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ON CRITICAL POINTS OF BLASCHKE PRODUCTS

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Abstract. We obtain an upper bound for the derivative of a Blaschke product, whose zeros lie in a certain Stolz-type region. We show that the derivative belongs to the space of analytic functions in the unit disk, introduced recently in [6]. As an outcome, we obtain a Blaschke-type condition for critical points of such Blaschke products.

1. Introduction

Given a sequence $\{z_n\} \subset \mathbb{D}$ subject to the Blaschke condition

$$\alpha := \sum_{n=1}^{\infty} (1 - |z_n|) < \infty, \tag{1}$$

let

$$B(z) = \prod_{n=1}^{\infty} b_n(z), \qquad b_n(z) = \frac{\overline{z}_n}{|z_n|} \frac{z_n - z}{1 - \overline{z}_n z}$$

be a Blaschke product with the zero set $Z(B) = \{z_n\}$. With no loss of generality we will assume that $B(0) \neq 0$.

One of the central problems with Blaschke products is that of the membership of their derivatives in classical function spaces, $B' \in X$. There is a vast literature on the problem, starting from investigations of P. Ahern and his collaborators [1, 2, 3, 4] and D. Protas [14] in 1970s, up to quite recent results of the Spanish school [8, 9, 10], see also [12, 15]. The above mentioned spaces X are primarily the Hardy spaces H^p, the Bergman spaces A^p, the Banach envelopes of the Hardy spaces B^p etc. Recall the definition of the Bergman spaces A^p, $p > 0$:

$$A^p = \{f \in \mathcal{A}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \, dxdy < \infty\}, \quad z = x + iy.$$

In this paper we will add to the list some new spaces $X = \mathcal{A}(E, \rho)$ of analytic functions in the unit disk, introduced recently in [6]. Given a closed set $E = \overline{E} \subset \mathbb{T}$ and $\rho > 0$, we say that an analytic function f belongs to $\mathcal{A}(E, \rho)$ if

$$|f(z)| \leq C_1 \exp\left(\frac{C_2}{d^\rho(z, E)}\right), \quad d(z, E) = \text{dist}(z, E)$$

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is the distance from $z \in \mathbb{D}$ to E, $C_{1,2}$ are positive constants.

The simplest and most general result drops out immediately from the Schwarz–Pick lemma for functions g from the unit ball of H^∞:

$$|g'(z)| \le \frac{1-|g(z)|^2}{1-|z|^2},$$

and states, that $g' \in A^p$ for all $0 < p < 1$. The result is sharp: there exists a Blaschke product B such that $B' \notin A^1$ (W. Rudin).

To proceed further, one should impose some additional restrictions either on absolute values $|z_n|$, stronger than (1), or on location, distribution of arguments of zeros z_n etc. For instance, a typical result of the first type is due to Protas [14]:

$$\sum_{n=1}^{\infty} (1-|z_n|)^r < \infty, \quad 0 < r < \frac{1}{2} \quad \Rightarrow \quad B' \in H^{1-r},$$

and $\frac{1}{2}$ is sharp.

We are more interested in the second direction, related to the location of zeros. A typical assumption here is that $Z(B)$ belongs to certain regions inside the unit disk.

Let $t \in \mathbb{T}$, $\gamma \ge 1$. Following [5, 3, 10] we introduce regions

$$R(t,\gamma,K) := \{\lambda \in \mathbb{D} : |t-\lambda|^\gamma \le K(1-|\lambda|)\}, \quad K \ge 1. \tag{2}$$

For $\gamma = 1$, $K > 1$ this is the standard Stolz angle. When $\gamma > 1$ the region touches the circle $\mathbb{T}$ at the vertex t with the power degree of tangency. The following result claims that B' belongs to H^p or A^p as soon as $Z(B) \subset R(t,1,K)$.

Theorem A. Let $Z(B) \subset R(t,1,K)$. Then

(1) $B' \in H^p$, $p < \frac{1}{2}$, and $\frac{1}{2}$ is sharp;

(2) $B' \in A^p$, $p < \frac{3}{2}$, and $\frac{3}{2}$ is sharp.

The first statement is proved in [9, Theorem 2.3], for the second one see [9, 8]. For related results in the case $Z(B) \subset R(t,\gamma,K)$ with $\gamma > 1$, see [10, Section 3].

We study the same problem for more general Stolz–type regions.

A function ϕ on the right half-line will be called a *model function*, if it is nonnegative, continuous and increasing, and

$$\phi(x) \le Cx, \qquad x \ge 0, \quad C = C(\phi) > 0. \tag{3}$$

We define a *Stolz angle associated with a model function* ϕ with the vertex at $t \in \mathbb{T}$ as

$$\mathcal{S}_\phi(t,K) = \mathcal{S}(t,K) := \{\lambda \in \mathbb{D} : \phi(|t-\lambda|) \le K(1-|\lambda|)\}, \quad K > 0. \tag{4}$$

Since $|t-\lambda| \le 2$ for $t, \lambda \in \overline{\mathbb{D}}$, it is clear that regions (2) are of the form (4) for an appropriate ϕ. Precisely, one can put $\phi(x) = x^\gamma$ for $0 \le x \le 2$, and

$\phi(x) = 2^{\gamma-1}x$ for $x \ge 2$. Next, given a closed set $E = \overline{E} \subset \mathbb{T}$ we define a *Stolz region, associated with a model function* ϕ *and the set* E, as

$$\mathcal{S}(E, K) := \{\lambda \in \mathbb{D} : \phi(d(\lambda, E)) \le K(1 - |\lambda|)\} = \bigcup_{t \in E} \mathcal{S}(t, K).$$

Here is our main result.

Theorem 1. *Let* B *be a Blaschke product such that* $Z(B) \subset \mathcal{S}(E, K)$. *Then*

$$|B'(z)| \le 2(2C + K)^2 \sum_{n=1}^{\infty} (1 - |z_n|)\, \phi^{-2}\left(\frac{d(z, E)}{6}\right). \tag{5}$$

For the standard Stolz angle and $E = \{t\}$ we take $\phi(x) = x$, so

$$|B'(z)| \le \frac{C_3}{|t - z|^2},$$

and part (1) in Theorem A follows. Similarly, for the region $R(t, \gamma, K)$, $\gamma > 1$, (5) implies

$$|B'(z)| \le \frac{C_4}{|t - z|^{2\gamma}},$$

and we come to the following result (cf. [10, Remark 1]).

Corollary 2. *If* $Z(B) \subset R(t, \gamma, K)$, $\gamma > 1$, *then* $B' \in H^p$ *for all* $p < 1/2\gamma$.

We are particularly interested in the model function $\phi(x) = \exp\{-x^{-\rho}\}$, $\rho > 0$. In this case (5) says that $B' \in \mathcal{A}(E, \rho)$.

Denote $Z(B') = \{z'_n\}$ the zero set of B'. Each result of the form $B' \in X$ provides some information about the critical points of B (zeros of B'), as long as the information about zero sets of functions from X is available. The most general condition applied to an arbitrary Blaschke product arises from the fact that $B' \in A^p$, $p < 1$, so (cf. [11, Theorem 4.7])

$$\sum_{n=1}^{\infty} \frac{1 - |z'_n|}{\left(\log \frac{1}{1 - |z'_n|}\right)^{1+\varepsilon}} < \infty, \qquad \forall \varepsilon > 0.$$

On the other hand there are Blaschke products B such that $\sum 1 - |z'_n| = \infty$ (see, e.g., [13]).

A Blaschke–type condition for zeros of functions from $\mathcal{A}(E, \rho)$ is given in a recent paper [6]. To present its main result we define, following P. Ahern and D. Clark [4, p.113], the type $\beta(E)$ of a closed subset E of the unit circle as

$$\beta(E) := \sup\{\beta \in \mathbb{R} : \ |E_x| = O(x^{\beta}), \ x \to 0\},$$

where

$$E_x := \{t \in \mathbb{T} : \ d(t, E) < x\}, \quad x > 0,$$

is an x-neighborhood of E, $|E_x|$ its normalized Lebesgue measure. For the equivalent definition and properties of the type see also [6, 7].

Theorem 3. *Given a closed set $E\subset\mathbb{T}$ and a Blaschke product B, assume that $Z(B)\subset\mathcal{S}_\phi(E,K)$, $\phi(x)=\exp\{-x^{-\rho}\}$, $\rho>0$. Then*

$$\sum_{n=1}^{\infty}(1-|z_n'|)d^{(\rho-\beta(E)+\varepsilon)_+}(z_n',E)<\infty, \qquad \forall\varepsilon>0,$$

$\beta(E)$ is the type of E, $(a)_+=\max(a,0)$.

It is clear that $\beta(E)=1$ for each finite set E.

Corollary 4. *Let $Z(B)\subset\mathcal{S}_\phi(t,K)$ with the same ϕ. Then*

$$\sum_{n=1}^{\infty}(1-|z_n'|)|t-z_n'|^{(\rho-1+\varepsilon)_+}<\infty, \qquad \forall\varepsilon>0.$$

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2. Main results

A model function ϕ is nonnegative and increasing, so for all $x,y,u\ge 0$

$$\phi\left(\frac{x+y+u}{3}\right)\le\phi(x)+\phi(y)+\phi(u). \tag{6}$$

We begin with the following result, which is similar to Vinogradov's lemma from [16].

Lemma 5. *Let $z\in\overline{\mathbb{D}}$, $t\in\mathbb{T}$ and $\lambda\in\mathcal{S}(t,K)$. Then*

$$\frac{1}{|1-\overline{\lambda}z|}\,\phi\left(\frac{|t-z|\lambda||}{3}\right)\le 2C+K. \tag{7}$$

Proof. With no loss of generality we assume that $t=1$. Since

$$\begin{aligned}|1-z|\lambda||&=\bigl|1-\overline{\lambda}z+z(\overline{\lambda}-|\lambda|)\bigr|\le|1-\overline{\lambda}z|+|\overline{\lambda}-|\lambda||\\&\le|1-\overline{\lambda}z|+(1-|\lambda|)+|1-\lambda|,\end{aligned}$$

then by (6)

$$\phi\left(\frac{|1-z|\lambda||}{3}\right)\le\phi(\bigl|1-\overline{\lambda}z\bigr|)+\phi(1-|\lambda|)+\phi(|1-\lambda|),$$

so

$$\frac{1}{|1-\overline{\lambda}z|}\,\phi\left(\frac{|1-z|\lambda||}{3}\right)\le A_1+A_2+A_3.$$

By (3), for the first two terms we have

$$A_1=\frac{\phi(\bigl|1-\overline{\lambda}z\bigr|)}{|1-\overline{\lambda}z|}\le C, \quad A_2\le\frac{\phi(1-|\lambda|)}{1-|\lambda|}\le C.$$

As for the third one,

$$A_3\le\frac{\phi(|1-\lambda|)}{1-|\lambda|}\le K$$

according to the assumption $\lambda\in\mathcal{S}(1,K)$. The proof is complete. □

Our main result gives a bound for the derivative B' in the case when the zero set $Z = Z(B) \subset \mathcal{S}(E, K)$.

Proof of Theorem 1. Denote $Z(t) := Z \bigcap \mathcal{S}(t, K)$, $t \in E$. Then there is an at most countable set $\{t_k\}_{k=1}^{\omega}$, $\omega \le \infty$, $t_k \in E$, so that $z_k \in Z(t_k)$, and $Z = \bigcup_k Z(t_k)$. It is clear that there is a disjoint decomposition

$$Z = \bigcup_k Z_k, \quad Z_k \neq \emptyset, \qquad Z_k \subset Z(t_k), \qquad Z_j \bigcap Z_k = \emptyset, \quad j \neq k.$$

Let us label the set Z in such a way that

$$Z = \{z_{kj}\}, \quad k = 1, 2, \dots, \omega, \quad j = 1, 2, \dots \omega_k, \quad \{z_{kj}\}_{j=1}^{\omega_k} \subset Z_k.$$

We proceed with the expression

$$B'(z) = \sum_{j=1}^{\infty} b'_n(z) B_n(z), \qquad B_n(z) = \frac{B(z)}{b_n(z)}\,,$$

so

$$|B'(z)| \le \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2}\,|B_n(z)| \le \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|1-\overline{z}_n z|^2} = \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1-|z_{kj}|^2}{|1-\overline{z}_{kj} z|^2}\,.$$

Note that $|t-z| \le 2\,|t - z|\lambda||$ for all $z, \lambda \in \mathbb{D}$ and $t \in \mathbb{T}$. Indeed,

$$|t - z|\lambda|| \ge 1 - |z\lambda| \ge 1 - |\lambda|$$

and

$$|t - z| \le |t - z|\lambda|| + |z|(1 - |\lambda|) \le 2\,|t - z|\lambda||\,,$$

as claimed. Hence

$$\phi\left(\frac{|t-z|}{6}\right) \le \phi\left(\frac{|t - z|\lambda||}{3}\right),$$

and by (7)

$$\begin{aligned}
\phi^2\left(\frac{d(z,E)}{6}\right)|B'(z)| &\le \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1-|z_{kj}|^2}{|1-\overline{z}_{kj} z|^2}\,\phi^2\left(\frac{|t_k - z|}{6}\right) \\
&\le \sum_{k=1}^{\omega} \sum_{j=1}^{\omega_k} \frac{1-|z_{kj}|^2}{|1-\overline{z}_{kj} z|^2}\,\phi^2\left(\frac{|t_k - z|z_{kj}||}{3}\right) \\
&\le 2(2C+K)^2 \sum_{n=1}^{\infty} (1-|z_n|),
\end{aligned}$$

which is (5). The proof is complete.

Theorem 3 is a direct consequence of Theorem 1 and [6, Theorem 3].

References

[1] P. Ahern, *The mean modulus of the derivateve of an inner function*, Indiana Univ. Math. J., **28** (1979), 311–347.

[2] P. Ahern, *The Poisson integral of a singular measure*, Can. J. Math. **35** (1983), 735–749.

[3] P. Ahern and D. Clark, *On inner functions with H^p derivatives*, Michigan Math. J., **21** (1974), 115–127.

[4] P. Ahern and D. Clark, *On inner functions with B^p derivatives*, Michigan Math. J., **23** (1976), 107–118.

[5] G. Cargo, *Angular and tangential limits of Blaschke products and their successive derivatives*, Canad. J. Math., **14** (1962), 334–348.

[6] S. Favorov and L. Golinskii, *A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators*, Amer. Math. Soc. Transl., **226** (2) (2009), 37–47.

[7] S. Favorov and L. Golinskii, *Blaschke-type conditions for analytic functions in the unit disk: inverse problems and local analogs*, preprint arXive:1007.3020v1 [math CV] 18 Jul 2010.

[8] D. Girela and J. Peláes, *On the membership in Bergman spaces of the derivative of A Blaschke product with zeros in a Stolz domain*, Canad. Math. Bull., **49** (3) (2006), 381–388.

[9] D. Girela, J. Peláes, and D. Vucotić, *Integrability of the derivative of a Blaschke product*, Proc. Edinburgh Math. Soc., **50** (2007), 673–687.

[10] D. Girela, J. Peláes, and D. Vucotić, *Interpolating Blaschke products: Stolz and tangential approach regions*, Constr. Approx., **27** (2008), 203–216.

[11] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces.* Graduate Texts in Mathematics, vol. 199. Springer-Verlag, New York, 2000.

[12] J. Mashreghi and M. Shabankhah, *Integral means of the logarithmic derivative of Blaschke products*, Comp. Meth. Func. Theory, **9** (2) (2009), 421–433.

[13] Ch. Pommerenke, *On the Green's function of Fuchsian groups*, Ann. Acad. Sci. Fenn., **2** (1976), 409–427.

[14] D. Protas, *Blaschke products with derivative in H^p and B^p*, Michigan Math. J., **20** (1973), 393–396.

[15] D. Protas, *Blaschke products with derivatives in function spaces*, preprint arXive:1001.5098v2 [math.CV] 10 Jun 2010.

[16] S. A. Vinogradov, *Multiplication and division in the space of analytic functions with area integrable derivative, and some related spaces*, Zap. Nauchn. Semin. POMI, **222** (1994), 45–77. (Russian)

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