FINITE GELFAND PAIR APPROACHES FOR EHRENFEST DIFFUSION MODEL

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Abstract. A classical diffusion model of Ehrenfest which consists of 2-urns and n-balls is realized by a finite Gelfand pair (H_n, S_n), where H_n is the hyperoctahedral group and S_n is the symmetric group. This fact can be generalized to multi-urn version by using Gelfand pairs of complex reflection groups .

Key Words: Finite Gelfand pair, Ehrenfest diffusion model, complex reflection group

1. introduction

There are two urns, the left one containing n-balls and the right one having no ball. We shuffle the balls according to a rule as follows; At each step one ball is chosen randomly and the ball is moved to the other urn chosen randomly. This process is called *Ehrenfest diffusion model*.

We consider a generalization of the above process by increasing the number of urns. Let $r \geq 2$. There are r distinct urns $U_0, U_1, \cdots, U_{r-1}$, initially U_0 contains distinct balls $B_1, \cdots, B_n$. Let us consider a similar diffusion process as above: At each step, we choose a ball randomly and transfer the ball into one of the other urns. Here we consider some kind of destinations of picked balls. The first one is any other urns without where it was (Section 3). Another two cases are considered in the Appendix of this paper; The second case is that a ball picked from U_i is moved to $U_{i+1 \pmod r}$ (Appendix 4.1). The last case is that a ball picked from U_i is moved to $U_{i+1 \pmod r}$ or $U_{i-1 \pmod r}$ (Appendix 4.2).

If U_j contains the ball B_i, then we define a function on the balls by $b_i = b(B_i) = j$. Then we can identify each configuration of our model with an element of a set

$$B(r, n) = \{(b_i \mid 1 \leq i \leq n) \mid 0 \leq b_i \leq r - 1\}.$$

In the next section, we see that the set $B(r, n)$ is an realization of a finite homogenous space $G(r, 1, n)/S_n$, where $G(r, 1, n)$ is an complex reflection group and S_n is a symmetric group. Indeed this pair of groups $(G(r, 1, n), S_n)$ is a Gelfand pair. Our main purpose is to analyze a stochastic space $B(r, n)$ by using this Gelfand pair. Further the book [2] is good introduction for an application of finite Gelfand pairs to probability theory.

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2. $B(r,n)$ AND $(G(r,1,n),S_n)$

In this section we introduce how to identify $B(r,n)$ with a certain finite and discrete homogenous space. Let S_n be the symmetric group and $G(r,1,n) = \mathbb{Z}/r\mathbb{Z} \wr S_n$ the complex reflection group. We denote an element of $G(r,1,n)$ by $(x_1,\cdots,x_n;\sigma)$, where $x_i \in \mathbb{Z}/r\mathbb{Z}$ and $\sigma \in S_n$. Under this notation, we remark that S_n is a subgroup $\{(0,\cdots,0;\sigma) \mid \sigma \in S_n\}$ of $G(r,1,n)$. We define an action of $G(r,1,n)$ on $B(r,n)$ by

$$x(b_i \mid 1 \leq i \leq n) = (x_i + b_{\sigma^{-1}(i)} \pmod r \mid 1 \leq i \leq n),$$

where $x = (x_1,\cdots,x_n;\sigma) \in G(r,1,n)$ and $(b_i \mid 1 \leq i \leq n) \in B(r,n)$.

Proposition 2.1. *This action is transitive on $B(r,n)$.*

Proof. We set $x = (r-b_1,\cdots,r-b_n;1)$. Then we have $x(b_i \mid 1 \leq i \leq n) = (0,\cdots,0)$. Furthermore this action is invertible. Therefore any two elements are transferred to each other by the action of $G(r,1,n)$. □

Proposition 2.2. *Put $I_0 = (0,\cdots,0) \in B(r,n)$. Then the stabilizer of I_0 is S_n.*

Proof. It is clear that the definition of the action. □

From these propositions, we can identify $B(r,n)$ with a finite homogenous space $G(r,1,n)/S_n$. Let $\mathbb{N} = \{0,1,2,\cdots\}$. For a composition $\mathbf{m} = (m_0,\cdots,m_{r-1})$ of n, let $|\mathbf{m}| = m_0 + \cdots + m_{r-1}$ be the size of $\mathbf{m}$. Put $N(r,n) = \{\mathbf{m} \in \mathbb{N}^r \mid |\mathbf{m}| = n\}$. For $\mathbf{m} \in N(r,n)$, we define a partition by $\lambda(\mathbf{m}) = (0^{m_0}1^{m_1}\cdots(r-1)^{m_{r-1}})$, i.e, m_i is regarded as the multiplicity of i. Let $m_\lambda(x_1,\cdots,x_n)$ be a monomial symmetric polynomial indexed by a partition λ. Let ξ be a primitive rth root of unity. We set

$$m_\lambda(\boldsymbol{\ell}) = m_\lambda(\underbrace{1,\cdots,1}_{\ell_0},\underbrace{\xi,\cdots,\xi}_{\ell_1},\cdots,\underbrace{\xi^{r-1},\cdots,\xi^{r-1}}_{\ell_{r-1}}),$$

for $\boldsymbol{\ell} = (\ell_0,\cdots,\ell_{r-1}) \in N(r,n)$.

Then the following theorem holds.

Theorem 2.3. [3]

(1) *A pair $(G(r,1,n),S_n)$ is a Gelfand pair.*
(2) *The permutation representation is decomposed as*

$$\mathbb{C}G(r,1,n)/S_n \sim \bigoplus_{\boldsymbol{k} \in N(r,n)} V(\boldsymbol{k}),$$

where $V(\boldsymbol{k})$ is an irreducible representation of $G(r,1,n)$ with $\dim V(\boldsymbol{k}) = \binom{n}{k_0,\cdots,k_{r-1}}$.
(3) *Let $\omega_{\boldsymbol{k}}$ be the zonal spherical function corresponding to $V(\boldsymbol{k})$. For $x = (x_1,\cdots,x_n;\sigma) \in G(r,1,n)$, we have*

$$\omega_{\boldsymbol{k}}(x) = \frac{m_{\lambda(\boldsymbol{k})}(\xi_1,\cdots,\xi_n)}{\binom{n}{k_0,\cdots,k_{r-1}}},$$

where $\xi_i = \xi^{x_i}$. Moreover, the table of the zonal spherical functions is given by

$$\left(\frac{m_{\lambda(\boldsymbol{k})}(\boldsymbol{\ell})}{\binom{n}{k_0,\cdots,k_{r-1}}} \right)_{\boldsymbol{k},\boldsymbol{\ell}\in N(r,n)}.$$

We denote by $\omega_{\boldsymbol{k},\boldsymbol{\ell}} = \frac{m_{\lambda(\boldsymbol{k})}(\boldsymbol{\ell})}{\binom{n}{k_0,\cdots,k_{r-1}}}$ the value of zonal spherical function indexed by $(\boldsymbol{k},\boldsymbol{\ell})$.

3. $B(r,n)$ AND $(G(r,1,n),S_n)$

We consider a stochastic processes on $B(r,n)$. The initial distribution is given by $\nu_0(b) = \begin{cases} 1, & b = I_0, \\ 0, & b \neq I_0. \end{cases}$ Let π be the uniform distribution on $B(r,n)$, i.e. $\pi \equiv \frac{1}{r^n}$. Similarly we denote by $\tilde{\pi}$ the uniform distribution on $G(r,1,n)$.

Let $P = (p(a,b))_{a,b\in B(r,n)}$ be a $G(r,1,n)$-invariant stochastic matrix on $B(r,n)$ i.e., $p(ga,gb) = p(a,b)$ for any $a,b \in B(r,n)$ and $x \in G(r,1,n)$. Then we define a function ν on $B(r,n)$ by $\nu(b) = p(b_0,b)$. Since the action of $G(r,1,n)$ is transitive, there exists $g \in G(r,1,n)$ such that $b = gb_0$. Put $\tilde{\nu}(g) = \frac{1}{n!}p(b_0,gb_0)$. It is easy to check that ν is a stochastic distribution and a bi-S_n invariant function on $G(r,1,n)$. Therefore $\tilde{\nu}$ can be expanded by the zonal spherical functions, say $\tilde{\nu} = \sum_{\boldsymbol{k}\in N(r,n)} a_{\boldsymbol{k}}\omega_{\boldsymbol{k}}$. Then the orthogonality relation of the zonal spherical functions gives us the following proposition.

Proposition 3.1. *Let $\boldsymbol{k} = (k_0,\cdots,k_{r-1}) \in N(r,n)$. Put $f_{\boldsymbol{k}} = \sum_{g\in G(r,1,n)} \tilde{\nu}(g)\overline{\omega_{\boldsymbol{k}}(g)}$. Then the coefficients a_k's are expressed by*

$$a_{\boldsymbol{k}} = \frac{\binom{n}{k_0,\cdots,k_{r-1}}}{r^n n!} f_{\boldsymbol{k}}.$$

Proof. Take an inner product of the both sides of $\tilde{\nu}$. □

The probability being in a state $b = gb_0$ $(g \in G(r,1,n))$ after N-steps iterate with a start point b_0 is

$$\begin{aligned} p^{(N)}(b) &= \sum_{b_0,\cdots,b_{k-1}\in B(r,n)} p(b_0,b_1)p(b_1,b_2)\cdots p(b_{N-1},b) \\ &= \sum_{b_0,\cdots,b_{N-1}\in B(r,n)} p(b_0,g_1b_0)p(g_1b_0,g_2b_0)\cdots p(g_{N-1}b_0,gb) \\ &= \sum_{x_1,\cdots,x_{N-1}\in G(r,1,n)} \tilde{\nu}(x_1)\tilde{\nu}(x_1^{-1}x_2)\cdots\tilde{\nu}(x_{N-1}^{-1}g) = \tilde{\nu}^{*N}(g) \text{ (Nth. convolution power)}, \end{aligned}$$

where we denote by g_i an element satisfying $b_i = g_ib_0$. By using a property of the zonal spherical functions $\omega_{\boldsymbol{k}} * \omega_{\boldsymbol{k}'} = \frac{|G(r,1,n)|}{\dim V(\boldsymbol{k})}\delta_{\boldsymbol{k}\boldsymbol{k}'}$, we have

$$\tilde{\nu}^{*N} = \sum_{\boldsymbol{k}\in N(r,n)} a_{\boldsymbol{k}}^N \omega_{\boldsymbol{k}}^{*N} = \sum_{\boldsymbol{k}\in N(r,n)} \frac{\binom{n}{k_0,\cdots,k_{r-1}}}{r^n n!} f_{\boldsymbol{k}}^N \omega_{\boldsymbol{k}}.$$

For stochastic distributions μ and μ' on a space X, the total variation distance is defined by

$$||\mu - \mu'||_{TV} = \frac{1}{2}\sum_{x\in X} |\mu(x) - \mu'(x)|.$$

Then the following estimate is known.

Proposition 3.2. [1, Corollary 4.9.2, pp. 144]

$$||\nu^{*N} - \pi||^2_{TV} = ||\tilde{\nu}^{*N} - \tilde{\pi}||^2_{TV} \leq \frac{1}{4}\sum \binom{n}{k_0, \cdots, k_{r-1}} |f_k|^{2N},$$

where $\boldsymbol{k} = (k_0, \cdots, k_{r-1})$ *runs over* $N(r, n)$ *except* $\boldsymbol{k} = (n, 0, \cdots, 0)$ *which corresponds to the trivial representation of* $G(r, 1, n)$*.*

We compute $f_{\boldsymbol{k}}$'s for three distinct stochastic matrices on $B(r, n)$. We define a function on $B(r, n) \times B(r, n)$ by

$$d_r(b, c) = \#\{i \mid b_i \neq c_i\},$$

where $b = (b_i \mid 1 \leq i \leq n)$ and $c = (c_i \mid 1 \leq i \leq n)$. We define a stochastic matrix by

$$p(x, y) = \begin{cases} \frac{1}{(r-1)n}, & d_r(x, y) = 1 \\ 0, & d_r(x, y) \neq 1. \end{cases}$$

Clearly the setting means the first way of shuffle explained in Section 1.

Proposition 3.3. *For* $\boldsymbol{k} = (k_0, \cdots, k_{r-1}) \in B(r, n)$*, we have* $f_{\boldsymbol{k}} = \frac{1}{r-1}\left(\frac{rk_0}{n} - 1\right)$

Proof. We define $g_{m,j} = (g_1, \cdots, g_n : 1) \in G(r, 1, n)$ by $g_i = \begin{cases} j & (i = m), \\ 0 & (i \neq m) \end{cases}$ for any $1 \leq m \leq n$ and $1 \leq j \leq r - 1$. Then we have $\nu(g) = \begin{cases} 1 & (g = g_{m,j}), \\ 0 & (g \neq g_{m,j}). \end{cases}$ Now we can compute $f_{\boldsymbol{k}}$ as follows.

$$\begin{aligned}
\sum_{g\in G(r,1,n)} \tilde{\nu}(g)\overline{\omega_{\boldsymbol{k}}(g)} &= \frac{1}{n(r-1)}\sum_{j=1}^{r-1}\sum_{m=1}^{n} \overline{\omega_{\lambda(\boldsymbol{k})}(g_{m,j})} \\
&= \frac{1}{r-1}\sum_{j=1}^{r-1} \frac{\sum_{i=0}^{r-1}\binom{n-1}{k_0,\cdots,k_i-1,\cdots,k_{r-1}}\zeta^{-ij}}{\binom{n}{k_0,\cdots,k_{r-1}}} = \frac{1}{r-1}\sum_{i=0}^{r-1}\frac{k_i}{n}\sum_{j=1}^{r-1}\zeta^{-ij} \\
&= \frac{1}{r-1}\left\{\frac{k_0(r-1)}{n} + \sum_{i=1}^{r-1}\frac{-k_i}{n}\right\} = \frac{1}{r-1}\left(\frac{rk_0}{n} - 1\right).
\end{aligned}$$

In the second equality we use Theorem 2.3 (3) and the following equation;

$$m_{\lambda(\boldsymbol{k})}(1, \cdots, 1, x) = \sum_{i=0}^{r-1}\binom{n-1}{k_0, \cdots, k_i - 1, \cdots, k_{r-1}} x^i.$$

□

From the proposition above, we can easily show $|f_k| \leq 1$ and $|f_k| = 1 \Leftrightarrow k_0 = \begin{cases} n & (r \geq 3) \\ 0 \text{ or } n & (r = 2). \end{cases}$

Theorem 3.4. *For $g = (x_1, \cdots, x_n : \sigma)$, we have*

$$\tilde{\nu}^{*N}(g) = \frac{1}{r^n n!} \sum_{k_0=0}^{n} \left(\frac{rk_0 - n}{n(r-1)} \right)^N e_{n-k_0}(\Phi_1, \cdots, \Phi_n),$$

where $\Phi_i = \xi^{x_i} + \xi^{2x_i} + \cdots + \xi^{(r-1)x_i} = \begin{cases} r-1 & (x_i = 1), \\ -1 & (x_i \neq 1) \end{cases}$ and $e_j(x_1, \cdots, x_n)$ is the j-th elementary symmetric polynomial.

Proof. For $\boldsymbol{k} = (k_0, k_1, \cdots, k_{r-1})$, we have the following generating function ([3])

$$\prod_{i=1}^{n} (1 + \Phi_i) = \sum_{k} \binom{n}{k_0, \cdots, k_{r-1}} \omega_{\boldsymbol{k}}(g).$$

We remark that $\prod_{i=1}^n (1+\Phi_i) = \sum_j e_j(\Phi_1, \cdots, \Phi_n)$ and $e_j(\Phi_1, \cdots, \Phi_n) = \sum_{\ell(\lambda(\boldsymbol{k}))=j} m_{\lambda(\boldsymbol{k})}(\xi_1, \cdots, \xi_n)$, where e_j is an elementary symmetric polynomial and $\ell(\lambda)$ is the length of λ. We compute

$$\begin{aligned}
\tilde{\nu}^{*N} &= \frac{1}{r^n n!} \sum_{\boldsymbol{k} \in N(r,n)} \frac{\binom{n}{k_0, \cdots, k_{r-1}}}{r^n n!} f_{\boldsymbol{k}}^N \omega_{\boldsymbol{k}} \\
&= \frac{1}{r^n n!} \sum_{k_0=0}^{n} \binom{n}{k_0} \left(\frac{rk_0 - n}{n(r-1)} \right)^N \sum_{k_1 + \cdots + k_{r-1} = n - k_0} \binom{n - k_0}{k_1, \cdots, k_{r-1}} \omega_{k_0, \cdots, k_{r-1}}(g) \\
&= \frac{1}{r^n n!} \sum_{k_0=0}^{n} \left(\frac{rk_0 - n}{n(r-1)} \right)^N \sum_{k_1 + \cdots + k_{r-1} = n - k_0} m_{\lambda(\boldsymbol{k})}(\xi_1, \cdots, \xi_n) \\
&= \frac{1}{r^n n!} \sum_{k_0=0}^{n} \left(\frac{rk_0 - n}{n(r-1)} \right)^N e_{n-k_0}(\Phi_1, \cdots, \Phi_n).
\end{aligned}$$

□

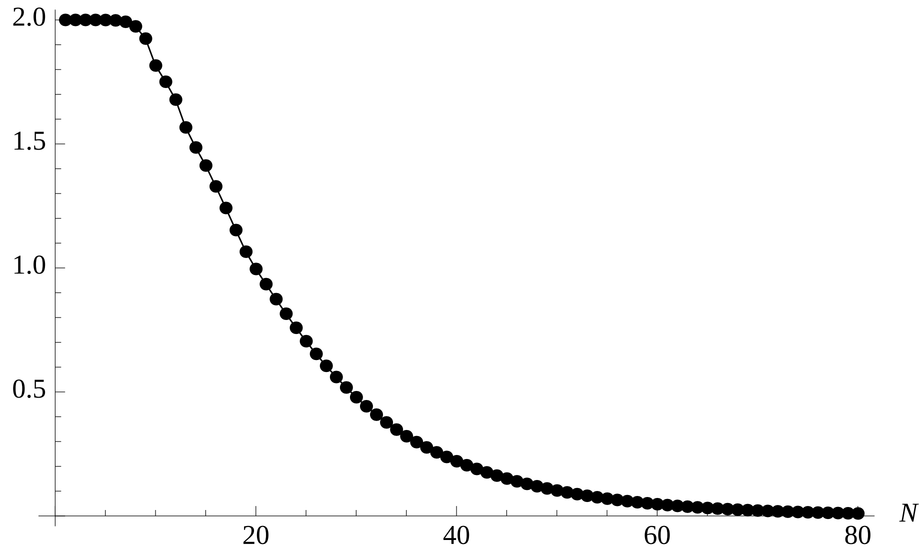
From this theorem, we have immediately the following corollary.

Corollary 3.5. *If $r > 2$, then $\lim_{N \to \infty} \tilde{\nu}^{*N} = \frac{1}{r^n n!}$. If $r = 2$, let ℓ be a number of balls in the urn 1, then*

$$\lim_{N \to \infty} \tilde{\nu}^{*2N} = \begin{cases} \frac{1}{2^{n-1} n!} & (\ell \equiv 0 \pmod 2), \\ 0 & (\ell \equiv 1 \pmod 2) \end{cases}, \quad \lim_{N \to \infty} \tilde{\nu}^{*2N-1} = \begin{cases} \frac{1}{2^{n-1} n!} & (\ell \equiv 1 \pmod 2), \\ 0 & (\ell \equiv 0 \pmod 2) \end{cases}.$$

We try to estimate an upper bound of $\tilde{\nu}^{*N}$. Before state a theorem, we see the following example.

Example 3.6. *We put $r = 3$ and $n = 20$. Then we have the following the graph of total variation distance.*



Here The horizontal axis is the number of shuffles N.

Theorem 3.7. *Put $N = \frac{n(r-1)}{2r}(\log r^n + c)$.*

(1) *In the case of $r = 2$, we have*

$$||\nu^{*N} - \pi||_{TV}^2 - 1/4 \leq \frac{1}{4}e^{-c}.$$

(2) *In the case of $r \geq 3$, we have*

$$||\nu^{*N} - \pi||_{TV}^2 \leq \frac{1}{4}e^{-c}.$$

Proof. We refer to Proposition 3.2 and compute

$$\begin{aligned}
||\nu^{*N} - \pi||_{TV}^2 &\leq \frac{1}{4} \sum_{\boldsymbol{k} \in N(r,k), k_0 \neq n} \binom{n}{k_0, \cdots, k_{r-1}} \left| \frac{1}{r-1}\left(\frac{rk_0}{n} - 1\right)\right|^{2N} \\
&= \frac{1}{4} \sum_{k_0=0}^{n-1} \binom{n}{k_0} \left| \frac{1}{r-1}\left(\frac{rk_0}{n} - 1\right)\right|^{2N} \sum_{k_1+\cdots+k_{r-1}=n-k_0} \binom{n-k_0}{k_1, \cdots, k_{r-1}} \\
&= \frac{1}{4} \sum_{k_0=0}^{n-1} \binom{n}{k_0} \left| \frac{1}{r-1}\left(\frac{rk_0}{n} - 1\right)\right|^{2N} (r-1)^{n-k_0} \\
&\leq \begin{cases} \frac{1}{4} + \frac{1}{4}e^{-\frac{4N}{n}} \sum_{k_0=1}^{n-1} \binom{n}{k_0} = \frac{1}{4} + \frac{1}{4}(2^n - 2)e^{-\frac{4N}{n}} \leq \frac{1}{4} + \frac{1}{4}2^n e^{-\frac{4N}{n}} & (r=2), \\ \frac{1}{4}e^{-\frac{2rN}{n(r-1)}} \sum_{k_0=0}^{n-1} \binom{n}{k_0}(r-1)^{n-k_0} = \frac{1}{4}(r^n - 1)e^{-\frac{2rN}{n(r-1)}} \leq \frac{1}{4}r^n e^{-\frac{2rN}{n(r-1)}} & (r \geq 3). \end{cases}
\end{aligned}$$

Here we use $|1 - a| \leq e^{-a}$ for $a \leq 1$. $\square$

Remark 3.8. *In the Theorem above, "1/4" is the limit of $||\nu^{*N} - \pi||_{TV}^2$ as $n \to \infty$ which comes from a consequence of Corollary 3.5.*

4. Appendix

Through this section, the urns $U_0, U_1, \cdots, U_{r-1}$ form a circle. Here we take up another shuffles and compute their Fourier coefficients.

4.1. **cyclic shuffle 1.** We consider the following type of shuffle; "The ball picked randomly is transferred into the next urn on the left." We define the following stochastic matrix on $B(r,n)$;

$$p(b,c) = \begin{cases} \frac{1}{n}, & d_r(b,c) = 1 \text{ and } b_i - c_i \in \{0, 1, r-1\} \ (1 \leq \forall i \leq n), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly this gives us the probabilities of the shuffle introduced above. Now we have

Proposition 4.1. *For* $k = (k_0, \cdots, k_{r-1}) \in B(r,n)$*, we have* $f_k = \sum_{i=0}^{r-1} \frac{k_i}{n} \zeta^{-i}$.

Proof. We define $g_{m,j} = (g_1, \cdots, g_n : 1) \in G(r,1,n)$ by $g_i = j\delta_{im}$.

$$\sum_{g \in G(r,1,n)} \tilde{\nu}(g) \overline{\omega_{\boldsymbol{k}}(g)} = \frac{1}{n} \sum_{m=1}^{n} \overline{\omega_{\lambda(\boldsymbol{k})}(g_{m,1})}$$
$$= \frac{\sum_{i=0}^{r-1} \binom{n-1}{k_0, \cdots, k_i - 1, \cdots, k_{r-1}} \zeta^{-i}}{\binom{n}{k_0, \cdots, k_{r-1}}} = \sum_{i=0}^{r-1} \frac{k_i}{n} \zeta^{-i}.$$

□

4.2. **cyclic shuffle 2.** Here a shuffle, "The ball picked randomly is transferred into the next urn on the right or left randomly," is considered. The stochastic matrix on $B(r,n)$ of this shuffle is given by

$$p(b,c) = \begin{cases} \frac{1}{2n}, & d_r(b,c) = 1 \text{ and } b_i - c_i \in \{0, \pm 1, \pm(r-1)\} \ (1 \leq \forall i \leq n), \\ 0, & \text{otherwise.} \end{cases}$$

Now we have

Proposition 4.2. *For* $k = (k_0, \cdots, k_{r-1}) \in B(r,n)$*, we have* $f_k = \frac{1}{r-1} \left(\frac{rk_0}{n} - 1 \right)$

Proof. We define $g_{m,j} = (g_1, \cdots, g_n : 1) \in G(r,1,n)$ by $g_i = j\delta_{im}$.

$$\sum_{g \in G(r,1,n)} \tilde{\nu}(g) \overline{\omega_{\boldsymbol{k}}(g)} = \frac{1}{2n} \sum_{m=1}^{n} \left(\overline{\omega_{\lambda(\boldsymbol{k})}(g_{m,1})} + \overline{\omega_{\lambda(k)}(g_{m,r-1})} \right)$$
$$= \frac{1}{2} \frac{\sum_{i=0}^{r-1} \binom{n-1}{k_0, \cdots, k_i - 1, \cdots, k_{r-1}} \left(\zeta^{-i} + \zeta^{i} \right)}{\binom{n}{k_0, \cdots, k_{r-1}}} = \frac{1}{2} \sum_{i=0}^{r-1} \frac{k_i}{n} \left(\zeta^{-i} + \zeta^{i} \right).$$

□

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