Tail behavior of stationary solutions of random difference equations: the case of regular matrices

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October 8, 2010

Given a sequence $(M_n, Q_n)_{n\geq 1}$ of i.i.d. random variables with generic copy (M, Q) such that M is a regular $d \times d$ matrix and Q takes values in $\mathbb{R}^d$, we consider the random difference equation (RDE) $R_n = M_n R_{n-1} + Q_n$, $n \geq 1$. Under suitable assumptions stated below, this equation has a unique stationary solution R such that, for some $\kappa > 0$ and some finite positive and continuous function K on $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$,

$$\lim_{t\to\infty} t^\kappa\, \mathbb{P}(xR > t) = K(x) \quad \text{for all } x \in S^{d-1}$$

holds true. A rather long proof of this result, originally stated by Kesten at the end of his famous article [12], was given by LePage [15]. The purpose of this article is to show how regeneration methods can be used to provide a much shorter argument (in particular for the positivity of K). It is based on a multidimensional extension of Goldie's implicit renewal theory developed in [9].

Keywords: Markov renewal theory; implicit renewal theory, Harris recurrence, regeneration, random operators and equations; stochastic difference equations; random dynamical systems

1 Introduction

Let $(M_n, Q_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with generic copy (M, Q) such that M is a real $d \times d$ matrix and Q takes values in $\mathbb{R}^d$. Suppose further that

$$\mathbb{E}\log^+ \|M\| < \infty \tag{A1}$$

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†Research supported by the Deutsche Forschungsgemeinschaft (SFB 878)

where $\|M\| := \sup_{|x|=1} |xM|$. Then, with $\Pi_n := M_1 \cdot ... \cdot M_n$, there exists $\beta \in [-\infty, \infty)$ such that

$$\beta := \lim_{n\to\infty} n^{-1} \log \|\Pi_n\| \quad \mathbb{P}\text{-a.s.}$$

and defines the Liapunov exponent of the RDE

$$R_n = M_n R_{n-1} + Q_n, \quad n \geq 1. \tag{1}$$

If β is negative and

$$\mathbb{E} \log^+ \|Q\| < \infty, \tag{A2}$$

then this recursive Markov chain has a unique stationary distribution which is given by the law of the almost surely convergent series

$$R := \sum_{n\geq 1} \Pi_{n-1} Q_n \tag{2}$$

and is also characterized as the unique solution to the stochastic fixed-point equation (SFPE)

$$Y \stackrel{d}{=} MY + Q \tag{3}$$

where $\stackrel{d}{=}$ means equality in law and where Y is understood to be independent of (M, Q). This by now standard result may easily be deduced from a more general one for iterations of random Lipschitz maps, see e.g. [8] or [6]. Our concern here is the tail behavior of R in the case when M takes almost surely values in $GL(d, \mathbb{R})$, the group of regular $d \times d$ matrices with real entries. For $x \in \mathbb{R}^d \setminus \{0\}$, we write $x^\sim$ for its projection on the unit sphere $S := S^{d-1}$, thus $x^\sim := |x|^{-1} x$. Lebesgue measure on the space of real $d \times d$-matrices, seen as $\mathbb{R}^{d^2}$, is denoted as $\lambda\!\!\!\lambda$ and the uniform distribution on S as $\lambda\!\!\!\lambda_S$. Finally, the open δ-balls in S and $GL(d, \mathbb{R})$ with centers x and A are denoted as $B_\delta(x)$ and $B_\delta(A)$, respectively.

Theorem 1.1. *Consider the RDE* (1) *and suppose that, in addition to* (A1), (A2) *and* $\beta < 0$, *the following assumptions hold:*

$$\mathbb{P}(M \in GL(d, \mathbb{R})) = 1. \tag{A3}$$

$$\max_{n\geq 1} \mathbb{P}((x\Pi_n)^\sim \in U) > 0 \textit{ for any } x \in S \textit{ and any open } \emptyset \neq U \subset S. \tag{A4}$$

$$\mathbb{P}(\Pi_{n_0} \in \cdot) \geq \gamma_0 \mathbf{1}_{B_c(\Gamma_0)} \lambda\!\!\!\lambda \textit{ for some } \Gamma_0 \in GL(d, \mathbb{R}),\ n_0 \in \mathbb{N} \textit{ and } c, \gamma_0 > 0. \tag{A5}$$

$$\mathbb{P}(Mv + Q = v) < 1 \textit{ for any column vector } v \in \mathbb{R}^d. \tag{A6}$$

There exists $\kappa_0 > 0$ *such that*

$$\mathbb{E} \inf_{x\in S} |xM|^{\kappa_0} \geq 1,\ \mathbb{E} \|M\|^{\kappa_0} \log^+ \|M\| < \infty \textit{ and } 0 < \mathbb{E} \|Q\|^{\kappa_0} < \infty. \tag{A7}$$

Then there exists a unique $\kappa \in (0, \kappa_0]$ *such that*

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E} \|\Pi_n\|^\kappa = 0, \tag{4}$$

and

$$\lim_{t\to\infty} t^\kappa\, \mathbb{P}(xR > t) = K(x) \quad \textit{for all } x \in S, \tag{5}$$

where K is a finite positive and continuous function on S.

Remark 1. This result (with one extra condition) was stated by Kesten at the end of his famous article [12] and later proved by Le Page [15] with the help of Kesten's Markov renewal theorem [13] (and without assuming (A5)). Markov renewal theory also plays an essential role in our approach, but we make use of a different Markov renewal theorem taken from [1] to show how proofs can be shortened considerably using Harris recurrence, which is the primary intention of this article. Condition (A5) plays a crucial role in obtaining (5) for all $x \in S$. Not assumed by LePage, he instead imposes the extra condition

$$\mathbb{E}\,\|Q\|^{\kappa_0+\varepsilon} < \infty \quad \text{for some } \varepsilon > 0$$

to arrive at the same conclusion. A similar result was also derived by Klüppelberg and Pergamenchtchikov [14] for a more specialized model. As further references, we mention related work by de Saporta et al. [5], by Guivarc'h [10] and, most recently, by Buraczewski et al. [3] who obtain more precise information on the tails of R under the restriction that M is a similarity (product of a dilation and an orthogonal transformation).

Remark 2. The most interesting ingredient to our approach may be roughly described as a suitable combination of Goldie's implicit renewal theory [9], lifted to the multidimensional situation, with the technique of sampling along "nice" regeneration epochs for the considered RDE (see Sections 6, 7 and 8).

Remark 3. Assumption (A6) is a condition on the dependence of M and Q (there is no need for independence), and asserts particularly that no Dirac measure solves the RDE. In fact our assumptions assure a priori, that $\operatorname{supp} R$ is unbounded in $\mathbb{R}^d$ (see Lemma 8.1).

Remark 4. Note that condition $\mathbb{E}\inf_{x\in S}|xM|^{\kappa_0} \geq 1$ in (A7) may be restated as

$$\mathbb{E}\,\lambda_d(MM^\top)^{\kappa_0/2} \geq 1,$$

where $\lambda_d(MM^\top)$ denotes the smallest of the d eigenvalues of the symmetric matrix $MM^\top$. This follows because $|xM| = (xMM^\top x^\top)^{1/2}$.

The further organization is as follows: Section 2 discusses the two central assumptions (A4) and (A5) in terms of their implications for obtaining Harris recurrence of an intrinsic Markov chain $(X_n)_{n\geq 0}$ on the sphere (see Section 5). We then proceed in Section 3 with some useful results concerning a whole class of SFPE that are solved by R and obtained via the use of stopping times. In particular, we explain how geometric sampling allows us to simplify some assumptions in Theorem 1.1 before proving it. Section 4 collects some facts about Harris recurrence and Markov renewal theory which are used in Section 5 to show that $\lim_{t\to\infty} t^{-\kappa}\,\mathbb{P}\,(\sup_{n\in\mathbb{N}}|xM_1\cdots M_n| > t)$ exists and is positive. This section further contains all necessary ingredients for the Markov renewal approach including the crucial measure change (harmonic transform) also used by Kesten. The proof of Theorem 1.1 is then provided in Sections 6, 7 and 8.

2 Minorization: Implications of (A4) and (A5)

It is useful to discuss at this early point the implications of the two conditions (A4) and (A5) in terms of the semigroup $(P^n)_{n\geq 1}$ of Markov transition kernels on S, defined by $P^n(x, A) :=$

$\mathbb{P}((x\Pi_n)^\sim \in A)$ for $x \in S$ and measurable $A \subset S$. The pertinent Markov chain being of interest here will be introduced in Section 5. For compact subsets C of $GL(d,\mathbb{R})$, we further define the substochastic kernels $P_C^n(x,\cdot) := \mathbb{P}((x\Pi_n)^\sim \in \cdot, \Pi_n \in C)$. Let I denote the identity matrix.

Lemma 2.1. *Suppose* (A4) *and* (A5). *Then for each $x \in S$ there exist $\delta, p > 0$, $m \in \mathbb{N}$, a compact subset C of $GL(d,\mathbb{R})$ and a probability measure ϕ on $B_\delta(x)$ such that*

$$P^m(y,\cdot) \geq P_C^m(y,\cdot) \geq p\phi \tag{MC}$$

for all $y \in S$.

Proof. Fix any $x \in S$. By (A4) and (A5), we can choose $n_1 \geq 1$, $\eta > 0$ and thereupon $0 < \delta < \eta$, $0 < \varsigma < c/2$ and a compact $B_1 \subset GL(d,\mathbb{R})$ in such a way that

(i) $\zeta := \inf_{y \in B_\delta(x)} P_{B_1}^{n_1}(y,U) > 0$, where $U := B_\eta\left((x\Gamma_0^{-1})^\sim\right)$;

(ii) $\Phi(A) := \int_{B_\varsigma(I)} \mathbf{1}_A((x\mathfrak{m})^\sim)\, \lambda\!\!\lambda(d\mathfrak{m})$ defines nonzero measure on $B_\delta(x)$;

(iii) for any $u \in U$, we have $u = (xF_u\Gamma_0^{-1})^\sim$ as well as $F_u\Gamma_0^{-1}B_{2\varsigma}(\Gamma_0) \supset B_\varsigma(I)$ for some unitary matrix F_u.

Put $C := B_1 \cdot \overline{B_{2\varsigma}(\Gamma_0)} := \{\Lambda_1\Lambda_2 : \Lambda_1 \in B_1, \Lambda_2 \in \overline{B_{2\varsigma}(\Gamma_0)}\}$, which is a compact subset of $GL(d,\mathbb{R})$ (as the continuous image of the compact $B_1 \times \overline{B_{2\varsigma}(\Gamma_0)}$). It then follows for any $y \in B_\delta(x)$ and measurable $A \subset S$ that

$$\begin{aligned}
P_C^{n_0+n_1}(y,A) &\geq \int_U \int_{B_{2\varsigma}(\Gamma_0)} \mathbf{1}_A((u\mathfrak{m})^\sim)\, \mathbb{P}(\Pi_{n_0} \in d\mathfrak{m})\, P_{B_1}^{n_1}(y,du) \\
&\geq \gamma_0 \int_U \int_{B_{2\varsigma}(\Gamma_0)} \mathbf{1}_A((u\mathfrak{m})^\sim)\, \lambda\!\!\lambda(d\mathfrak{m})\, P_{B_1}^{n_1}(y,du) \\
&= \gamma_0 \int_U \int_{B_{2\varsigma}(\Gamma_0)} \mathbf{1}_A((xF_u\Gamma_0^{-1}\mathfrak{m})^\sim)\, \lambda\!\!\lambda(d\mathfrak{m})\, P_B^{n_1}(y,du) \\
&\geq \gamma_0 |\det(\Gamma_0)|^d\, P_{B_1}^{n_1}(y,U) \int_{B_\varsigma(I)} \mathbf{1}_A((x\mathfrak{m})^\sim)\, \lambda\!\!\lambda(d\mathfrak{m}) \\
&\geq \gamma_0\zeta |\det(\Gamma_0)|^d \int_{B_\varsigma(I)} \mathbf{1}_A((x\mathfrak{m})^\sim)\, \lambda\!\!\lambda(d\mathfrak{m})
\end{aligned} \tag{6}$$

which proves (MC) for all $y \in B_\delta(x)$ with $m = n_0 + n_1$ and $\phi := \Phi(B_\delta(x))^{-1}\Phi$.

In order to extend (MC) to all $y \in S$, observe that for any y, we can pick $\varepsilon(y) > 0$, $n_2(y) \geq 1$ and compact $B^y \subset GL(d,\mathbb{R})$ such that

$$\inf_{z \in B_{\varepsilon(y)}(y)} P_{B^y}^{n_2(y)}(z, B_\delta(x)) > 0.$$

By compactness, $S = \bigcup_{i=1}^k B_{\varepsilon(y_i)}(y_i)$ for suitable $y_1,...,y_k$, and a straightforward argument then shows that $\inf_{y \in S} P_{B_2}^{n_2}(y, B_\delta(x)) > 0$ for a suitable $n_2 \geq \max_{i=1,\ldots,k} n_2(y_i)$ and with $B_2 := \bigcup_{i=1}^k B^{y_i}$. It is now readily seen with the help of property (i) that

(iv) $\xi := \inf_{y \in S} P_B^{n_1+n_2}(y,U) > 0$, where $B := B_1 \cdot B_2$.

By estimating $P_C^{n_0+n_1+n_2}$ with $C := B \cdot \overline{B_{2\varsigma}(\Gamma_0)}$ following (6) and utilizing (iv) instead of (i), we finally obtain (MC) for all $y \in S$ (with the same ϕ and $m = n_0 + n_1 + n_2$). Further details can be omitted. □

Remark 5. It is useful for later purposes (see Lemma 5.6) to point out that (MC) is "embedded" in a bivariate condition, obtained via consideration of the bivariate extensions $H^n(x,\cdot) := \mathbb{P}(((x\Pi_n)^\sim, \Pi_n) \in \cdot)$ and $H_C^n(x,\cdot) := \mathbb{P}(((x\Pi_n)^\sim, \Pi_n) \in \cdot, \Pi_n \in C)$ of $P^n(x,\cdot)$ and $P_C^n(x,\cdot)$, respectively. Keeping notation and settings from above and with $\gamma_1 := |\det(\Gamma_0)|^{-d} \max_{\mathfrak{m}\in B} |\det(\mathfrak{m})|^d$ finite, a similar estimation as in (6) leads to

$$\begin{aligned}
&H_C^{n_0+n_1+n_2}(y, D) \\
&\quad \geq \gamma_0 \int\limits_{U\times B}\int\limits_{B_{2\varsigma}(\Gamma_0)} \mathbf{1}_D((u\mathfrak{m}_0)^\sim, \mathfrak{m}_1\mathfrak{m}_0)\ \lambda\!\!\!\lambda(d\mathfrak{m}_0)\, H_B^{n_1+n_2}(y, d(u,\mathfrak{m}_1)) \\
&\quad = \gamma_0 \int\limits_{U\times B}\int\limits_{B_{2\varsigma}(\Gamma_0)} \mathbf{1}_D((xF_u\Gamma_0^{-1}\mathfrak{m}_0)^\sim, \mathfrak{m}_1\mathfrak{m}_0)\ \lambda\!\!\!\lambda(d\mathfrak{m}_0)\, H_B^{n_1+n_2}(y, d(u,\mathfrak{m}_1)) \\
&\quad \geq \frac{\gamma_0}{\gamma_1} \int\limits_{U\times B}\int\limits_{B_{\varsigma}(I)} \mathbf{1}_D((x\mathfrak{m}_0)^\sim, \mathfrak{m}_1\Gamma_0F_u^{-1}\mathfrak{m}_0)\ \lambda\!\!\!\lambda(d\mathfrak{m}_0)\, H_B^{n_1+n_2}(y, d(u,\mathfrak{m}_1))
\end{aligned}$$

and thus to the bivariate minorization condition

$$H_C^{n_0+n_1+n_2}(y,\cdot) \geq q\,\psi(y,\cdot) \tag{BMC}$$

for all $y \in S$, some $q > 0$ and a probability kernel $\psi(y,\cdot)$ on $S \times C$. It contains (MC) as a special case, for $P^{n_0+n_1+n_2}(y,\cdot) = H^{n_0+n_1+n_2}(y,\cdot\times C)$ and $\psi(\cdot\times C) = \phi$.

3 The stopped RDE and geometric sampling

Geometric sampling and, more generally, the use of stopping times for $(M_n, Q_n)_{n\geq 1}$ provides a useful technique in our subsequent analysis and is thus briefly discussed next.

3.1 R remains solution to the stopped equation

Let $(\mathcal{G}_n)_{n\geq 0}$ be a filtration such that $(M_n, Q_n)_{n\geq 1}$ is adapted to it and $(M_k, Q_k)_{k>n}$ is independent of $\mathcal{G}_n$ for any $n \geq 0$. Consider any a.s. finite stopping time τ with respect to $(\mathcal{G}_n)_{n\geq 0}$ which, by suitable choice of the latter, includes the case that τ and $(M_n, Q_n)_{n\geq 1}$ are independent (pure randomization). Then it is readily checked that R defined in (2) satisfies

$$R = \Pi_\tau R^\tau + Q^\tau \tag{7}$$

where

$$Q^n := \sum_{k=1}^{n} \Pi_{k-1} Q_k \quad \text{and} \quad R^n := \sum_{k>n} \left(\prod_{j=n+1}^{k-1} M_j\right) Q_k$$

for $n \geq 1$. But since $(M_{\tau+n}, Q_{\tau+n})_{n\geq 1}$ is a copy of $(M_n, Q_n)_{n\geq 1}$ and independent of $(M_n, Q_n)_{1\leq n\leq \tau}$ and τ, it follows that R^τ is independent of (Π_τ, Q^τ) with $R^\tau \stackrel{d}{=} R$. In other words, (the law of) R also solves the stopped SFPE

$$Y \stackrel{d}{=} \Pi_\tau Y + Q^\tau \tag{8}$$

and provides a stationary distribution to the RDE

$$Y_n = M_n' Y_{n-1} + Q_n', \quad n \geq 1, \tag{9}$$

where $(M_n', Q_n')_{n\geq 1}$ is a sequence of i.i.d. copies of (Π_τ, Q^τ). Uniqueness follows if (A1), (A2) persist to hold for the "stopped pair" (Π_τ, Q^τ) together with

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_{\sigma_n}\| < 0 \ \mathbb{P}\text{-a.s.}$$

where $(\sigma_n)_{n\geq 0}$ denotes a zero-delayed renewal process such that $\sigma_1 = \tau$ and

$$\left(\sigma_n - \sigma_{n-1}, (M_k, Q_k)_{\sigma_{n-1}<k\leq\sigma_n}\right), \quad n \geq 1$$

are i.i.d. For stopping times τ with finite mean this is indeed easily verified and we state the result (without proof) in the following lemma.

Lemma 3.1. *The law of R forms the unique solution to the SFPE* (8) *whenever* $\mathbb{E}\,\tau < \infty$.

In order for finding the tail behavior of R, we are now allowed to do so within the framework of any stopped SFPE (8) with finite mean τ. The idea is to pick τ in such a way that (Π_τ, Q^τ) has nice additional properties compared to (M, Q). Geometric sampling provides a typical example that will be used hereafter and therefore discussed next. Another use of this technique appears in Section 8.

3.2 Geometric sampling

Suppose now that $(\sigma_n)_{n\geq 0}$ is independent of $(M_n, Q_n)_{n\geq 1}$ with geometric(1/2) increments, that is $\mathbb{P}(\tau = n) = 1/2^n$ for each $n \geq 1$. Then not only Lemma 3.1 holds true but also the following result:

Lemma 3.2. *If (M, Q) satisfies the assumption of Theorem 1.1 and thus* (MC)*, then so does (Π_τ, Q^τ) with $n = n_0 = m = 1$ in* (A4), (A5) *and* (MC)*. Also,* $\lim_{n\to\infty} n^{-1} \log \mathbb{E}\,\|\Pi_{\sigma_n}\|^\kappa = 0$ *implies* $\lim_{n\to\infty} n^{-1} \log \mathbb{E}\,\|\Pi_n\|^\kappa = 0$*, i.e.* (4).

Proof. That (A1), (A2) and $\lim_{n\to\infty} n^{-1} \log \|\Pi_{\sigma_n}\| < 0$ $\mathbb{P}$-a.s. persist to hold under any finite mean stopping time τ has already been pointed out before Lemma 3.1. As for (A3) to (A5), we just note that $\mathbb{P}(\Pi_\tau \in \cdot) = \sum_{k\geq 1} 2^{-n} \mathbb{P}(\Pi_n \in \cdot)$. Assumption (A6) ensures that the law of R is nondegenerate. But since R is also the unique solution to (8), (A6) must hold for (Π_τ, Q^τ) as well. Finally,

$$\mathbb{E} \inf_{x\in S} |x\Pi_\tau| = \sum_{n\geq 1} 2^{-n}\, \mathbb{E} \inf_{x\in S} |x\Pi_n|$$

in combination with

$$\begin{aligned}\mathbb{E}\inf_{x\in S}|x\Pi_n|^{\kappa_0} &= \mathbb{E}\left(\inf_{x\in S}|(xM_1\cdot\ldots\cdot M_{n-1})^{\sim}M_n|^{\kappa_0}\cdot|xM_1\cdot\ldots\cdot M_{n-1}|^{\kappa_0}\right)\\ &\ge \mathbb{E}\left(\inf_{x\in S}|xM_n|^{\kappa_0}\cdot\inf_{x\in S}|x\Pi_{n-1}|^{\kappa_0}\right)\\ &= \mathbb{E}\inf_{x\in S}|xM_n|^{\kappa_0}\ \mathbb{E}\inf_{x\in S}|x\Pi_{n-1}|^{\kappa_0}\\ &= \ldots = \left(\mathbb{E}\inf_{x\in S}|xM|^{\kappa_0}\right)^n \ge 1 \qquad (10)\end{aligned}$$

for each $n\ge 1$ shows the first assertion of (A7) for (Π_τ, Q^τ). The remaining two moment assertions are again easily verified by standard estimates. We therefore omit further details. Finally, suppose that $\lim_{n\to\infty} n^{-1}\log\mathbb{E}\,\|\Pi_{\sigma_n}\|^\kappa = 0$ By subadditivity, $\xi := \lim_{n\to\infty} n^{-1}\log\mathbb{E}\,\|\Pi_n\|^\kappa = \inf_{n\ge 1} n^{-1}\log\mathbb{E}\,\|\Pi_n\|^\kappa$ exists in $[-\infty,\infty)$. Since

$$\frac{1}{n}\log\mathbb{E}\,\|\Pi_{\sigma_n}\|^\kappa = \frac{1}{n}\log\sum_{k\ge n}\mathbb{E}\,\|\Pi_k\|^\kappa\ \mathbb{P}(\sigma_n=k) \ge \frac{1}{n}\sum_{k\ge n}\mathbb{P}(\sigma_n=k)\ \log\mathbb{E}\,\|\Pi_k\|^\kappa$$

it is not difficult to see that $\xi > -\infty$. But then we further infer for any $\varepsilon>0$ and all sufficiently large n that

$$\frac{1}{n}\log\mathbb{E}\,\|\Pi_{\sigma_n}\|^\kappa \ge \frac{1}{n}\log\sum_{k\ge n}e^{k(\xi-\varepsilon)}\ \mathbb{P}(\sigma_n=k) = \log\mathbb{E}\,e^{(\xi-\varepsilon)\tau}$$

and thus $\xi\le 0$ upon taking $n\to\infty$ and then $\varepsilon\to 0$. By doing the same in the reverse inequality

$$\frac{1}{n}\log\mathbb{E}\,\|\Pi_{\sigma_n}\|^\kappa \le \frac{1}{n}\log\sum_{k\ge n}e^{k(\xi+\varepsilon)}\ \mathbb{P}(\sigma_n=k) = \log\mathbb{E}\,e^{(\xi+\varepsilon)\tau}$$

finally shows $\xi=0$ as claimed in (4). □

In view of the previous lemma we can now make the **standing assumption** that

$$\text{(A4),(A5),(MC) and (BMC) hold with } n_0=n=m=1. \qquad \text{(SA)}$$

4 Harris recurrence and Markov renewal theory

4.1 Strongly aperiodic Harris chains

Here and in the following subsection let S be a general separable metric space with Borel-σ-algebra $\mathcal{S}$. A Markov chain $(X_n)_{n\ge 0}$ on S is called *strongly aperiodic Harris chain*, if there exists a set $\mathfrak{R}\in\mathcal{S}$, called *regeneration set*, such that $\mathbb{P}_x(X_n\in\mathfrak{R}\text{ infinitely often}) = 1$ for all $x\in S$ (recurrence) and, furthermore,

$$\inf_{x\in\mathfrak{R}} P(x,\cdot) \ge p\,\phi \qquad (11)$$

for some $p>0$, $r\in\mathbb{N}$ and a probability measure ϕ with $\phi(\mathfrak{R})=1$. Strong aperiodicity refers to the fact that P and not P^m for some $m\ge 2$ satsifies (11). If S itself is regenerative then

$(X_n)_{n\geq 0}$ is called *Doeblin chain*. A strongly aperiodic Harris chain $(X_n)_{n\geq 0}$ possesses a nice regenerative structure as shown by the following regeneration lemma due to Athreya and Ney [2].

Lemma 4.1. *On a possibly enlarged probability space, one can redefine $(X_n)_{n\geq 0}$ together with an increasing sequence $(\sigma_n)_{n\geq 0}$ of random epochs such that the following conditions are fulfilled under any $\mathbb{P}_x$, $x \in S$:*

(R1) There is a filtration $\mathcal{G} = (\mathcal{G}_n)_{n\geq 0}$ such that $(X_n)_{n\geq 0}$ is Markov adapted and each σ_n a stopping time with respect to $\mathcal{G}$.

(R2) $(\sigma_n - \sigma_1)_{n\geq 1}$ forms a zero-delayed renewal sequence with increment distribution $\mathbb{P}_\phi(\sigma_1 \in \cdot)$ and is independent of σ_1.

(R3) $\forall\ k \geq 1$, $(X_{\sigma_k+n})_{n\geq 0}$ is independent of $(X_j)_{0\leq j\leq \sigma_k - 1}$ with distribution $\mathbb{P}_\phi((X_n)_{n\geq 0} \in \cdot)$.

The σ_n, called *regeneration epochs*, are obtained by the following coin-tossing procedure: If ν_n, $n \geq 1$, denote the successive return times of the chain to $\mathfrak{R}$, then at each such ν_n a p-coin is tossed. If head comes up, then X_{ν_n+1} is generated according to ϕ, while it is generated according to $(1-p)^{-1}(P(X_{\nu_n}, \cdot) - p\phi)$ otherwise. Hence, the $\sigma_n - 1$ are those return epochs at which the coin toss produces a head.

4.2 Markov renewal theory

Let $(X_n, U_n)_{n\geq 0}$ be a temporally homogeneous Markov chain on $S \times \mathbb{R}$ such that

$$\mathbb{P}\left((X_{n+1}, U_{n+1}) \in A \times B | X_n, U_n\right) = P(X_n, A \times B) \quad \text{a.s.}$$

for all $n \geq 0$ and a transition kernel P. Then the associated sequence $(X_n, V_n)_{n\geq 0}$ with $V_n = V_{n-1} + U_n$ for $n \geq 1$ is also a Markov chain and called *Markov Random Walk (MRW) with driving chain* $(X_n)_{n\geq 0}$. This extends the notion of classical random walk with i.i.d. increments because, conditioned on $(X_n)_{n\geq 0}$, the U_n are independent, but no longer identically distributed. In fact, the conditional distribution of U_n given $(X_k)_{k\geq 0}$ is of the form $Q((X_{n-1}, X_n), \cdot)$ for each $n \geq 1$ and a suitable stochastic kernel Q. The MRW is called *d-arithmetic*, if there exists a minimal $d > 0$ and a measurable function $\gamma : S \to [0, d)$ such that

$$\mathbb{P}\left(U_1 - \gamma(x) + \gamma(y) \in d\,\mathbb{Z}\,|X_0 = x, X_1 = y\right) = 1$$

for $P_\pi((X_0, X_1) \in \cdot)$ almost all $(x, y) \in S^2$, and *nonarithmetic* otherwise. As usual, for any distribution λ on S, $\mathbb{P}_\lambda$ means $\mathbb{P}_\lambda((X_0, V_0) \in \cdot) = \lambda \otimes \delta_0$. The Markov renewal measure $\sum_{n\geq 0} \mathbb{P}_\lambda((X_n, V_n) \in \cdot)$ associated with the given MRW under $\mathbb{P}_\lambda$ is denoted as $\mathbb{U}_\lambda$.

Being enough for our purposes, we focus hereafter on the case when the driving chain is a strongly aperiodic Harris chain on compact state space and thus having a a unique stationary distribution, denoted as π.

Defining the first exit time $\tau(t) := \inf\{n \geq 0 : V_n > t\}$ consider the residual lifetime process $R_t := (V_{\tau(t)} - t)\mathbf{1}_{\{\tau(t)<\infty\}}$ and the jump process $Z(t) := X_{\tau(t)}\mathbf{1}_{\{\tau(t)<\infty\}}$. A measurable function

$g : S \times \mathbb{R} \to \mathbb{R}$ is called *π-directly Riemann integrable* if

$$g(x,\cdot) \text{ is } \lambda\text{-a.e. continuous for } \pi\text{-almost all } x \in S \tag{12}$$

$$\text{and} \quad \int_S \sum_{n\in\mathbb{Z}} \sup_{t\in[n\delta,(n+1)\delta)} |g(x,t)|\,\pi(dx) < \infty \quad \text{for some } \delta > 0, \tag{13}$$

where λ denotes Lebesgue measure on $\mathbb{R}$. The following Markov renewal theorem (MRT) is the main result of [1]:

Theorem 4.2. *Let $(X_n, V_n)_{n\geq 0}$ be a nonarithmetic MRW with strongly aperiodic Harris driving chain $(X_n)_{n\geq 0}$ with stationary distribution π. Let $\alpha := \mathbb{E}_\pi V_1 > 0$. If $g : S \times \mathbb{R} \to \mathbb{R}$ is π-directly Riemann integrable then, for π-almost all $x \in S$,*

$$g * \mathbb{U}_x(t) := \mathbb{E}_x \left(\sum_{n\geq 0} g(X_n, t - V_n) \right) \to \frac{1}{\alpha} \int_S \int_{\mathbb{R}} g(u,v)\,dv\,\pi(du). \tag{14}$$

as $t \to \infty$. Moreover, if $f : S \times (0,\infty) \to (0,\infty)$ is bounded and continuous, then

$$\lim_{t\to\infty} \mathbb{E}_x \left(f(Z(t), R(t)) \mathbf{1}_{\{\tau(t)<\infty\}} \right) = L(f) \tag{15}$$

for π-almost all $x \in S$ and some constant $L(f) > 0$.

Remark 6. The following extension of the above result follows directly upon inspection of the coupling proof given in [1, Section 7]: If ϕ is any minorizing distribution for the transition kernel of the Harris driving chain $(X_n)_{n\geq 0}$, then $g * \mathbb{U}_\phi(t)$ is a bounded function and converges to the limit given in (14). This fact will be used in Section 7.

Remark 7. Note that $(V_n)_{n\geq 0}$ satisfies the strong law of large numbers, viz.

$$\lim_{n\to\infty} \frac{V_n}{n} = \alpha \quad \mathbb{P}_x\text{-a.s.} \tag{16}$$

for all $x \in S$.

5 Measure change and tail behaviour of $\sup_{n\geq 1} |x\Pi_n|$

Returning to the model described in the Introduction, we proceed with a short account of the ideas in [12] and [15]. Recall that $S = S^{d-1}$ and define

$$X_n := (X_0\Pi_n)^\sim \quad \text{and} \quad U_n := \log |X_{n-1}M_n|.$$

Since $U_n = \log|X_0\Pi_n| - \log|X_0\Pi_{n-1}|$, the sequence $(X_n, V_n)_{n\geq 0}$ forms a MRW on $S \times \mathbb{R}$ with initial values (X_0, V_0), and

$$\{\tau(t) < \infty\} = \left\{ \sup_{n\geq 1} \log |X_0\Pi_n| > t \right\}.$$

Under the assumptions of Theorem 1.1 (with $n_0 = n = 1$ in (A4) and (A5)), $(X_n)_{n\ge 0}$ is easily seen to be a strongly aperiodic Harris chain with transition kernel $P = P^1$ defined in Section 2. However, $(X_n, V_n)_{n\ge 0}$ does not satisfy the conditions of the MRT, since by (16),

$$\alpha = \lim_{n\to\infty} \frac{\log |X_0 \Pi_n|}{n} \le \lim_{n\to\infty} \frac{\log \|\Pi_n\|}{n} = \beta < 0 \quad \mathbb{P}\text{-a.s.}$$

A MRW with positive drift is indeed obtained after a change of measure (harmonic transform) for which it is crucial that $\mathbb{P}(\log \|M\| > 0) > 0$ which in turn follows from assumption (A7).

Theorem 5.1. *Under the assumptions of Theorem 1.1, there exist $\kappa \in (0, \kappa_0]$ and a positive continuous function $r : S \to (0, \infty)$ such that*

$$\begin{aligned} &{}^{\kappa}\mathbb{E}_x f(X_0, V_0, X_1, V_1, \ldots, X_n, V_n) \\ &\qquad := \frac{1}{r(x)}\, \mathbb{E}_x \Big(r(X_n) e^{\kappa V_n} f\left(X_0, V_0, X_1, V_1, \ldots, X_n, V_n\right) \Big), \end{aligned} \tag{17}$$

for all bounded continuous functions f and all $n \ge 0$, defines a distribution $\mathbb{P}^\kappa$. Under $\mathbb{P}^\kappa$, $(X_n, V_n)_{n\ge 0}$ is a MRW with positive drift and satisfies the assumptions of Theorem 4.2. The constant κ is the unique value such that (4) holds true, i.e.

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E}\, \|\Pi_n\|^{\varkappa} = 0.$$

5.1 Proof of Theorem 5.1: Choice of κ and r

Defining the positive operators

$$T_\varkappa : \mathcal{C}(S) \to \mathcal{C}(S), \quad f(x) \mapsto \mathbb{E}\left(e^{\varkappa \log |xM|} f((xM)^\sim) \right), \quad \varkappa \in (0, \kappa_0],$$

we must find κ such that T_κ has maximal eigenvalue 1 with positive eigenfunction r. Due to our standing assumption, $T_\varkappa$ is even strictly positive in the sense that $T_\varkappa f$ is everywhere positive whenever $0 \ne f \ge 0$. Indeed, for any such f, the set $U_f = \{f > 0\}$ is nonempty and open by continuity whence, using (A4) with $n = 1$, we infer

$$T_\varkappa f(x) \ge \int |y|^\varkappa\, \mathbf{1}_{U_f}(y^\sim)\, f(y^\sim)\; \mathbb{P}(xM \in dy) > 0$$

for all $x \in S$. The strict positivity will enable us to provide an elegant proof of the important Lemma 5.4 below.

Lemma 5.2. *Let $\varrho(\varkappa)$ be the spectral radius of $T_\varkappa$, i.e. $\varrho(\varkappa) = \lim_{n\to\infty} \|T_\varkappa^n\|^{1/n}$. Then $T_\varkappa$ has an eigenvalue of maximal modulus equal to $\varrho(\varkappa)$.*

Proof. The adjoint operator $T_\varkappa^* : \mathcal{C}(S)^* \to \mathcal{C}(S)^*$, $\mathcal{C}(S)^*$ being the space of regular bounded signed measures on $\mathcal{S}$, is weakly compact, i.e. it maps bounded sets to weakly sequentially compact sets. This follows by Prokhorov's theorem because

$$\|T_\varkappa^*\| = \|T_\varkappa\| \le \mathbb{E}\, \|M\|^\varkappa < \infty$$

and S is compact. By [7, Theorem VI.4.8], $T_\varkappa$ is then weakly compact as well, and by [7, Corollary VI.7.5], $T_\varkappa^2$ is compact. Hence, by [7, Lemmata VII.4.5 & 6], the spectrum of $T_\varkappa$ is pure point (maybe except for 0) and $T_\varkappa$ possesses an eigenvalue $\lambda_\varkappa$ that is maximal in modulus, i.e. $|\lambda_\varkappa| = \varrho(\varkappa)$. ∎

The following argument shows the existence of $\kappa \in (0, \kappa_0]$ with $\varrho(\kappa) = 1$. As one can readily verify by induction, $T_\kappa^n f(x) = \mathbb{E}\left(e^{\kappa \log|x\Pi_n|} f((x\Pi_n)^\sim)\right)$, and we infer $\varrho(\kappa_0) \geq 1$ upon choosing $f = \mathbf{1}_S$ and using (10). If $\varrho(\kappa_0) = 1$ we are done, so suppose that $\varrho(\kappa_0) > 1$ and thus $\|T_{\kappa_0}^n\| = \sup_{x\in S} \mathbb{E}\left(e^{\kappa_0 \log|x\Pi_n|}\right) > 1$ for all sufficiently large n. Since $\varkappa \mapsto \|T_\varkappa^n f\|$ is log-convex and thus continuous on $(0, \kappa_0)$ and lower semicontinuous on $(0, \kappa_0]$ for each $f \in \mathcal{C}(S)$ and $n \geq 1$ (use Hölder's inequality), the same holds true for $\varkappa \mapsto \|T_\varkappa^n\|$ as its pointwise supremum. It follows that $\|T_{\kappa_1}^n\| > 1$ for some $\kappa_1 \in (0, \kappa_0)$ and all sufficiently large n and therefore $\varrho(\kappa_1) \geq 1$. But we also have $\varrho(\varkappa) < 1$ for some $\varkappa \in (0, \kappa_0)$ because $\beta < 0$ (and by the Furstenberg-Kesten theorem). Hence, $\varrho(\kappa) = 1$ for some $\kappa \in (0, \kappa_0)$ which is unique by strict convexity of ϱ. That κ also satisfies (4) follows from the following more general lemma.

Lemma 5.3. *For each $\varkappa \in (0, \kappa_0]$,*

$$\varrho(\varkappa) = \lim_{n\to\infty} (\mathbb{E}\,\|\Pi_n\|^\varkappa)^{1/n}.$$

Proof. Obviously, $\varrho(\varkappa) = \lim_{n\to\infty} \sup_{x\in S} (\mathbb{E}\,|x\Pi_n|)^{1/n} \leq \liminf_{n\to\infty} (\mathbb{E}\,\|\Pi_n\|^\varkappa)^{1/n}$. For the converse note that, by [4, Prop. 3.2], $Z_{x_0} := \inf_{n\geq 0} \|\Pi_n\|^{-1} |x_0\Pi_n| > 0$ a.s. for any $x_0 \in S$, whence

$$\sup_{x\in S} \mathbb{E}\,|x\Pi_n|^\varkappa \geq \mathbb{E}\,\|\Pi_n\|^\varkappa \frac{\mathbb{E}\,|x_0\Pi_n|^\varkappa}{\mathbb{E}\,\|\Pi_n\|^\varkappa} \geq \mathbb{E}\,\|\Pi_n\|^\varkappa \frac{\mathbb{E}\,Z_{x_0} \|\Pi_n\|^\varkappa}{\mathbb{E}\,\|\Pi_n\|^\varkappa}$$

and therefore (using Jensen's inequality)

$$\varrho(\varkappa) \geq \limsup_{n\to\infty} (\mathbb{E}\,\|\Pi_n\|^\varkappa)^{1/n} \lim_{n\to\infty} \frac{\mathbb{E}\,Z_{x_0}^{1/n} \|\Pi_n\|^\varkappa}{\mathbb{E}\,\|\Pi_n\|^\varkappa} = \limsup_{n\to\infty} (\mathbb{E}\,\|\Pi_n\|^\varkappa)^{1/n}.$$

which completes the proof. □

Lemma 5.4. *Let $\varrho(\kappa) = 1$. Then T_κ has maximal eigenvalue 1 with one-dimensional eigenspace containing a positive eigenfunction r which further is symmetric, i.e. $r(x) = r(-x)$ for all $x \in S$.*

Proof. The following argument goes back to Karlin [11, Section 5] and hinges on the strict positivity of T_κ. By Lemma 5.2, T_κ has eigenvalue λ with $|\lambda| = 1$. Let f be a corresponding eigenfunction. Obviously, $T_\kappa f = \lambda f$ implies

$$T_\kappa |f| \geq |f|.$$

Suppose that $T_\kappa |f| - |f| \neq 0$. Then, by the strict positivity of T_κ, we have that $T_\kappa(T_\kappa |f| - |f|)$ is positive and thus $> \eta$ for some $\eta > 0$ chosen small such that, furthermore, $T_\kappa |f| < 1/\eta$ (S compact). From this we further infer

$$T_\kappa^2 |f| - T_\kappa |f| > \eta > \eta^2 T_\kappa |f|, \quad \text{hence } T_\kappa^2 |f| > (1+\eta^2) T_\kappa |f|$$

and thereby $T_\kappa^n T_\kappa |f| > (1+\eta^2)^n T_\kappa |f|$ for all $n \geq 1$ upon iteration. Consequently, $\|T_\kappa^n\| > (1+\eta^2)^n$ for all $n \geq 1$ and thus $\varrho(\kappa) > 1$, a contradiction that leads to the conclusion that $T_\kappa |f| = |f|$ und thus that $r := |f|$ is a positive eigenfunction for the eigenvalue 1.

Now, suppose there is another eigenfunction g, linearly independent of r and w.l.o.g. real-valued (for, if g is an eigenfunction, then so are its real and imaginary parts if nontrivial).

Pick ε such that $h := r + \varepsilon g$ is nonnegative, but $h(x) = 0$ for some x. By linear independence, h does not vanish everywhere. Since it is again an eigenfunction, the strict positivity of T_κ implies that it must be positive everywhere which is a contradiction. Hence r must be the unique eigenfunction modulo scalars.

Finally, we must prove the asserted symmetry of r. To this end note first that T_κ maps symmetric functions to symmetric functions. Its weak compactness entails that T_κ^2 is a compact operator [7, Corollary VI.7.5] and thus maps bounded sequences to sequences with (strongly) convergent subsequences. As a consequence, any accumulation point g of the bounded sequence $n^{-1}T_\kappa \sum_{k=1}^n T_\kappa^k \mathbf{1}_S$, $n \geq 1$, is a *continuous* positive symmetric function with $Tg = g$ and thus a multiple of r. Hence, r must be symmetric. $\square$

Now ${}^\kappa\widehat{P}f(x,t) := r(x)^{-1}\mathbb{E}\left(|xM|^\kappa f((xM)^\sim, t + \log|xM|)r((xM)^\sim)\right)$ defines a Markov transition kernel on $S \times \mathbb{R}$ corresponding to $({}^\kappa\mathbb{P}_x((X_n, V_n)_{n\geq 0} \in \cdot))_{x\in S}$ as defined by (17). Its associated "marginal"

$$ {}^\kappa Pf(x) := \frac{1}{r(x)}\mathbb{E}\left(|xM|^\kappa f((xM)^\sim)r((xM)^\sim)\right) \tag{18} $$

is the transition kernel of $(X_n)_{n\geq 0}$ under $({}^\kappa\mathbb{P}_x)_{x\in S}$.

5.2 Proof of Theorem 5.1: Checking the assumptions of the MRT

This section corresponds to [12, Proposition 2], but provides a much shorter proof, even if technical details not mentioned here had been included. A random variable $T \geq 0$ is called geometrically bounded hereafter if it has exponentially decreasing tails.

Lemma 5.5. *Suppose* (A4), (A5) *and* (SA). *Then* $(X_n)_{n\geq 0}$ *is a strongly aperiodic Doeblin chain on* S *under* ${}^\kappa\mathbb{P}$ *for every pair* (r,κ) *such that* (17) *defines a probability measure (including the trivial case* $(\mathbf{1}_S, 0)$*). Moreover, for every* $x \in S$ *there is some* $\delta > 0$ *such that* $B_\delta(x)$ *is a regeneration set.*

Proof. Let $x \in S$. By (MC) with $m = 1$, we know that $P(y,\cdot) \geq P_C(y,\cdot) \geq p\phi$ for suitable δ, p, C, ϕ and all $y \in S$ (see Lemma 2.1, especially (ii) in the proof for the definition of ϕ). In particular, $\inf_{y\in S} P(y, B_\delta(x)) \geq p\phi(B_\delta(x)) = p > 0$ which gives (uniformly) geometrically bounded times to reach $B_\delta(x)$ from any $y \in S$ under $\mathbb{P}$, that is, $B_\delta(x)$ is regenerative with respect to P. In order to get the same with respect to ${}^\kappa P$, we first note that

$$ \gamma_1 := \min_{z_1,z_2\in S} \frac{r(z_1)}{r(z_2)} \quad \text{and} \quad \gamma_2 := \min_{z\in S,\, \mathfrak{m}\in C} |z\mathfrak{m}|^\kappa $$

are clearly both positive (here the compactness of the set $C \subset GL(d,\mathbb{R})$ enters in a crucial way). But then we infer

$$ \begin{aligned} {}^\kappa P(y,A) \geq {}^\kappa P_C(y,A) &\geq \frac{1}{r(y)} \int_C |y\mathfrak{m}|^\kappa\, \mathbf{1}_A((y\mathfrak{m})^\sim)\, r((y\mathfrak{m})^\sim)\, \mathbb{P}(M \in d\mathfrak{m}) \\ &\geq \gamma_1\gamma_2 \int_C \mathbf{1}_A((y\mathfrak{m})^\sim)\, \mathbb{P}(M \in d\mathfrak{m}) \\ &= \gamma_1\gamma_2 P_C(y,A) \geq p\gamma_1\gamma_2\phi(A) \end{aligned} $$

for any measurable $A \subset B_\delta(x)$ and any $y \in S$, and this shows that $B_\delta(x)$ is regenerative with respect to ${}^\kappa P$ as well. □

In view of the previous result and Lemma 2.1 we infer that $(X_n)_{n\ge 0}$ is a strongly aperiodic Harris chain under both $\mathbb{P}$ and ${}^\kappa\mathbb{P}$, in fact with the same minorizing probability measure ϕ on a regenerative δ-ball $B_\delta(x)$. *The stationary distribution of this chain under* ${}^\kappa\mathbb{P}$ *is always denoted as* π *hereafter.*

Regarding the regeneration scheme resulting from the minorization condition (MC) with $m = 1$, we point out the following important supplement that hinges on the bivariate extension (BMC) stated in Remark 5 (with $n_0 + n_1 + n_2 = 1$ due to our standing assumption).

Lemma 5.6. *Let $B_\delta(x)$ be regenerative with minorizing probability measure ϕ as defined in Lemma 2.1. Then under both, $\mathbb{P}$ and ${}^\kappa\mathbb{P}$, there exists a sequence of regeneration epochs $(\sigma_n)_{n\ge 0}$ as described in Lemma 4.1 such that $X_{\sigma_n} \stackrel{d}{=} \phi$ and*

$$\inf_{x\in S} |xM_{\sigma_n}| \ge \mathfrak{c} \quad \mathbb{P}\text{-}a.s. \tag{19}$$

for some $\mathfrak{c} > 0$ and all $n \ge 1$.

Proof. We just note that, by (BMC), we may generate $(X_{\sigma_n}, M_{\sigma_n})$ given $X_{\sigma_n - 1} = y$ at any regeneration epoch σ_n according to $\psi(y,\cdot)$ having first marginal ϕ, thus $X_{\sigma_n} \stackrel{d}{=} \phi$. Moreover, $M_{\sigma_n} \in C$ $\mathbb{P}$-a.s. for a compact $C \subset GL(d,\mathbb{R})$ which entails $\|M_{\sigma_n}^{-1}\| \le \mathfrak{c}^{-1}$ $\mathbb{P}$-a.s. for some constant $\mathfrak{c} > 0$. Since

$$\inf_{x\in S} |xM_{\sigma_n}| = \inf_{x\in S} \frac{|xM_{\sigma_n}|}{\left|xM_{\sigma_n}M_{\sigma_n}^{-1}\right|} = \frac{1}{\sup_{x\in S} |xM_{\sigma_n}^{-1}|} = \frac{1}{\|M_{\sigma_n}^{-1}\|},$$

we finally infer (19). □

In order to ensure that $(X_n, V_n)_{n\ge 0}$ is nonarithmetic, Kesten [12] imposes an additional assumption involving so-called feasible matrices. But in view of assumption (A5) it should not take by surprise that this is not needed. The following lemma provides the confirmation.

Lemma 5.7. *Suppose* (A4), (A5) *and* (SA). *Then $(X_n, V_n)_{n\ge 0}$ is nonarithmetic under ${}^\kappa\mathbb{P}_\pi$.*

Proof. If our claim failed there would be some $d > 0$ and a function $\gamma : S \to \mathbb{R}$ such that

$${}^\kappa\mathbb{P}_x(\log|xM| \in \gamma(x) - \gamma((xM)^\sim) + d\,\mathbb{Z}, (xM)^\sim \in U) = {}^\kappa\mathbb{P}_x((xM)^\sim \in U) > 0$$

for all $x \in S$ and open $U \subset S$. Since, by Lemma 5.5, ${}^\kappa\mathbb{P}_x(y,\cdot) \ge p\phi$ for some $p > 0$ and with ϕ as defined in Lemma 2.1 (for a regenerative δ-ball about x), this would further entail

$$\int_{B_\varsigma(I)} \mathbf{1}_{\gamma(x)-\gamma((x\mathfrak{m})^\sim)+d\,\mathbb{Z}}(\log|x\mathfrak{m}|)\,\mathbf{1}_{B_\delta(x)}((x\mathfrak{m})^\sim)\;\lambda\!\!\lambda(d\mathfrak{m}) = \phi(B_\delta(x)) = 1.$$

But the left-hand side being clearly zero, we arrive at a contradiction. □

For the proof of Theorem 5.1, it finally remains to verify that $(X_n, V_n)_{n\ge 0}$ has positive drift under ${}^\kappa\mathbb{P}_\pi$. The subsequent argument simplifies the original one given by Kesten [12].

Lemma 5.8. *Under ${}^{\kappa}\mathbb{P}_{\pi}$, $(X_n, V_n)_{n\ge 0}$ has positive drift, given by*

$$\alpha := {}^{\kappa}\mathbb{E}_{\pi}\left(\frac{V_n}{n}\right) = \frac{1}{n}\int \frac{1}{r(x)}\,\mathbb{E}\,|x\Pi_n|^{\kappa}\log|x\Pi_n|\,r((x\Pi_n)^{\sim})\ \pi(dx)$$

for any $n \ge 1$.

Proof. For each $n \ge 1$, the function

$$g_n(\varkappa) := \int \frac{1}{r(x)}\,\mathbb{E}\,|x\Pi_n|^{\varkappa}\,r((x\Pi_n)^{\sim})\ \pi(dx)$$

is finite and thus convex for $\varkappa \in [0, \kappa_0]$. Moreover $g_n(\kappa) = 1$ and $g_n'(\kappa) = \alpha n$. By convexity, α is positive if we can show that $g_n(\varkappa) < 1$ for some n and some $\varkappa < \kappa$. To this end pick any $\varkappa < \kappa$ and recall that $\varrho(\varkappa) < 1$. It follows that

$$g_n(\varkappa) = \int \frac{T_{\varkappa}^n r(x)}{r(x)}\,\pi(dx) = \int \frac{\|r\|_\infty}{r(x)}\,T_{\varkappa}^n\left(\frac{r(x)}{\|r\|_\infty}\right)\,\pi(dx) \le C\|T_{\varkappa}^n\|$$

for some $C \in (0,\infty)$ and all $n \ge 1$. As $\|T_{\varkappa}^n\|^{1/n} \to \varrho(\varkappa)$, we infer $g_n(\varkappa) \to 0$ and thus the desired result. ☐

5.3 Tail behavior of $\sup_{n\ge 1}|x\Pi_n|$

With the help of the MRT 4.2, we are now able to prove the following result on the tail behavior of $\sup_{n\ge 1}|x\Pi_n|$.

Proposition 5.9. *Under the conditions of Theorem 1.1 and with r as defined in Lemma 5.4,*

$$\lim_{t\to\infty} t^{\kappa}\,\mathbb{P}\left(\sup_{n\ge 1}|x\Pi_n| > t\right) = L\,r(x),$$

for π-almost all $x \in S$ and some $L > 0$.

Proof. The function $f : S \times (0,\infty) \to (0,\infty)$, $(y,s) \mapsto e^{-\kappa s}/r(y)$ is bounded and continuous whence, by an application of the MRT,

$$L(f) := \lim_{t\to\infty} {}^{\kappa}\mathbb{E}_x\left(f(Z(t),R(t))\mathbf{1}_{\{\tau(t)<\infty\}}\right)$$

exists and is positive. On the other hand, we have

$$\begin{aligned}
{}^{\kappa}\mathbb{E}_x\left(f(Z(t),R(t))\mathbf{1}_{\{\tau(t)<\infty\}}\right) &= \sum_{n\ge 1} {}^{\kappa}\mathbb{E}_x\left(f(X_n, V_n - t)\,\mathbf{1}_{\{\tau(t)=n\}}\right)\\
&= \sum_{n\ge 1} {}^{\kappa}\mathbb{E}_x\left(\frac{1}{r(X_n)}\,e^{-\kappa V_n + \kappa t}\,\mathbf{1}_{\{\tau(t)=n\}}\right)\\
&= \frac{e^{\kappa t}}{r(x)}\sum_{n\ge 1}\mathbb{E}_x\left(\frac{1}{r(X_n)}\,e^{-\kappa V_n}\,r(X_n)\,e^{\kappa V_n}\,\mathbf{1}_{\{\tau(t)=n\}}\right)\\
&= \frac{e^{\kappa t}}{r(x)}\,\mathbb{P}_x(\tau(t)<\infty)\\
&= \frac{e^{\kappa t}}{r(x)}\,\mathbb{P}\left(\sup_{n\ge 1}\ \log|x\Pi_n| > t\right), \qquad (20)
\end{aligned}$$

which provides the asserted result upon subtituting e^t by t. ☐

6 Proof of Theorem 1.1: Implicit Markov renewal theory

We now turn to the proof of our main result Theorem 1.1, all assumptions of which will therefore be in force throughout, in fact in strengthened form given by our standing assumption.

Embarking on ideas by Goldie [9] and Le Page [15], a comparison of the distribution functions of xR and xMR will enable us to make use of a Markov modulated version of Goldie's implicit renewal theory. This will prove that $K(x) = \lim_{t\to\infty} t^{\kappa}\mathbb{P}\left(xR > t\right)$ exists for π-almost all $x \in S$.

We start with a simple lemma, stated without proof, which is just Lemma 9.3 in [9] adapted to our situation.

Lemma 6.1. *Let $x \in S$. If $K(x) := \lim_{t\to\infty} t^{-1}\int_0^t s^{\kappa}\,\mathbb{P}\left(xR > s\right) ds$ exists and is finite, then so does $\lim_{t\to\infty} t^{\kappa}\,\mathbb{P}\left(xR > t\right)$ and equals $K(x)$ as well.*

Substituting t' for e^t and a change of variables show that $t'^{-1}\int_0^{t'} s^{\kappa}\,\mathbb{P}\left(xR > s\right) ds$ equals $e^{-t}\int_{-\infty}^{t} e^{(\kappa+1)s}\,\mathbb{P}\left(xR > e^s\right) ds$ which is the form needed in the next result which provides us with the basic renewal theoretic identity.

Lemma 6.2. *For all $t \in \mathbb{R}$,*

$$\frac{e^{-t}}{r(x)}\int_{-\infty}^{t} e^{(\kappa+1)s}\,\mathbb{P}\left(xR > e^s\right) ds = \sum_{n\geq 0}\int \hat{g}(y, t-u)\,{}^{\kappa}\mathbb{P}_x\left(X_n \in dy, V_n \in du\right), \tag{21}$$

where $\hat{g}(y,t) := \int_{-\infty}^{t} e^{-(t-s)} g(y,s)\,ds$ is the exponential smoothing of

$$g(y,s) = \frac{e^{\kappa s}}{r(y)}\left[\mathbb{P}\left(yR > e^s\right) - \mathbb{P}\left(yMR > e^s\right)\right].$$

Proof. For arbitrary $n \in \mathbb{N}$, $x \in S$ and $s \in \mathbb{R}$, consider the following telescoping sum for $\mathbb{P}\left(xR > e^s\right)$ (recalling independence of R, M, and $(M_n)_{n\geq 1}$)

$$\begin{aligned}
&\sum_{k=1}^{n}\left[\mathbb{P}\left(x\Pi_{k-1}R > e^s\right) - \mathbb{P}\left(x\Pi_k R > e^s\right)\right] + \mathbb{P}\left(x\Pi_n R > e^s\right)\\
&= \sum_{k=1}^{n}\left[\mathbb{P}_x\left(e^{V_{k-1}}X_{k-1}R > e^s\right) - \mathbb{P}_x\left(e^{V_{k-1}}X_{k-1}M_k R > e^s\right)\right] + \mathbb{P}_x\left(e^{V_n}X_n R > e^s\right)\\
&= \sum_{k=0}^{n-1}\left[\mathbb{P}_x\left(e^{V_k}X_k R > e^s\right) - \mathbb{P}_x\left(e^{V_k}X_k MR > e^s\right)\right] + \mathbb{P}_x\left(e^{V_n}X_n R > e^s\right)\\
&= \sum_{k=0}^{n-1}\int \mathbb{P}\left(yR > e^{s-u}\right) - \mathbb{P}\left(yMR > e^{s-u}\right)\,\mathbb{P}_x(X_k \in dy, V_k \in du)\\
&\qquad\qquad + \mathbb{P}_x\left(e^{V_n}X_n R > e^s\right)
\end{aligned}$$

Multiply by $e^{\kappa s}/r(x) > 0$ to obtain

$$\begin{aligned}
&\frac{e^{\kappa s}}{r(x)}\,\mathbb{P}\left(xR > e^{s}\right) \\
&\quad = \sum_{k=0}^{n-1} \int \frac{e^{\kappa(s-u)}}{r(x)} \left[\mathbb{P}\left(yR > e^{s-u}\right) - \mathbb{P}\left(yMR > e^{s-u}\right)\right] \\
&\qquad\qquad \times \frac{r(y)}{r(y)}\, e^{\kappa u}\, \mathbb{P}_x(X_k \in dy, V_k \in du) + \frac{e^{\kappa s}}{r(x)}\,\mathbb{P}_x\left(e^{V_n} X_n R > e^{s}\right) \\
&\quad = \sum_{k=0}^{n-1} \int g(y, s-u)\, \mathbb{P}_x^{\kappa}(X_k \in dy, V_k \in du) + \frac{e^{\kappa s}}{r(x)}\,\mathbb{P}_x\left(X_n R > e^{s-V_n}\right).
\end{aligned}$$

Convolution with a standard exponential distribution then gives

$$\begin{aligned}
\int_{-\infty}^{t} e^{-(t-s)} \frac{1}{r(x)} e^{\kappa s}\, \mathbb{P}\left(xR > e^{s}\right) ds &= \sum_{k=0}^{n-1} \int \hat{g}(y, t-u)\, {}^{\kappa}\mathbb{P}_x\left(X_k \in dy, V_k \in du\right) \\
&\quad + \int_{-\infty}^{t} e^{-(t-s)}\, \frac{e^{\kappa s}}{r(x)}\,\mathbb{P}_x\left(X_n R > e^{s-V_n}\right) ds
\end{aligned}$$

By the Cauchy-Schwarz inequality, $|X_n R|^2 \le |X_n|\,|R| = |R|$ and thus

$$\mathbb{P}_x\left(X_n R > e^{s-V_n}\right) \le \mathbb{P}_x\left(|X_n R| > e^{s-V_n}\right) \le \mathbb{P}_x\left(|R| > e^{2(s-V_n)}\right).$$

But the last term converges to 0 as $n \to \infty$ for any $t > 0$, because $\lim_{n\to\infty} V_n = -\infty$ $\mathbb{P}_x$-a.s. Hence assertion (21) follows by an appeal to the dominated convergence theorem. □

Obviously, if $\mathbb{U}_x^{\kappa} := \sum_{n\ge 0} {}^{\kappa}\mathbb{P}_x((X_n, V_n) \in \cdot)$, then the right-hand side of (21) equals $\hat{g} * U_x^{\kappa}(t)$ for x outside a π-null set N provided that sum and integral may be interchanged for $x \notin N$. But the latter follows if we can prove hereafter that $\hat{g}$ is π-directly Riemann integrable which will also be the crucial condition that ensures applicability of the MRT 4.2. Indeed, if $\hat{g}$ has this property, then, by Equation (5.8) and Lemma A.5 in [1],

$$\hat{g} * \mathbb{U}_x^{\kappa}(t) = {}^{\kappa}\mathbb{E}_x\left(\sum_{k\ge 0} \hat{g}(X_k, t-V_k)\right) = \int \sum_{k\ge 0} \hat{g}(y, t-u)\, {}^{\kappa}\mathbb{P}_x\left(X_k \in dy, V_k \in du\right) < \infty$$

for all $t \in \mathbb{R}$ and π-almost all $x \in S$. Split $\hat{g}$ in positive and negative part. This yields two π-null sets N_1, N_2 such that $\hat{g}^+ * \mathbb{U}_x^{\kappa}(t)$ and $\hat{g}^- * \mathbb{U}_x^{\kappa}(t)$ are finite for all $x \in (N_1 \cup N_2)^c$ and all $t \in \mathbb{R}$. By Fubini's theorem, sum and integral in (21) may be interchanged for all $x \in (N_1 \cup N_2)^c$. This is enough because the MRT asserts convergence of $\hat{g} * \mathbb{U}_x^{\kappa}(t)$ only for x outside a π-null set.

Instead of π-direct Riemann integrability of $\hat{g}$ we will actually show the stronger property that

$$\sup_{y\in S} \sum_{n\in\mathbb{Z}} \sup_{t\in[n\delta,(n+1)\delta)} |\hat{g}(y,t)| < \infty, \tag{22}$$

which can be done by resorting to the methods of Goldie [9, proof of Theorem 4.1] which are only summarized here. Let $L_1(\mathbb{R})$ as usual be the space of Lebesgue integrable functions.

Lemma 6.3. *If $f \in L_1(\mathbb{R})$ and $\hat{f}(t) := \int_{-\infty}^{t} e^{-(t-u)} f(u)\, du$, then*

$$\sum_{n\in\mathbb{Z}} \sup_{t\in[n\delta,(n+1)\delta)} \left|\hat{f}(t)\right| \le \delta e^{2\delta} \int |f(t)|\, dt < \infty$$

for any $\delta > 0$.

Proof. W.l.o.g. let f be nonnegative. For all $\varepsilon > 0$ and $t \in \mathbb{R}$, we have

$$\hat{f}(t+\varepsilon) \ge e^{-t-\varepsilon} \int_{-\infty}^{t} e^{u} f(u)\, du = e^{-\varepsilon} \hat{f}(t).$$

Consequently, with $m_{n,\delta} := \sup_{t\in[n\delta,(n+1)\delta)} \hat{f}(t)$,

$$\delta \sum_{n\in\mathbb{Z}} m_{n,\delta} \le \delta e^{\delta} \sum_{n\in\mathbb{Z}} \hat{f}(n\delta) \le \delta e^{2\delta} \sum_{n\in\mathbb{Z}} \int_{(n-1)\delta}^{n\delta} f(t)\, dt = \delta e^{2\delta} \int f(t)\, dt < \infty$$

for any $\delta > 0$. □

In view of the previous lemma, it suffices to show for (22) that $\int |g(y,s)|\, ds$ is uniformly bounded in y. First observe that (cf. [9, Corollary 2.4])

$$\begin{aligned}
\int_{\mathbb{R}} |g(y,s)|\, ds &= \int_{\mathbb{R}} \frac{e^{\kappa s}}{r(y)} \left|\mathbb{P}\left(yR > e^{s}\right) - \mathbb{P}\left(yMR > e^{s}\right)\right| ds \\
&= \int_{\mathbb{R}} \frac{e^{\kappa s}}{r(y)} \left|\mathbb{P}\left(yMR + yQ > e^{s}\right) - \mathbb{P}\left(yMR > e^{s}\right)\right| ds \\
&= \frac{1}{\kappa r(y)} \mathbb{E}\left|((yMR+yQ)^{+})^{\kappa} - ((yMR)^{+})^{\kappa}\right|.
\end{aligned}$$

Then a case-by-case analysis with respect to the signs of yMR and yQ yields that

$$\sup_{y\in S} \frac{1}{\kappa r(y)} \mathbb{E}\left|((yMR+yQ)^{+})^{\kappa} - ((yMR)^{+})^{\kappa}\right| < \infty,$$

see [9, Theorem 4.1]. Now we are ready to prove

Lemma 6.4. *For π-almost all $x \in S$,*

$$\lim_{t\to\infty} \frac{e^{-t}}{r(x)} \int_{-\infty}^{t} e^{(\kappa+1)s}\, \mathbb{P}\left(xR > e^{s}\right) ds = K_0,$$

where $K_0 := \frac{1}{\alpha\kappa} \int_S \frac{1}{r(y)} \mathbb{E}\left(((yR)^{+})^{\kappa} - ((yMR)^{+})^{\kappa}\right) \pi(dy) < \infty$ and α as before denotes the ${}^{\kappa}\mathbb{P}$-almost sure limit of $n^{-1}V_n$.

Proof. Since $\hat{g}$ is π-directly Riemann integrable, we may exchange sum and integral in (21) for π-almost all $x \in S$ and apply the MRT. This tells us that the right-hand side of (21) has the finite limit

$$\begin{aligned}
&\frac{1}{\alpha}\int_S\int_{\mathbb{R}} e^{-t}\int_{-\infty}^{t}\frac{e^{(\kappa+1)s}}{r(y)}\left[\mathbb{P}\left(yR>e^s\right)-\mathbb{P}\left(yMR>e^s\right)\right]ds\ dt\,\pi(dy)\\
&=\frac{1}{\alpha}\int_S\int_{\mathbb{R}} g(y,t)\ dt\ \pi(dy)\\
&=\frac{1}{\alpha}\int_S\frac{1}{r(y)}\int_{\mathbb{R}} e^{\kappa t}\left[\mathbb{P}\left(yR>e^t\right)-\mathbb{P}\left(yMR>e^t\right)\right]\ dt\ \pi(dy)\\
&=\frac{1}{\alpha}\int_S\frac{1}{r(y)}\int_0^{\infty} u^{\kappa-1}\left[\mathbb{P}\left(yR>u\right)-\mathbb{P}\left(yMR>u\right)\right]\ du\ \pi(dy)\\
&=\frac{1}{\alpha\kappa}\int_S\frac{1}{r(y)}\mathbb{E}\left(((yR)^+)^\kappa-((yMR)^+)^\kappa\right)\pi(dy)
\end{aligned}$$

for π-almost all x. ☐

7 Proof of Theorem 1.1: Assertion (5) holds for all $x \in S$

So far we have proved our main assertion (5) (except for the positivity of $K(x)$) for π-almost all $x \in S$ and thus for all x from a dense subset of S (this is a direct consquence of (A4)). By employing a refined renewal argument, we will now remove this restriction. To this end, we fix an arbitrary $x \in S$ and $\delta > 0$ so small that $B_\delta(x)$ is regenerative (for P and ${}^\kappa\mathbb{P}$) with minorizing distribution ϕ as defined in Section 2 and associated sequence $(\sigma_n)_{n\geq 1}$ of regeneration epochs when using (BMC), which ensures $M_{\sigma_n} \in C$ for a compact $C \subset GL(d,\mathbb{R})$. Put $\sigma := \sigma_1$. The task is to show that $\hat{g} * \mathbb{U}_x^\kappa(t)$ converges to K_0, and we begin by pointing out that

$$\hat{g} * \mathbb{U}_x^\kappa(t) = {}^\kappa\mathbb{E}_x\left(\sum_{k\geq 0}\hat{g}(X_k, t-V_k)\right) = G(x,t) + \hat{g} * \mathbb{U}_{\psi(x)}^\kappa(t) \tag{23}$$

where $\psi(x)(\cdot) := {}^\kappa\mathbb{P}_x((X_\sigma, V_\sigma) \in \cdot)$ and

$$G(x,t) := {}^\kappa\mathbb{E}_x\left(\sum_{k=0}^{\sigma-1}\hat{g}(X_k, t-V_k)\right), \quad (x,t) \in S \times \mathbb{R}\,.$$

As for this last function, we now prove:

Lemma 7.1. *The function G is bounded and satisfies* $\lim_{t\to\infty} G(y,t) = 0$ *for all* $y \in S$.

Proof. By (22), $C := \sup\{|\hat{g}(y,t)| : y \in S, t \in \mathbb{R}\} < \infty$, and since $(X_n)_{n\geq 0}$ is a strongly aperiodic Doeblin chain, we infer

$$\sup_{y\in S, t\in\mathbb{R}} |G(y,t)| \leq C \sup_{y\in S} {}^\kappa\mathbb{E}_y\sigma < \infty.$$

Just note that the time it takes to hit the regenerative ball $B_\delta(x)$ pertaining to σ from any y is geometrically bounded (uniformly in y) and that a geometric number of coin tosses (see Lemma 4.1 and thereafter) of such times determines σ. Turning to the convergence assertion, we point out that, again by property (22), $\lim_{t\to\infty} \hat{g}(y,t) = 0$ for all $y \in S$, which implies the desired result by an appeal to the dominated convergence theorem. □

In view of (23), we are now left with a proof of $\hat{g} * \mathbb{U}^\kappa_{\psi(x,\cdot)}(t) \to K_0$ defined in Lemma 6.4. This requires two further lemmata.

Lemma 7.2. *The sequence $(X_\sigma, (X_n, U_n)_{n>\sigma})$ is independent of $(X_{\sigma-1}, V_{\sigma-1})$ under ${}^\kappa\mathbb{P}_x$ with distribution given by ${}^\kappa\mathbb{P}_\phi((X_0, (X_n, U_n)_{n\geq 1}) \in \cdot)$.*

Proof. The independence follows immediately when observing that, by regeneration, $(X_\sigma, (X_n, M_n)_{n>\sigma})$ is independent of $(X_{\sigma-1}, V_{\sigma-1})$ under ${}^\kappa\mathbb{P}_x$ in combination with $U_k = \log|X_{k-1}M_k|$ for each $k \geq 1$. The proof is completed by the observation that ${}^\kappa\mathbb{P}_x((X_\sigma, (X_n, M_n)_{n>\sigma}) \in \cdot) = {}^\kappa\mathbb{P}_\phi((X_0, (X_n, M_n)_{n\geq 1}) \in \cdot)$. □

Lemma 7.3. *The random variable $U_\sigma = \log|X_{\sigma-1}M_\sigma|$ is a.s. bounded, that is taking values in some finite interval $[s_*, s^*]$.*

Proof. We have that $\|M_\sigma\|^{-1} \leq |U_\sigma| \leq \|M_\sigma\|$ ${}^\kappa\mathbb{P}_{\psi(x,\cdot)}$-a.s. Hence the assertion follows because, as already noted above, ${}^\kappa\mathbb{P}_{\psi(x)}(M_\sigma \in C) = 1$ for some compact $C \subset GL(d,\mathbb{R})$. □

Define $V_{\sigma,n} := V_{\sigma+n} - V_\sigma$ for $n \geq 0$ and then

$$h(x,s,t) := {}^\kappa\mathbb{E}_x\left(\sum_{k\geq 0} \hat{g}(X_{\sigma+k}, t - s - V_{\sigma-1} - V_{\sigma,k})\right)$$

for $s,t \in \mathbb{R}$. Lemma 7.2 implies

$$h(x,s,t) = \int_{\mathbb{R}} \hat{g} * \mathbb{U}^\kappa_\phi(t-s-r)\, {}^\kappa\mathbb{P}_x(V_{\sigma-1} \in dr).$$

As $\hat{g}$ satisfies (22), we infer from the MRT 4.2 and the subsequent remark that $\hat{g} * \mathbb{U}^\kappa_\phi(t)$ is bounded and converges to K_0. By the dominated convergence theorem, the same limit holds for $\lim_{t\to\infty} h(x,s,t)$ for all s.

Finally, the connection between $h(x,s,t)$ and $\hat{g} * \mathbb{U}_{\psi(x)}(t)$ becomes apparent after the following observations: By Lemma 7.3, $U_\sigma = \log|X_{\sigma-1}M_\sigma|$ is taking values in some finite interval $[s_*, s^*]$. Hence we can estimate $\hat{g} * \mathbb{U}_{\psi(x)}(t)$ by

$$\inf_{s\in[s_*,s^*]} h(x,s,t) \leq \hat{g} * \mathbb{U}_{\psi(x)}(t) \leq \sup_{s\in[s_*,s^*]} h(x,s,t)$$

and thus arrive at the desired conclusion that $\lim_{t\to\infty} \hat{g} * \mathbb{U}_{\psi(x)}(t) = K_0$.

8 Proof of Theorem 1.1: The limit $K(x)$ is positive

A combination of Lemma 6.1, Lemma 6.4 and the result of Section 7 renders convergence of $t^\kappa\,\mathbb{P}(xR > t)$ to the continuous function

$$K(x) := K_0\, r(x) = \frac{r(x)}{\alpha\kappa} \int_S \frac{1}{r(y)} \mathbb{E}\left(((yR)^+)^\kappa - ((yMR)^+)^\kappa \right) \pi(dy)$$

for all $x \in S$. To complete the proof of Theorem 1.1, it remains to show that K or, equivalently, K_0 is positive, which is the topic of this final section.

Clearly, it suffices to show that $\limsup_{t\to\infty} t^\kappa \mathbb{P}\left(xR > t\right) > 0$ for some $x \in S$. Notice that, as r is symmetric (Lemma 5.4), the same holds true for $K(x)$, hence

$$\lim_{t\to\infty} \mathbb{P}\left(xR > t\right) = \lim_{t\to\infty} \mathbb{P}\left(-xR > t\right) = \frac{1}{2} \lim_{t\to\infty} \mathbb{P}\left(|xR| > t\right).$$

So it is enough to show that $\limsup_{t\to\infty} t^\kappa \mathbb{P}\left(|xR| > t\right) > 0$ for all x. To this end we need the following lemma, originally due to Le Page [15, Lemma 3.11], which ensures that R and its "marginals" xR for any $x \in S$ have unbounded support. It is this result where the nondegeneracy assumption (A6), unused so far, enters in a crucial way. We postpone the proof until the end of this section.

Lemma 8.1. *For all $x \in S$ and $t \in \mathbb{R}$,*

$$\mathbb{P}\left(xR \le t\right) < 1. \tag{24}$$

What this lemma shows is that, fixing any $x_0 \in S$, we can choose $\xi > 0$ and then sufficiently small $\zeta, \eta \in (0, 1)$ and $\delta \in (0, \zeta)$ such that

$$\mathbb{P}\left(yR > \xi\right) \ge \eta \quad \text{and} \quad \mathbb{P}\left(yR < (1-\zeta)\xi\right) \ge \eta \tag{25}$$

for all $y \in B_\delta(x_0)$. Notice that

$$\inf_{x,y \in B_\delta(x_0)} xy > 1 - \delta. \tag{26}$$

We continue with a decomposition of xR with respect to entrances of $(x\Pi_k)^\sim$ into $B_\delta(x_0)$. Let $\mathcal{G} = (\mathcal{G}_n)_{n\ge0}$ be a filtration as described at the beginning of Subsection 3.1 and consider an increasing sequence $(\sigma_n)_{n\ge1}$ of $\mathcal{G}$-stopping times such that $\mathbb{P}\left(X_{\sigma_n} \in B_\delta(x_0)\right) = 1$ for each $n \ge 1$. Note that (25) particularly holds for $y = X_{\sigma_n} = (x\Pi_{\sigma_n})^\sim$. Recall from Subsection 3.1 the definition of Q^n and R^n as well as $R^\tau \stackrel{d}{=} R$ for any a.s. finite stopping time τ with respect to $(\mathcal{F}_n)_{n\ge0}$, the natural filtration of $(M_n, Q_n)_{n\ge1}$.

Lemma 8.2. *Given any $x_0 \in S$ and sufficiently small $0 < \delta < 1$, it holds true that*

$$\mathbb{P}\left(|xR| > t\right) \ge \eta\, \mathbb{P}\left(\sup_{n\ge1} |xQ^{\sigma_n} + \xi\, x\Pi_{\sigma_n} y| > t\right)$$

for all $y \in B_\delta(x_0)$.

Proof. This is an extension of Levy's inequality and inspired by [9, Prop. 4.2]. Since $R^{\sigma_k} \stackrel{d}{=} R$ for all $k \geq 1$ we see that (25) holds for R^{σ_k} as well. We show first that

$$\mathbb{P}\left(xR > t\right) \geq \eta\, \mathbb{P}\left(\sup_{n\geq 1} xQ^{\sigma_n} + \xi\, x\Pi_{\sigma_n} y > t\right)$$

and will consider $\mathbb{P}\left(-xR > t\right)$ in a second step. Define

$$C_k := \left\{\max_{1\leq j<k}\left(xQ^{\sigma_j} + \xi\, x\Pi_{\sigma_j} y\right) \leq t,\; xQ^{\sigma_k} + \xi\, x\Pi_{\sigma_k} y > t\right\}$$
$$\text{and}\quad D_k := \left\{x\Pi_{\sigma_k} R^{\sigma_k} > \xi\, x\Pi_{\sigma_k} y\right\}.$$

By (26), $0 < (x\Pi_{\sigma_k})^{\sim}\, y \leq 1$ for all $y \in B_\delta(x_0)$, giving

$$D_k = \left\{(x\Pi_{\sigma_k})^{\sim}\, R^{\sigma_k} > \xi\, (x\Pi_{\sigma_k})^{\sim}\, y\right\} \supset \left\{(x\Pi_{\sigma_k})^{\sim}\, R^{\sigma_k} > \xi\right\}$$

and thus $\mathbb{P}\left(D_k|\mathcal{F}_{\sigma_k}\right) \geq \eta$ $\mathbb{P}$-a.s. In combination with $\sum_{k=1}^n (C_k \cap D_k) \subset \{xR > t\}$ and $C_k \in \mathcal{F}_{\sigma_k}$, this implies

$$\mathbb{P}\left(xR > t\right) \geq \sum_{k=1}^{n} \int_{C_k} \mathbb{P}\left(D_k|\mathcal{F}_{\sigma_k}\right) d\,\mathbb{P} \geq \eta\, \mathbb{P}\left(\bigcup_{k=1}^{n} C_k\right),$$

and thus $\mathbb{P}\left(xR > t\right) \geq \eta\, \mathbb{P}\left(\sup_{n\geq 1}\left(xQ^{\sigma_n} + \xi x\Pi_{\sigma_n} y\right) > t\right)$ by letting $n \to \infty$.

Turning to the respective inequality for $\mathbb{P}\left(-xR > t\right)$, define

$$C_k' := \left\{\max_{1\leq j<k}\left(-xQ^{\sigma_j} - \xi\, x\Pi_{\sigma_j} y\right) \leq t,\; -xQ^{\sigma_k} - \xi\, x\Pi_{\sigma_k} y > t\right\}$$
$$\text{and}\quad D_k' := \left\{-x\Pi_{\sigma_k} R^{\sigma_k} > -\xi\, x\Pi_{\sigma_k} y\right\} = \left\{(x\Pi_{\sigma_k})^{\sim}\, R^{\sigma_k} < \xi\, (x\Pi_{\sigma_k})^{\sim}\, y\right\}.$$

Again by (26), $(x\Pi_{\sigma_k})^{\sim}\, y \geq 1 - \delta > 1 - \zeta$ for all $y \in B_\delta(x_0)$, giving

$$D_k' \supset \left\{(x\Pi_{\sigma_k})^{\sim}\, R^{\sigma_k} < (1-\zeta)\xi\right\}$$

and thus $\mathbb{P}\left(D_k|\mathcal{F}_{\sigma_k}\right) \geq \eta$ $\mathbb{P}$-a.s. Now reasoning as above,

$$\mathbb{P}\left(-xR > t\right) \geq \eta \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k=1}^{n} C_k'\right) = \eta\, \mathbb{P}\left(\sup_{n\geq 1}\left(-xQ^{\sigma_n} - \xi x\Pi_{\sigma_n} y\right) > t\right)$$

The desired result hence follows by a combination of this inequality with the one obtained for $\mathbb{P}\left(xR > t\right)$. ☐

Proposition 8.3. *For π-almost all $x \in S$, $\lim_{t\to\infty} t^\kappa\, \mathbb{P}_x(|xR| > t)$ is positive.*

Proof. By Lemma 5.5, $(X_n)_{n\geq 0}$ is a strongly aperiodic Doeblin chain (under $\mathbb{P}$ as well as ${}^\kappa\mathbb{P}$) with $B_\delta(x)$ being regenerative for any $x \in S$ and suitable $\delta = \delta(x) > 0$. Pick any regenerative $B_\delta(x_0)$ and let σ_n to be the associated n^{th} regeneration time as defined in the Regeneration Lemma 4.1 with $X_{\sigma_n} \stackrel{d}{=} \phi$. Define also $\Pi_{j,k} := M_j \cdot \ldots \cdot M_k$, $Q^{j,n} := \sum_{k=j}^n \Pi_{j,k-1} Q_k$ and (with $\sigma_0 := 0$ and any $y \in B_\delta(x_0)$)

$$T_n = xQ^{\sigma_n} + \xi\, x\Pi_{\sigma_n} y,$$
$$\Delta_n := Q^{\sigma_{n-1}+1,\sigma_n} - \xi\left(I - \Pi_{\sigma_{n-1}+1,\sigma_n}\right) y,$$
$$U_n = x\Pi_{\sigma_{n-1}}\Delta_n$$

for $n \geq 1$. Then $T_n = T_{n-1} + U_n$ and $\{\sup_{n\geq 1} |T_n| > t\} \supset \{\sup_{n\geq 2} |U_n| > 2t\}$. Since $\mathbb{P}(X_{\sigma_n} \in B_\delta(x_0)) = 1$ for each $n \geq 1$, Lemma 8.2 provides us with

$$\mathbb{P}(|xR| > t) \geq \eta\, \mathbb{P}\left(\sup_{n\geq 1} |T_n| > t\right)$$

for some $\eta > 0$. Also, by Lemma 5.6, $\inf_{y\in S} |yM_{\sigma_n}| \geq \mathfrak{c}$ a.s. for all $n \geq 1$ and a suitable $\mathfrak{c} > 0$. Hence, for all $t > 0$,

$$\begin{aligned}
\mathbb{P}(|xR| > t) &\geq \eta\, \mathbb{P}\left(\sup_{n\geq 2} |U_n| \geq 2t\right) \\
&= \eta\, \mathbb{P}\left(\sup_{n\geq 1} |x\Pi_{\sigma_n-1}|\, |(x\Pi_{\sigma_n-1})^{\sim} M_{\sigma_n}|\, |X_{\sigma_n}\Delta_{n+1}| \geq 2t\right) \\
&\geq \eta \sum_{n\geq 1} \mathbb{P}\left(\bigcap_{k=1}^{n-1} A_k,\ |x\Pi_{\sigma_n-1}| > \frac{2t}{\mathfrak{c}\varepsilon}, |X_{\sigma_n}\Delta_{n+1}| > \varepsilon\right) \\
&\geq \eta \sum_{n\geq 1} \mathbb{P}\left(\bigcap_{k=1}^{n-1} A_k,\ |x\Pi_{\sigma_n-1}| > \frac{2t}{\mathfrak{c}\varepsilon}\right) \mathbb{P}_\phi(|X_0\Delta_1| > \varepsilon) \qquad \text{(use (R3))} \\
&\geq \eta\, \mathbb{P}_\phi(|X_0\Delta_1| > \varepsilon)\, \mathbb{P}\left(\sup_{n\geq 1} |x\Pi_{\sigma_n-1}| > \frac{2t}{\mathfrak{c}\varepsilon}\right),
\end{aligned}$$

where $A_k = \{|x\Pi_{\sigma_k-1}| \leq 2t/(\mathfrak{c}\varepsilon)\}$ for $k \geq 1$ and some fixed $0 < \varepsilon < 1$. Two subsequent lemmata will show positivity of $\mathbb{P}_\phi(|X_0\Delta_1| > \varepsilon)$ (Lemma 8.4) and $\limsup_{t\to\infty} \mathbb{P}\left(\sup_{n\geq 1} |x\Pi_{\sigma_n-1}| > 2t/(\mathfrak{c}\varepsilon)\right)$ (Lemma 8.5), and this clearly yields the desired conclusion. □

Lemma 8.4. *In the situation of Proposition 8.3, there exist $\varepsilon > 0$ and $y \in B_\delta(x_0)$ such that (notice here the dependence of Δ_1 on y)*

$$\mathbb{P}_\phi(|X_0\Delta_1| > \varepsilon) > 0.$$

Proof. Suppose that $X_0\Delta_1 = X_0(Q^\sigma - \xi\,(I - \Pi_\sigma)\, y) = 0$ $\mathbb{P}_\phi$-a.s. for all $y \in B_\delta(x_0)$, where $\sigma := \sigma_1$. Then the same holds true for all y in the convex hull of $B_\delta(x_0)$ (as a subset of $\mathbb{R}^d$) which contains a basis of $\mathbb{R}^d$. Consequently, the range of $Q^\sigma - \xi\,(I - \Pi_\sigma)$ and $\{tX_0 : t \in \mathbb{R}\}$ are orthogonal $\mathbb{P}_\phi$-a.s. On the other hand, $\sigma_n^{-1} \log \|\Pi_{\sigma_n}\| \to 0$ $\mathbb{P}_\phi$-a.s. implies $\mathbb{P}_\phi(\|\Pi_\sigma\| < 1) > 0$ and thus that $Q^\sigma - \xi\,(I - \Pi_\sigma)$ has full range $\mathbb{R}^d$ on a set of positive probability under $\mathbb{P}_\phi$. This contradicts our starting assumption and the lemma is proved. □

Lemma 8.5. *Let $x \in S$ be such that $\lim_{t\to\infty} t^\kappa\, \mathbb{P}(\sup_{n\in\mathbb{N}} |x\Pi_n| > t)$ exists and is positve. Then there exists a regenerative $B_\delta(x_0)$, $x_0 \in S$ and $\delta > 0$, satisfying* (26) *and with associated regeneration epochs σ_n, $n \geq 1$, such that*

$$\limsup_{t\to\infty} t^\kappa\, \mathbb{P}\left(\sup_{n\in\mathbb{N}} |x\Pi_{\sigma_n-1}| > t\right) > 0.$$

Proof. Since S is compact and every $x \in S$ the center of a regenerative ball satisfying (26), we can find a finite covering $S = \bigcup_{i=1}^k B_\delta(x_i)$ by such balls. Let τ_n^i, $n \geq 1$, denote the associated

$\mathbb{P}$-a.s. finite return epochs of $(X_n)_{n\ge 0}$ to $B_\delta(x_i)$ and note that $\bigcup_{i=1}^k \{\tau_n^i : n \ge 1\} = \mathbb{N}$ $\mathbb{P}$-a.s. It follows for all $t > 0$

$$\mathbb{P}\left(\sup_{n\ge 1} |x\Pi_n| > t\right) \le \sum_{i=1}^k \mathbb{P}\left(\sup_{n\ge 1} \left|x\Pi_{\tau_n^i}\right| > t\right)$$

and then by Fatou's lemma and Proposition 5.9

$$\sum_{i=1}^k \limsup_{t\to\infty} t^\kappa\, \mathbb{P}\left(\sup_{n\ge 1} \left|x\Pi_{\tau_n^i}\right| > t\right) \ge \lim_{t\to\infty} t^\kappa\, \mathbb{P}\left(\sup_{n\ge 1} |x\Pi_n| > t\right) > 0$$

for π-almost all $x \in S$. Consequently, $\limsup_{t\to\infty} t^\kappa \mathbb{P}\left(\sup_{n\ge 1} \left|x\Pi_{\tau_n^i}\right| > t\right) > 0$ for at least one i. Now let $(\sigma_n)_{n\ge 1}$ be the pertinent sequence of regeneration epochs as defined in Subsection 4.1. Introducing i.i.d. Bernoulli(p) variables $J_1, J_2, ...$ ($p \in (0,1]$ given by the minorization condition valid on $B(x_i, \delta)$), which are independent of all other occurring variables, we may assume that the $\sigma_n - 1$ are defined as exactly those τ_k^i at which $J_k = 1$ holds true. As a consequence, $\sup_{n\ge 1} |x\Pi_{\sigma_n - 1}| = \sup_{n\ge 1} |x\Pi_{\tau_n}^i| \, \mathbf{1}_{\{J_n = 1\}}$ and thus

$$t^\kappa\, \mathbb{P}\left(\sup_{n\ge 1} |x\Pi_{\sigma_n - 1}| > t\right) \ge p \cdot t^\kappa\, \mathbb{P}\left(\sup_{n\ge 1} \left|x\Pi_{\tau_n^i}\right| > t\right).$$

for all $t > 0$. This yields our assertion. □

Finally, we provide the proof of Lemma 8.1.

of Lemma 8.1 . We first show that $\operatorname{supp} R$ is not a compact subset of $\mathbb{R}^d$. Use (7) to infer for each $n \ge 1$,

$$\Pi_n \operatorname{supp} R + Q^n = \operatorname{supp} R \quad \mathbb{P}\text{-a.s.}$$

and thus also ${}^\kappa\mathbb{P}$-a.s., for $\mathbb{P}$ and ${}^\kappa\mathbb{P}$ are equivalent probability measures on each $\mathcal{F}_n = \sigma((M_j, Q_j)_{1\le j\le n})$, $n \ge 1$. Now assume, that $\operatorname{supp} R$ is bounded. By (A6), there exist at least two distinct $x_1, x_2 \in \operatorname{supp} R$. Defining $v := x_1 - x_2$, it then follows that for all $n \ge 1$ and some $C \in (0, \infty)$

$$|\Pi_n v| \le |\Pi_n x_1 + Q^n| + |\Pi_n x_2 + Q^n| < C \quad {}^\kappa\mathbb{P}\text{-a.s.}$$

and thereupon for all $x \in S$

$$C \ge |x\Pi_n v| = |x\Pi_n|\, |(x\Pi_n)^\sim v| \quad {}^\kappa\mathbb{P}\text{-a.s.}$$

By Lemma 5.5, the return times σ_n of $(x\Pi_n)^\sim$ in $B_\delta(v)$ are ${}^\kappa\mathbb{P}$-a.s.-finite, yielding

$$\limsup_{n\to\infty} |x\Pi_{\sigma_n}| \le \frac{C}{X_{\sigma_n} v} \le \frac{C}{1-\delta} \quad {}^\kappa\mathbb{P}\text{-a.s.}$$

for all $x \in S$, where (26) should be recalled for the final bound. Consequently,

$$\limsup_{n\to\infty} \frac{V_{\sigma_n}}{\sigma_n} = \limsup_{n\to\infty} \frac{1}{\sigma_n} |X_0 \Pi_{\sigma_n}| = 0 \quad {}^\kappa\mathbb{P}_\pi\text{-a.s.}$$

which contradicts Lemma 5.8.

Having thus shown that $\operatorname{supp} R$ is not compact in $\mathbb{R}^d$, there exist sequences $(x_n)_{n\geq 1} \subset \operatorname{supp} R$ with $\lim_{n\to\infty} |x_n| = \infty$ whence, by compactness of S, the following set is nonempty:

$$D := \{y \in S \ : \ \exists\, (x_n)_{n\geq 1} \subset \operatorname{supp} R,\ \lim_{n\to\infty} |x_n| = \infty,\ \lim_{n\to\infty} x_n^{\sim} = y\}.$$

Now suppose that $\mathbb{P}\,(x_0 R \leq t_0) = 1$ for some $(x_0, t_0) \in S \times \mathbb{R}$. For any $y_0 \in D$, choose a sequence $(x_n)_{n\geq 1} \subset \operatorname{supp} R$ such that $x_n^{\sim} \to y_0$. It follows that $x_0 x_n < t_0$ for all n and thereby (since $|x_n| \to \infty$), that $x_0 y_0 \leq 0$ for all $y_0 \in D$. On the other hand, $(Mx_n + Q)^{\sim} \to My_0$ for the unbounded sequence $(Mx_n + Q)_{n\geq 1}$ (which is $\mathbb{P}$-a.s. a subset of $\operatorname{supp} R$) implies $My_0 \in D$ $\mathbb{P}$-a.s. and therefore

$$\mathbb{P}\,(x_0 M y_0 \leq 0) = \mathbb{P}\,((x_0 M)^{\sim}\, y_0 \leq 0) = 1,$$

in particular $\mathbb{P}(x_0 M \notin B_\delta(y_0)) = 0$ for sufficiently small $\delta > 0$ which is a contradiction to (A4) (with $n_0 = 1$). □

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