[kdV_g] = 0$ in $H_{dR}^2(M)$, then there is a periodic magnetic geodesic in every energy level (see [4]). Concerning non exact magnetic forms positive functions k are of special interest, since the magnetic form is symplectic in this case. For $k > 0$ a closed magnetic geodesic exists in every energy level, if (M, g) is a flat torus [2, 10] or if (M, g) is a sphere S^2 with nonnegative curvature [16]. The (essentially) only nonexistence result for closed magnetic geodesics is based on an old result of Hedlund [8].

Example (Horocycle flow [6]). *Let (M, g) be a compact hyperbolic surface of constant curvature $K_g \equiv -1$ and $k \equiv 1$.*

- (1) *If $0 < c < 1$, then E_c contains a contractible closed magnetic geodesic.*
- (2) *There are no closed magnetic geodesics in E_1 .*
- (3) *If $c > 1$, there are no contractible closed magnetic geodesics in E_c , but any non trivial free homotopy class of closed curves can be represented by one.*

The existence question for closed magnetic geodesics on hyperbolic surfaces for non constant functions k is poorly understood. We shall show: If (M, g) is a compact hyperbolic surface with Gaussian curvature $K_g \geq -1$ and $k \geq 1$ a positive function, then there is a contractible closed magnetic geodesics in E_c for all $0 < c < 1$. The example of the horocycle flow shows that this existence result is sharp.

We consider curves, that are Alexandrov embedded.

Definition 1.1. *(oriented Alexandrov embedded) Let $B \subset \mathbb{R}^2$ denote the open ball of radius 1 centered at $0 \in \mathbb{R}^2$. An immersion $\gamma \in C^1(\partial B, M)$ will be called oriented Alexandrov embedded, if there is an immersion $F \in C^1(\bar{B}, M)$, such that $F|_{\partial B} = \gamma$ and F is orientation preserving in the sense that*

$$\langle DF|_x x, J_g(\gamma(x))\dot{\gamma}(x) \rangle_{T_{\gamma(x)}S^2, g} > 0$$

for all $x \in \partial B$.

We shall prove

Theorem 1.2. *Let (M, g) be a smooth, compact, orientable surface with negative Euler characteristic and $k \in C^\infty(M)$ a positive function. Assume there is $K_0 > 0$ such that k and the Gaussian curvature K_g of (M, g) satisfy*

$$k > (K_0)^{\frac{1}{2}} \text{ and } K_g \geq -K_0.$$

Then there is an oriented Alexandrov embedded curve $\gamma \in C^2(S^1, M)$ that solves (1.2) and the number of such solutions is at least $-\chi(M)$ provided they are all nondegenerate.

The equivalence between (1.1) and (1.2) leads to

Corollary 1.3. *Let (M, g) be a smooth, compact, orientable surface with negative Euler characteristic and $k \in C^\infty(M)$ a positive function. Assume there is $K_0 > 0$ such that k and the Gaussian curvature K_g of (M, g) satisfy*

$$k \geq (K_0)^{\frac{1}{2}} \text{ and } K_g \geq -K_0.$$

Then every energy level E_c with $0 < c < 1$ contains an oriented Alexandrov embedded closed magnetic geodesic and the number of such closed magnetic geodesics in E_c is at least $-\chi(M)$ provided they are all nondegenerate.

The proof of our existence results is organized as follows. We consider solutions to (1.2) as zeros of the vector field $X_{k,g}$ defined on the Sobolev space $H^{2,2}(S^1, M)$: For $\gamma \in H^{2,2}(S^1, M)$ we let $X_{k,g}(\gamma)$ be the unique weak solution of

$$(-D_{t,g}^2 + 1)X_{k,g}(\gamma) = -D_{t,g}\dot{\gamma} + |\dot{\gamma}|_g k(\gamma) J_g(\gamma) \dot{\gamma} \quad (1.3)$$

in $T_\gamma H^{2,2}(S^1, M)$. The uniqueness implies that any zero of $X_{k,g}$ is a weak solution of (1.2) which is a classical solution in $C^2(S^1, M)$ applying standard regularity theory.

After setting up notation in Section 2 and introducing the classes of maps and spaces needed for our analysis we recall in Section 3 the definition and properties of the S^1 -equivariant Poincaré-Hopf index defined in [15],

$$\chi_{S^1}(X_{k,g}, \mathcal{M}_A) \in \mathbb{Z},$$

where \mathcal{M}_A is the set of oriented Alexandrov embedded regular curves in $H^{2,2}(S^1, M)$.

From the uniformization theorem (M, g) is isometric to $(\mathbb{H}/\Gamma, e^\varphi g_0)$, where Γ is a group of isometries of the standard hyperbolic plane (\mathbb{H}, g_0) acting freely and properly discontinuously and φ is a function in $C^\infty(\mathbb{H}/\Gamma, \mathbb{R})$. Since the problem of prescribing geodesic curvature is invariant under isometries we may assume without loss of generality that

$$(M, g) = (\mathbb{H}/\Gamma, e^\varphi g_0).$$

In Section 4 we analyze the *unperturbed problem* with $k \equiv k_0 > 0$ and $g = g_0$: We compute the set of oriented Alexandrov embedded zeros of X_{k_0, g_0} and the image and kernel of the corresponding linearizations. The perturbative

analysis in Section 5, which carries over from [15], is used to compute the degree of the unperturbed problem in Section 6: For large positive constants k_0 and the standard metric g_0 we shall show that

$$\chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) = -\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M .

Section 7 contains the apriori estimate which implies that under the assumptions of Theorem 1.2 the set of solutions to (1.2) is compact in \mathcal{M}_A . The homotopy invariance of the S^1 -equivariant Poincaré-Hopf index then leads to the identity

$$\chi_{S^1}(X_{k, g}, \mathcal{M}_A) = \chi_{S^1}(X_{k_0, g_{can}}, \mathcal{M}_A) = -\chi(M).$$

The resulting proof of Theorem 1.2 is given in Section 8.

2. PRELIMINARIES

It is convenient for the functional analytic setting to assume that M is embedded in some \mathbb{R}^{q_M} . We consider for $m \in \mathbb{N}_0$ the set of Sobolev functions

$$H^{m,2}(S^1, M) := \{\gamma \in H^{m,2}(S^1, \mathbb{R}^{q_M}) : \gamma(t) \in M \text{ for a.e. } t \in S^1.\}$$

For $m \geq 1$ the set $H^{m,2}(S^1, M)$ is a sub-manifold of the Hilbert space $H^{m,2}(S^1, \mathbb{R}^{q_M})$ and is contained in $C^{m-1}(S^1, \mathbb{R}^{q_M})$. Hence, if $m \geq 1$ then $\gamma \in H^{m,2}(S^1, M)$ satisfies $\gamma(t) \in M$ for all $t \in S^1$. In this case the tangent space at $\gamma \in H^{m,2}(S^1, M)$ is given by

$$T_\gamma H^{m,2}(S^1, M) := \{V \in H^{m,2}(S^1, \mathbb{R}^{q_M}) : V(t) \in T_{\gamma(t)}M \text{ for all } t \in S^1\}.$$

For $m = 0$ the set $H^{0,2}(S^1, M) = L^2(S^1, M)$ fails to be a manifold. We define for $\gamma \in H^{1,2}(S^1, M)$ the space $T_\gamma L^2(S^1, M)$ by

$$T_\gamma L^2(S^1, M) := \{V \in L^2(S^1, \mathbb{R}^{q_M}) : V(t) \in T_{\gamma(t)}M \text{ for a.e. } t \in S^1\}.$$

A metric g on M induces a metric on $H^{m,2}(S^1, M)$ for $m \geq 1$ by setting for $\gamma \in H^{m,2}(S^1, M)$ and $V, W \in T_\gamma H^{m,2}(S^1, S^2)$

$$\begin{aligned} \langle W, V \rangle_{T_\gamma H^{m,2}(S^1, S^2), g} := & \int_{S^1} \left\langle \left((-1)^{\lfloor \frac{m}{2} \rfloor} (D_{t,g})^m + 1 \right) V(t), \right. \\ & \left. \left((-1)^{\lfloor \frac{m}{2} \rfloor} (D_{t,g})^m + 1 \right) W(t) \right\rangle_{\gamma(t), g} dt, \end{aligned}$$

where $\lfloor m/2 \rfloor$ denotes the largest integer that does not exceed $m/2$.

Since g and k are smooth, $X_{k,g}$ is a smooth vector field (see [15, 20, Sec. 6]) on the set $H_{reg}^{2,2}(S^1, M)$ of regular curves,

$$H_{reg}^{2,2}(S^1, M) := \{\gamma \in H^{2,2}(S^1, M) : \dot{\gamma}(t) \neq 0 \text{ for all } t \in S^1\}.$$

From [15] there holds

$$\begin{aligned} & (-D_{t,g}^2 + 1)D_g X_{k,g}|_\gamma(V) \\ &= -D_{t,g}^2 V - R_g(V, \dot{\gamma})\dot{\gamma} + |\dot{\gamma}|_g^{-1} \langle D_{t,g} V, \dot{\gamma} \rangle_g k(\gamma) J_g(\gamma) \dot{\gamma} \\ & \quad + |\dot{\gamma}|_g (k'(\gamma)V) J_g(\gamma) \dot{\gamma} + |\dot{\gamma}|_g k(\gamma) \left((D_g J_g|_\gamma V) \dot{\gamma} + J_g(\gamma) D_{t,g} V \right). \end{aligned} \quad (2.1)$$

We note that (see also [21, Thm. 6.1])

$$(-D_{t,g}^2 + 1)D_g X_{k,g}|_\gamma(V) = (-D_{t,g}^2 + 1)V + T(V),$$

where T is a linear map from $T_\gamma H^{2,2}(S^1, M)$ to $T_\gamma L^2(S^1, M)$ that depends only on the first derivatives of V and is therefore compact. Taking the inverse $(-D_{t,g}^2 + 1)^{-1}$ we deduce that $D_g X_{k,g}|_\gamma$ is the form *identity + compact* and thus a Rothe map (see [15]).

The vector field $X_{k,g}$ as well as the set of solutions to (1.2) is invariant under a circle action: For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $\gamma \in H^{2,2}(S^1, M)$ we define $\theta * \gamma \in H^{2,2}(S^1, M)$ by

$$\theta * \gamma(t) = \gamma(t + \theta).$$

Moreover, for $V \in T_\gamma H^{2,2}(S^1, M)$ we let

$$\theta * V := V(\cdot + \theta) \in T_{\theta * \gamma} H^{2,2}(S^1, M).$$

Then $X_{k,g}(\theta * \gamma) = \theta * X_{k,g}(\gamma)$ for any $\gamma \in H^{2,2}(S^1, M)$ and $\theta \in S^1$. Thus, any zero gives rise to a S^1 -orbit of zeros. We call γ a *prime* curve, if the isotropy group $\{\theta \in S^1 : \theta * \gamma = \gamma\}$ of γ is trivial.

For $m \geq 1$ the exponential map $Exp_g : TH^{m,2}(S^1, M) \rightarrow H^{m,2}(S^1, M)$ is defined for $\gamma \in H^{m,2}(S^1, M)$ and $V \in T_\gamma H^{m,2}(S^1, M)$ by

$$Exp_{\gamma,g}(V)(t) := Exp_{\gamma(t),g}(V(t)),$$

where $Exp_{z,g}$ denotes the exponential map on (M, g) at $z \in M$. Due to its pointwise definition

$$\theta * Exp_{\gamma,g}(V)(t) = Exp_{\theta * \gamma,g}(\theta * V)(t).$$

We shall find solutions to (1.2) in the class of oriented Alexandrov embedded curves. Let $\gamma \in H^{2,2}(S^1, M)$ be an oriented Alexandrov embedded curves with corresponding oriented immersion F from B to M . If we equip B with the metric F^*g induced by F , then the outer normal $N_B(x)$ at $x \in \partial B$ with respect to F^*g satisfies

$$DF|_x N_B(x) = N_\gamma(x)$$

where $N_\gamma(x)$ denotes the normal to the curve γ at $x \in \partial B$ defined by

$$N_\gamma(x) := |\dot{\gamma}(x)|^{-1} J_g(\gamma(x)) \dot{\gamma}(x).$$

In [16] the following two basic properties of oriented Alexandrov embedded curves are shown.

Lemma 2.1.

- (1) Let (γ_n) in $C^2(\partial B, M)$ be a sequence of immersions, which are oriented Alexandrov embedded, such that (γ_n) converges to an immersion γ_0 in $C^2(\partial B, M)$ with strictly positive geodesic curvature. Then γ_0 is oriented Alexandrov embedded.
- (2) The set of regular, oriented Alexandrov embedded curves is open in $H^{2,2}(S^1, M)$.

Property (1) and (2) are given in [16] for closed curves in S^2 . Since the analysis in the proof of (1) and (2) is done in tubular neighborhoods of closed curves, properties (1) and (2) continue to hold if S^2 is replaced by a general surface M .

3. THE S^1 -POINCARÉ-HOPF INDEX

In [15] a S^1 -equivariant Poincaré-Hopf index or S^1 -degree is introduced for equivariant vector fields on subsets of $H^{2,2}(S^1, S^2)$. The S^1 -degree is based on an equivariant version of the Sard-Smale lemma [15, Lem 3.9], which depends on an appropriate change of a vector field locally around its critical orbits. It's merely a matter of form to extend this local argument, when S^2 is replaced by a general surface M . We give a short account of the definition and properties of the S^1 -degree for equivariant vector fields on subsets of $H^{2,2}(S^1, M)$.

We define a C^2 equivariant vector field W_g on $H^{2,2}(S^1, M)$ by

$$W_g(\gamma) = (-(D_{t,g})^2 + 1)^{-1}\dot{\gamma}, \text{ for } \gamma \in H^{2,2}(S^1, M).$$

We will compute the S^1 -Poincaré-Hopf index for the following class of vector fields.

Definition 3.1. Let \mathcal{M} be an open S^1 -invariant subset of prime curves in $H^{2,2}(S^1, M)$. A C^2 vector field X on \mathcal{M} is called (\mathcal{M}, g, S^1) -admissible, if

- (1) X is S^1 -equivariant, i.e. $X(\theta * \gamma) = \theta * X(\gamma)$ for all $(\theta, \gamma) \in S^1 \times \mathcal{M}$.
- (2) X is proper in \mathcal{M} , i.e. the set $\{\gamma \in \mathcal{M} : X(\gamma) = 0\}$ is compact,
- (3) X is orthogonal to W_g , i.e. $\langle X(\gamma), W_g(\gamma) \rangle_{T_\gamma H^{2,2}(S^1, M)} = 0$ for all $\gamma \in \mathcal{M}$.
- (4) X is a Rothe field, i.e. if $X(S^1 * \gamma) = 0$ then $D_g X|_\gamma$ and $\text{Proj}_{\langle W_g(\gamma) \rangle^\perp} \circ D_g X|_\gamma$ are Rothe maps in $\mathcal{L}(T_\gamma H^{2,2}(S^1, M))$ and $\mathcal{L}(\langle W_g(\gamma) \rangle^\perp)$, respectively.
- (5) X is elliptic, i.e. there is $\varepsilon > 0$ such that for all finite sets of charts

$$\{(Exp_{\gamma_i, g}, B_{2\delta_i}(0)) : \gamma_i \in H^{4,2}(S^1, M) \text{ for } 1 \leq i \leq n\},$$

and finite sets

$$\{W_i \in T_{\gamma_i} H^{4,2}(S^1, M) : \|W_i\|_{T_{\gamma_i} H^{4,2}(S^1, M)} < \varepsilon \text{ for } 1 \leq i \leq n\},$$

there holds: If $\alpha \in \bigcap_{i=1}^n \text{Exp}_{\gamma_i, g}(B_{\delta_i}(0)) \subset H^{2,2}(S^1, M)$ satisfies

$$X(\alpha) = \sum_{i=1}^n \text{Proj}_{\langle W_g(\alpha) \rangle^\perp} \circ D\text{Exp}_{\gamma_i, g}|_{\text{Exp}_{\gamma_i, g}^{-1}(\alpha)}(W_i)$$

then α is in $H^{4,2}(S^1, M)$.

It is shown in [15] that $X_{k,g}$ satisfies properties (3) – (4). Hence, $X_{k,g}$ is (\mathcal{M}, g, S^1) -admissible if and only if $X_{k,g}$ is proper in \mathcal{M} . Note that the regularity property (5), taking $W_i = 0$, shows that any zero of X belongs to $H^{4,2}(S^1, M)$. Furthermore, for $\gamma \in H^{4,2}(S^1, M)$ the map $\theta \mapsto \theta * \gamma$ is C^2 from S^1 to $H^{2,2}(S^1, M)$. Hence, if $X(\gamma) = 0$ then

$$0 = D_\theta(X(\theta * \gamma))|_{\theta=0} = D_g X|_\gamma(\dot{\gamma}),$$

such that the kernel of $D_g X|_\gamma$ is nontrivial. If X is a vector field orthogonal to W_g and $X(\gamma) = 0$, then

$$0 = D(\langle X(\alpha), W_g(\alpha) \rangle_{T_\alpha H^{2,2}(S^1, M), g})|_\gamma = \langle D_g X|_\gamma, W_g(\gamma) \rangle_{T_\gamma H^{2,2}(S^1, M), g}$$

where the various curvature terms and terms containing derivatives of W_g vanish as $X(\gamma) = 0$. Thus, $X(\gamma) = 0$ implies

$$D_g X|_\gamma : T_\gamma H^{2,2}(S^1, M) \rightarrow \langle W_g(\gamma) \rangle^\perp, \quad (3.1)$$

and the projection $\text{Proj}_{\langle W_g(\gamma) \rangle^\perp}$ in (4) is unnecessary.

Definition 3.2. Let \mathcal{M} be an open S^1 -invariant subset of prime curves in $H^{2,2}(S^1, M)$, $S^1 * \gamma \subset \mathcal{M}$, and X a (\mathcal{M}, g, S^1) -admissible vector field on \mathcal{M} . The orbit $S^1 * \gamma$ is called a critical orbit of X , if $X(\gamma) = 0$. The orbit $S^1 * \gamma$ is called a nondegenerate critical orbit of X , if $X(\gamma) = 0$ and

$$D_g X|_\gamma : \langle W_g(\gamma) \rangle^\perp \rightarrow \langle W_g(\gamma) \rangle^\perp$$

is an isomorphism.

Note that if $\gamma \in H^{4,2}(S^1, M) \subset H^{2,2}(S^1, M)$ then $\dot{\gamma} \notin \langle W_g(\gamma) \rangle^\perp$.

Definition 3.3. Let $\{g_t : t \in [0, 1]\}$ be a family of smooth metrics on M , which induces a corresponding family of metrics on $H^{2,2}(S^1, M)$, still denoted by g_t . Let \mathcal{M} be an open S^1 -invariant subset of prime curves in $H^{2,2}(S^1, M)$ and X_0, X_1 two vector-fields on \mathcal{M} such that X_i is (\mathcal{M}, g_i, S^1) -admissible for $i = 0, 1$. A C^2 family of vector-fields $X(t, \cdot)$ on \mathcal{M} for $t \in [0, 1]$ is called a (\mathcal{M}, g_t, S^1) -homotopy between X_0 and X_1 , if

- $X(0, \cdot) = X_0$ and $X(1, \cdot) = X_1$,
- $\{(t, \gamma) \in [0, 1] \times \mathcal{M} : X(t, \gamma) = 0\}$ is compact,
- $X_t := X(t, \cdot)$ is (\mathcal{M}, g_t, S^1) -admissible for all $t \in [0, 1]$.

We write (\mathcal{M}, g, S^1) -homotopy, if the family of metrics $\{g_t\}$ is constant.

Note that, if $\{k_t \in C^\infty(M, \mathbb{R}) : t \in [0, 1]\}$ is a C^2 family of smooth function, then $t \mapsto X_{k_t, g_t}$ is a (\mathcal{M}, g_t, S^1) -homotopy, if and only if the set

$$\{(t, \gamma) \in [0, 1] \times \mathcal{M} : X_{k_t, g_t}(\gamma) = 0\}$$

is compact.

We let \mathcal{M} be an open S^1 -invariant subset of prime curves in $H^{2,2}(S^1, M)$ and X a (\mathcal{M}, g, S^1) -admissible vector field on \mathcal{M} . The local S^1 -degree of an isolated, nondegenerate critical orbit $S^1 * \gamma_0$ is defined by

$$\deg_{loc, S^1}(X, S^1 * \gamma_0) := \text{sgn} D_g X|_{\gamma_0},$$

where $\text{sgn} D_g X|_{\gamma_0}$ is the sign of the Rothe map $D_g X|_{\gamma_0}$ in $\mathcal{L}(\langle W_g(\gamma) \rangle^\perp)$. Since $D_g X_{k, g}|_{\gamma_0}$ is of the form *identity* + *compact*, in the above situation $\text{sgn} D_g X_{k, g}|_{\gamma_0}$ is given by the usual Leray-Schauder degree.

Using an equivariant version of the Sard-Smale lemma a S^1 -equivariant Poincaré-Hopf index

$$\chi(X, \mathcal{M}) \in \mathbb{Z}$$

is defined in [15] with the following properties.

Lemma 3.4.

- (1) *If X is (\mathcal{M}, g, S^1) -admissible with only finitely many critical orbits, that are all nondegenerate, then*

$$\chi_{S^1}(X, \mathcal{M}) := \sum_{\{S^1 * \gamma \subset \mathcal{M} : X(S^1 * \gamma) = 0\}} \deg_{loc, S^1}(X, S^1 * \gamma).$$

- (2) *If X_0 and X_1 are (\mathcal{M}, g_t, S^1) -homotop, then $\chi(X_0, \mathcal{M}) = \chi(X_1, \mathcal{M})$.*

4. THE UNPERTURBED PROBLEM

Let $\mathbb{H} \subset \mathbb{R}^3$ be the standard hyperbolic plane

$$\mathbb{H} := \{(\xi^1, \xi^2, \tau) \in \mathbb{R}^3 : \tau^2 - |\xi|^2 = 1 \text{ and } \tau > 0\}$$

with metric g_0 induced by the Minkowski metric g_m ,

$$g_m := (d\xi^1)^2 + (d\xi^2)^2 - (d\tau)^2 = \langle \cdot, \cdot \rangle_m.$$

We choose the orientation on \mathbb{H} such that $J_{g_0}(y)$ is given for $y \in \mathbb{H}$ by

$$J_{g_0}(y)(v) := y \times_m v \text{ for all } v \in T_y \mathbb{H},$$

where \times_m denotes the twisted cross product in \mathbb{R}^3 ,

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \times_m \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} := \begin{pmatrix} v^3 w^2 - v^2 w^3 \\ v^1 w^3 - v^3 w^1 \\ v^1 w^2 - v^2 w^1 \end{pmatrix}.$$

The twisted cross product \times_m is related to the usual cross product \times in \mathbb{R}^3 by $v \times_m w = I_{2,1}v \times I_{2,1}w$, where $I_{2,1}$ is given by

$$I_{2,1} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and satisfies for $a, b, c, d \in \mathbb{R}^3$

$$\begin{aligned} \langle (a \times_m b), a \rangle_m &= 0 = \langle (a \times_m b), b \rangle_m, \\ a \times_m (b \times_m c) &= -b \langle a, c \rangle_m + c \langle a, b \rangle_m, \\ \langle (a \times_m b), (c \times_m d) \rangle_m &= -\langle a, c \rangle_m \langle b, d \rangle_m + \langle b, c \rangle_m \langle a, d \rangle_m \end{aligned}$$

We fix a compact, orientable Riemannian surface (M, g_0) ,

$$M := \mathbb{H}/\Gamma,$$

where $\Gamma \subset SO(2,1)_+$ is a group of oriented isometries acting freely and properly discontinuously on \mathbb{H} . Concerning the metric we will be sloppy and denote by g_0 the metric on \mathbb{H} as well as the induced metric on \mathbb{H}/Γ . The unperturbed problem on M is given by

$$D_{t,g_0} \dot{\gamma} = |\dot{\gamma}|_{g_0} k_0 J_{g_0}(\gamma) \dot{\gamma}, \quad (4.1)$$

where k_0 is a positive constant.

We shall compute the S^1 -degree of the unperturbed equation (4.1) in three steps. *Step 1:* We compute explicitly the set \mathcal{Z}_M of Alexandrov embedded solutions in $H^{2,2}(S^1, M)$ to (4.1) and show that \mathcal{Z}_M is a finite dimensional, nondegenerate manifold, in the sense that we have for all $\tilde{\alpha} \in \mathcal{Z}_M$

$$\begin{aligned} T_{\tilde{\alpha}} \mathcal{Z}_M &= \text{kernel}(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}}), \\ T_{\tilde{\alpha}} H^{2,2}(S^1, M) &= T_{\tilde{\alpha}} \mathcal{Z}_M \oplus R(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}}). \end{aligned}$$

Step 2: In Section 5 we perform a finite dimensional reduction of a slightly perturbed problem: We consider for $k_1 \in C^\infty(M, \mathbb{R})$, which will be chosen later, and $\varepsilon \in \mathbb{R}$, which is assumed to be very small, the perturbed vector field $X_{g_0, \varepsilon}$ defined by

$$\begin{aligned} X_{g_0, \varepsilon}(\gamma) &:= (-D_{t, g_0}^2 + 1)^{-1} (-D_{t, g_0} \dot{\gamma} + |\dot{\gamma}|_{g_0} (k_0 + \varepsilon k_1(\gamma)) J_{g_0}(\gamma) \dot{\gamma}) \\ &= X_{k_0, g_0}(\gamma) + \varepsilon K_1(\gamma), \end{aligned}$$

where the vector field K_1 is given by

$$K_1(\gamma) := (-D_{t, g_0}^2 + 1)^{-1} |\dot{\gamma}|_{g_0} (k_1(\gamma) J_{g_0}(\gamma) \dot{\gamma}).$$

We show that if $S^1 * \tilde{\alpha}_0 \subset \mathcal{Z}_M$ is a nondegenerate critical orbit of the vector field $\tilde{\alpha} \mapsto P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha})$ on \mathcal{Z}_M , where $P_1(\tilde{\alpha})$ is a projection onto $T_{\tilde{\alpha}} \mathcal{Z}_M$ defined below, then for any $0 < \varepsilon \ll 1$ there is a unique nondegenerate critical orbit $S^1 * \tilde{\gamma}(\varepsilon)$ of $X_{g_0, \varepsilon}$ such that $\tilde{\gamma}(\varepsilon)$ converges to $\tilde{\alpha}_0$ as $\varepsilon \rightarrow 0^+$ and

$$\text{deg}_{loc, S^1}(X_{g_0, \varepsilon}, S^1 * \tilde{\gamma}(\varepsilon)) = -\text{deg}_{loc}(P_1(\cdot) \circ K_1(\cdot), S^1 * \tilde{\alpha}_0).$$

Step 3: In Section 6 we choose a Morse function $k_1 \in C^\infty(M, \mathbb{R})$ with critical points

$$\{\tilde{w}_i \in M : 1 \leq i \leq n\}.$$

We show that if $k_0 \gg 1$ is large, then $P_1(\cdot) \circ K_1(\cdot)$ has exactly n critical orbits $\{S^1 * \tilde{\alpha}_{i,k_0} : 1 \leq i \leq n\}$ such that for $1 \leq i \leq n$

$$\deg_{loc}(P_1(\cdot) \circ K_1(\cdot), S^1 * \tilde{\alpha}_{i,k_0}) = \deg_{loc}(\nabla k_1, \tilde{w}_i).$$

This yields the formula $\chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) = -\chi(M)$, where \mathcal{M}_A is the subset of $H^{2,2}(S^1, M)$ consisting of Alexandrov embedded, regular curves.

Step 1: The prescribed geodesic curvature equation with $k \equiv k_0$ on (\mathbb{H}, g_0) is given by

$$Proj_{\gamma^\perp, g_m} \ddot{\gamma} = |\dot{\gamma}|_m k_0 \gamma \times_m \dot{\gamma}, \quad (4.2)$$

where $\gamma \in H^{2,2}(S^1, \mathbb{H})$, $\dot{\gamma}$ and $\ddot{\gamma}$ are the usual derivatives of γ considered as a curve in \mathbb{R}^3 , $|\dot{\gamma}|_m$ is the Minkowski norm of $\dot{\gamma}$ in (\mathbb{R}^3, g_m) .

If $k_0 > 1$ then there is a unique $r = r(k_0) > 0$ such that

$$k_0 = \frac{\sqrt{1+r^2}}{r}.$$

We call a triple of vectors $\{v_0, v_1, w\}$ in \mathbb{R}^3 a positive oriented orthonormal system with respect to g_m , if

$$\begin{aligned} \langle v_0, v_1 \rangle_m &= \langle v_0, w \rangle_m = \langle v_1, w \rangle_m = 0, \\ \langle v_0, v_0 \rangle_m &= \langle v_1, v_1 \rangle_m = -\langle w, w \rangle_m = 1, \\ v_0 \times_m v_1 &= w. \end{aligned}$$

We define for $\lambda > 0$ and a positive oriented orthonormal system $\{v_0, v_1, w\}$ the function $\alpha \in C^\infty(\mathbb{R}, \mathbb{H})$ by

$$\alpha(t, \lambda, v_0, v_1, w) := \sqrt{1+r^2}w + r \cos(\lambda r^{-1}t)v_1 + r \sin(\lambda r^{-1}t)v_0 \quad (4.3)$$

A direct calculation shows that $\alpha(\cdot, \lambda, v_0, v_1, w)$ solves (4.2). We fix $(\gamma_0, \tilde{v}_0) \in T\mathbb{H}$ with $\tilde{v}_0 \neq 0$ and define the parameter $\lambda := |\tilde{v}_0|_m$ and the positive oriented orthonormal system (v_0, v_1, w) by

$$v_0 := \lambda^{-1}\tilde{v}_0, \quad v_1 := -r\gamma_0 - \sqrt{1+r^2}(v_0 \times_m \gamma_0), \quad w := v_0 \times_m v_1.$$

Then $\alpha(\cdot, \lambda, v_0, v_1, w)$ satisfies the initial conditions

$$\alpha(0, \lambda, v_0, v_1, w) = \gamma_0, \quad \dot{\alpha}(0, \lambda, v_0, v_1, w) = \tilde{v}_0,$$

and we deduce that all non constant solutions of (4.2) are obtained in this way. Since we are only interested in solutions in $H^{2,2}(S^1, \mathbb{H})$ we get an extra condition on λ , i.e. the 1-periodicity leads to

$$\lambda \in 2\pi\mathbb{N}r.$$

Lemma 4.1. *The oriented Alexandrov embedded solutions in $H^{2,2}(S^1, \mathbb{H})$ of equation (4.2) are given by the set of simple solutions*

$$\mathcal{Z}_{\mathbb{H}} := \left\{ \alpha(\cdot, 2\pi r, v_0, v_1, w) : \right. \\ \left. \{v_0, v_1, w\} \text{ is a pos. orth. system in } (\mathbb{R}^3, g_m) \right\}.$$

Proof. From the analysis above the periodic solutions to (4.2) are given by

$$\left\{ \alpha(\cdot, 2\pi nr, v_0, v_1, w) : n \in \mathbb{N} \text{ and} \right. \\ \left. \{v_0, v_1, w\} \text{ is a pos. orth. system in } (\mathbb{R}^3, g_m) \right\}.$$

We fix $n \in \mathbb{N}$ and a positive orthonormal system $\{v_0, v_1, w\}$ and write

$$\gamma_n := \alpha(\cdot, 2\pi nr, v_0, v_1, w).$$

Assume γ_n is oriented Alexandrov embedded and let F_n be the corresponding immersion. Since \mathbb{H} is diffeomorphic to \mathbb{R}^2 we may assume that γ_1 is a simple curve in the plane (\mathbb{R}^2, δ) with standard metric δ . If we apply the Gauß-Bonnet formula to $(B, F_n^* \delta)$ and the embedded curve γ_1 in the plane, we obtain

$$\begin{aligned} 2\pi &= \int_{\partial B} k_{F_n^* \delta} dS_{F_n^* \delta} + \int_B K_{F_n^* \delta} dA_{F_n^* \delta} \\ &= \int_{\gamma_n} k_{\delta} dS_{\delta} = n \int_{\gamma_1} k_{\delta} dS_{\delta} = n2\pi, \end{aligned}$$

which is only possible for $n = 1$.

The curve γ_1 is oriented Alexandrov embedded using polar coordinates and

$$[0, 2\pi] \times [0, 1] \ni (t, s) \mapsto \sqrt{1 + s^2 r^2} w + sr \cos(t) v_1 + sr \sin(t) v_0.$$

□

The Lorentz transformations $SO(2, 1)_+$ of (\mathbb{R}^3, g_m) ,

$$SO(2, 1)_+ := \{A \in O(2, 1) : A(\mathbb{H}) \subset \mathbb{H} \text{ and } \det A = 1\},$$

correspond to the oriented isometries of (\mathbb{H}, g_0) and act on solutions: if γ solves (4.2) so does $A \circ \gamma$ for any $A \in SO(2, 1)_+$. We have

$$A \circ \alpha(\cdot, \lambda, v_0, v_1, w) = \alpha(\cdot, \lambda, A(v_0), A(v_1), A(w)).$$

Moreover, there holds,

$$\alpha(\cdot, 2\pi r, v_0, v_1, w) = \theta * \alpha(\cdot, 2\pi r, v'_0, v'_1, w') \quad (4.4)$$

for some $\theta \in S^1$ if and only if $w = w'$. Consequently, the critical orbits of (4.2) in \mathbb{H} , $\{S^1 * \gamma : \gamma \in \mathcal{Z}_{\mathbb{H}}\}$, are parametrized by $w \in \mathbb{H}$ and correspond to “circles” with radius r around the center w in \mathbb{H} .

We let π_M be the natural projection, $\pi_M : \mathbb{H} \rightarrow \mathbb{H}/\Gamma$. Any point $z \in \mathbb{H}$ admits a neighborhood $U = B_{\delta}(z)$ such that $\pi_M|_U : U \rightarrow \pi_M(U)$ is an isometry. From (4.3) there is $C_{k_0} > 1$ such that if $k_0 \geq C_{k_0}$ then any solution to (4.2) on \mathbb{H} passing through z remains in U . For M is compact $C_{k_0} = C_{k_0}(\Gamma)$ and $\delta > 0$ may be chosen independently of z . Equation (1.2) is

invariant under isometries, hence the set of solutions to (4.1) with $k_0 \geq C_{k_0}$ is given by

$$\begin{aligned} & \{ \pi_M \circ \alpha(\cdot, 2\pi r, v_0, v_1, w) : \\ & \{v_0, v_1, w\} \text{ is a pos. orth. system in } (\mathbb{R}^3, g_m) \}. \end{aligned}$$

Moreover, we have

Lemma 4.2. *If $k_0 \geq C_{k_0}$, then the oriented Alexandrov embedded solutions in $H^{2,2}(S^1, M)$ of equation (4.1) are given by the set of simple solutions*

$$\begin{aligned} \mathcal{Z}_M := & \{ \tilde{\alpha} = \pi_M \circ \alpha(\cdot, 2\pi r, v_0, v_1, w) : \\ & \{v_0, v_1, w\} \text{ is a pos. orth. system in } (\mathbb{R}^3, g_m) \}. \end{aligned}$$

Proof. We fix $n \in \mathbb{N}$ and a positive orthonormal system $\{v_0, v_1, w\}$ and write

$$\gamma_n := \pi_M \circ \alpha(\cdot, 2\pi nr, v_0, v_1, w)$$

From the above analysis any periodic solution to (4.1) on (M, g_0) is of this form. Hence, it is enough to show that γ_n is oriented Alexandrov embedded, if and only if $n = 1$.

Concatenating the immersion in the proof of Lemma 4.1 with π_M we deduce that γ_1 is oriented Alexandrov embedded. Suppose γ_n is oriented Alexandrov embedded with an immersion $F_n : B \rightarrow M$. From the homotopy lifting property of the covering $\pi_M : \mathbb{H} \rightarrow M$ we may lift F_n to see that $\alpha(\cdot, 2\pi nr, v_0, v_1, w)$ is oriented Alexandrov embedded in \mathbb{H} . From Lemma 4.1 this is only possible for $n = 1$. \square

From (4.4) we find

$$\pi_M \circ \alpha(\cdot, 2\pi r, v_0, v_1, w) = \theta * \pi_M \circ \alpha(\cdot, 2\pi r, v'_0, v'_1, w')$$

for some $\theta \in S^1$ if and only if $\pi_M(w) = \pi_M(w')$, such that the critical orbits of (4.2) in M are parametrized by $w \in M$ and correspond to projections on M of “circles” in \mathbb{H} .

In the following we always assume that

$$k_0 \geq C_{k_0}.$$

We denote by $X_{k_0, g_0, \mathbb{H}}$ the vector field on $H^{2,2}(S^1, \mathbb{H})$ corresponding to equation (4.2). We fix a solution $\alpha = \alpha(\cdot, 2\pi r, v_0, v_1, w)$ of (4.2) and note that for $V \in T_\alpha H^{2,2}(S^1, \mathbb{H})$

$$R_{g_0}(V, \dot{\alpha})\dot{\alpha} = -V|\dot{\alpha}|_m^2 + \langle V, \dot{\alpha} \rangle_m \dot{\alpha}.$$

By (2.1) a vector field W is contained in the kernel of $D_{g_0} X_{k_0, g_0, \mathbb{H}}|_\alpha$ if and only if W is a periodic solution of

$$\begin{aligned} 0 = & -D_{t, g_0}^2 W + W|\dot{\alpha}|_m^2 - \langle W, \dot{\alpha} \rangle_m \dot{\alpha} \\ & + |\dot{\alpha}|_m^{-1} \langle D_{t, g_0} W, \dot{\alpha} \rangle_m k_0 (\alpha \times_m \dot{\alpha}) + |\dot{\alpha}|_m k_0 (\alpha \times_m D_{t, g_0} W). \end{aligned} \quad (4.5)$$

Due to the geometric origin of equation (4.2) and the $SO(2, 1)_+$ invariance we find that

$$\begin{aligned} W_0(t, v_0, v_1, w) &:= t\dot{\alpha}, \\ W_1(t, v_0, v_1, w) &:= \dot{\alpha} = 2\pi r(-\sin(2\pi t)v_1 + \cos(2\pi t)v_0), \\ W_2(t, v_0, v_1, w) &:= (1 + r^2)^{\frac{1}{2}}v_1 + r\cos(2\pi t)w, \\ W_3(t, v_0, v_1, w) &:= (1 + r^2)^{\frac{1}{2}}v_0 + r\sin(2\pi t)w, \end{aligned} \quad (4.6)$$

solve (4.5). In the sequel, we will omit the dependence of W_i on (v_0, v_1, w) , if there is no possibility of confusion. The initial values of W_0, \dots, W_3

$$\begin{aligned} W_0(0, v_0, v_1, w) &= 0, \quad D_{t, g_0}W_0(0, v_0, v_1, w) = 2\pi r v_0, \\ W_1(0, v_0, v_1, w) &= 2\pi r v_0, \quad D_{t, g_0}W_1(0, v_0, v_1, w) = -4\pi^2 r^3 k_0(k_0 v_1 + w), \\ W_2(0, v_0, v_1, w) &= r k_0 v_1 + r w, \quad D_{t, g_0}W_2(0, v_0, v_1, w) = 0, \\ W_3(0, v_0, v_1, w) &= r k_0 v_0, \quad D_{t, g_0}W_3(0, v_0, v_1, w) = -2\pi r^3(k_0 v_1 + w). \end{aligned}$$

are a basis of $(T_{\alpha(0)}\mathbb{H})^2$, such that any solution to (4.5) is a linear combination of W_0, \dots, W_3 . As only W_1, \dots, W_3 are periodic, we obtain

$$\text{kernel}(D_{g_0}X_{k_0, g_0, \mathbb{H}}|_{\alpha}) = \langle W_1, W_2, W_3 \rangle. \quad (4.7)$$

We fix a neighborhood U of $\alpha(0)$ as above, where $\pi_M : U \rightarrow \pi_M(U)$ is an isometry. Then $\alpha \in H^{2,2}(S^1, U)$ and π_M induces isomorphisms

$$\begin{aligned} \pi_M : H^{2,2}(S^1, U) &\rightarrow H^{2,2}(S^1, \pi_M(U)), \quad \alpha \mapsto \pi_M \circ \alpha, \\ (\pi_M)_* : T_{\alpha}H^{2,2}(S^1, \mathbb{H}) &\rightarrow T_{\pi_M \circ \alpha}H^{2,2}(S^1, M), \quad V \mapsto d\pi_M|_{\alpha}V, \end{aligned}$$

where $(\pi_M)_*$ is an isometry. Moreover, there holds on $H^{2,2}(S^1, U)$

$$\begin{aligned} (\pi_M)_* \circ X_{g_0, k_0, \mathbb{H}} &= X_{g_0, k_0} \circ \pi_M, \\ (\pi_M)_* \circ D_{g_0}X_{g_0, k_0, \mathbb{H}}|_{\alpha} &= D_{g_0}X_{g_0, k_0}|_{\pi_M \circ \alpha} \circ (\pi_M)_*. \end{aligned} \quad (4.8)$$

Since $\mathcal{Z}_{\mathbb{H}}$ and \mathcal{Z}_M are three dimensional submanifolds of $H^{2,2}(S^1, \mathbb{H})$ and $H^{2,2}(S^1, M)$, respectively, we have for $\alpha \in \mathcal{Z}_{\mathbb{H}}$ and $\tilde{\alpha} = \pi_M \circ \alpha \in \mathcal{Z}_M$

$$\begin{aligned} T_{\alpha}\mathcal{Z}_{\mathbb{H}} &= \text{kernel}(D_{g_0}X_{k_0, g_0, \mathbb{H}}|_{\alpha}) = \langle W_1, W_2, W_3 \rangle, \\ T_{\tilde{\alpha}}\mathcal{Z}_M &= \text{kernel}(D_{g_0}X_{k_0, g_0}|_{\tilde{\alpha}}) \\ &= \langle \tilde{W}_i := (\pi_M)_* \circ W_i : 1 \leq i \leq 3 \rangle. \end{aligned}$$

To compute the image of $D_{g_0}X_{k_0, g_0, \mathbb{H}}|_{\alpha}$ we note that $\{\dot{\alpha}, \alpha \times_m \dot{\alpha}\}$ is an orthogonal system in $T_{\alpha}\mathbb{H}$ for any $t \in S^1$. Thus any $V \in T_{\alpha}H^{2,2}(S^1, \mathbb{H})$ may be written as

$$V = \lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times_m \dot{\alpha})$$

for some functions $\lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R})$. Using the fact that

$$D_{t, g_0}\dot{\alpha} = |\dot{\alpha}|_m k_0 (\alpha \times_m \dot{\alpha}) \quad \text{and} \quad D_{t, g_0}(\alpha \times_m \dot{\alpha}) = -|\dot{\alpha}|_m k_0 \dot{\alpha},$$

we obtain

$$\begin{aligned} D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha}(V) &= (-D_{t, g_0}^2 + 1)^{-1} \left((-\lambda_1'' + 2\pi\sqrt{1+r^2}\lambda_2') \dot{\alpha} \right. \\ &\quad \left. + (-\lambda_2'' - (2\pi)^2\lambda_2)(\alpha \times_m \dot{\alpha}) \right). \end{aligned} \quad (4.9)$$

Concerning W_1, \dots, W_3 and W_{g_0} we find

$$\begin{aligned} W_1(t) &= \dot{\alpha}(t), \\ W_2(t) &= -\frac{1}{2\pi r} \left(\sqrt{1+r^2} \sin(2\pi t) \dot{\alpha}(t) + \cos(2\pi t) (\alpha \times_m \dot{\alpha}) \right), \\ W_3(t) &= -\frac{1}{2\pi r} \left(-\sqrt{1+r^2} \cos(2\pi t) \dot{\alpha}(t) + \sin(2\pi t) (\alpha \times_m \dot{\alpha}) \right) \\ W_{g_0}(\alpha) &= (1 + |\dot{\alpha}|_m^2 k_0^2)^{-1} \dot{\alpha} = (1 + 4\pi^2(1+r^2))^{-1} W_1. \end{aligned} \quad (4.10)$$

Lemma 4.3. *If $r \neq (2\pi)^{-1}$, then we have for $\alpha \in \mathcal{Z}_{\mathbb{H}}$*

$$\begin{aligned} \{0\} &= \langle W_1, W_2, W_3 \rangle \cap R(D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha}), \\ \langle W_1 \rangle^{\perp} &= \langle W_2, W_3 \rangle \oplus R(D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha}) \end{aligned}$$

Proof. For $\lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R})$ we have

$$\begin{aligned} &(-D_{t, g_0}^2 + 1)(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times_m \dot{\alpha})) \\ &= \left(-\lambda_1'' + 4\pi\sqrt{1+r^2}\lambda_2' + (4\pi^2(1+r^2) + 1)\lambda_1 \right) \dot{\alpha} \\ &\quad + \left(-\lambda_2'' - 4\pi\sqrt{1+r^2}\lambda_1' + (4\pi^2(1+r^2) + 1)\lambda_2 \right) \alpha \times_m \dot{\alpha} \end{aligned}$$

Hence we get by direct calculations

$$\begin{aligned} (-D_{t, g_0}^2 + 1)(W_1) &= (4\pi^2(1+r^2) + 1)\dot{\alpha}, \\ (-D_{t, g_0}^2 + 1)(-2\pi r W_2) &= \sqrt{1+r^2}(4\pi^2 r^2 + 1) \sin(2\pi t) \dot{\alpha} \\ &\quad + (-4\pi^2 r^2 + 1) \cos(2\pi t) (\alpha \times_m \dot{\alpha}), \end{aligned} \quad (4.11)$$

$$\begin{aligned} (-D_{t, g_0}^2 + 1)(-2\pi r W_3) &= -\sqrt{1+r^2}(4\pi^2 r^2 + 1) \cos(2\pi t) \dot{\alpha} \\ &\quad + (-4\pi^2 r^2 + 1) \sin(2\pi t) (\alpha \times_m \dot{\alpha}). \end{aligned} \quad (4.12)$$

Consequently, by (3.1), (4.10), and the above computations W_1 is orthogonal to $\langle W_2, W_3 \rangle$ and to $R(D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha})$ in $T_{\alpha} H^{2,2}(S^1, \mathbb{H})$. As in $L^2(S^1, \mathbb{R})$

$$\lambda_2'' + (2\pi)^2 \lambda_2 \perp_{L^2} \langle \cos(2\pi t), \sin(2\pi t) \rangle, \quad \langle \lambda_1'', \lambda_2' \rangle \perp_{L^2} \text{const},$$

we get from (4.9) and the fact that $1 - 4\pi^2 r^2 \neq 0$

$$\begin{aligned} \{0\} &= (-D_{t, g_0}^2 + 1)(\langle W_1, W_2, W_3 \rangle) \\ &\quad \cap (-D_{t, g_0}^2 + 1) D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha}(T_{\alpha} H^{2,2}(S^1, \mathbb{H})) \end{aligned}$$

and the claim follows for $D_{g_0} X_{k_0, g_0, \mathbb{H}}|_{\alpha}$ is a Fredholm operator of index 0. \square

Moreover, we see for $\alpha \in \mathcal{Z}_{\mathbb{H}}$

$$\begin{aligned}
R(D_{g_0} X_{k_0, g_0, \mathbb{H}} | \alpha) &= \{(-D_{t, g_0}^2 + 1)^{-1} ((-\lambda_1'' + 2\pi\sqrt{1+r^2}\lambda_2')\dot{\alpha} \\
&\quad - (\lambda_2'' + (2\pi)^2\lambda_2)(\alpha \times_m \dot{\alpha})) : \lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R})\} \\
&= \{(-D_{t, g_0}^2 + 1)^{-1} (\lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha})) : \lambda_i \in L^2(S^1, \mathbb{R}), \\
&\quad \lambda_1 \perp_{L^2} 1, \lambda_2 \perp_{L^2} \langle \cos(2\pi t), \sin(2\pi t) \rangle\} \\
&= \langle (\alpha \times_m \dot{\alpha}) \rangle \oplus E_+, \tag{4.13}
\end{aligned}$$

where E_+ is given by

$$\begin{aligned}
E_+ &= \{(-D_{t, g_0}^2 + 1)^{-1} (\lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha})) : \\
&\quad \lambda_i \in L^2(S^1, \mathbb{R}), \lambda_1 \perp_{L^2} 1, \lambda_2 \perp_{L^2} \langle 1, \cos(2\pi t), \sin(2\pi t) \rangle\}
\end{aligned}$$

We have for $V = \lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha})$ in $T_{\alpha}H^{2,2}(S^1, \mathbb{H})$

$$D_{g_0} X_{k_0, g_0, \mathbb{H}} | \alpha(V) \in E_+ \iff \lambda_2 \perp_{L^2} 1 \iff V \perp_{L^2} (\alpha \times_m \dot{\alpha}).$$

We fix

$$V = (-D_{t, g_0}^2 + 1)^{-1} (\lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha})) \in E_+.$$

Then

$$\begin{aligned}
&\int_{S^1} \langle V, \alpha \times_m \dot{\alpha} \rangle_m \\
&= \int_{S^1} \langle (-D_{t, g_0}^2 + 1)^{-1} (\lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha})), \alpha \times_m \dot{\alpha} \rangle_m \\
&= \int_{S^1} \langle \lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha}), (-D_{t, g_0}^2 + 1)^{-1} (\alpha \times_m \dot{\alpha}) \rangle_m \\
&= (4\pi^2(1+r^2) + 1)^{-1} \int_{S^1} \langle \lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha}), \alpha \times_m \dot{\alpha} \rangle_m = 0.
\end{aligned}$$

Consequently, $D_{g_0} X_{k_0, g_0, \mathbb{H}} | \alpha(E_+) = E_+$.

E_+ is L^2 -orthogonal to $\alpha \times_m \dot{\alpha}$ and $\dot{\alpha}$, we may thus write

$$V = (\nu_1 + f_1)\dot{\alpha} + (\nu_2 + f_2)(\alpha \times_m \dot{\alpha}),$$

where

$$\nu_1, \nu_2 \perp_{L^2} \langle 1, \sin(2\pi \cdot), \cos(2\pi \cdot) \rangle \text{ and } f_1, f_2 \in \langle \sin(2\pi \cdot), \cos(2\pi \cdot) \rangle.$$

Then

$$\begin{aligned}
&\langle (-D_{t, g_0}^2 + 1)D_{g_0} X_{k_0, g_0, \mathbb{H}} | \alpha(V), V \rangle_{L^2} \\
&= \int_{S^1} (\nu_1')^2 - 2\pi\sqrt{1+r^2}\nu_1'\nu_2 + (\nu_2')^2 - 4\pi^2(\nu_2)^2 \\
&\quad \int_{S^1} (f_1')^2 - 2\pi\sqrt{1+r^2}f_1'f_2. \tag{4.14}
\end{aligned}$$

Since $\nu_2 \perp_{L^2} \langle 1, \cos(2\pi \cdot), \sin(2\pi \cdot) \rangle$ we have

$$\int_{S^1} (\nu_2')^2 - 4\pi^2(\nu_2)^2 \geq \int_{S^1} 16\pi^2(\nu_2)^2$$

and for $0 < r \leq 1$

$$\begin{aligned} & \int_{S^1} (\nu_1')^2 - 2\pi\sqrt{1+r^2}\nu_1'\nu_2 + (\nu_2')^2 - 4\pi^2(\nu_2)^2 \\ & \geq \int_{S^1} (\nu_1')^2 - \frac{1}{4}(\nu_1')^2 - 4\pi^2(1+r^2)(\nu_2)^2 + (\nu_2')^2 - 4\pi^2(\nu_2)^2 \\ & \geq \int_{S^1} \frac{3}{4}(\nu_1')^2 + 4\pi^2(\nu_2)^2. \end{aligned}$$

Concerning the remaining term in (4.14) we note that as $(-D_{t,g_0}^2 + 1)$ maps

$$\{\lambda_1\dot{\alpha} + \lambda_2(\alpha \times_m \dot{\alpha}) : \lambda_1, \lambda_2 \in \langle \sin(2\pi \cdot), \cos(2\pi \cdot) \rangle\}$$

into itself and $V \in E_+$ there holds

$$f_1\dot{\alpha} + f_2(\alpha \times_m \dot{\alpha}) \in (-D_{t,g_0}^2 + 1)^{-1} \langle \cos(2\pi \cdot)\dot{\alpha}, \sin(2\pi \cdot)\dot{\alpha} \rangle.$$

Hence, by explicit computations there are $x, y \in \mathbb{R}$ satisfying

$$\begin{aligned} f_1(t) &= x \cos(2\pi t) + y \sin(2\pi t), \\ f_2(t) &= \frac{8\pi^2\sqrt{1+r^2}}{4\pi^2(2+r^2)+1} (y \cos(2\pi t) - x \sin(2\pi t)), \end{aligned}$$

such that

$$\int_{S^1} (f_1')^2 - 2\pi\sqrt{1+r^2}f_1'f_2 = \frac{2\pi^2(1-4\pi^2r^2)}{4\pi^2(2+r^2)+1} (x^2 + y^2).$$

This shows that if $r < (2\pi)^{-1}$, then

$$\langle (-D_{t,g_0}^2 + 1)D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha}(V), V \rangle_{L^2} > 0 \text{ for all } V \in E_+ \setminus \{0\},$$

and the homotopy

$$[0, 1] \ni s \mapsto (1-s)(D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha})|_{E_+} + s \text{id}|_{E_+}$$

is admissible. We use the decomposition in (4.13) and

$$D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha}(\alpha \times_m \dot{\alpha}) = -\frac{4\pi^2}{4\pi^2(1+r^2)+1}(\alpha \times_m \dot{\alpha})$$

to see that under the assumption $r < (2\pi)^{-1}$

$$(D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha})|_{R(D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha})} \sim \begin{pmatrix} -1 & 0 \\ 0 & \text{id}|_{E_+} \end{pmatrix}.$$

Consequently, for $r < (2\pi)^{-1}$

$$\text{sgn}(D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha})|_{R(D_{g_0}X_{k_0,g_0,\mathbb{H}}|_{\alpha})} = -1. \quad (4.15)$$

We remark that the formula for the degree continues to hold for $r > (2\pi)^{-1}$. From (4.8) and the fact that $(\pi_M)_*$ is an isometry we obtain for $\tilde{\alpha} \in \mathcal{Z}_M$

$$\begin{aligned} \{0\} &= \langle \tilde{W}_1, \tilde{W}_2, \tilde{W}_3 \rangle \cap R(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}}), \\ \langle \tilde{W}_1 \rangle^\perp &= \langle \tilde{W}_2, \tilde{W}_3 \rangle \oplus R(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}}), \\ -1 &= \text{sgn}(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}}) |_{R(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}})}. \end{aligned} \quad (4.16)$$

We fix $\tilde{\alpha}_0 \in \mathcal{Z}_M$ and a parametrization φ of \mathcal{Z}_M , which maps an open neighborhood of 0 in $\langle \tilde{W}_1(\tilde{\alpha}_0), \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle$ into \mathcal{Z}_M , such that

$$\varphi(0) = \tilde{\alpha}_0 \text{ and } D\varphi|_0 = id.$$

As \mathcal{Z}_M consists of smooth functions, \mathcal{Z}_M is a sub-manifold of $H^{m,2}(S^1, M)$ for $1 \leq m < \infty$. We define a map Φ from an open neighborhood \mathcal{U} of 0 in

$$T_{\tilde{\alpha}_0} H^{2,2}(S^1, M) = \langle \tilde{W}_1(\tilde{\alpha}_0), \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle \oplus \text{Range}(D_{g_0} X_{k_0, g_0} |_{\tilde{\alpha}_0})$$

to $H^{2,2}(S^1, M)$ by

$$\Phi(W, U) := \text{Exp}_{\tilde{\alpha}_0, g_0}(\text{Exp}_{\tilde{\alpha}_0, g_0}^{-1}(\varphi(W)) + U). \quad (4.17)$$

Then (Φ, \mathcal{U}) is a chart of $H^{2,2}(S^1, M)$ around $\tilde{\alpha}_0$ such that \mathcal{U} is an open neighborhood of 0 in $T_{\tilde{\alpha}_0} H^{2,2}(S^1, M)$, $\Phi(0) = \tilde{\alpha}_0$, and

$$D\Phi|_0 = id, \quad \Phi^{-1}(\mathcal{Z}_M \cap \Phi(\mathcal{U})) = \mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0), \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle.$$

From the properties of $\text{Exp}_{\tilde{\alpha}_0, g_0}$ the map Φ is a chart of $H^{k,2}(S^1, M)$ around $\tilde{\alpha}_0$ for any $1 \leq k \leq 4$ and shrinking \mathcal{U} we may assume that

$$T_{\Phi(V)} H^{1,2}(S^1, M) = \langle \frac{d}{dt} \Phi(V) \rangle \oplus D\Phi|_V(\langle \dot{\alpha}_0 \rangle^{\perp, H^{1,2}}), \quad (4.18)$$

$$T_{\Phi(V)} H^{2,2}(S^1, M) = \langle W_{g_0}(\Phi(V)) \rangle \oplus D\Phi|_V(\langle W_{g_0}(\tilde{\alpha}_0) \rangle^\perp), \quad (4.19)$$

$$\text{Proj}_{\langle W_{g_0}(\Phi(V)) \rangle^\perp} \circ D\Phi|_V : \langle W_{g_0}(\tilde{\alpha}_0) \rangle^\perp \xrightarrow{\cong} \langle W_{g_0}(\Phi(V)) \rangle^\perp, \quad (4.20)$$

and the norm of the projections in (4.18) and (4.19) as well as the norm of the map in (4.20) and its inverse are uniformly bounded with respect to V .

5. THE PERTURBATIVE ANALYSIS

For $\tilde{\alpha}_0 \in \mathcal{Z}_M$ the vectors $\tilde{W}_1(\tilde{\alpha}_0)$ and $W_{g_0}(\tilde{\alpha}_0)$ are collinear and we use $\langle \tilde{W}_1(\tilde{\alpha}_0) \rangle$ instead of $\langle W_{g_0}(\tilde{\alpha}_0) \rangle$ in the analysis below.

We define a S^1 -invariant vector bundle $SH^{2,2}(S^1, M)$ by

$$SH^{2,2}(S^1, M) := \{(\gamma, V) \in TH^{2,2}(S^1, M) : \gamma \neq \text{const}, V \in \langle W_g(\gamma) \rangle^\perp\}.$$

As in [15, Sec. 4] we obtain a chart Ψ for the bundle $SH^{2,2}(S^1, M)$ around $(\tilde{\alpha}_0, 0)$ by,

$$\begin{aligned} \Psi : \mathcal{U} \times \mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp &\rightarrow SH^{2,2}(S^1, M), \\ \Psi(V, U) &:= (\Phi(V), \text{Proj}_{\langle W_{g_0}(\Phi(V)) \rangle^\perp} \circ D\Phi|_V(U)). \end{aligned}$$

We define

$$X_{g_0, \varepsilon}^\Phi : \mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp \rightarrow \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp$$

by

$$X_{g_0, \varepsilon}^\Phi(V) := Proj_2 \circ \Psi^{-1}(\Phi(V), X_{g_0, \varepsilon}(\Phi(V))).$$

As in [15, Lem. 3.5] it is easy to see that

$$\begin{aligned} V \in \mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp \text{ is a (nondegenerate) zero of } X_{g_0, \varepsilon}^\Phi \text{ if and only if} \\ S^1 * \Phi(V) \text{ is a (nondegenerate) critical orbit of } X_{g_0, \varepsilon}, \end{aligned} \quad (5.1)$$

and if $X_{g_0, \varepsilon}^\Phi(V) = 0$, then

$$D_{g_0} X_{g_0, \varepsilon}^\Phi|_V = A_V^{-1} \circ D_{g_0} X_{g_0, \varepsilon}|_{\Phi(V)} \circ D\Phi|_V, \quad (5.2)$$

where the isomorphism $A_V : \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp \rightarrow \langle W_{g_0}(\Phi(V)) \rangle^\perp$ is given by

$$A_V = Proj_{\langle W_{g_0}(\Phi(V)) \rangle^\perp} \circ D\Phi|_V.$$

From Lemma 4.3 we may assume

$$\mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp = \mathcal{U}_1 \times \mathcal{U}_2,$$

where \mathcal{U}_1 and \mathcal{U}_2 are open neighborhoods of 0 in $\langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle$ and $R(D_{g_0} X_{k_0, g_0}|_{\tilde{\alpha}_0})$. We denote for $\tilde{\alpha} \in \mathcal{Z}_M$ by $P_2(\tilde{\alpha})$ the projection onto $R(D_{g_0} X_{g_0, 0}|_{\tilde{\alpha}})$ with respect to the decomposition

$$\langle \tilde{W}_1(\tilde{\alpha}) \rangle^\perp = \langle \tilde{W}_2(\tilde{\alpha}), \tilde{W}_3(\tilde{\alpha}) \rangle \oplus R(D_{g_0} X_{k_0, g_0}|_{\tilde{\alpha}}),$$

and by $P_1(\tilde{\alpha})$ the projection onto $\langle \tilde{W}_2(\tilde{\alpha}), \tilde{W}_3(\tilde{\alpha}) \rangle$. Moreover, for $W \in \mathcal{U}_1$ we define for $i = 1, 2$

$$P_i^\Phi(W) := (A_W)^{-1} \circ P_i(\Phi(W)) \circ A_W.$$

The projections $P_1^\Phi(W)$ and $P_2^\Phi(W)$ correspond to the decomposition

$$\langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp = \langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle \oplus R(D_{g_0} X_{g_0, 0}^\Phi|_W), \quad (5.3)$$

as we have for $W \in \mathcal{U}_1$

$$D_{g_0} X_{g_0, 0}^\Phi|_W = A_W^{-1} \circ D_{g_0} X_{g_0, 0}|_{\Phi(W)} \circ A_W.$$

Moreover, for $\tilde{\alpha} \in \mathcal{Z}_M$ the vector field $K_1(\tilde{\alpha})$ is orthogonal to $\tilde{W}_1(\tilde{\alpha})$ and we may define a vector field on \mathcal{Z}_M by

$$\mathcal{Z}_M \ni \tilde{\alpha} \mapsto P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha}) \in \langle \tilde{W}_2(\tilde{\alpha}), \tilde{W}_3(\tilde{\alpha}) \rangle.$$

Note that $P_1(\cdot) \circ K_1(\cdot)$ is S^1 -equivariant, i.e.

$$\theta * (P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha})) = P_1(\theta * \tilde{\alpha}) \circ K_1(\theta * \tilde{\alpha}) \text{ for all } (\theta, \tilde{\alpha}) \in S^1 \times \mathcal{Z}_M.$$

If $P_1(\tilde{\alpha}_0) \circ K_1(\tilde{\alpha}_0) = 0$ for some $\tilde{\alpha}_0 \in \mathcal{Z}_M$ differentiating the identity

$$0 \equiv \langle P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha}), \tilde{W}_1(\tilde{\alpha}) \rangle$$

we find that the covariant derivative $D_{\mathcal{Z}_M}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_0}$ maps

$$T_{\tilde{\alpha}_0}\mathcal{Z}_M = \langle \tilde{W}_1(\tilde{\alpha}_0), \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle$$

to $\langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle$ and the S^1 equivariance leads to

$$D_{\mathcal{Z}_M}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_0}(\tilde{W}_1(\tilde{\alpha}_0)) = 0.$$

Consequently, we say that $S^1 * \tilde{\alpha}_0 \in \mathcal{Z}_M$ is a nondegenerate zero orbit of $P_1(\cdot) \circ K_1(\cdot)$, if $P_1(\tilde{\alpha}_0) \circ K_1(\tilde{\alpha}_0) = 0$ and

$$D_{\mathcal{Z}_M}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_0} : \langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle \rightarrow \langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle$$

is invertible.

Using the above notation the perturbative analysis done in [15] carries over and we state the following four results without proof (see [15, Lem. 5.2-5.5]).

Lemma 5.1. *For $\tilde{\alpha}_0 \in \mathcal{Z}$ after possibly shrinking \mathcal{U} there are $\varepsilon_0 > 0$ and*

$$\begin{aligned} U &\in C^2([-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1, \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp), \\ R &\in C^2([-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1, \langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle), \end{aligned}$$

such that for all $(\varepsilon, W) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1$

$$\begin{aligned} R(\varepsilon, W) &= X_{g_0, \varepsilon}^\Phi(W + U(\varepsilon, W)), \\ 0 &= P_1^\Phi(W) \circ U(\varepsilon, W), \\ O(\varepsilon)_{\varepsilon \rightarrow 0} &= \|U(\varepsilon, W)\| + \|D_W U(\varepsilon, W)\| + \|R(\varepsilon, W)\| + \|D_W R(\varepsilon, W)\|, \\ R(\varepsilon, W) &= \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + o(\varepsilon)_{\varepsilon \rightarrow 0}, \\ U(\varepsilon, W) &= -\varepsilon(D_{g_0} X_{g_0, 0}^\Phi|_W)^{-1} \circ P_2^\Phi(W) \circ K_1^\Phi(W) + o(\varepsilon)_{\varepsilon \rightarrow 0}. \end{aligned}$$

Moreover, the functions $U(\varepsilon, W)$ and $R(\varepsilon, W)$ are unique, in the sense that, if (ε, W, U, R) in $[-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1 \times \mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp \times \mathcal{U}_1$ satisfies

$$X_{g_0, \varepsilon}^\Phi(W + U) = R \text{ and } P_1^\Phi(W)(U) = 0,$$

then $U = U(\varepsilon, W)$ and $R = R(\varepsilon, W)$.

Lemma 5.2. *Under the assumptions of Lemma 5.1 we have as $\varepsilon \rightarrow 0$*

$$X_{g_0, \varepsilon}^\Phi(W + U(\varepsilon, W)) = \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)_{\varepsilon \rightarrow 0},$$

where K_1^Φ is the vector-field K_1 in the coordinates Φ , i.e.

$$K_1^\Phi = X_{g_0, 1}^\Phi - X_{g_0, 0}^\Phi.$$

Lemma 5.3. *Under the assumptions of Lemma 5.1 suppose 0 is a nondegenerate zero of the vector-field $P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)$, in the sense that $P_1^\Phi(0) \circ K_1^\Phi(0) = 0$ and*

$$D_W(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0 \in \mathcal{L}(\langle W_2(\tilde{\alpha}_0), W_3(\tilde{\alpha}_0) \rangle)$$

is an isomorphism. Then, after possibly shrinking ε_0 and \mathcal{U} , for any $0 < \varepsilon \leq \varepsilon_0$ there is a unique $W(\varepsilon) \in \mathcal{U}_1$ such that

$$\begin{aligned} X_{g_0, \varepsilon}^\Phi(W(\varepsilon) + U(\varepsilon, W(\varepsilon))) &= 0, \\ W(\varepsilon) &\rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Moreover, $V(\varepsilon) := W(\varepsilon) + U(\varepsilon, W(\varepsilon))$ is the only zero of $X_{g_0, \varepsilon}^\Phi$ in $\mathcal{U} \cap \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp$ and is nondegenerate with

$$\text{sgn}(DX_{g_0, \varepsilon}^\Phi|_{V(\varepsilon)}) = -\det(DW(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0).$$

Lemma 5.4. *Under the assumptions of Lemma 5.1 suppose $\tilde{\alpha}_0$ is a nondegenerate zero of the vector-field $P_1(\cdot) \circ K_1(\cdot)$ on \mathcal{Z}_M , in the sense that $P_1(\tilde{\alpha}_0) \circ K_1(\tilde{\alpha}_0) = 0$ and*

$$D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_0} \in \mathcal{L}(\langle W_2(\tilde{\alpha}_0), W_3(\tilde{\alpha}_0) \rangle)$$

is an isomorphism. Then for any $0 < \varepsilon < \varepsilon_0$ there is $\tilde{\gamma}(\varepsilon) \in \Phi(\mathcal{U})$ satisfying

$$X_{g_0, \varepsilon}(\tilde{\gamma}(\varepsilon)) = 0 \text{ and } \tilde{\gamma}(\varepsilon) \rightarrow \tilde{\alpha}_0 \text{ as } \varepsilon \rightarrow 0.$$

Moreover, $S^1 * \tilde{\gamma}(\varepsilon)$ is the unique critical orbit of $X_{g_0, \varepsilon}$ in $\Phi(\mathcal{U})$ and is nondegenerate with

$$\text{deg}_{\text{loc}, S^1}(X_{g_0, \varepsilon}, S^1 * \tilde{\gamma}(\varepsilon)) = -\det(D_{\mathcal{Z}_M}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_0}).$$

6. THE COMPUTATION OF THE DEGREE

In order to compute the S^1 -degree of $X_{g_0, \varepsilon}$ we choose a smooth Morse function $k_1 \in C^\infty(M, \mathbb{R})$. The corresponding vector-field K_1 on $H^{2,2}(S^1, M)$ is given by

$$K_1(\gamma) = (-D_{t, g_0}^2 + 1)^{-1}(|\dot{\gamma}|_{g_0} k_1(\tilde{\gamma}) J_{g_0}(\gamma) \dot{\gamma}).$$

We note that for $\tilde{\alpha} = \pi_M \circ \alpha(\cdot, 2\pi|r|, v_0, v_1, w) \in \mathcal{Z}_M$ and $r > 0$ small enough we have

$$K_1(\tilde{\alpha}) = 2\pi r (-D_{t, g_0}^2 + 1)^{-1} (k_1(\tilde{\alpha})(\varphi_M)_*(\alpha \times_m \dot{\alpha})).$$

Consequently, from (4.11), (4.12), and (4.13)

$$P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha}) = \sigma_2(\tilde{\alpha}) \tilde{W}_2(\tilde{\alpha}) + \sigma_3(\tilde{\alpha}) \tilde{W}_3(\tilde{\alpha}), \quad (6.1)$$

where $\sigma_2(\tilde{\alpha}), \sigma_3(\tilde{\alpha}) \in \mathbb{R}$ are defined by the condition that

$$2\pi r k_1(\tilde{\alpha}) - \frac{\sigma_2(\tilde{\alpha})}{2\pi r} (1 - 4\pi^2 r^2) \cos(2\pi \cdot) - \frac{\sigma_3(\tilde{\alpha})}{2\pi r} (1 - 4\pi^2 r^2) \sin(2\pi \cdot)$$

is L^2 -orthogonal to $\langle \cos(2\pi \cdot), \sin(2\pi \cdot) \rangle$. Hence,

$$\begin{aligned} \sigma_2(\tilde{\alpha}) &= \frac{8\pi^2 r^2}{1 - 4\pi^2 r^2} \int_0^1 k_1 \circ \tilde{\alpha}(t) \cos(2\pi t) dt, \\ \sigma_3(\tilde{\alpha}) &= \frac{8\pi^2 r^2}{1 - 4\pi^2 r^2} \int_0^1 k_1 \circ \tilde{\alpha}(t) \sin(2\pi t) dt. \end{aligned}$$

In the following we are interested in the asymptotics of σ_2 and σ_3 as $r \rightarrow 0^+$ or equivalently as $k_0 \rightarrow \infty$. There holds

$$\begin{aligned} & \frac{1 - 4\pi^2 r^2}{8\pi^2 r^2} \sigma_2(\tilde{\alpha}) \\ &= \int_0^1 \left(k_1 \circ \pi_M(w) + r dk_1|_{\pi_M(w)} \cos(2\pi t) (\pi_M)_* v_1 \right. \\ & \quad \left. + r dk_1|_{\pi_M(w)} \sin(2\pi t) (\pi_M)_* v_0 + O(r^2) \right) \cos(2\pi t) dt \\ &= \frac{1}{2} r dk_1|_{\pi_M(w)} (\pi_M)_* v_1 + O(r^2), \end{aligned} \quad (6.2)$$

and analogously we find

$$\frac{1 - 4\pi^2 r^2}{8\pi^2 r^2} \sigma_3(\tilde{\alpha}) = \frac{1}{2} r dk_1|_{\pi_M(w)} (\pi_M)_* v_0 + O(r^2). \quad (6.3)$$

From the above expansion we easily deduce

Lemma 6.1. *For all $\delta > 0$ there is $r_0 > 0$ such that for all $0 < r \leq r_0$ and*

$$\tilde{\alpha} = \pi_M(\sqrt{1 + r^2} w + r \cos(2\pi t) v_1 + r \sin(2\pi t) v_0) \in \mathcal{Z}_M$$

satisfying $P_1(\tilde{\alpha}) \circ K_1(\tilde{\alpha}) = 0$ there holds

$$\pi_M(w) \in \bigcup_{i=1}^n B_\delta(\tilde{w}_i),$$

where $\{\tilde{w}_i : 1 \leq i \leq n\}$ denotes the set of critical points of k_1 in M .

Fix $w_0 \in \mathbb{H}$ and a positive orthonormal system $\{v_0, v_1, w_0\}$ in (\mathbb{R}^3, m) such that $\pi_M(w_0)$ is a critical point of k_1 in M . We choose $\delta > 0$, a parametrization

$$w : B_1(0) \subset \mathbb{R}^2 \rightarrow B_\delta(w_0) \subset \mathbb{H}, \quad (x, y) \mapsto w(x, y),$$

and smooth maps $v_0, v_1 : B_1(0) \rightarrow \mathbb{R}^3$ such that $\{v_0(x, y), v_1(x, y), w(x, y)\}$ is orthonormal for all $(x, y) \in B_1(0)$ and

$$(v_0(0, 0), v_1(0, 0), w(0, 0)) = (v_0, v_1, w_0), \quad \frac{\partial w}{\partial x}|_{(0,0)} = v_1, \quad \frac{\partial w}{\partial y}|_{(0,0)} = v_0. \quad (6.4)$$

Shrinking $\delta > 0$ we may assume that $\pi_M \circ \varphi_w$ parametrizes M and that $(x, y) \mapsto \tilde{\alpha}(x, y)$ is an injective immersion from $B_1(0)$ to \mathcal{Z}_M , where

$$\tilde{\alpha}(x, y) := \pi_M(\sqrt{1 + r^2} w(x, y) + r \cos(2\pi \cdot) v_1(x, y) + r \sin(2\pi \cdot) v_0(x, y)).$$

From (4.6) and (6.4) we get as $r, \delta \rightarrow 0^+$

$$\begin{aligned} \frac{\partial}{\partial x} \tilde{\alpha}|_{(x,y)} &= \tilde{W}_2(\tilde{\alpha}(x, y)) + O(r) + O(\delta) \\ \frac{\partial}{\partial y} \tilde{\alpha}|_{(x,y)} &= \tilde{W}_3(\tilde{\alpha}(x, y)) + O(r) + O(\delta). \end{aligned} \quad (6.5)$$

Define $H : B_1(0) \rightarrow \mathbb{R}^2$ by

$$H(x, y) := (\sigma_2(\tilde{\alpha}(x, y)), \sigma_3(\tilde{\alpha}(x, y))).$$

By (6.2) and (6.3) we have as $r \rightarrow 0^+$

$$\begin{aligned} \frac{H(x, y)}{8\pi^2 r^3} &:= (dk_1|_{\pi_M(w(x, y))}(\pi_M)_*v_1(x, y), dk_1|_{\pi_M(w(x, y))}(\pi_M)_*v_0(x, y)) \\ &\quad + O(r). \end{aligned}$$

Since

$$\frac{d}{dx}\pi_M \circ w|_{0,0} = (\pi_M)_*v_1(0, 0), \quad \frac{d}{dy}\pi_M \circ w|_{0,0} = (\pi_M)_*v_0(0, 0)$$

we find for small values of $\delta > 0$ and $r > 0$

$$\begin{aligned} \deg(H, B_1(0), 0) &= \deg(\nabla(k_1 \circ \pi_M \circ w), B_1(0), 0) \\ &= \deg(\nabla k_1, B_\delta(\pi_M(w_0)), 0) = \deg_{loc}(\nabla k_1, \pi_M(w_0)), \end{aligned}$$

and the set of zeros of H in $B_1(0)$ is non-empty. Fix a zero $(x_0, y_0) \in B_1(0)$ of H . Then

$$dH|_{(x_0, y_0)} = \begin{pmatrix} \frac{\partial}{\partial x}(\sigma_2 \circ \tilde{\alpha})|_{(x_0, y_0)} & \frac{\partial}{\partial y}(\sigma_2 \circ \tilde{\alpha})|_{(x_0, y_0)} \\ \frac{\partial}{\partial x}(\sigma_3 \circ \tilde{\alpha})|_{(x_0, y_0)} & \frac{\partial}{\partial y}(\sigma_3 \circ \tilde{\alpha})|_{(x_0, y_0)} \end{pmatrix}.$$

From (6.2), (6.3), and the fact that $H(x_0, y_0) = 0$ we get

$$dk_1|_{\pi_M(w(x_0, y_0))} = O(r).$$

Thus, we have as $r \rightarrow 0^+$

$$\begin{aligned} &dk_1|_{\tilde{\alpha}(x_0, y_0)(t)} \frac{\partial}{\partial x} \tilde{\alpha}|_{(x_0, y_0)}(t) \\ &= dk_1|_{\pi_M(w(x_0, y_0))}(\pi_M)_* \frac{\partial}{\partial x} w|_{(x_0, y_0)} \\ &\quad + r \left(\nabla \frac{\partial}{\partial r} \tilde{\alpha}(x_0, y_0)(t)|_{r=0} dk_1|_{\pi_M(w(x_0, y_0))}(\pi_M)_* \frac{\partial}{\partial x} w|_{(x_0, y_0)} \right. \\ &\quad \left. + dk_1|_{\pi_M(w(x_0, y_0))} \nabla \frac{\partial}{\partial r} \tilde{\alpha}(x_0, y_0)(t)|_{r=0} \frac{\partial}{\partial x} \tilde{\alpha}(x_0, y_0)(t)|_{r=0} \right) + O(r^2) \\ &= dk_1|_{\pi_M(w(x_0, y_0))}(\pi_M)_* \frac{\partial}{\partial x} w|_{(x_0, y_0)} \\ &\quad + r (\nabla dk_1)|_{\pi_M(w(x_0, y_0))} \left((\pi_M)_* \frac{\partial}{\partial x} w|_{(x_0, y_0)}, \right. \\ &\quad \left. \cos(2\pi t)(\pi_M)_*v_1(x_0, y_0) + \sin(2\pi t)(\pi_M)_*v_0(x_0, y_0) \right) + O(r^2). \end{aligned}$$

Using (6.4), this leads to, as $r, \delta \rightarrow 0^+$

$$\begin{aligned}
& \frac{\partial}{\partial x}(\sigma_2 \circ \tilde{\alpha})|_{(x_0, y_0)} \\
&= \int_0^1 dk_1|_{\tilde{\alpha}(x_0, y_0)(t)} \frac{\partial}{\partial x} \tilde{\alpha}|_{(x_0, y_0)}(t) \cos(2\pi t) dt \\
&= \frac{r}{2} (\nabla dk_1)|_{\pi_M(w(x_0, y_0))} \left((\pi_M)_* \frac{\partial}{\partial x} w|_{(x_0, y_0)}, (\pi_M)_* v_1(x_0, y_0) \right) + O(r^2) \\
&= \frac{r}{2} (\nabla dk_1)|_{\pi_M(w_0)} \left((\pi_M)_* v_1, (\pi_M)_* v_1 \right) + O(r^2) + O(r\delta).
\end{aligned}$$

Analogously, we may compute the remaining partial derivatives of H and we find for small values of $\delta > 0$ and $r > 0$

$$\operatorname{sgn} \det(dH|_{(x_0, y_0)}) = \operatorname{sgn} \det(\nabla dk_1|_{\pi_M(w_0)}) = \operatorname{deg}_{loc}(\nabla k_1, \pi_M(w_0)), \quad (6.6)$$

such that (x_0, y_0) is the unique zero of H in $B_1(0)$. From (6.1) we see that

$$P_1(\tilde{\alpha}(x_0, y_0)) \circ K_1(\tilde{\alpha}(x_0, y_0)) = 0,$$

by (6.5) we obtain as $r, \delta \rightarrow 0^+$

$$\begin{aligned}
& \nabla_{\tilde{W}_2} (P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}(x_0, y_0)} \\
&= \left(d\sigma_2|_{\tilde{\alpha}(x_0, y_0)} \tilde{W}_2(\tilde{\alpha}(x_0, y_0)) \right) \tilde{W}_2(\tilde{\alpha}(x_0, y_0)) \\
&\quad + \left(d\sigma_3|_{\tilde{\alpha}(x_0, y_0)} \tilde{W}_2(\tilde{\alpha}(x_0, y_0)) \right) \tilde{W}_3(\tilde{\alpha}(x_0, y_0)) \\
&= \frac{\partial}{\partial x}(\sigma_2 \circ \tilde{\alpha})|_{(x_0, y_0)} \tilde{W}_2(\tilde{\alpha}(x_0, y_0)) \\
&\quad + \frac{\partial}{\partial x}(\sigma_3 \circ \tilde{\alpha})|_{(x_0, y_0)} \tilde{W}_3(\tilde{\alpha}(x_0, y_0)) + O(r) + O(\delta).
\end{aligned}$$

Concerning the covariant derivative of $P_1(\cdot) \circ K_1(\cdot)$ in direction \tilde{W}_3 we have to replace $\frac{\partial}{\partial x}$ by $\frac{\partial}{\partial y}$ in the above formula. Consequently, from (6.6)

$$\operatorname{sgn} \det(D_{\mathcal{Z}_M} (P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}(x_0, y_0)}) = \operatorname{deg}_{loc}(\nabla k_1, \pi_M(w_0)).$$

Thus we arrive at the following

Lemma 6.2. *Let $\{\tilde{w}_i : 1 \leq i \leq n\}$ denote the set of critical points of k_1 in M . Then there is $r_0 > 0$ such that for all $0 < r \leq r_0$ the set of critical orbits of $P_1(\cdot) \circ K_1(\cdot)$ is given by $\{S^1 * \tilde{\alpha}_{i,r} : 1 \leq i \leq n\}$, where*

$$\tilde{\alpha}_{i,r} = \pi_M(\sqrt{1+r^2}w_{i,r} + r \cos(2\pi t)v_{1,i,r} + r \sin(2\pi t)v_{0,i,r}) \in \mathcal{Z}_M.$$

Moreover, we have for $1 \leq i \leq n$

$$\begin{aligned}
& \pi_M(w_{i,r}) \rightarrow \tilde{w}_i \text{ as } r \rightarrow 0^+, \\
& \operatorname{sgn} \det(D_{\mathcal{Z}_M} (P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_{i,r}}) = \operatorname{deg}_{loc}(\nabla k_1, \tilde{w}_i).
\end{aligned}$$

Proof. From Lemma 6.1 and the analysis of H we may choose $\delta > 0$ and $r_0 > 0$ such that the union $\cup_{i=1}^n B_\delta(\tilde{w}_i)$ is disjoint and for every i and $0 < r \leq r_0$ there is a unique $\pi_M(w_{i,r}) \in B_\delta(\tilde{w}_i)$ corresponding to a critical orbit

$S^1 * \tilde{\alpha}_{i,r}$. Moreover, if $r \rightarrow 0^+$ we may shrink $\delta > 0$, which yields together with the uniqueness of $\pi_M(w_{i,r})$ the claimed asymptotic. \square

Lemma 6.3. *Let \mathcal{M}_A be the set of oriented Alexandrov embedded regular curves in $H^{2,2}(S^1, M)$. There is $C_{k_0} > 0$ such that for all $k_0 \geq C_{k_0}$ we have*

$$\chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) = -\chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M .

Proof. We choose a Morse function k_1 on M with nondegenerate critical points $\{\tilde{w}_i : 1 \leq i \leq n\}$. From Lemma 6.2 we obtain $C_{k_0} > 0$ such that for all $k_0 \geq C_{k_0}$ the critical orbits of $P_1(\cdot) \circ K_1(\cdot)$ are given by $\{S^1 * \tilde{\alpha}_{i, k_0} : 1 \leq i \leq n\}$ satisfying

$$\text{sgn det}(D_{\mathcal{Z}_M}(P_1(\cdot) \circ K_1(\cdot))|_{\tilde{\alpha}_{i, k_0}}) = \text{deg}_{loc}(\nabla k_1, \tilde{w}_i).$$

We fix $k_0 \geq C_{k_0}$. By Lemma 5.3 there is $\varepsilon > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $1 \leq i \leq n$ there is $\tilde{\gamma}_i(\varepsilon) \in \Phi(\mathcal{U}_i)$ satisfying

$$X_{g_0, \varepsilon}(\tilde{\gamma}_i(\varepsilon)) = 0 \text{ and } \tilde{\gamma}_i(\varepsilon) \rightarrow \tilde{\alpha}_{i, k_0} \text{ as } \varepsilon \rightarrow 0.$$

Moreover, $S^1 * \tilde{\gamma}_i(\varepsilon)$ is the unique critical orbit of $X_{g_0, \varepsilon}$ in $\Phi(\mathcal{U}_i)$ and is nondegenerate with

$$\text{deg}_{loc, S^1}(X_{g_0, \varepsilon}, S^1 * \tilde{\gamma}_i(\varepsilon)) = -\text{deg}_{loc}(\nabla k_1, \tilde{w}_i). \quad (6.7)$$

To show that there is an open neighborhood \mathcal{U} of \mathcal{Z}_M and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the critical orbits of $X_{g_0, \varepsilon}$ in \mathcal{U} are given exactly by $\{\tilde{\gamma}_i(\varepsilon) : 1 \leq i \leq n\}$ we argue by contradiction. Suppose there are $\varepsilon_n \rightarrow 0^+$ and a sequence $(\tilde{\alpha}_n)$ of zeros of X_{g_0, ε_n} that converges to \mathcal{Z}_M but $\tilde{\alpha}_n \notin \{\tilde{\gamma}_i(\varepsilon) : 1 \leq i \leq n\}$. Up to a subsequence we may assume

$$\tilde{\alpha}_n \rightarrow \tilde{\alpha}_0 \in \mathcal{Z}_M$$

as $n \rightarrow \infty$. For large n we use the chart Φ around $\tilde{\alpha}_0$ given in (4.17). From the existence of a slice of the S^1 -action (see [15, Lem. 3.1]) we get sequences $\theta_n \in \mathbb{R}/\mathbb{Z}$ and $V_n \in \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp$ converging to 0 such that

$$\theta_n * \tilde{\alpha}_n = \Phi(V_n).$$

Note that from the S^1 -invariance and by construction

$$X_{g_0, \varepsilon_n}(\theta_n * \tilde{\alpha}_n) = 0 \text{ and } X_{g_0, \varepsilon_n}^\Phi(V_n) = 0.$$

We consider the map

$$\Lambda : \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp = \langle \tilde{W}_2(\tilde{\alpha}_0), \tilde{W}_3(\tilde{\alpha}_0) \rangle \oplus R(D_{g_0} X_{g_0, 0}^\Phi|_0) \rightarrow \langle \tilde{W}_1(\tilde{\alpha}_0) \rangle^\perp,$$

defined by

$$\Lambda(W, V) := W + D_{g_0} X_{g_0, 0}^\Phi|_W(V).$$

From (5.3) the map Λ is a diffeomorphism locally around $(0, 0)$, hence we may decompose

$$V_n = \Phi^{-1}(\theta_n * \alpha_n) = W_n + U_n,$$

where $W_n \in \langle \tilde{W}_1(\alpha_0) \rangle^\perp$ and $U_n \in R(D_{g_0} X_{k_0, g_0}^\Phi |_{W_n})$ converge to 0 as $n \rightarrow \infty$. From the uniqueness part of Lemma 5.1, as $X_{g_0, \varepsilon_n}^\Phi(W_n + U_n) = 0$, we get $U_n = U(\varepsilon_n, W_n)$. By Lemma 5.2 we see that

$$P_1(\tilde{\alpha}_0) \circ K_1(\tilde{\alpha}_0) = 0.$$

Consequently, $S^1 * \alpha_0 \in \{S^1 * \tilde{\alpha}_{i, k_0} : 1 \leq i \leq n\}$. From the uniqueness part in Lemma 5.3 we finally arrive at the contradiction

$$S^1 * \tilde{\alpha}_n \in \{S^1 * \tilde{\gamma}_i(\varepsilon_n) : 1 \leq i \leq n\}.$$

From the definition of the S^1 -equivariant Poincaré-Hopf index, the classification of Alexandrov embedded zeros of X_{k_0, g_0} , and (6.7) there holds for small $\varepsilon > 0$

$$\begin{aligned} \chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) &= \chi_{S^1}(X_{k_0, g_0}, \mathcal{U}) = \chi_{S^1}(X_{g_0, \varepsilon}, \mathcal{U}) \\ &= \sum_{i=1}^n \deg_{loc, S^1}(X_{g_0, \varepsilon}, S^1 * \tilde{\gamma}_i(\varepsilon)) \\ &= - \sum_{i=1}^n \deg_{loc}(\nabla k_1, \tilde{w}_i) = -\chi(M). \end{aligned}$$

□

7. THE APRIORI ESTIMATE

We fix a continuous family of metrics $\{g_t : t \in [0, 1]\}$ on M and a continuous family of positive continuous function $\{k_t : t \in [0, 1]\}$ on M . We assume that there is $K_0 > 0$, such that the Gaussian curvature K_{g_t} of each metric g_t on M and the functions $\{k_t\}$ satisfy

$$K_{g_t} \geq -K_0, \quad (7.1)$$

$$k_{inf} := \inf\{k_t(x) : (x, t) \in M \times [0, 1]\} > (K_0)^{\frac{1}{2}}. \quad (7.2)$$

We let X_t be the vector field on $H^{2,2}(S^1, M)$ defined by

$$X_t := X_{k_t, g_t}.$$

We denote by $\mathcal{M}_A \subset H^{2,2}(S^1, M)$ the set

$$\mathcal{M}_A := \{\gamma \in H_{reg}^{2,2}(S^1, M) : \gamma \text{ is prime and oriented Alexandrov embedded.}\}.$$

We shall show that the set

$$X^{-1}(0) := \{(\gamma, t) \in \mathcal{M}_A \times [0, 1] : X_t(\gamma) = 0\}$$

is compact in $\mathcal{M}_A \times [0, 1]$. Fix $(\gamma, t) \in X^{-1}(0)$. Then there is an oriented immersion $F : \overline{B} \rightarrow M$ with $F|_{\partial B} = \gamma$. We denote by F^*g_t the induced metric on B .

Lemma 7.1. *For any $(\gamma, t) \in X^{-1}(0)$ there is $\varphi \in C^2(\overline{B}, \mathbb{R})$ satisfying*

$$\begin{aligned} -\Delta_{F^*g_t}\varphi + K_{F^*g_t} + K_0e^\varphi &= 0 \text{ in } B, \\ \partial_\nu\varphi &= 0 \text{ on } \partial B, \end{aligned} \quad (7.3)$$

where ν denotes the unit normal oriented to the outside.

Moreover, there is $C_0 > 0$, which may be chosen independently of $(\gamma, t) \in X^{-1}(0)$, such that

$$0 \geq \varphi \geq -C_0.$$

Proof. To show the existence of a solution φ we use the method of upper and lower solutions (see also [9]). The function $\varphi_+ \equiv 0$ satisfies

$$-\Delta_{F^*g_t}\varphi_+ + K_{F^*g_t} + K_0e^{\varphi_+} = K_{F^*g_t} + K_0 \geq 0,$$

from (7.1) and the fact that F is a local isometry. Hence, φ_+ is a supersolution of (7.3). To find a subsolution, we let $\varphi_1 \in C^\infty(M, \mathbb{R})$ be defined as the solution to the linear equation

$$-\Delta_{g_t}\varphi_1 + K_{g_t} - 2\pi\chi(M)\text{vol}(M, g_t)^{-1} = 0 \text{ in } M, \quad \int_M \varphi_1 dg_t = 0.$$

By standard elliptic estimates using a Green's function on (M, g_t) (see [3, Thm 4.13]) we have

$$\begin{aligned} \sup_M |\varphi_1| &\leq C(g_t) \left(\sup_M |K_{g_t}| - 2\pi\chi(M)\text{vol}(M, g_t)^{-1} \right) \\ &\leq C_1, \end{aligned}$$

because $\{g_t : t \in [0, 1]\}$ is a compact set of smooth metrics. We may choose $C_2 > 1$ such that we have for all $t \in [0, 1]$

$$-C_2 \leq \ln \left(-2\pi\chi(M)K_0\text{vol}(M, g_t)^{-1} \right).$$

Since F is a local isometry, there holds

$$\Delta_{F^*g_t}(\varphi_1 \circ F) = (\Delta_{g_t}\varphi_1) \circ F.$$

We define $\varphi_- \in C^2(B, \mathbb{R})$ by

$$\varphi_- := \varphi_1 \circ F - C_1 - C_2$$

and get

$$-\Delta_{F^*g_t}\varphi_- + K_{F^*g_t} + K_0e^{\varphi_-} = 2\pi\chi(M)\text{vol}(M, g_t)^{-1} + K_0e^{\varphi_1 \circ F - C_1 - C_2} \leq 0.$$

Hence φ_- is a subsolution of (7.3) satisfying

$$-C_0 := -(2C_1 + C_2) \leq \varphi_- < \varphi_+.$$

Using a version of the method of upper and lower solutions given in [14] we find a solution φ to (7.3) satisfying $\varphi_- \leq \varphi \leq \varphi_+$. \square

We consider B equipped with the metric $h_t := e^\varphi F^* g_t$. Then the Gaussian curvature K_{h_t} and the geodesic curvature k_{h_t} of ∂B with respect to (B, h_t) are given by (see [3, Sec 5.8.2])

$$K_{h_t} \equiv -K_0 \text{ and } k_{h_t} = k_{F^* g_t} e^{-\frac{\varphi}{2}}.$$

Consequently, since $0 \leq \varphi$,

$$\inf_{\partial B} k_{h_t} \geq \inf_{\partial B} k_{F^* g_t} \geq k_{\text{inf}}.$$

The Gauss-Bonnet formula applied to (B, h_t) gives

$$2\pi = - \int_B K_0 dh_t + \int_{\partial B} k_{h_t} dS_{h_t} \geq -K_0 A(B, h_t) + k_{\text{inf}} L(\partial B, h_t),$$

where $A(B, h_t)$ denotes the area of B and $L(\partial B, h_t)$ the length of ∂B with respect to h_t . The isoperimetric inequality (see [13, Thm 4.3]) yields

$$L(\partial B, h_t)^2 \geq 4\pi A(B, h_t) + K_0 A(B, h_t)^2 \geq K_0 A(B, h_t)^2.$$

Thus we arrive at

$$2\pi \geq -(K_0)^{\frac{1}{2}} L(\partial B, h_t) + k_{\text{inf}} L(\partial B, h_t).$$

This yields

$$L(\gamma, g_t) = L(\partial B, F^* g_t) \leq e^{C_0} L(\partial B, h_t) \leq e^{C_0} \frac{2\pi}{k_{\text{inf}} - (K_0)^{\frac{1}{2}}}.$$

Using again the Gauss-Bonnet formula we see

$$\begin{aligned} 2\pi &= - \int_B K_0 dh_t + \int_{\partial B} k_{h_t} dS_{h_t} \\ &\leq e^{\frac{C_0}{2}} (\sup\{k_t(x) : (x, t) \in M \times [0, 1]\}) L(\partial B, h_t) \\ &\leq e^{2C_0} (\sup\{k_t(x) : (x, t) \in M \times [0, 1]\}) L(\gamma, g_t). \end{aligned}$$

Consequently, there is $C > 0$, such that

$$C \leq L(\gamma, g_t) \leq C^{-1}, \tag{7.4}$$

for all $(\gamma, t) \in X^{-1}(0)$.

Fix a sequence $(\gamma_n, t_n)_{n \in \mathbb{N}}$ in $X^{-1}(0)$. As a solution each γ_n is parameterized proportional to its arc-length. From (7.4), (γ_n) is uniformly bounded in $C^1(S^1, M)$. Using the equation (1.2) we obtain a uniform bound of (γ_n) in $C^3(S^1, M)$, such that we may extract a subsequence, still denoted by $(\gamma_n, t_n)_{n \in \mathbb{N}}$, which converges in $C^2(S^1, M) \times [0, 1]$ to (γ_0, t_0) . The convergence in $C^2(S^1, M)$ and the lower bound in (7.4) imply that $X_{t_0}(\gamma_0) = 0$ and that γ_0 is an immersion. By Lemma 2.1 the curve γ_0 is oriented Alexandrov embedded and hence $(\gamma_0, t_0) \in X^{-1}(0)$. This shows that

Lemma 7.2. *Under the assumptions (7.1) and (7.2) the set $X^{-1}(0)$ is compact.*

8. EXISTENCE RESULTS

We give the proof of our main existence result.

Proof of Theorem 1.2. From the uniformization theorem (M, g) is isometric to $(\mathbb{H}/\Gamma, e^\varphi g_0)$, where $\Gamma \subset O(2, 1)_+$ is a group of isometries acting freely and properly discontinuously and φ is a function in $C^\infty(\mathbb{H}/\Gamma, \mathbb{R})$. Due to the invariance of (1.2) under isometries we may assume without loss of generality that

$$(M, g) = (\mathbb{H}/\Gamma, e^\varphi g_0).$$

We consider the family of metrics $\{g_t := e^{t\varphi} g_0 : t \in [0, 1]\}$ and choose a large constant $k_0 \gg 1$, such that

$$k_0 > \left(-\inf\{K_{g_t}(x) : (x, t) \in M \times [0, 1]\} \right)^{\frac{1}{2}} + \inf_M k + C_{k_0},$$

where K_{g_t} denotes the Gaussian curvature of the metric g_t given by

$$K_{g_t} = e^{-t\varphi} (-t\Delta_{g_0}(\varphi) + 2).$$

From Lemma 7.2 the homotopy

$$[0, 1] \ni t \mapsto X_{k_0, g_t}$$

is $(\mathcal{M}_A, g_t, S^1)$ -admissible. By Lemma 6.3 and the homotopy invariance of the S^1 -equivariant Poincaré-Hopf index we obtain

$$-\chi(M) = \chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) = \chi_{S^1}(X_{k_0, g}, \mathcal{M}_A).$$

For $t \in [0, 1]$ we define $k_t \in C^\infty(M, \mathbb{R})$ by

$$k_t(x) := (1 - t)k_0 + tk(x).$$

Then

$$\inf\{k_t(x) : (x, t) \in M \times [0, 1]\} = \inf_M k > \left(-\inf_M K_g \right)^{\frac{1}{2}}.$$

From Lemma 7.2 the homotopy

$$[0, 1] \ni t \mapsto X_{k_t, g}$$

is (\mathcal{M}_A, g, S^1) -admissible and there holds

$$\chi_{S^1}(X_{k_t, g}, \mathcal{M}_A) = \chi_{S^1}(X_{k_0, g}, \mathcal{M}_A) = -\chi(M).$$

This gives the claim. \square

REFERENCES

- [1] V. I. Arnold. The first steps of symplectic topology. *Uspekhi Mat. Nauk*, 41(6(252)):3–18, 229, 1986.
- [2] V. I. Arnold. On some problems in symplectic topology. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 1–5. Springer, Berlin, 1988.
- [3] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

- [4] Gonzalo Contreras, Leonardo Macarini, and Gabriel P. Paternain. Periodic orbits for exact magnetic flows on surfaces. *Int. Math. Res. Not.*, (8):361–387, 2004.
- [5] V. L. Ginzburg. New generalizations of Poincaré’s geometric theorem. *Funktsional. Anal. i Prilozhen.*, 21(2):16–22, 96, 1987.
- [6] Viktor L. Ginzburg. On closed trajectories of a charge in a magnetic field. An application of symplectic geometry. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 131–148. Cambridge Univ. Press, Cambridge, 1996.
- [7] Viktor L. Ginzburg and Başak Z. Gürel. Periodic orbits of twisted geodesic flows and the Weinstein-Moser theorem. *Comment. Math. Helv.*, 84(4):865–907, 2009.
- [8] Gustav A. Hedlund. Fuchsian groups and transitive horocycles. *Duke Math. J.*, 2(3):530–542, 1936.
- [9] Jerry L. Kazdan and F. W. Warner. Curvature functions for compact 2-manifolds. *Ann. of Math. (2)*, 99:14–47, 1974.
- [10] V. V. Kozlov. Calculus of variations in the large and classical mechanics. *Uspekhi Mat. Nauk*, 40(2(242)):33–60, 237, 1985.
- [11] S. P. Novikov. The Hamiltonian formalism and a multivalued analogue of Morse theory. *Uspekhi Mat. Nauk*, 37(5(227)):3–49, 248, 1982.
- [12] S. P. Novikov and I. A. Taimanov. Periodic extremals of multivalued or not everywhere positive functionals. *Dokl. Akad. Nauk SSSR*, 274(1):26–28, 1984.
- [13] Robert Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978.
- [14] Klaus Schmitt. Revisiting the method of sub- and supersolutions for nonlinear elliptic problems. In *Proceedings of the Sixth Mississippi State-UBA Conference on Differential Equations and Computational Simulations*, volume 15 of *Electron. J. Differ. Equ. Conf.*, pages 377–385, San Marcos, TX, 2007. Southwest Texas State Univ.
- [15] Matthias Schneider. Closed magnetic geodesics on S^2 . Preprint, [arXiv:0808.4038 \[math.DG\]](https://arxiv.org/abs/0808.4038), 2008.
- [16] Matthias Schneider. Alexandrov embedded closed magnetic geodesics on S^2 . Preprint, [arXiv:0903.1128 \[math.DG\]](https://arxiv.org/abs/0903.1128), 2009.
- [17] I. A. Taimanov. Non-self-intersecting closed extremals of multivalued or not everywhere-positive functionals. *Izv. Akad. Nauk SSSR Ser. Mat.*, 55(2):367–383, 1991.
- [18] I. A. Taimanov. Closed extremals on two-dimensional manifolds. *Uspekhi Mat. Nauk*, 47(2(284)):143–185, 223, 1992.
- [19] I. A. Taimanov. The type numbers of closed geodesics. *Regul. Chaotic Dyn.*, 15(1):84–100, 2010.
- [20] A. J. Tromba. A general approach to Morse theory. *J. Differential Geometry*, 12(1):47–85, 1977.
- [21] A. J. Tromba. The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree. *Adv. in Math.*, 28(2):148–173, 1978.

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