Finsler connection preserving angle in dimensions $N \geq 3$

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Abstract

We show that if a Finsler space is conformally automorphic to a Riemannian space and the automorphism is positively homogeneous with respect to tangent vectors, then the indicatrix of the Finsler space is a space of constant curvature. In this case, the Finslerian two-vector angle can explicitly be found, which gives rise to simple and explicit representation for the connection preserving the angle in the indicatrix-homogeneous case. The connection is metrical and the Finsler space is obtainable from the Riemannian space by means of the parallel deformation. Since also the transitivity of covariant derivative holds, in such Finsler spaces the metrical non-linear angle-preserving connection is the respective export of the metrical linear Riemannian connection. From the commutators of covariant derivatives the associated curvature tensor is found. In case of the $\mathcal{FS}$-space, the explicit example of the conformally automorphic transformation can be developed, which entails the explicit connection coefficients and the metric function of the Finsleroid type.

1. Motivation and description

In any dimension $N \geq 3$ the Finsler metric function F geometrizes the tangent bundle TM over the base manifold M such that at each point $x \in M$ the tangent space T_xM is endowed with the curvature tensor constructed from the respective Finslerian metric tensor $g_{\{x\}}(y)$ by means of the conventional rule of the Riemannian geometry considering y to be the variable argument. There arises the Riemannian space $\mathcal{R}_{\{x\}} = \{g_{\{x\}}(y), T_xM\}$ supported by the point $x \in M$ such that T_xM plays the role of the base manifold for the space. In the Riemannian limit of the Finsler space, the spaces $\mathcal{R}_{\{x\}}$ are Euclidean spaces, so that the tensor $g_{\{x\}}(y)$ is independent of y. The conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ can naturally be taken to treat as the next level of generality of the Finsler space. Can the metrical connection preserving the two-vector angle be introduced on that level?

The deformation of the Riemannian space to the Finsler space proves to be convenient invention to apply. Namely, in the particular case when the Riemannian space can be deformated to the Finsler space characterized by the conformally flat structure of the spaces $\mathcal{R}_{\{x\}}$ the positive and clear answer to the above question can be arrived at.

Given an N-dimensional Riemannian space $\mathcal{R}^N = (M, S)$, where S denotes the Riemannian metric function, one may endeavor to obtain a Finsler space $\mathcal{F}^N = (M, F)$ by applying an appropriate *deformation* $\mathcal{C}$ of the space $\mathcal{R}^N$. The notation F stands for the Finsler metric function. The base manifold M is keeping the same for both the spaces, $\mathcal{R}^N$ and $\mathcal{F}^N$.

We assume that the transformation $\mathcal{C}$ is *restrictive*, in the sense that no point $x \in M$ is shifted under the transformation, so that in each tangent space T_xM the deformation maps tangent vectors $y \in T_xM$ into the tangent vectors of the same T_xM:

$$y = \mathcal{C}(x, \bar{y}), \quad y, \bar{y} \in T_xM. \tag{1.1}$$

In general, this transformation is non-linear with respect to $\bar{y}$. Non-singularity and sufficient smoothness are always implied.

We may evidence in the Riemannian space $\mathcal{R}^N$ the *metrical linear Riemannian connection* $\mathcal{RL}$, which in terms of local coordinates $\{x^i\}$ introduced in M is given by

$$\mathcal{RL} = \{L^m{}_j, L^m{}_{ij}\} : \qquad L^m{}_j = -a^m{}_{ij}y^i, \quad L^m{}_{ij} = a^m{}_{ij}, \tag{1.2}$$

with $a^m{}_{ij} = a^m{}_{ij}(x)$ standing for the Christoffel symbols constructed from the Riemannian metric tensor $a_{mn}(x)$ of the space $\mathcal{R}^N$. The indices $i, j, ...$ are specified on the range $(1, \ldots, N)$. The respective covariant derivative ∇ can be introduced in the natural way, namely by means of the definition (4.18) which uses the operator

$$d_i^{\rm Riem} = \frac{\partial}{\partial x^i} + L^k{}_i \frac{\partial}{\partial y^k}, \tag{1.3}$$

to act on tensors considered on the tangent bundle underlined the space $\mathcal{R}^N$. In the space, the scalar product $\langle y_1, y_2\rangle_{\{x\}}^{\rm Riem} = a_{mn}(x)y_1^m y_2^n$ of two vectors y_1, y_2 supported by a fixed point $x \in M$ is linear with respect to each vector, which gives rise to the profound meaning of the connection (1.2) to preserve the product under the entailed parallel transports of the entered vectors along curves running on M.

In the Finsler space, the scalar product is essentially non-linear object with respect to the entered vectors, so that we may hope to meet similar preservation property in the Finslerian domains if only we apply the connection which is non-linear, in the sense that the involved connection coefficients depend on tangent vectors y in non-linear way. With this hope, we need the *metrical non-linear Finsler connection* $\mathcal{FN}$, such that

$$\mathcal{FN} = \{N^m{}_j, D^m{}_{ij}\} : \qquad N^m{}_j = N^m{}_j(x,y), \qquad D^m{}_{ij} = D^m{}_{ij}(x,y). \tag{1.4}$$

The adjective "metrical" means that the action of the entailed covariant derivative $\mathcal{D}$ on the Finsler metric function, and also on the Finsler metric tensor, yields identically zero. The coefficients $N^m{}_j$ and $D^m{}_{ij}$ are assumed to be positively homogeneous regarding the dependence on vectors y, respectively of degree 1 and degree 0.

Accordingly, the most important object what should be lifted from the Riemannian to Finslerian space is the two-vector angle, to be denoted by $\alpha_{\{x\}}(y_1, y_2)$, where $y_1, y_2 \in T_xM$. Like to the Riemannian geometry proper, the underlined idea is to measure the angle by means of length of the respective geodesic arcs evidenced on the indicatrix.

The Finsler space endows the vector pair y_1, y_2 with the scalar product

$$\langle y_1, y_2 \rangle_{\{x\}} = F(x, y_1) F(x, y_2) \alpha_{\{x\}}(y_1, y_2)$$

on analogy of the Riemannian geometry.

The non-linear deformation

$$\mathcal{FN} = \mathcal{C} \cdot \mathcal{RL} \tag{1.5}$$

of the Riemannian connection may exist to yield the Finsler connection $\mathcal{FN}$ which preserves the Finslerian two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ under the associated parallel transports of the vectors y_1, y_2.

In the theory of Finsler spaces, the key objects, the connection included, were introduced and studied on the basis of various convenient sets of axioms (see [1-5] and references therein). Regarding the significance of the angle notion, the important farther step was made in [6] were in processes of studying implications of the two-vector angle defined by area, the theorem was proved which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angle. This Tamássy's theorem substantiates the idea to develop the Finsler connection from the Finsler two-vector angle, possibly on the analogy of the Riemannian geometry.

To meet new methods of applications, the interesting chain of linear connections was introduced and studied in [3]. It was emphasized that in the Riemannian geometry we have naturally the metrical and linear connection. We depart from this connection to develop the Finsler connection.

Namely, we shall confine our attention to the case when the space $\mathcal{F}^N$ is obtainable from the space $\mathcal{R}^N$ by means of the *conformal automorphism*, according to the definition (2.1) of Section 2. We shall also assume that under the used transformations the Finslerian indicatrix $\mathcal{IF}_{\{x\}} \in T_xM$ and the Riemannian sphere $\mathcal{S}_{\{x\}} \in T_xM$ are in correspondence (according to (2.2)).

Additionally, we subject the $\mathcal{C}$-transformations to the condition of positive homogeneity with respect to tangent vectors y, denoting the degree of homogeneity by H. We call the H the *degree of conformal automorphism*.

Remarkably, such Finsler spaces of dimensions $N \geq 3$ can be characterized by the condition that the indicatrix is a space of constant curvature (see Proposition 2.1). *The indicatrix curvature value $\mathcal{C}_{Ind.}$ is the square of the degree of conformal automorphism,* that is,

$$\mathcal{C}_{\text{Ind.}} \equiv H^2 \tag{1.6}$$

(indicated in (2.3)). The condition has been realized, the Finslerian two-vector angle $\alpha_{\{x\}}(y_1,y_2)$ proves to be a factor of the angle $\alpha^{\rm Riem}_{\{x\}}(y_1,y_2)$ operative traditionally in the Riemannian space, namely the simple equality

$$\alpha_{\{x\}}(y_1,y_2)=\frac{1}{H(x)}\alpha^{\rm Riem}_{\{x\}}(y_1,y_2) \tag{1.7}$$

(see (2.31)-(2.32)) is obtained.

The equality

$$S(x,\bar{y})=(F(x,y))^H \tag{1.8}$$

(see (2.10)) is arisen, which validates the indicatrix correspondence principle (2.2).

We set forth the conventional requirement of preservation of the Finsler metric function $F(x,y)$, namely

$$d_iF=0 \tag{1.9}$$

with

$$d_i=\frac{\partial}{\partial x^i}+N^k{}_i(x,y)\frac{\partial}{\partial y^k}. \tag{1.10}$$

With the definition

$$\mathcal{D}y^n \ := dy^n-N^n{}_j(x,y)dx^j \tag{1.11}$$

of covariant displacement of the tangent vector, the parallel transport of the vector means the vanishing

$$\mathcal{D}y^n=0. \tag{1.12}$$

We apply this observation to the two-vector angle $\alpha_{\{x\}}(y_1,y_2)$: the coefficients $N^k{}_i(x,y)$ fulfill the *angle preservation equation*

$$d_i\alpha_{\{x\}}(y_1,y_2)=0, \qquad y_1,y_2\in T_xM \tag{1.13}$$

under the parallel displacements of the entered vectors y_1 and y_2, if the involved operator d_i is taken to read

$$d_i=\frac{\partial}{\partial x^i}+N^k{}_i(x,y_1)\frac{\partial}{\partial y_1^k}+N^k{}_i(x,y_2)\frac{\partial}{\partial y_2^k}. \tag{1.14}$$

The $N^k{}_i(x,y)$ thus appeared can naturally be interpreted as the *coefficients of the nonlinear connection produced by angle.*

In this way we fulfill the canonical geometrical principle: the angle $\alpha_{\{x\}}(y_1,y_2)$ formed by two vectors y_1 and y_2 is left unchanged under the parallel displacements of the vectors y_1 and y_2, namely $\mathcal{D}\alpha \stackrel{\rm def}{=} (dx^i)d_i\alpha=0$, for $d_i\alpha=0$.

In general the indicatrix curvature value $\mathcal{C}_{\rm Ind.}$ may depend on the points $x\in M$. We say that the space $\mathcal{F}^N$ is *indicatrix-homogeneous*, if the value is a constant. In view of the result $\mathcal{C}_{\rm Ind.}\equiv H^2$ (indicated in (2.3)), such spaces can be characterized by the condition that the degree H of conformal automorphism is independent of x.

It proves that *in the indicatrix-homogeneous case of the studied space $\mathcal{F}^N$ the equations* (1.13)–(1.14) *can explicitly be solved for the coefficients* $N^k{}_i$ (see Proposition 2.2 and Note placed thereafter in Section 2).

From the obtained coefficients $N^k{}_m$ given by (2.36), the entailed coefficients

$$N^k{}_{mn}=\frac{\partial N^k{}_m}{\partial y^n}, \qquad N^k{}_{mnj}=\frac{\partial N^k{}_{mn}}{\partial y^j} \tag{1.15}$$

can straightforwardly be evaluated (Section 3). Let us use the coefficients to construct the covariant derivative $\mathcal{D}_m g_{nj}$ of the Finsler metric tensor $g_{nj} = g_{nj}(x,y)$ of the considered space $\mathcal{F}^N$, namely

$$\mathcal{D}_m g_{nj} \ := d_m g_{nj} + N^k{}_{mj} g_{kn} + N^k{}_{mn} g_{kj}, \tag{1.16}$$

where d_m is given by (1.10). It proves that the covariant derivative introduced by (1.16) with the coefficients $N^k{}_m$ given by (2.36) possesses the property

$$\mathcal{D}_m g_{nj} = 0 \tag{1.17}$$

in the indicatrix-homogeneous case. The property can be verified by straightforward substitutions which result in the vanishing

$$y_k N^k{}_{mnj} = 0 \tag{1.18}$$

(see Proposition 3.1).

It is amazing but the fact that the last vanishing is an implication of the identity $y^k C_{knj} = 0$ shown by the Cartan tensor $C_{knj} = (1/2)\partial g_{kn}/\partial y^j$. Indeed, additional evaluation leads to the result

$$N^k{}_{mnj} = -\mathcal{D}_m C^k{}_{nj} \tag{1.19}$$

in the indicatrix-homogeneous case (see Proposition 3.2), where

$$\mathcal{D}_m C^k{}_{nj} \ := d_m C^k{}_{nj} - N^k{}_{mt} C^t{}_{nj} + N^t{}_{mn} C^k{}_{tj} + N^t{}_{mj} C^k{}_{nt}. \tag{1.20}$$

The coefficients $N_{kmnj} = g_{kh} N^h{}_{mnj}$ can be written as

$$N_{kmnj} = -\mathcal{D}_m C_{knj} \tag{1.21}$$

and, therefore, they are *symmetric* with respect to the subscripts k, n, j.

Thus, with the identification

$$D^k{}_{in}(x,y) = -N^k{}_{in}(x,y), \tag{1.22}$$

in the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type (that is, when $H = const$) the metrical angle-preserving connection (1.4) is given by the coefficients $\{N^k_i, D^k{}_{in}\}$ found explicitly. Recollecting the assumed homogeneity of the coefficients, from (1.22) we infer the equality

$$D^k{}_{im} y^m = -N^k{}_i. \tag{1.23}$$

Realizing the $\mathcal{C}$-transformation locally by $y^i = y^i(x,t)$ with $t^n \equiv \bar{y}^n$ (see (2.11)) and applying the Riemannian operator

$$d_i^{\text{Riem}} = \frac{\partial}{\partial x^i} - a^k{}_{ih} t^h \frac{\partial}{\partial t^k}$$

(cf. (1.3)) to the field $y^i(x,t)$, it is possible to conclude that

$$N^n{}_i = d_i^{\text{Riem}} y^n \tag{1.24}$$

(see (2.47)). This representation of the coefficients $N^n{}_i$ possesses a clear geometrical and tensorial meaning and is alternative (and equivalent) to the representation (2.36). The derivation of the representation (1.24) uses the formula (1.23).

According to Proposition 2.3, the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type is obtained from the Riemannian space $\mathcal{R}^N$ by means of the *parallel deformation*.

Since also the transitivity of covariant derivative holds, namely $\mathcal{D}_n t^i = 0$ (see (2.39)), and $g_{kh} = \mathcal{C}^m_k \mathcal{C}^n_h a_{mn}$ (see (2.25)), we should conclude that in the Finsler space $\mathcal{F}^N$ of the indicatrix-homogeneous type the metrical angle-preserving connection is the $\mathcal{C}$-export of the metrical linear Riemannian connection (1.2) applied conventionally in the background Riemannian space $\mathcal{R}^N$.

In Section 4 we perform the attentive comparison between the commutators of the involved Finsler covariant derivative $\mathcal{D}$ and the commutators of the underlined Riemannian covariant derivative ∇, assuming $H = const$. By this method, we derive the associated curvature tensor $\rho_k{}^n{}_{ij}$. The skew-symmetry $\rho_{mnij} = -\rho_{nmij} = -\rho_{mnji}$ holds. The covariant derivative $\mathcal{D}_l$ of the tensor fulfills the cyclic identity, completely similar to the Riemannian case in which the cyclic identity is valid for the derivative $\nabla_l a_k{}^n{}_{ij}$ of the Riemannian curvature tensor $a_k{}^n{}_{ij}$. The tensor $M^n{}_{ij} = -y^k \rho_k{}^n{}_{ij}$ proves to be transitive to the Riemannian tensor $-y^n_t t^h a_h{}^t{}_{ij}$, namely the equality $M^n{}_{ij} = -y^n_t t^h a_h{}^t{}_{ij}$ holds. The very tensor $\rho_k{}^n{}_{ij}$ is not transitive to the Riemannian precursor $a_h{}^m{}_{ij}$, instead the more general equality

$$\rho_k{}^n{}_{ij} = -(1-H)\frac{1}{F}(l_k\delta^n_m - l^n g_{mk})M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k$$

is obtained. We observe that the difference between the curvature tensor $\rho_k{}^n{}_{ij}$ and the transitive term $y^n_m a_h{}^m{}_{ij} t^h_k$ is proportional to $(1-H)$. Squaring the tensor yields the sum

$$\rho^{knij}\rho_{knij} = a^{knij}a_{knij} + \frac{2}{S^2}\left(\frac{1}{H^2}-1\right) t^l a_l{}^{nij} t^h a_{hnij},$$

which is the $\mathcal{F}^N$-extension of the Riemannian term $a^{knij}a_{knij}$. The difference $\rho^{knij}\rho_{knij} - a^{knij}a_{knij}$ is proportional to $(H^{-2}-1)$.

In Section 5 we develop an explicit and attractive particular case, namely we present the explicit example (5.27) of the conformally automorphic transformation (2.1), specializing the Finsler space to be the $\mathcal{FS}$-space. The space is endowed with the Finsler metric function F which is constructed from a Riemannian metric function $S = \sqrt{a_{ij}(x)y^iy^j}$ and an 1-form $b = b_i(x)y^i$ according to the functional dependence

$$F(x,y) = \Phi\,(x; b, S, y)\,, \tag{1.25}$$

where Φ is a sufficiently smooth scalar function. In step-by-step way, we derive the coefficients $N^m{}_i$ specified by (2.36), obtaining the explicit representation (5.72)–(5.75). It proves that the suitability of the proposed transformation (5.27) imposes the severe restriction on the Finsler metric function, namely the function must be of the Finsleroid type (described in [7]). In the restricted case which implies independence of the function $\Phi\,(x; b, S, y)$ of x, assuming also that the Riemannian norm of the 1-form b is a constant, the obtained coefficients $N^m{}_i$ straightforwardly entail the vanishing set $\mathcal{D}_n F = \mathcal{D}_n y_j = \mathcal{D}_n g_{ij} = 0$ (see (5.96)-(5.98)), together with the angle preservation (1.13). Simplifying coefficients $N^m{}_i$ culminates in the representation (5.102). The initial transformation (5.27) reduces to (5.106), for it proves possible to find explicitly the involved functions ϱ and μ.

In Conclusions, Section 6, we emphasize several important ideas.

In Appendices A-E we present the explicit evaluations which are required to verify the validity of the formulated propositions.

2. Main observations

Below, *any dimension* $N \ge 3$ is allowable.

Let M be an N-dimensional C^∞ differentiable manifold, T_xM denote the tangent space to M at a point $x \in M$, and $y \in T_xM\backslash 0$ mean tangent vectors. Suppose we are given on the tangent bundle TM a Riemannian metric S. Denote by $\mathcal{R}^N = (M, S)$ the obtained N-dimensional Riemannian space. Let additionally a Finsler metric function F be introduced on this TM, yielding a Finsler space $\mathcal{F}^N = (M, F)$. We shall study the Finsler space $\mathcal{F}^N$ specified according to the following definition.

INPUT DEFINITION. The space $\mathcal{F}^N$ is *conformally automorphic* to the Riemannian space $\mathcal{R}^N$:

$$\mathcal{F}^N = \mathcal{C} \cdot \mathcal{R}^N \tag{2.1}$$

such that in each tangent space T_xM the $\mathcal{C}$-automorphism transforms conformally the metric produced by the Finsler metric to the Euclidean metric entailed by the Riemannian metric. It is assumed that the applied $\mathcal{C}$-transformations do not influence any point $x \in M$ of the base manifold M and that they are invertible. It is also natural to require that the $\mathcal{C}$-transformations send unit vectors to unit vectors:

$$\mathcal{IF}_{\{x\}} = \mathcal{C} \cdot \mathcal{S}_{\{x\}}. \tag{2.2}$$

Additionally, we subject the $\mathcal{C}$-transformations to the condition of positive homogeneity with respect to tangent vectors y, denoting the degree of homogeneity by H. We call the H the *degree of conformal automorphism.*

The existence of such spaces is explained by the following proposition.

Proposition 2.1. *A Finsler space is of the claimed type $\mathcal{F}^N$ if and only if the indicatrix of the Finsler space is a space of constant curvature. Denoting the indicatrix curvature value by $\mathcal{C}_{Ind.}$, the equality*

$$\mathcal{C}_{Ind.} \equiv H^2, \qquad H > 0, \tag{2.3}$$

is obtained. The relevant conformal multiplier is given by p^2 with

$$p = \frac{1}{H} F^{1-H}. \tag{2.4}$$

The proposition is of the local meaning in both the base manifold and the tangent space. The validity of the proposition can be verified by simple straightforward evaluations (which are presented in Appendix A).

The value $\mathcal{C}_{\text{Ind.}}$ may vary from point to point of the manifold M, so that in general $H = H(x)$.

We take $\mathcal{C}_{\text{Ind.}} > 0$. Extension of the proposition to negative value of $\mathcal{C}_{\text{Ind.}}$ would be a straightforward task.

On every punctured tangent space $T_xM \setminus 0$, the Finsler metric function F is assumed to be positive, and also positively homogeneous of degree 1:

$$F(x, ky) = kF(x, y), \qquad k > 0, \forall y. \tag{2.5}$$

Therefore, the conformal factor $p^2 = \left(F^{1-H}/H\right)^2$ possesses this kind of homogeneity with degree $2(1-H)$. For a given function F we can construct the covariant tangent vector $\hat{y} = \{y_i\}$ and the Finslerian metric tensor $\{g_{ij}\}$ in the ordinary way:

$$y_i := \frac{1}{2}\frac{\partial F^2}{\partial y^i}, \qquad g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}.$$

The contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, where δ stands for the Kronecker symbol.

Let the $\mathcal{C}$-transformation (2.1) be assigned locally by means of the differentiable functions

$$\bar{y}^m = \bar{y}^m(x, y), \tag{2.6}$$

subject to the required homogeneity

$$\bar{y}^m(x, ky) = k^H \bar{y}^m(x, y), \qquad k > 0, \forall y. \tag{2.7}$$

This entails the identity

$$\bar{y}^m_k y^k = H\bar{y}^m, \tag{2.8}$$

where $\bar{y}^m_k = \partial\bar{y}^m/\partial y^k$. Fulfilling the conformal automorphism (2.1) means locally

$$g_{mn}(x, y) = c_{ij}(x, \bar{y})\bar{y}^i_m \bar{y}^j_n, \qquad c_{ij} = p^2 a_{ij}(x). \tag{2.9}$$

Contracting the g_{mn} by $y^m y^n$ and noting the involved homogeneity together with the value (2.4) of p, we get the equality

$$S(x, \bar{y}) = (F(x, y))^H, \tag{2.10}$$

where $S = \sqrt{a_{mn}(x)\bar{y}^m\bar{y}^n}$.

Denote by

$$y^i = y^i(x, t), \qquad t^n \equiv \bar{y}^n, \tag{2.11}$$

the inverse transformation, so that

$$y^i(x, kt) = k^{1/H} y^i(x, t), \qquad k > 0, \forall t, \tag{2.12}$$

and

$$y^i_n t^n = \frac{1}{H} y^i, \tag{2.13}$$

where $y^i_n = \partial y^i/\partial t^n$. The inverse to (2.9) reads:

$$g_{kh} y^k_m y^h_n = c_{mn}. \tag{2.14}$$

The following useful relations can readily be arrived at:

$$y_m y^m_n = \frac{F^2}{HS^2} t_n \equiv \frac{1}{H} F^{2(1-H)} t_n, \qquad t_n = a_{nh} t^h, \tag{2.15}$$

and

$$y_m y^m_{nl} t^l_j + g_{mj} y^m_n = 2\left(\frac{1}{H} - 1\right) F^{-2H} y_j t_n + \frac{1}{H} F^{2(1-H)} a_{nh} t^h_j,$$

where $t^l_j = \bar{y}^l_j, y^m_{nl} = \partial y^m_n/\partial y^l$. Alternatively,

$$t_h t^h_n = \frac{HS^2}{F^2} y_n \equiv HF^{2(H-1)} y_n \tag{2.16}$$

and

$$t_h t^h_{nu} y^u_i + a_{hi} t^h_n = 2(H-1)F^{-2} t_i y_n + HF^{2(H-1)} g_{nu} y^u_i, \tag{2.17}$$

where $t^h_{nu} = \partial t^h_n/\partial y^u$. We may also write

$$t_h t^h_{ni} + H^2 F^{2(H-1)} g_{ni} = 2(H-1)F^{-2} HF^{2(H-1)} y_i y_n + HF^{2(H-1)} g_{ni},$$

or

$$t_h t^h_{ni} = H(1-H)F^{2(H-1)}(g_{ni} - 2l_n l_i). \tag{2.18}$$

From (2.14) it follows that $g_{nm} y^m_i = p^2 t^j_n a_{ij}$.

Differentiating (2.9) with respect to y^k yields the following representation for the Cartan tensor $C_{mnk} = (1/2)\partial g_{mn}/\partial y^k$:

$$2C_{mnk} = (1-H)\frac{2}{F} l_k g_{mn} + p^2(t^i_{mk} t^j_n + t^i_m t^j_{nk}) a_{ij}. \tag{2.19}$$

Contracting this tensor by y^n results in the equality

$$p^2 t^i_{mk} t^j a_{ij} = \left(\frac{1}{H} - 1\right)(h_{km} - l_k l_m), \tag{2.20}$$

where the vanishing $C_{mnk} y^n = 0$ and the homogeneity identity (2.8) have been taken into account.

Symmetry of the tensor C_{mnk} demands

$$(1-H)\frac{2}{F}(l_k g_{mn} - l_m g_{kn}) + p^2(t^i_m t^j_{nk} - t^i_k t^j_{nm}) a_{ij} = 0, \tag{2.21}$$

so that we may alternatively write

$$C_{mnk} = (1-H)\frac{1}{F}(l_k g_{mn} + l_n g_{mk} - l_m g_{nk}) + p^2 t^i_m t^j_{nk} a_{ij}. \tag{2.22}$$

Contracting the last tensor by g^{nk} yields

$$2C_m = (1-H)\frac{2}{F} l_m + g^{nk} p^2 (t^i_{nk} t^j_m + t^i_n t^j_{mk}) a_{ij} \equiv 2C_{mnk} g^{nk},$$

from which it ensues that

$$2C_m = (1-H)\frac{2}{F} l_m + 2g^{nk} p^2 t^i_{nk} t^j_m a_{ij} + g^{nk} p^2 (t^i_n t^j_{mk} - t^i_m t^j_{nk}) a_{ij},$$

or

$$2C_m = (1-H)\frac{2}{F} l_m + 2g^{nk} p^2 t^i_{nk} t^j_m a_{ij} - (1-H) g^{nk} \frac{2}{F}(l_m g_{nk} - l_n g_{mk}) a_{ij}.$$

It is also convenient to use the representation

$$FC_m = -(N-2)(1-H)l_m + Fg^{nk}p^2 t^i_{nk} t^j_m a_{ij}. \tag{2.23}$$

The space $\mathcal{F}^N$ is obtainable from the Riemannian space $\mathcal{R}^N$ by means of the deformation (1.1) which, owing to (2.2) and (2.9), can be presented by the *conformal deformation tensor*

$$\mathcal{C}^m_k := p\bar{y}^m_k, \tag{2.24}$$

so that

$$g_{kh} = \mathcal{C}^m_k \mathcal{C}^n_h a_{mn}. \tag{2.25}$$

The zero-degree homogeneity

$$\mathcal{C}^m_n(x, ky) = \mathcal{C}^m_n(x, y), \qquad k > 0, \forall y, \tag{2.26}$$

holds, together with

$$\mathcal{C}^m_n y^n = F^{1-H} \bar{y}^m. \tag{2.27}$$

The indicatrix correspondence (2.2) is a direct implication of the equality $S = F^H$ (see (2.10)). We may apply the considered transformation (2.6) to the unit vectors:

$$l = \mathcal{C} \cdot L: \quad l^i = y^i(x, L); \qquad L = \mathcal{C}^{-1} \cdot l: \quad L^i = t^i(x, l), \tag{2.28}$$

where $l^i = y^i/F(x,y)$ and $L^i = t^i/S(x,t)$ are components of the respective Finslerian and Riemannian unit vectors, which possess the properties $F(x,l) = 1$ and $S(x,L) = 1$. We have $L^m = t^m(x,l)$. On the other hand, from (2.4) and (2.9) it just follows that

$$g_{mn}(x,l) = \frac{1}{H^2} a_{ij}(x) t^i_m(x,l) t^j_n(x,l), \tag{2.29}$$

so that under the transformation (2.28) we have

$$g_{mn}(x,l) dl^m dl^n = \frac{1}{H^2} a_{ij}(x) dL^i dL^j. \tag{2.30}$$

No support vector enters the right-hand part in the previous equality (2.30). Therefore, any two nonzero tangent vectors $y_1, y_2 \in T_xM$ in a fixed tangent space T_xM form the $\mathcal{F}^N$*-space angle*

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \arccos \lambda, \tag{2.31}$$

where the scalar

$$\lambda = \frac{a_{mn}(x) t^m_1 t^n_2}{S_1 S_2}, \qquad \text{with} \quad t^m_1 = t^m(x, y_1) \quad \text{and} \quad t^m_2 = t^m(x, y_2), \tag{2.32}$$

is of the entire Riemannian meaning in the space $\mathcal{R}^N$; the notation

$$S_1 = \sqrt{a_{mn}(x) t^m_1 t^n_1}, \qquad S_2 = \sqrt{a_{mn}(x) t^m_2 t^n_2}$$

has been used.

From (2.32) it follows that

$$\frac{\partial\lambda}{\partial x^i}=\frac{a_{mn,i}t_1^mt_2^n}{S_1S_2}+\frac{1}{S_1S_2}a_{mn}\left(\frac{\partial t_1^m}{\partial x^i}t_2^n+t_1^m\frac{\partial t_2^n}{\partial x^i}\right)$$

$$-\frac{1}{2}\lambda\left[\frac{1}{S_1S_1}\left(a_{mn,i}t_1^mt_1^n+2a_{mn}\frac{\partial t_1^m}{\partial x^i}t_1^n\right)+\frac{1}{S_2S_2}\left(a_{mn,i}t_2^mt_2^n+2a_{mn}\frac{\partial t_2^m}{\partial x^i}t_2^n\right)\right], \tag{2.33}$$

where $a_{mn,i}=\partial a_{mn}/\partial x^i$, and

$$\frac{\partial\lambda}{\partial y_1^k}=\left[\frac{a_{mn}t_2^n}{S_1S_2}-\frac{a_{mn}t_1^n}{S_1S_1}\lambda\right]t_{1k}^m,\qquad \frac{\partial\lambda}{\partial y_2^k}=\left[\frac{a_{mn}t_1^n}{S_2S_1}-\frac{a_{mn}t_2^n}{S_2S_2}\lambda\right]t_{2k}^m. \tag{2.34}$$

Let the coefficients $N^k{}_i$ be subjected to the equation

$$d_i\lambda=0, \tag{2.35}$$

where d_i is the operator (1.10).

It is possible to establish the validity of the following proposition.

Proposition 2.2. *When $d_iF=0$ and $H=const$, the equation (2.35) can be solved for the coefficients $N^m{}_n$, yielding*

$$N^m{}_n=-y_i^m\left(\frac{\partial t^i}{\partial x^n}+a^i{}_{kn}t^k\right). \tag{2.36}$$

See Appendix B.

In (2.36), the $a^i{}_{kn}=a^i{}_{kn}(x)$ are the Christoffel symbols

$$a^i{}_{kn}=\frac{1}{2}a^{ih}\Big(\frac{\partial a_{hk}}{\partial x^n}+\frac{\partial a_{hn}}{\partial x^k}-\frac{\partial a_{kn}}{\partial x^h}\Big) \tag{2.37}$$

of the Riemannian space $\mathcal{R}^N$.

Note. When $H=const$, from (2.31) it just follows that the angle $\alpha_{\{x\}}(y_1,y_2)$ fulfills the vanishing which is completely similar to (2.35), namely the vanishing (1.13) claimed in Section 1.

With the covariant derivative

$$\mathcal{D}_nt^i\ :=d_nt^i+a^i{}_{kn}t^k \tag{2.38}$$

the representation (2.36) can be interpreted as the manifestation of the *transitivity*

$$\mathcal{D}_nt^i=0 \tag{2.39}$$

of the connection under the conformal automorphism (2.1).

By differentiating (2.39) with respect to y^m we may conclude that the covariant derivative

$$\mathcal{D}_nt^i_m\ :=d_nt^i_m-D^h{}_{nm}t^i_h+a^i{}_{nl}t^l_m,\qquad D^h{}_{nm}=-N^h{}_{nm}, \tag{2.40}$$

vanishes identically:

$$\mathcal{D}_n t^i_m = 0. \tag{2.41}$$

Since $y^n_k t^k_j = \delta^n_j$, the previous identity can be transformed to

$$d^{\rm Riem}_i y^n_k + D^n{}_{is} y^s_k - a^h{}_{ik} y^n_h = 0, \tag{2.42}$$

which can be interpreted as the covatiant derivative vanishing:

$$\mathcal{D}_i y^n_k = 0. \tag{2.43}$$

This formula entails

$$\mathcal{D}_i y^n = 0 \tag{2.44}$$

(because of (2.39)), where

$$\mathcal{D}_i y^n \ := d^{\rm Riem}_i y^n + D^n{}_{is} y^s. \tag{2.45}$$

Here, y^n mean the functions $y^n(x,t)$ introduced by (2.11). We have used the Riemannian operator

$$d^{\rm Riem}_i = \frac{\partial}{\partial x^i} + L^k{}_i \frac{\partial}{\partial t^k}, \qquad L^k{}_i = -a^k{}_{ih} t^h \tag{2.46}$$

(cf. (1.3)).

Since $D^n{}_{is} y^s = -N^n{}_i$, from (2.44)-(2.45) we may conclude that the representation

$$N^n{}_i = d^{\rm Riem}_i y^n \equiv \frac{\partial y^n(x,t)}{\partial x^i} + y^n_h L^h{}_i \tag{2.47}$$

is valid which is alternative to (2.36).

Let us verify (2.42). We have

$$0 = y^n_k \left(\frac{\partial t^k_j}{\partial x^i} + N^h{}_i t^k_{hj} - D^h{}_{ij} t^k_h + a^k{}_{il} t^l_j \right)$$

$$= -t^k_j \left(\frac{\partial y^n_k}{\partial x^i} + y^n_{kh} \frac{\partial t^h}{\partial x^i} \right) + y^n_k \left(N^h{}_i t^k_{hj} - D^h{}_{ij} t^k_h + a^k{}_{il} t^l_j \right).$$

Contracting this by y^j_m yields

$$0 = \frac{\partial y^n_m}{\partial x^i} + y^n_{mh} \frac{\partial t^h}{\partial x^i} - y^j_m y^n_k N^h{}_i t^k_{hj} + D^n{}_{ij} y^j_m - y^n_k a^k{}_{im}.$$

Take $N^h{}_i$ from (2.36):

$$0 = \frac{\partial y^n_m}{\partial x^i} + y^n_{mh} \frac{\partial t^h}{\partial x^i} + y^j_m y^n_k y^h_l t^k_{hj} \left(\frac{\partial t^l}{\partial x^i} + a^l{}_{ji} t^j \right) + D^n{}_{ij} y^j_m - y^n_k a^k{}_{im}.$$

This vanishing is tantamount to the considered (2.42), for $y^j_m y^n_k y^h_l t^k_{hj} = -y^n_{ml}$.

Owing to the equalities (2.4), (2.24), and (2.43), we are entitled to formulate the following proposition.

Proposition 2.3. *When $d_i F = 0$ and $H = const$, the deformation tensor* (2.24) *is parallel*

$$\mathcal{D}_n \mathcal{C}^m_k = 0, \tag{2.48}$$

where

$$\mathcal{D}_n \mathcal{C}^m_k = d_n \mathcal{C}^m_k - D^h{}_{nk} \mathcal{C}^m_h + a^m{}_{nl} \mathcal{C}^l_k. \tag{2.49}$$

With these observations, it is possible to develop a direct method to induce the connection in the Finsler space $\mathcal{F}^N$ from the metrical linear Riemannian connection (1.2) meaningful in the background Riemannian space $\mathcal{R}^N$.

The coefficients $N^k{}_i(x, y)$ can also be obtained by means of the transitivity map

$$\{N^k{}_i\} = \mathcal{C} \cdot \{L^k{}_i\}. \tag{2.50}$$

Indeed, with an arbitrary differentiable scalar $w(x, y)$, we can apply the transformation $\{y^i = y^i(x, t),\ t^n \equiv \bar{y}^n\}$ indicated in (2.11) and consider the $\mathcal{C}$-transform

$$W(x, t) = w(x, y), \qquad \text{which entails} \quad \frac{\partial W}{\partial t^n} = y^k_n \frac{\partial w}{\partial y^k}, \tag{2.51}$$

thereafter postulating that the $\mathcal{C}$-transformation is *covariantly transitive*, namely

$$\left(\frac{\partial}{\partial x^i} + N^k{}_i(x, y)\frac{\partial}{\partial y^k}\right) w(x, y) = \left(\frac{\partial}{\partial x^i} + L^k{}_i(x, t)\frac{\partial}{\partial t^k}\right) W(x, t). \tag{2.52}$$

Since the field w is arbitrary, the last equality is fulfilled if and only if

$$N^k{}_i = d^{\text{Riem}}_i y^k \equiv \frac{\partial y^k(x, t)}{\partial x^i} + y^k_h L^h{}_i. \tag{2.53}$$

This is the representation which is required to realize the map (2.50). We have again arrived at the coefficients (2.47).

With the knowledge of the coefficients $N^k{}_i(x, y)$, we can use the formulas (2.40) and (2.41) to express the Finslerian connection coefficients $D^h{}_{nm}$ through the Riemannian Christoffel symbols $a^i{}_{nl}$. Thus we have induced the connection in the Finsler space $\mathcal{F}^N$ from the metrical linear Riemannian connection (1.2) meaningful in the background Riemannian space $\mathcal{R}^N$.

It can readily be noted that the transitivity property (2.52) can straightforwardly be extended to scalars dependent on two vectors. Namely, if

$$W(x, t_1, t_2) = w(x, y_1, y_2), \tag{2.54}$$

then

$$\left(\frac{\partial}{\partial x^i} + N^k_{1i}\frac{\partial}{\partial y^k_1} + N^k_{2i}\frac{\partial}{\partial y^k_2}\right) w(x, y_1, y_2) = \left(\frac{\partial}{\partial x^i} + L^k_{1i}\frac{\partial}{\partial t^k_1} + L^k_{2i}\frac{\partial}{\partial t^k_2}\right) W(x, t_1, t_2), \tag{2.55}$$

where $N^k_{1i} = N^k{}_i(x, y_1)$, $N^k_{2i} = N^k{}_i(x, y_2)$, $L^k_{1i} = L^k{}_i(x, t_1)$, $L^k_{2i} = L^k{}_i(x, t_2)$.

3. Properties of connection coefficients

The derivative coefficients (1.15) can straightforwardly be evaluated from (2.36). We obtain explicitly

$$N^k{}_{mn} = -y^k_{sl} t^l_n T^s_m - y^k_s T^s_{n,m} \quad \text{with} \qquad T^s_m = \frac{\partial t^s}{\partial x^m} + a^s{}_{mh} t^h, \quad T^s_{n,m} = \frac{\partial t^s_n}{\partial x^m} + a^s{}_{mh} t^h_n, \tag{3.1}$$

which entails the contractions

$$y_kN^k{}_{mn}=-\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_nt_s+\frac{1}{H}F^{2(1-H)}a_{sh}t^h_n-g_{ln}y^l_s\right)T^s_m-\frac{F^2}{HS^2}t_sT^s_{n,m} \quad (3.2)$$

and

$$y_kN^k{}_{mn}+g_{ln}N^l{}_m=-\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_nt_s+\frac{1}{H}F^{2(1-H)}a_{sh}t^h_n\right)T^s_m-\frac{F^2}{HS^2}t_sT^s_{n,m},$$

together with

$$y_kN^k{}_{mni}+g_{ki}N^k{}_{mn}+g_{ln}N^l{}_{mi}+2C_{lni}N^l{}_m$$

$$=-\left(\frac{1}{H}-1\right)2F^{-2H}\left[(g_{ni}-2Hl_nl_i)t_s+(y_na_{sl}t^l_i+y_ia_{sl}t^l_n)\right]T^s_m-\frac{1}{H}F^{2(1-H)}a_{sh}t^h_{ni}T^s_m$$

$$-\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_nt_s+\frac{1}{H}F^{2(1-H)}a_{sh}t^h_n\right)T^s_{i,m}$$

$$-\left(2\left(\frac{1}{H}-1\right)F^{-2H}y_it_s+\frac{1}{H}F^{2(1-H)}a_{sh}t^h_i\right)T^s_{n,m}-\frac{1}{H}F^{2(1-H)}t_s\left(\frac{\partial t^s_{ni}}{\partial x^m}+a^s{}_{mh}t^h_{ni}\right). \quad (3.3)$$

The attentive calculation of the entered terms (carried out in Appendix C) leads to the following remarkable result.

Proposition 3.1. *If the coefficients $N^k{}_m$ are taking according to Proposition* 2.2, then the vanishing $y_kN^k{}_{mnj}=0$ holds identically.

In performing involved calculation it is necessary to note that in view of (2.15) and (2.36), we can write

$$d_mF=\frac{\partial F}{\partial x^m}+N^k{}_n\frac{\partial F}{\partial y^k}=\frac{\partial F}{\partial x^m}+N^k{}_nl_k=\frac{\partial F}{\partial x^m}-\frac{1}{FH}F^{2(1-H)}t_sT^s_m$$

so that, because of $d_mF=0$, the equality

$$\frac{\partial F}{\partial x^m}=\frac{1}{FH}F^{2(1-H)}t_sT^s_m \quad (3.4)$$

is valid.

It is also possible to evaluate the covariant derivative $\mathcal{D}_mC^k{}_{nj}$ (see (1.20)), using the equality $d_mg_{hn}=-N^t{}_{mh}g_{tn}-N^t{}_{mn}g_{th}$ entailed by the metricity (1.16). This way leads to the following result.

Proposition 3.2. *The representation $N^k{}_{mnj}=-\mathcal{D}_mC^k{}_{nj}$ is valid, whenever $d_mF=$ 0 and $H=const.$*

Proof of this proposition can be arrived at during a long chain of straightforward substitutions (see Appendix D).

4. Entailed curvature tensor

Throughout the present section we assume that $H = const$. Given a tensor $w^n{}_k = w^n{}_k(x,y)$ of the tensorial type (1,1), commuting the covariant derivative

$$\mathcal{D}_i w^n{}_k \ := d_i w^n{}_k + D^n{}_{ih} w^h{}_k - D^h{}_{ik} w^n{}_h \tag{4.1}$$

yields the equality

$$\left[\mathcal{D}_i\mathcal{D}_j - \mathcal{D}_j\mathcal{D}_i\right] w^n{}_k = M^h{}_{ij}\frac{\partial w^n{}_k}{\partial y^h} - E_k{}^h{}_{ij} w^n{}_h + E_h{}^n{}_{ij} w^h{}_k \tag{4.2}$$

with the tensors

$$M^n{}_{ij} \ := d_i N^n{}_j - d_j N^n{}_i \tag{4.3}$$

and

$$E_k{}^n{}_{ij} \ := d_i D^n{}_{jk} - d_j D^n{}_{ik} + D^m{}_{jk} D^n{}_{im} - D^m{}_{ik} D^n{}_{jm}. \tag{4.4}$$

When the choice $D^k{}_{in} = -N^k{}_{in}$ is made (cf. (1.23)), the tensor (4.3) can be written in the form

$$M^n{}_{ij} = \frac{\partial N^n{}_j}{\partial x^i} - \frac{\partial N^n{}_i}{\partial x^j} - N^h{}_i D^n{}_{jh} + N^h{}_j D^n{}_{ih}. \tag{4.5}$$

By applying the commutation rule (4.2) to the particular choices $\{F, y^n, y_k, g_{nk}\}$ and noting the vanishing $\{\mathcal{D}_i F = \mathcal{D}_i y^n = \mathcal{D}_i y_k = \mathcal{D}_i g_{nk} = 0\}$, we obtain the identities

$$y_n M^n{}_{ij} = 0, \qquad y^k E_k{}^n{}_{ij} = -M^n{}_{ij}, \qquad y_n E_k{}^n{}_{ij} = M_{kij}, \tag{4.6}$$

and

$$E_{mnij} + E_{nmij} = 2C_{mnh} M^h{}_{ij} \quad \text{with} \quad C_{mnh} = \frac{1}{2}\frac{\partial g_{mn}}{\partial y^h}. \tag{4.7}$$

Differentiating (4.5) with respect to y^k and using the equality $N^j{}_i = -D^j{}_{ik} y^k$ yield

$$E_k{}^n{}_{ij} = -\frac{\partial M^n{}_{ij}}{\partial y^k}. \tag{4.8}$$

The cyclic identity

$$\mathcal{D}_k M^n{}_{ij} + \mathcal{D}_j M^n{}_{ki} + \mathcal{D}_i M^n{}_{jk} = 0 \tag{4.9}$$

is valid, where

$$\mathcal{D}_k M^n{}_{ij} = d_k M^n{}_{ij} + D^n{}_{kt} M^t{}_{ij} - a^s{}_{ki} M^n{}_{sj} - a^s{}_{kj} M^n{}_{is}. \tag{4.10}$$

It proves pertinent to replace in the commutator (4.2) the partial derivative $\partial w^n{}_k/\partial y^h$ by the definition

$$\mathcal{S}_h w^n{}_k \ := \frac{\partial w^n{}_k}{\partial y^h} + C^n{}_{hk} w^h{}_k - C^m{}_{hk} w^n{}_m \tag{4.11}$$

which has the meaning of the covariant derivative in the tangent space supported by the point $x \in M$. In particular,

$$\mathcal{S}_h g_{nk} \ := \frac{\partial g_{nk}}{\partial y^h} - C^m{}_{hn} g_{mk} - C^m{}_{hk} g_{nm} = 0.$$

With the *curvature tensor*

$$\rho_k{}^n{}_{ij} \ := E_k{}^n{}_{ij} - M^h{}_{ij} C^n{}_{hk}, \tag{4.12}$$

the commutator (4.2) takes on the form

$$(\mathcal{D}_i \mathcal{D}_j - \mathcal{D}_j \mathcal{D}_i)\, w^n{}_k = M^h{}_{ij} \mathcal{S}_h w^n{}_k - \rho_k{}^h{}_{ij} w^n{}_h + \rho_h{}^n{}_{ij} w^h{}_k. \tag{4.13}$$

We denote $\rho_{knij} = g_{mn}\rho_k{}^m{}_{ij}$. The skew-symmetry

$$\rho_{mnij} = -\rho_{nmij} \tag{4.14}$$

holds (cf. (4.7)). The tensor obeys also the cyclic identity

$$\mathcal{D}_l \rho_k{}^n{}_{ij} + \mathcal{D}_j \rho_k{}^n{}_{li} + \mathcal{D}_i \rho_k{}^n{}_{jl} = 0, \tag{4.15}$$

where

$$\mathcal{D}_l \rho_k{}^n{}_{ij} = d_l \rho_k{}^n{}_{ij} + D^n{}_{lt}\rho_k{}^t{}_{ij} - D^t{}_{lk}\rho_t{}^n{}_{ij} - a^s{}_{li}\rho_k{}^n{}_{sj} - a^s{}_{lj}\rho_k{}^n{}_{is}.$$

Let us realize the action of the $\mathcal{C}$-transformation (2.1)-(2.2) on tensors by the help of the *transitivity rule,* that is,

$$\{w^n{}_m(x,y)\} = \mathcal{C}\cdot\{W^n{}_m(x,t)\}: \quad w^n{}_m = y^n_h t^j_m W^h{}_j, \tag{4.16}$$

where $W^n{}_m$ is a tensor of type (1,1). The metrical linear connection $\mathcal{RL}$ introduced by (1.2) may be used to define the covariant derivative ∇ in $\mathcal{R}^N$ according to the conventional rule:

$$\nabla_i W^n{}_m = \frac{\partial W^n{}_m}{\partial x^i} + L^k{}_i \frac{\partial W^n{}_m}{\partial t^k} + L^n{}_{hi} W^h{}_m - L^h{}_{mi} W^n{}_h, \tag{4.17}$$

which can be written shortly with the help of the operator d_i^{Riem} defined by (1.3), namely

$$\nabla_i W^n{}_m = d_i^{\text{Riem}} W^n{}_m + L^n{}_{hi} W^h{}_m - L^h{}_{mi} W^n{}_h. \tag{4.18}$$

We have

$$\nabla_i S = 0, \qquad \nabla_i y^j = 0, \qquad \nabla_i a_{mn} = 0. \tag{4.19}$$

Due to the nullifications $\mathcal{D}_i y^n_h = 0$ and $\mathcal{D}_i t^j = 0$ (see (2.39) and (2.43)), we have the transitivity property

$$\mathcal{D}_i w^n{}_m = y^n_h t^j_m \nabla_i W^h{}_j \tag{4.20}$$

for the covariant derivatives.

In the commutator

$$[\nabla_i \nabla_j - \nabla_j \nabla_i]\, W^n{}_k = -y^m a_m{}^h{}_{ij} \frac{\partial W^n{}_k}{\partial y^h} - a_k{}^h{}_{ij} W^n{}_h + a_h{}^n{}_{ij} W^h{}_k \tag{4.21}$$

the associated Riemannian curvature tensor is constructed in the ordinary way

$$a_n{}^i{}_{km} = \frac{\partial a^i{}_{nm}}{\partial x^k} - \frac{\partial a^i{}_{nk}}{\partial x^m} + a^u{}_{nm} a^i{}_{uk} - a^u{}_{nk} a^i{}_{um}. \tag{4.22}$$

With the ordinary Riemannian covariant derivative

$$\nabla_k a_h{}^t{}_{ij} = \frac{\partial a_h{}^t{}_{ij}}{\partial x^k} + a^t{}_{ku} a_h{}^u{}_{ij} - a^u{}_{kh} a_u{}^t{}_{ij} - a^u{}_{ki} a_h{}^t{}_{uj} - a^u{}_{kj} a_h{}^t{}_{iu}, \tag{4.23}$$

the cyclic identity

$$\nabla_k a_m{}^n{}_{ij} + \nabla_j a_m{}^n{}_{ki} + \nabla_i a_m{}^n{}_{jk} = 0 \tag{4.24}$$

holds.

Under these conditions, by comparing the Finslerian commutator (4.13) with the Riemannian precursor (4.21), we obtain

$$M^n{}_{ij} = -y^n_t t^h a_h{}^t{}_{ij} \tag{4.25}$$

and

$$E_k{}^n{}_{ij} = y^n_h t^h_{km} M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k, \tag{4.26}$$

together with

$$\rho_k{}^n{}_{ij} = \left(y^n_h t^h_{km} - C^n{}_{mk}\right) M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k.$$

Inserting here the tensor $C^n{}_{mk}$ taken from (2.22) and noting the vanishing $l_m M^m{}_{ij} = 0$ (see (4.6)), we get

$$\rho_k{}^n{}_{ij} = \left(y^n_h t^h_{km} - (1-H)\frac{1}{F}(l_k \delta^n_m + l^n g_{mk}) - p^2 t^l_m t^h_{rk} a_{lh} g^{nr}\right) M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k.$$

Let us lower here the index n and use the equality $g_{nm} y^m_i = p^2 t^j_n a_{ij}$ (ensued from (2.14)). This yields

$$\rho_{knij} = \left(p^2 t^l_n t^h_{km} a_{lh} - (1-H)\frac{1}{F}(l_k g_{mn} + l_n g_{mk}) - p^2 t^l_m t^h_{nk} a_{lh}\right) M^m{}_{ij} + p^2 a_{ml} t^l_n a_h{}^m{}_{ij} t^h_k.$$

Next, we use here the skew-symmetry relation (2.21), obtaining

$$\rho_{knij} = \left((1-H)\frac{2}{F}(l_n g_{mk} - l_m g_{kn}) - (1-H)\frac{1}{F}(l_k g_{mn} + l_n g_{mk})\right) M^m{}_{ij} + p^2 a_{ml} t^l_n a_h{}^m{}_{ij} t^h_k,$$

or

$$\rho_{knij} = -(1-H)\frac{1}{F}(l_k M_{nij} - l_n M_{kij}) + p^2 a_{hlij} t^h_k t^l_n, \tag{4.27}$$

where $a_{hlij} = a_{lr} a_h{}^r{}_{ij}$. Finally, we return the index n to the upper position, arriving at

$$\rho_k{}^n{}_{ij} = -(1-H)\frac{1}{F}(l_k \delta^n_m - l^n g_{mk}) M^m{}_{ij} + y^n_m a_h{}^m{}_{ij} t^h_k. \tag{4.28}$$

The totally contravariant representation

$$\rho^{knij} = g^{pk} a^{mi} a^{nj} \rho_p{}^n{}_{mn}$$

reads

$$\rho^{knij} = -(1-H)\frac{1}{F}(l^k M^{nij} - l^n M^{kij}) + \frac{1}{p^2} y^k_h y^n_r a^{hrij}, \tag{4.29}$$

where $a^{hrij} = a^{hl} a^{mi} a^{nj} a_l{}^r{}_{mn}$ and $M^{mij} = a^{hi} a^{nj} M^m{}_{hn}$.

Similarly, we can conclude from (4.25) that the tensor

$$M_{nij} = g_{nm} M^m{}_{ij}$$

reads

$$M_{nij} = -p^2 t^h t^m_n a_{hmij}. \tag{4.30}$$

Squaring yields

$$M^{nij} M_{nij} = p^2 t^l a_l{}^{nij} t^h a_{hnij}. \tag{4.31}$$

From the representations (4.27) and (4.30) it follows directly that the cyclic identities (4.9) and (4.15) are consequences of the Riemannian cyclic identity (4.24), for $\mathcal{D}_l F = \mathcal{D}_l l_k = \mathcal{D}_l t^h_k = \mathcal{D}_l p = \mathcal{D}_l t^m = 0$.

Now we square the ρ-tensor:

$$\rho^{knij}\rho_{knij} = (1-H)^2 \frac{2}{F^2} M^{nij} M_{nij} - 2(1-H)\frac{1}{F}(l^k M^{nij} - l^n M^{kij}) p^2 a_{hlij} t^h_k t^l_n + a^{knij} a_{knij}$$

$$= (1-H)^2 \frac{2}{F^2} M^{nij} M_{nij} - 2(1-H)H\frac{1}{F^2} p^2 (a_{hlij} t^h t^l_n M^{nij} - a_{hlij} t^h_k t^l M^{kij}) + a^{knij} a_{knij},$$

or

$$\rho^{knij}\rho_{knij} = (1-H)^2 \frac{2p^2}{F^2} t^l a_l{}^{nij} t^h a_{hnij} + 2(1-H)\frac{Hp^2}{F^2}(a_{hlij} t^h t^r a_r{}^{lij} - a_{hlij} t^l t^r a_r{}^{hij}) + a^{knij} a_{knij},$$

which is

$$\rho^{knij}\rho_{knij} = a^{knij} a_{knij} + \frac{2}{S^2}\left(\frac{1}{H^2} - 1\right) t^l a_l{}^{nij} t^h a_{hnij}. \tag{4.32}$$

Because of the transitivity (4.20), from (4.25) it follows that

$$\mathcal{D}_l M^n{}_{ij} = -y^n_t t^h \nabla_l a_h{}^t{}_{ij}. \tag{4.33}$$

From (4.28) we can conclude that

$$\mathcal{D}_l \rho_k{}^n{}_{ij} = (1-H)\frac{1}{F}(l_k \delta^n_m - l^n g_{mk}) y^m_t t^h \nabla_l a_h{}^t{}_{ij} + y^n_m t^h_k \nabla_l a_h{}^m{}_{ij}. \tag{4.34}$$

It is also convenient to use the representation

$$\rho_{knij} = T_{kn}{}^{hm} a_{hmij}, \tag{4.35}$$

where

$$T_{kn}{}^{hm} = p^2 \left[\frac{1}{2}(t^h_k t^m_n - t^m_k t^h_n) + (1-H)\frac{1}{F^2}(y_k t^h t^m_n - y_n t^h t^m_k)\right]. \tag{4.36}$$

Since

$$\mathcal{D}_l T_{kn}{}^{hm} = 0, \tag{4.37}$$

we have the relation

$$\mathcal{D}_l \rho_{knij} = T_{kn}{}^{hm} \nabla_l a_{hmij}. \tag{4.38}$$

5. $\mathcal{FS}$-space example of the space $\mathcal{F}^N$

Let us also assume that the manifold M admits a non–vanishing 1-form $b = b(x, y)$ and denote by

$$c = ||b||_{\text{Riemannian}} \tag{5.1}$$

the respective Riemannian norm value, assuming

$$0 < c < 1. \tag{5.2}$$

With respect to natural local coordinates x^i we have the local representations

$$a^{ij}(x)b_i(x)b_j(x) = c^2(x), \qquad b = b_i(x)y^i. \tag{5.3}$$

The reciprocity $a^{in}a_{nj} = \delta^i{}_j$ is assumed, where $\delta^i{}_j$ stands for the Kronecker symbol. The covariant index of the vector b_i will be raised by means of the Riemannian rule $b^i = a^{ij}b_j$, which inverse reads $b_i = a_{ij}b^j$.

We shall use also the normalized vectors

$$\widetilde{b}_i = \frac{1}{c}b_i, \qquad \widetilde{b}^i = \frac{1}{c}b^i = a^{ij}\widetilde{b}_j, \qquad a^{mn}\widetilde{b}_m\widetilde{b}_n = 1. \tag{5.4}$$

We get

$$a_{ij}y^iy^j - b^2 > 0 \tag{5.5}$$

and may conveniently use the variable

$$q := \sqrt{a_{ij}y^iy^j - b^2}. \tag{5.6}$$

Obviously, the inequality

$$q^2 \geq \frac{1-c^2}{c^2}\,b^2 \tag{5.7}$$

is valid.

We also introduce the tensor

$$r_{ij}(x) \ := a_{ij}(x) - b_i(x)b_j(x) \tag{5.8}$$

to have the representation

$$q = \sqrt{r_{ij}y^iy^j}. \tag{5.9}$$

The equalities

$$r_{ij}b^j = (1-c^2)b_i, \qquad r_{in}r^{nj} = r^j{}_i - (1-c^2)b^jb_i \tag{5.10}$$

hold.

In evaluations it is convenient to use the variables

$$u_i \ := a_{ij}y^j, \qquad v^i \ := y^i - bb^i, \qquad v_m \ := u_m - bb_m = r_{mn}y^n \equiv a_{mn}v^n. \tag{5.11}$$

We have

$$r_{ij} = \frac{\partial v_i}{\partial y^j}, \qquad \frac{\partial b}{\partial y^i} = b_i, \qquad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}, \qquad v_ib^i = v^ib_i = (1-c^2)b, \tag{5.12}$$

and

$$u_iv^i = v_iy^i = q^2, \qquad r_{in}v^n = v_i - (1-c^2)bb_i, \qquad v_kv^k = q^2 - (1-c^2)b^2. \tag{5.13}$$

With the variable

$$w = \frac{q}{b}, \qquad b > 0, \tag{5.14}$$

we obtain

$$\frac{\partial w}{\partial y^i} = \frac{qe_i}{b^2}, \qquad e_i = -b_i + \frac{b}{q^2}v_i, \tag{5.15}$$

and $y^ie_i = 0$.

The Finsler metric function F of the $\mathcal{FS}$-space is given by (1.25). When $b > 0$, we can conveniently use the *generating metric function* $V = V(x, w)$ to have the representation

$$F = bV(x, w). \tag{5.16}$$

The unit vector $l_m = \partial F/\partial y^m$ is given by

$$l_m = b_mV + we_mV', \qquad V' = \frac{\partial V}{\partial w}. \tag{5.17}$$

It proves convenient to use the quantities

$$\tau = \frac{wV}{V'}, \tag{5.18}$$

$$\widetilde{q} = \sqrt{q^2 + \left(1 - \frac{1}{c^2}\right)b^2}, \tag{5.19}$$

and

$$\widetilde{w} = \frac{\widetilde{q}}{b}, \qquad b > 0. \tag{5.20}$$

There are the useful equalities

$$\tau = \widetilde{\tau} = \frac{\widetilde{w}V}{\widetilde{V}'}, \qquad \widetilde{V} = V, \qquad \widetilde{V}' = \frac{\partial \widetilde{V}}{\partial \widetilde{w}}, \qquad b^ml_m = c^2V\left(1 - \frac{\widetilde{w}^2}{\tau}\right).$$

We say that the $\mathcal{FS}$-space is *special*, if $\partial\widetilde{V}/\partial x^n = 0$, that is when

$$\widetilde{V} = \widetilde{V}(\widetilde{w}). \tag{5.21}$$

Take two differentiable scalar functions

$$C = C(x), \quad C_1 = C_1(x), \qquad C > 0, \quad C > |C_1|, \tag{5.22}$$

and construct the scalars

$$H = \sqrt{C^2 - (C_1)^2} \tag{5.23}$$

and

$$\breve{k} = \sqrt{\frac{C - C_1}{C + C_1}}. \tag{5.24}$$

Let a positive function $\mu = \mu(x, y)$ be specified according to

$$\sqrt{\mu} = \frac{H}{2\breve{k}}\left[1 + \breve{k}^2 + (1 - \breve{k}^2)\cos\varrho\right], \tag{5.25}$$

where $\varrho = \varrho(x,y)$ is an input scalar. We can write

$$\sqrt{\mu} = C + C_1 \cos\varrho. \tag{5.26}$$

Consider the transformation $t^m = t^m(x,y)$ with

$$t^m = \left[i^m \sin\varrho + \frac{1}{2\breve{k}}[1 - \breve{k}^2 + (1 + \breve{k}^2)\cos\varrho]\widetilde{b}^m\right]\frac{H}{\sqrt{\mu}}F^H, \tag{5.27}$$

where

$$i^m = \left(y^m - \frac{1}{c^2}bb^m\right)\frac{1}{\widetilde{q}}. \tag{5.28}$$

We have

$$b_m i^m = 0, \qquad a_{mn}i^m i^n = 1, \qquad a_{mn}y^m i^n = \widetilde{q}, \tag{5.29}$$

and

$$b^* = \frac{1}{2\breve{k}}\left[1 - \breve{k}^2 + (1 + \breve{k}^2)\cos\varrho\right]\frac{H}{\sqrt{\mu}}S, \tag{5.30}$$

where

$$S = \sqrt{a_{mn}t^m t^n}, \qquad b^* = t^m \breve{b}_m, \qquad \breve{b}_m = \widetilde{b}_m, \qquad \breve{b}^m = \widetilde{b}^m. \tag{5.31}$$

We get also the equality

$$b^* = (C_1 + C\cos\varrho)\frac{1}{\sqrt{\mu}}S. \tag{5.32}$$

The functions (5.27) obviously fulfill the H-degree homogeneity condition (2.7). The validity of the equality $S = F^H$ (see (2.10)) can readily be verified.

The property

$$t^m(x, b(x)) \sim b^m(x) \tag{5.33}$$

holds.

The following useful equalities can readily be obtained:

$$\cos\varrho = -\frac{(1 - \breve{k}^2)S - (1 + \breve{k}^2)b^*}{(1 + \breve{k}^2)S - (1 - \breve{k}^2)b^*},$$

$$\sqrt{\mu} = \frac{2H\breve{k}S}{(1 + \breve{k}^2)S - (1 - \breve{k}^2)b^*}, \tag{5.34}$$

$$\cos\varrho = -\frac{\sqrt{\mu}}{2H\breve{k}S}[(1 - \breve{k}^2)S - (1 + \breve{k}^2)b^*],$$

and

$$\sin^2\varrho = 4\breve{k}^2\frac{S^2 - (b^*)^2}{\left[(1 + \breve{k}^2)S - (1 - \breve{k}^2)b^*\right]^2},$$

together with

$$\frac{\sin^2\varrho}{\mu}=\frac{1}{H^2}\frac{S^2-(b^*)^2}{S^2}. \tag{5.35}$$

Differentiating (5.28) yields

$$\frac{\partial i^m}{\partial y^k}=\left(\delta^m_k-\frac{1}{c^2}b_kb^m\right)\frac{1}{b\widetilde{w}}-\frac{1}{b}b_ki^m-\frac{1}{b}\frac{w^2}{\widetilde{w}^2}i^me_k, \tag{5.36}$$

which entails

$$y^k\frac{\partial i^m}{\partial y^k}=0,\qquad b^k\frac{\partial i^m}{\partial y^k}=0,\qquad i^na_{nm}\frac{\partial i^m}{\partial y^k}=0,\qquad b_m\frac{\partial i^m}{\partial y^k}=0,\qquad y^na_{nm}\frac{\partial i^m}{\partial y^k}=0. \tag{5.37}$$

We can use the relations

$$b_k=\frac{b}{F}l_k-w^2\frac{1}{\tau}e_k,\qquad \widetilde{w}i_k=w^2e_k+\widetilde{w}^2b_k=w^2e_k+\widetilde{w}^2\left(\frac{b}{F}l_k-w^2\frac{1}{\tau}e_k\right),$$

so that

$$i_k-\frac{1}{F}b\widetilde{w}l_k=\frac{w^2}{\widetilde{w}}\left(1-\frac{\widetilde{w}^2}{\tau}\right)e_k,$$

where $i_k=a_{kn}i^n$.

We have also

$$a_{mh}\frac{\partial i^m}{\partial y^k}=\left(a_{hk}-\frac{1}{c^2}b_kb_h\right)\frac{1}{b\widetilde{w}}-\frac{1}{b}b_ki_h-\frac{1}{b}\frac{w^2}{\widetilde{w}^2}i_he_k,$$

which entails

$$a_{mh}\frac{\partial i^m}{\partial y^k}=\left(a_{hk}-\frac{1}{c^2}b_kb_h-i_ki_h\right)\frac{1}{b\widetilde{w}}. \tag{5.38}$$

With these observations, from (5.27) we find that the derivative coefficients $t^m_k=\partial t^m/\partial y^k$ can be given by

$$\frac{1}{H}t^m_k=\left[\cos\varrho\, i^m-\frac{1}{2\breve{k}}(1+\breve{k}^2)\sin\varrho\,\breve{b}^m\right]\varrho'\frac{w}{b}e_k\frac{F^H}{\sqrt{\mu}}+\sin\varrho\frac{\partial i^m}{\partial y^k}\frac{F^H}{\sqrt{\mu}}+\frac{1}{F}l_kt^m-\frac{1}{H\sqrt{\mu}}\frac{\partial\sqrt{\mu}}{\partial y^k}t^m.$$

Since

$$\frac{\partial\sqrt{\mu}}{\partial y^k}=-\frac{H}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_k,$$

we obtain the explicit representation

$$\frac{1}{H}\sqrt{\mu}t^m_k=\left[\cos\varrho\, i^m-\frac{1}{2\breve{k}}(1+\breve{k}^2)\sin\varrho\,\breve{b}^m\right]\varrho'\frac{w}{b}e_kF^H$$

$$+\sin\varrho\frac{\partial i^m}{\partial y^k}F^H+\left[\sqrt{\mu}\frac{1}{F}l_k+\frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_k\right]t^m. \tag{5.39}$$

The identity $t^m_ky^k=Ht^m$ can readily be verified.

We can straightforwardly evaluate the contraction $a_{mn}t^m_k t^n_h$, which leads to the expression which is a linear combination of g_{kh}, $e_k e_h$, $l_k l_h$, and $e_k l_h + e_h l_k$. To obtain the conformal result, the terms $l_k l_h$ are to be canceled, which proves possible if and only if the function μ is taken to be

$$\mu = \frac{1}{\widetilde{w}^2}\tau \sin^2 \varrho, \tag{5.40}$$

which entails

$$\frac{\widetilde{w}^2}{\tau} = \frac{1}{H^2}\frac{S^2-(b^*)^2}{S^2} \tag{5.41}$$

(see (5.35)). With the choice of μ according to (5.40), using the representation (5.39) leads straightforwardly to the equality

$$\frac{1}{H^2}a_{mn}t^m_k t^n_h = \frac{1}{\tau}\frac{\widetilde{w}^2}{\sin^2 \varrho}\left(\frac{\partial \varrho}{\partial w}\right)^2 \frac{w^2}{b^2}e_k e_h F^{2H}$$

$$+\frac{1}{\tau}F^{2H}\left(-\frac{\tau - w(\tau' - w)}{\tau} + 1 - \frac{w^2}{\widetilde{w}^2}\right) w^2 e_k e_h \frac{1}{b^2} + F^{2(H-1)}g_{kh} \tag{5.42}$$

(see Appendix E).

Subjecting the ϱ to the equation

$$\frac{\partial \varrho}{\partial \widetilde{w}} = \frac{1}{\widetilde{w}}\sqrt{\frac{\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w})}{\widetilde{\tau}}}\sin \varrho, \tag{5.43}$$

where $\widetilde{\tau}' = \partial\tau/\partial\widetilde{w}$, is necessary and sufficient to reduce the right-hand part in (5.42) to the conformal representation, namely we obtain simply

$$\frac{1}{H^2}a_{mn}t^m_k t^n_h = F^{2(H-1)}g_{kh}. \tag{5.44}$$

Comparing the last representation with the formulas (2.1), (2.4), and (2.9) makes us conclude that the following assertion is valid.

Proposition 5.1. *With choosing the function μ to be given by* (5.40) *and subjecting the function ϱ to the equation* (5.43), *the transformation $t^m = t^m(x, y)$ introduced by* (5.27) *fulfills the conformal automorphism* (2.1)-(2.2).

From (5.28) it follows that

$$\frac{1}{b}y^m = \frac{\sqrt{\tau}}{HS}(t^m - b^*\breve{b}^m) + \frac{1}{c^2}b^m,$$

so that we can write (5.27) to read

$$t^m - b^*\breve{b}^m = \frac{HF^H}{\sqrt{\tau}}\left[\frac{1}{b}y^m - \frac{1}{c^2}b^m\right].$$

On the other hand, using the equality $S = F^H$ in (5.41) yields

$$S^2-(b^*)^2=\frac{H^2F^{2H}}{\tau}\widetilde{w}^2. \tag{5.45}$$

Thus, it is valid that

$$\frac{1}{\widetilde{w}}\left(\frac{1}{b}y^m-\frac{1}{c^2}b^m\right)=\frac{1}{\sqrt{S^2-(b^*)^2}}\left(t^m-b^*\breve{b}^m\right). \tag{5.46}$$

Let us evaluate the explicit form of the respective coefficients ${N^m}_n$ proposed by (2.36). Accordingly, we assume $d_iF=0$ and $H=const$. Since $S=F^H$, we have $d_iS=0$. From (2.38)–(2.39) it follows that $d_it^n=-{a^n}_{ih}t^h$.

Also, $d_ib^*=d_i(t^m\widetilde{b}_m)=-{a^m}_{ih}t^h\widetilde{b}_m+t^m(\nabla_i\widetilde{b}_m+{a^r}_{im}\widetilde{b}_r)=t^m\nabla_i\widetilde{b}_m$, so that

$$\frac{1}{\sqrt{S^2-(b^*)^2}}d_ib^*=\left[\frac{1}{\widetilde{w}}\left(\frac{1}{b}y^m-\frac{1}{c^2}b^m\right)+\frac{1}{\sqrt{S^2-(b^*)^2}}b^*\breve{b}^m\right]\nabla_i\widetilde{b}_m.$$

Henceforth, we assume $c=const$. In this case $b^m\nabla_ib_m=0$, and therefore

$$\frac{1}{\sqrt{S^2-(b^*)^2}}d_ib^*=\frac{1}{\widetilde{w}}\frac{1}{b}y^m\nabla_i\widetilde{b}_m.$$

Under these conditions, applying the operator d_i to (5.46) yields

$$-\frac{1}{\widetilde{w}^2}d_i\widetilde{w}\left(\frac{1}{b}y^m-\frac{1}{c^2}b^m\right)+\frac{1}{\widetilde{w}}\left(\frac{1}{b}d_iy^m-\frac{1}{b^2}y^md_ib-\frac{1}{c^2}(\nabla_ib^m-{a^m}_{ih}b^h)\right)$$

$$=b^*d_ib^*\frac{1}{(S^2-(b^*)^2)\sqrt{S^2-(b^*)^2}}\left(t^m-b^*\breve{b}^m\right)$$

$$+\frac{1}{\sqrt{S^2-(b^*)^2}}\left(d_it^m-(d_ib^*)\breve{b}^m-b^*(\nabla_i\breve{b}^m-{a^m}_{ih}\widetilde{b}^h)\right),$$

or

$$-\left[\frac{1}{\widetilde{w}}d_i\widetilde{w}+\frac{1}{S^2-(b^*)^2}b^*d_ib^*\right]\left(\frac{1}{b}y^m-\frac{1}{c^2}b^m\right)+\frac{1}{b}d_iy^m-\frac{1}{b^2}y^md_ib-\frac{1}{c^2}(\nabla_ib^m-{a^m}_{ih}b^h)$$

$$=\frac{\widetilde{w}}{\sqrt{S^2-(b^*)^2}}\left(-{a^m}_{ih}\left(b^*\breve{b}^h+\frac{HF^H}{\sqrt{\tau}}\left(\frac{1}{b}y^h-\frac{1}{c^2}b^h\right)\right)-b^*(\nabla_i\breve{b}^m-{a^m}_{ih}\widetilde{b}^h)\right)-\frac{1}{b}(y^h\nabla_i\breve{b}_h)\breve{b}^m.$$

Using the new variable

$$W=\frac{b^*}{S}, \tag{5.47}$$

we have

$$\cos\varrho=-\frac{1-\breve{k}^2-(1+\breve{k}^2)W}{1+\breve{k}^2-(1-\breve{k}^2)W} \tag{5.48}$$

and

$$\sin^2\varrho = 4\breve{k}^2\frac{1-W^2}{\left[1+\breve{k}^2-(1-\breve{k}^2)W\right]^2}, \tag{5.49}$$

which entails

$$\frac{\partial\cos\varrho}{\partial W} = 4\breve{k}^2\frac{1}{\left[1+\breve{k}^2-(1-\breve{k}^2)W\right]^2}. \tag{5.50}$$

In the equality

$$\frac{1}{\widetilde{w}}\frac{\partial\widetilde{w}}{\partial W} = \frac{1}{\sin\varrho}\frac{\partial\varrho}{\partial W}P$$

we use (5.49) and (5.50), obtaining

$$\frac{1}{\widetilde{w}}\frac{\partial\widetilde{w}}{\partial W} = -\frac{1}{1-W^2}P,$$

where the notation

$$P = \sqrt{\frac{\widetilde{\tau}}{\widetilde{\tau}-\widetilde{w}(\widetilde{\tau}'-\widetilde{w})}}$$

has been used.

Therefore, in the special case (see (5.21)) of the $\mathcal{FS}$-space, we have

$$d_i\widetilde{w} = \frac{\partial\widetilde{w}}{\partial W}d_iW = -\frac{\widetilde{w}}{1-W^2}Pd_iW = -\frac{\widetilde{w}}{1-W^2}P\frac{1}{S}d_ib^*$$

and can write

$$\frac{1}{S}(d_ib^*)\frac{1}{1-W^2}(P-W)\left(\frac{1}{b}y^m-\frac{1}{c^2}b^m\right)+\frac{1}{b}d_iy^m-\frac{1}{b^2}y^md_ib-\frac{1}{c^2}\nabla_ib^m$$

$$=\frac{\widetilde{w}}{\sqrt{S^2-(b^*)^2}}\left(-a^m{}_{ih}\frac{HF^H}{\sqrt{\tau}}\frac{1}{b}y^h-b^*\nabla_i\breve{b}^m\right)-\frac{1}{b}(y^h\nabla_i\breve{b}_h)\breve{b}^m$$

and

$$d_iy^m = -\frac{1}{S}(d_ib^*)\frac{1}{1-W^2}(P-W)\left(y^m-\frac{1}{c^2}bb^m\right)+\frac{1}{b}y^md_ib+\frac{1}{c^2}b\nabla_ib^m$$

$$+\frac{\widetilde{w}}{\sqrt{S^2-(b^*)^2}}\left(-a^m{}_{ih}\frac{HF^H}{\sqrt{\tau}}y^h-bb^*\nabla_i\breve{b}^m\right)-(y^h\nabla_i\breve{b}_h)\breve{b}^m.$$

Taking into account the equality

$$d_ib = -\frac{b\widetilde{w}}{\tau}d_i\widetilde{w}$$

which is valid in the special case of the $\mathcal{FS}$-space, we obtain

$$d_iy^m = -\frac{1}{S}(d_ib^*)\frac{1}{1-W^2}(P-W)\left(y^m-\frac{1}{c^2}bb^m\right)+\frac{\widetilde{w}^2}{\tau}\frac{1}{S}(d_ib^*)\frac{1}{1-W^2}Py^m$$

$$+\frac{\sqrt{\tau}}{HF^H}\left(-a^m{}_{ih}\frac{HF^H}{\sqrt{\tau}}y^h - bb^*\nabla_i\breve{b}^m\right) - (y^h\nabla_i\breve{b}_h)\breve{b}^m + \frac{1}{c^2}b\nabla_i b^m$$

$$= -\frac{1}{b\widetilde{w}}y^h\nabla_i\breve{b}_h\frac{1}{\sqrt{1-W^2}}(P-W)\left(y^m - \frac{1}{c^2}bb^m\right)$$

$$+\frac{\widetilde{w}^2}{\tau}\frac{1}{b\widetilde{w}}y^h\nabla_i\breve{b}_h\frac{1}{\sqrt{1-W^2}}Py^m - b\frac{\sqrt{\tau}}{H}W\nabla_i\breve{b}^m - (y^h\nabla_i\breve{b}_h)\breve{b}^m + \frac{1}{c^2}b\nabla_i b^m - a^m{}_{ih}y^h.$$

Here,

$$1 - W^2 = H^2\frac{\widetilde{w}^2}{\tau}.$$

Therefore,

$$d_iy^m = -\frac{1}{b\widetilde{w}}y^h\nabla_i\breve{b}_h\frac{1}{H}\frac{\sqrt{\tau}}{\widetilde{w}}(P-W)\left(y^m - \frac{1}{c^2}bb^m\right) + \frac{\widetilde{w}^2}{\tau}\frac{1}{b\widetilde{w}}y^h\nabla_i\breve{b}_h\frac{1}{H}\frac{\sqrt{\tau}}{\widetilde{w}}Py^m$$

$$-b\frac{\sqrt{\tau}}{H}W\nabla_i\breve{b}^m - (y^h\nabla_i\breve{b}_h)\breve{b}^m + \frac{1}{c^2}b\nabla_i b^m - a^m{}_{ih}y^h. \tag{5.51}$$

Noting the equality

$$N^m{}_i = d_iy^m \tag{5.52}$$

leads to

$$N^m{}_i = \frac{1}{H}\frac{1}{b\widetilde{w}}(y^h\nabla_i\widetilde{b}_h)F\alpha^m - \frac{1}{H}\sqrt{\tau - H^2\widetilde{w}^2}\,b\widetilde{\beta}^m_i - (y^h\nabla_i\widetilde{b}_h)\widetilde{b}^m + \widetilde{b}\nabla_i\widetilde{b}^m - a^m{}_{ih}y^h, \tag{5.53}$$

where

$$F\alpha^m = \frac{\widetilde{w}}{\sqrt{\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w})}}\left[y^m - \frac{\tau}{\widetilde{w}^2}\left(y^m - \widetilde{b}\widetilde{b}^m\right)\right] \tag{5.54}$$

and

$$\widetilde{\beta}^m_i = \nabla_i\widetilde{b}^m - \frac{1}{b^2\widetilde{w}^2}(y^h\nabla_i\widetilde{b}_h)\left(y^m - \widetilde{b}\widetilde{b}^m\right), \tag{5.55}$$

which can also be written in the form

$$N^m{}_i = \frac{1}{H}\frac{1}{\widetilde{q}}(y^h\nabla_i\widetilde{b}_h)F\alpha^m - \frac{1}{H}\sqrt{B - H^2\widetilde{q}^2}\,\widetilde{\beta}^m_i - (y^h\nabla_i\widetilde{b}_h)\widetilde{b}^m + \widetilde{b}\nabla_i\widetilde{b}^m - a^m{}_{ih}y^h, \tag{5.56}$$

with

$$F\alpha^m = \frac{\widetilde{w}}{\sqrt{\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w})}}\left[y^m - \frac{B}{\widetilde{q}^2}\left(y^m - \widetilde{b}\widetilde{b}^m\right)\right], \tag{5.57}$$

$$\widetilde{\beta}_i^m = \nabla_i \widetilde{b}^m - \frac{1}{\widetilde{q}^2}(y^h \nabla_i \widetilde{b}_h)\left(y^m - \widetilde{b}\widetilde{b}^m\right), \tag{5.58}$$

and

$$B = b^2 \tau. \tag{5.59}$$

Thus we have

Proposition 5.2. *If in the special case of the $\mathcal{FS}$-space with $c = const$ the transformation* (5.27) *results in the conformally automorphic space, then the coefficients* (2.36) *can explicitly be given by means of the representation* (5.56)-(5.59).

It is easy to verify that

$$\alpha^m l_m = 0, \qquad \widetilde{\beta}_i^m b_m = 0, \qquad \widetilde{\beta}_i^m l_m = 0, \qquad \widetilde{\beta}_i^m \alpha_m = 0. \tag{5.60}$$

By contracting (5.56) we find

$$l_m N^m{}_i = -(y^h \nabla_i b_h)\left(1 - \frac{1+w^2}{\tau}\right) V - l_m a^m{}_{ih} y^h. \tag{5.61}$$

From this result it follows that

$$\frac{\partial F}{\partial x^k} + l_m N^m{}_k = 0. \tag{5.62}$$

Indeed, denoting

$$s_k \ := y^m \nabla_k b_m, \tag{5.63}$$

we get

$$\frac{\partial q}{\partial x^k} = -\frac{b}{q}(s_k + y^m b_h a^h{}_{mk}) + \frac{1}{q} y^m y^n \frac{\partial a_{mn}}{\partial x^k} \tag{5.64}$$

and

$$\frac{\partial w}{\partial x^k} = -\frac{1}{q}(s_k + y^m b_h a^h{}_{mk}) - \frac{q}{b^2}(s_k + y^m b_h a^h{}_{mk}) + \frac{1}{bq} y^m y^n \frac{\partial a_{mn}}{\partial x^k},$$

or

$$\frac{\partial w}{\partial x^k} = -\frac{1}{b^2 q} S^2 (s_k + y^m b_h a^h{}_{mk}) + \frac{1}{bq} y^m y^n \frac{\partial a_{mn}}{\partial x^k}. \tag{5.65}$$

The equality

$$\widetilde{w}\frac{\partial \widetilde{w}}{\partial x^k} = w \frac{\partial w}{\partial x^k} \tag{5.66}$$

can appropriately be used.

In the special case $F = b\widetilde{V}(\widetilde{w})$ (see (5.21)) of the $\mathcal{FS}$-space we have

$$\frac{\partial F}{\partial x^k} = \left(\widetilde{V} - \frac{1}{bq} S^2 \frac{w}{\widetilde{w}} \widetilde{V}'\right) s_k + \left(\widetilde{V} - \frac{1}{bq} S^2 \frac{w}{\widetilde{w}} \widetilde{V}'\right) y^m b_h a^h{}_{mk} + \frac{1}{q}\frac{w}{\widetilde{w}} \widetilde{V}' y^m y^n \frac{\partial a_{mn}}{\partial x^k},$$

or

$$\frac{\partial F}{\partial x^k} = \left(\widetilde{V} - \frac{S^2}{b^2}\frac{1}{\widetilde{w}} \widetilde{V}'\right) s_k + \left(\widetilde{V} - \frac{S^2}{b^2}\frac{1}{\widetilde{w}} \widetilde{V}'\right) y^m b_h a^h{}_{mk} + \frac{1}{b\widetilde{w}} \widetilde{V}' y^m y^n \frac{\partial a_{mn}}{\partial x^k}. \tag{5.67}$$

In terms of the function $\widetilde{\tau}$, we can write

$$\frac{\partial F}{\partial x^k} = \widetilde{V}\left(1 - \frac{1+w^2}{\widetilde{\tau}}\right) s_k + \widetilde{V}\left(1 - \frac{1+w^2}{\widetilde{\tau}}\right) y^m b_h a^h{}_{mk} + \frac{\widetilde{V}}{b\widetilde{\tau}} y^m y^n \frac{\partial a_{mn}}{\partial x^k}. \tag{5.68}$$

With this equality the validity of the vanishing (5.62) can readily be verified.

The following proposition is valid.

Proposition 5.3. *The transformation* (5.27) *entails the conformal automorphism* (2.1) *iff*

$$\tau = \breve{C}^2 + 2\breve{C}\sqrt{1-H^2}\,\widetilde{w} + \widetilde{w}^2. \tag{5.69}$$

It follows that

$$\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w}) = \breve{C}^2.$$

In these formulas, $\breve{C}$ is an integration scalar $\breve{C} = \breve{C}(x)$. It can readily be seen that when $|\breve{C}| \neq 1$, the entailed Finsler metric function F can vanish at various values of tangent vectors y. To agree with the condition that F vanishes only at zero-vectors $y = 0$, we admit strictly the values $\breve{C} = 1$ and $\breve{C} = -1$. In this case we can write the above τ as follows:

$$\tau = 1 + g\widetilde{w} + \widetilde{w}^2, \quad -2 < g < 2. \tag{5.70}$$

Generally, the g may depend on x. We obtain

$$B - H^2\widetilde{q}^2 = \left(b + \frac{1}{2}g\widetilde{q}\right)^2. \tag{5.71}$$

In this case the coefficients (5.56) take on the form

$$N^m{}_i = \frac{1}{h}\frac{1}{\widetilde{q}} F m^m \widetilde{s}_i - \frac{1}{h}\left(b + \frac{1}{2}g\widetilde{q}\right)\widetilde{\beta}^m_i - \widetilde{b}^m \widetilde{s}_i + \widetilde{b}\nabla_i \widetilde{b}^m - a^m{}_{ih} y^h, \tag{5.72}$$

with

$$m^m = \frac{1}{\widetilde{q}F}\left[\widetilde{q}^2\widetilde{b}^m - (b + g\widetilde{q})\left(y^m - \widetilde{b}\widetilde{b}^m\right)\right] \equiv \frac{1}{\widetilde{q}F}\left[B^2\widetilde{b}^m - (b + g\widetilde{q})\,y^m\right], \tag{5.73}$$

$$\widetilde{\beta}^m_i = \nabla_i \widetilde{b}^m - \frac{1}{\widetilde{q}^2}\left(y^m - \widetilde{b}\widetilde{b}^m\right)\widetilde{s}_i, \tag{5.74}$$

$m^m = \alpha^m$, and

$$\widetilde{s}_i = y^h \nabla_i \widetilde{b}_h. \tag{5.75}$$

Note. We used the input representation $F = bV(x,w)$ (see (5.16)) at $b > 0$. All the performed calculations can be repeated word-for-word in the negative case $b < 0$. The above representation (5.72)-(5.75) obtained for the coefficients $N^m{}_i$ embraces both the cases $b > 0$ and $b < 0$.

The last three terms in (5.72) are linear with respect to the tangent vectors y.

The function τ given by (5.70) represents the $\mathcal{F}\mathcal{F}^{PD}_g$-Finsleroid space described in the paper [7]. To comply with the representations used in [7], we should replace the notation H by the notation h:

$$h=\sqrt{1-\frac{g^2}{4}}. \tag{5.76}$$

The g plays the role of the characteristic parameter. The $\mathcal{F}\mathcal{F}^{PD}_g$-Finsleroid metric function K is given as it follows:

$$K=\sqrt{B}\,J, \qquad \text{with} \quad J=\mathrm{e}^{-\frac{1}{2}g\chi}, \tag{5.77}$$

where

$$\chi=\frac{1}{h}\Bigl(-\arctan\frac{G}{2}+\arctan\frac{L}{hb}\Bigr), \text{ if } b\ge 0; \quad \chi=\frac{1}{h}\Bigl(\pi-\arctan\frac{G}{2}+\arctan\frac{L}{hb}\Bigr), \text{ if } b\le 0, \tag{5.78}$$

with the function $L=\widetilde{q}+(g/2)b$ fulfilling the identity

$$L^2+h^2b^2=B. \tag{5.79}$$

B is the function given by (5.71):

$$B=b^2+gb\widetilde{q}+\widetilde{q}^2; \tag{5.80}$$

$G=g/h$. The definition range

$$0\le\chi\le\frac{1}{h}\pi$$

is of value to describe all the tangent space. The normalization in (5.78) is such that

$$\chi\Big|_{y=b}=0. \tag{5.81}$$

The quantity χ can conveniently be written as

$$\chi=\frac{1}{h}f \tag{5.82}$$

with the function

$$f=\arccos\frac{A(x,y)}{\sqrt{B(x,y)}}, \qquad A=b+\frac{1}{2}g\widetilde{q}, \tag{5.83}$$

ranging as follows:

$$0\le f\le\pi. \tag{5.84}$$

The function K is the solution for the equation (5.70).

The Finsleroid-axis vector b^i relates to the value $f=0$, and the opposed vector $-b^i$ relates to the value $f=\pi$:

$$f=0 \quad\sim\quad y=b; \qquad f=\pi \quad\sim\quad y=-b. \tag{5.85}$$

The normalization is such that

$$K(x,b(x))=1 \tag{5.86}$$

(notice that $\widetilde{q}=0$ at $y^i=b^i$). The positive (not absolute) homogeneity holds: $K(x,\gamma y)=\gamma K(x,y)$ for any $\gamma>0$ and all admissible (x,y).

The entailed components $y_i := (1/2)\partial K^2/\partial y^i$ of the covariant tangent vector $\hat{y} = \{y_i\}$ can be found in the simple form

$$y_i = (u_i + g\widetilde{q}b_i)J^2, \tag{5.87}$$

where $u_i = a_{ij}y^j$.

Under these conditions, we obtain the $\mathcal{FF}^{PD}_g$-Finsleroid space

$$\mathcal{FF}^{PD}_g := \{M;\ a_{ij}(x);\ b_i(x);\ g(x);\ K(x,y)\}. \tag{5.88}$$

Definition. Within any tangent space T_xM, the metric function $K(x,y)$ produces the $\mathcal{FF}^{PD}_g$-*Finsleroid*

$$\mathcal{FF}^{PD}_{g;\{x\}} := \{y \in \mathcal{FF}^{PD}_{g;\{x\}} : y \in T_xM, K(x,y) \le 1\}. \tag{5.89}$$

Definition. The $\mathcal{FF}^{PD}_g$-*Indicatrix* $\mathcal{IF}^{PD}_{g;\{x\}} \subset T_xM$ is the boundary of the $\mathcal{FF}^{PD}_g$-Finsleroid, that is,

$$\mathcal{IF}^{PD}_{g\{x\}} := \{y \in \mathcal{IF}^{PD}_{g\{x\}} : y \in T_xM, K(x,y) = 1\}. \tag{5.90}$$

Definition. The scalar $g(x)$ is called the *Finsleroid charge*. The 1-form $b = b_i(x)y^i$ is called the *Finsleroid–axis* 1-*form*.

It can readily be seen that

$$\det(g_{ij}) = \left(\frac{K^2}{B}\right)^N \det(a_{ij}) > 0, \qquad A^iA_i = \frac{N^2g^2}{4},$$

where $A_i = KC_i$.

Note. The representation (5.72)-(5.75) obtained for the coefficients $N^m{}_i$ coincides exactly with the representation (6.53) of [7]. Considering the vector $C_i = g^{mn}C_{imn}$, the equality

$$m^m = \frac{C^m}{\sqrt{g^{kh}C_kC_h}} \tag{5.91}$$

holds exactly with the vector m^m given by the representation (5.73) (which is equivalent to the representation (A.46) proposed in [7]).

Let us verify Proposition 5.3. With the variable $W = b^*/S$ (see (5.47)) we can write the equation (5.43) as follows:

$$\frac{1}{\sin\varrho}\frac{\partial\varrho}{\partial W}\frac{\partial W}{\partial\widetilde{w}}\widetilde{w} = \sqrt{\frac{\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w})}{\widetilde{\tau}}}. \tag{5.92}$$

Let us introduce the function j by means of the equality

$$\tau = j\widetilde{w}^2. \tag{5.93}$$

We obtain $[\widetilde{\tau} - \widetilde{w}(\widetilde{\tau}' - \widetilde{w})]/\widetilde{\tau} = [1 - j - \widetilde{w}\widetilde{j}']/j$ and

$$\left(\frac{1}{\sin^2\varrho}\frac{\partial\cos\varrho}{\partial W}\right)^2\left(\frac{\partial W}{\partial\widetilde{w}}\widetilde{w}\right)^2=\frac{1}{j}-1-\frac{1}{j}\frac{\partial j}{\partial W}\frac{\partial W}{\partial\widetilde{w}}\widetilde{w}. \tag{5.94}$$

Using (5.49) and (5.50) together with

$$j=H^2\frac{1}{1-W^2} \tag{5.95}$$

(see (5.41)), we can write the equation (5.94) in the form

$$\left(\frac{1}{1-W^2}\frac{\partial W}{\partial\widetilde{w}}\widetilde{w}\right)^2+2W\left(\frac{1}{1-W^2}\frac{\partial W}{\partial\widetilde{w}}\widetilde{w}\right)=\frac{1}{H^2}(1-W^2)-1,$$

which can conveniently be written as follows:

$$\left(\frac{1}{1-W^2}\frac{\partial W}{\partial\widetilde{w}}\widetilde{w}+W\right)^2=\left(\frac{1}{H^2}-1\right)(1-W^2).$$

It proves convenient to go over to the variable W^2:

$$\left(\frac{1}{1-W^2}\frac{\partial W^2}{\partial\widetilde{w}}\widetilde{w}+2W^2\right)^2=4\left(\frac{1}{H^2}-1\right)(1-W^2)W^2.$$

Since

$$W^2=1-\frac{H^2\widetilde{w}^2}{\tau}$$

(see (5.45)), we get

$$\left(-\frac{\tau}{\widetilde{w}^2}\frac{\partial\frac{\widetilde{w}^2}{\tau}}{\partial\widetilde{w}}\widetilde{w}+2\left(1-\frac{H^2\widetilde{w}^2}{\tau}\right)\right)^2=4(1-H^2)\frac{\widetilde{w}^2}{\tau}\left(1-\frac{H^2\widetilde{w}^2}{\tau}\right),$$

or

$$\left(\widetilde{w}\widetilde{\tau}'-2\tau+2(\tau-H^2\widetilde{w}^2)\right)^2=4(1-H^2)\widetilde{w}^2(\tau-H^2\widetilde{w}^2).$$

Simplifying leaves us with the equation

$$(\widetilde{\tau}'-2H^2\widetilde{w})^2=4(1-H^2)(\tau-H^2\widetilde{w}^2),$$

which can readily be solved to yield (5.69). Proposition 5.3 is valid.

The coefficients (5.72) show the properties

$$u_kN^k{}_n=-\frac{1}{h}gqy^j\nabla_nb_j-u_ka^k{}_{nj}y^j,\qquad b_kN^k{}_n=\frac{1}{h}(1-h)y^j\nabla_nb_j-b_ka^k{}_{nj}y^j$$

(where $u_k=a_{kn}y^n$), and

$$d_nb\equiv\frac{\partial b}{\partial x^n}+b_kN^k{}_n=\frac{1}{h}y^j\nabla_nb_j,\qquad d_nq\equiv\frac{\partial q}{\partial x^n}+\frac{1}{q}v_kN^k{}_n=-\frac{1}{hq}(b+gq)y^j\nabla_nb_j,$$

together with

$$d_n\left(\frac{q}{b}\right) = -\frac{1}{b^2qh}By^j\nabla_n b_j, \quad d_n B = -\frac{g}{qh}By^j\nabla_n b_j, \quad d_n\frac{B}{b^2} = -\frac{2q+gb}{b^3qh}By^j\nabla_n b_j.$$

With these formulas it is possible to verify directly the validity of the vanishing

$$\mathcal{D}_n K \ := \frac{\partial K}{\partial x^n} + N^m{}_n l_m = 0, \tag{5.96}$$

$$\mathcal{D}_n y_j \ := \frac{\partial y_j}{\partial x^n} + N^m{}_n g_{mj} - D^m{}_{nj} y_m = 0, \tag{5.97}$$

and

$$\mathcal{D}_n g_{ij} \ := \frac{\partial g_{ij}}{\partial x^n} + 2N^m{}_n C_{mji} - D^m{}_{nj} g_{mi} - D^m{}_{ni} g_{mj} = 0, \tag{5.98}$$

where $D^m{}_{nj} = -\partial N^m{}_n/\partial y^j$.

The identity

$$l_h N^h{}_i = -K\frac{g\widetilde{q}}{B}(y^h\nabla_i\widetilde{b}_h) - l_t a^t{}_{ih} y^h \tag{5.99}$$

coming from (5.72) is useful to take into account when considering the vanishing (5.96). The vanishing (5.97) can be obtained directly by differentiating (5.96) with respect to y^j.

Using (5.96) and (5.99), we can modify the representation (5.72) by evaluating the sum

$$N^m{}_i + l^m\frac{\partial K}{\partial x^i} = N^m{}_i - l^m N^h{}_i l_h = \frac{g\widetilde{q}}{B}y^m\widetilde{s}_i + l^m l_t a^t{}_{ih} y^h$$

$$+\frac{1}{h}\frac{1}{\widetilde{q}}Km^m\widetilde{s}_i - \frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\widetilde{\beta}_i^m - \widetilde{b}^m\widetilde{s}_i + \widetilde{b}\nabla_i\widetilde{b}^m - a^m{}_{ih}y^h.$$

We insert here (5.74), getting

$$N^m{}_i = -l^m\frac{\partial K}{\partial x^i} + \frac{g\widetilde{q}}{B}y^m\widetilde{s}_i + \frac{1}{h}\frac{1}{\widetilde{q}}Km^m\widetilde{s}_i$$

$$+\left[\widetilde{b} - \frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\right]\nabla_i\widetilde{b}^m + \frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\frac{1}{\widetilde{q}^2}\widetilde{s}_i\left(y^m - \widetilde{b}\widetilde{b}^m\right) - \widetilde{b}^m\widetilde{s}_i - h_t^m a^t{}_{ij}y^j.$$

Let us introduce the tensor

$$\widetilde{\eta}^{kn} = a^{kn} - \widetilde{b}^k\widetilde{b}^n - \frac{1}{\widetilde{q}^2}\widetilde{v}^k\widetilde{v}^n, \qquad \widetilde{v}^k = y^k - \widetilde{b}\widetilde{b}^k. \tag{5.100}$$

We come to

$$N^m{}_i = -l^m\frac{\partial K}{\partial x^i} + \frac{g\widetilde{q}}{B}y^m\widetilde{s}_i + \frac{1}{h}\frac{1}{\widetilde{q}}Km^m\widetilde{s}_i$$

$$+\left[\widetilde{b} - \frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\right]\widetilde{\eta}^{mj}\nabla_i\widetilde{b}_j + \left[\widetilde{b} - \frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\right]\frac{1}{\widetilde{q}^2}\widetilde{v}^m\widetilde{s}_i$$

$$+\frac{1}{h}\left(b+\frac{1}{2}g\widetilde{q}\right)\frac{1}{\widetilde{q}^2}\widetilde{s}_i\widetilde{v}^m-\widetilde{b}^m\widetilde{s}_i-h^m_t a^t{}_{ij}y^j.$$

In this way, with the tensor

$$\mathcal{H}^{mj} := g^{mj}-l^m l^j-m^m m^j, \tag{5.101}$$

we arrive at the representation

$$N^m{}_i=-l^m\frac{\partial K}{\partial x^i}+\left[\left(\widetilde{b}-\frac{1}{h}\left(b+\frac{g}{2}\widetilde{q}\right)\right)\mathcal{H}^{mj}\frac{K^2}{B}+\left(\frac{1}{h\widetilde{q}}-\frac{b^2+\widetilde{q}^2}{\widetilde{q}B}\right)Km^m y^j\right]\nabla_i\widetilde{b}_j-h^m_t a^t{}_{ij}y^j, \tag{5.102}$$

where $h^m_t=\delta^m_t-l^m l_t$ and m^m is the vector (5.73).

The equality $\mathcal{H}^{mj}=(B/K^2)\widetilde{\eta}^{mj}$ holds.

In the dimension $N=2$ we would have $\mathcal{H}^{mj}=0$.

Regarding regularity of the global y-dependence, it should be noted that the $\mathcal{FF}^{PD}_g$-Finsleroid metric function K given by the formulas (5.76)-(5.80) involves the scalar $\widetilde{q}=\sqrt{\widetilde{r}_{mn}y^m y^n}$ with $\widetilde{r}_{mn}=a_{mn}-\widetilde{b}_m\widetilde{b}_n$. Since the 1-form $\widetilde{b}$ is of the unit norm $||\widetilde{b}||=1$, the scalar $\widetilde{q}$ is zero when $y=b$ or $y=-b$, that is, in the directions of the north pole or the south pole of the Finsleroid. The derivatives of K may involve the fraction $1/\widetilde{q}$ which gives rise to the *pole singularities* when $\widetilde{q}=0$. This just happens in the right-hand part of the representation (5.102) for the coefficients $N^m{}_i$.

Therefore, we may apply the coefficients on but the b-slit tangent bundle $\mathcal{T}_bM := TM\setminus 0\setminus b\setminus -b$ (obtained by deleting out in $TM\setminus 0$ all the directions which point along, or oppose, the directions given rise to by the 1-form b), on which the coefficients $N^m{}_i$, as well as the function K, are smooth of the class C^∞ regarding the y-dependence.

On the punctured tangent bundle $TM\setminus 0$, the metric function K is smooth globally of the class C^2 and not of the class C^3 regarding the y-dependence.

In the case (5.70) the equation (5.43) can readily be solved, yielding

$$\rho=f, \tag{5.103}$$

where f is the function which was indicated in (5.83). We obtain

$$\sin\varrho=\frac{h\widetilde{q}}{\sqrt{B}},\qquad \cos\varrho=\frac{b+\frac{1}{2}g\widetilde{q}}{\sqrt{B}}. \tag{5.104}$$

The representation (5.40) entails

$$\mu=h^2, \tag{5.105}$$

so that from (5.26) we may conclude that $C_1=0$. The transformation (5.27) reduces to

$$t^m=\left[h(y^m-\widetilde{b}\widetilde{b}^m)+\left(b+\frac{1}{2}g\widetilde{q}\right)\widetilde{b}^m\right]\frac{K^h}{\sqrt{B}}. \tag{5.106}$$

Thus we have

Proposition 5.4. *In the $\mathcal{FF}^{PD}_g$-Finsleroid space the transformation* (5.106) *performs the conformally automorphic transformation. When $h=const$ and $c=const$, the coefficients* (2.36) *can explicitly be given by means of the representation* (5.101)-(5.102).

In the remainder of the present section, we take $c = 1$, that is, $||b||_{\text{Riemannian}} = 1$. Using (5.73), we can transform (5.106) to the expansion

$$t^m = (T_1 l^m + T_2 m^m)\frac{K^2}{B}\frac{K^{h-1}}{\sqrt{B}} \tag{5.107}$$

with respect to the frame $\{l^m, m^m\}$, where

$$T_1 = -(1-h)q^2 + B + \frac{1}{2}gq(b+gq), \qquad T_2 = \left((1-h)b + \frac{1}{2}gq\right)q. \tag{5.108}$$

The t^m of (5.106) is equivalent to the ζ^m of (6.26) of [7]: $t^m \equiv \zeta^m$. The coefficients (5.102) are equivalent to (6.62) of [7]. Therefore, with the substitution $\zeta^m = t^m$ all the relations among curvature tensors which were established in [7] are applicable to the approach developed in the present section, including the following:

$$\frac{B}{K^2}M_{nij} = \left((1-h)b + \frac{1}{2}gq\right)\frac{1}{h}b_l a_n{}^l{}_{ij} - \left(\frac{g}{2q}v_n + (1-h)b_n\right)\frac{1}{h}y^t b_l a_t{}^l{}_{ij} - a_{tnij}y^t$$

and

$$\frac{B}{K^2}M^{nij}M_{nij} = \left(\frac{1}{h}\left((1-h)b + \frac{g}{2}q\right)b_h a^{nhij} - a_h{}^{nij}y^h\right)\left(\frac{1}{h}\left((1-h)b + \frac{g}{2}q\right)b_l a_n{}^l{}_{ij} - a_{tnij}y^t\right).$$

If we take λ from (2.32) and the coefficients $N^k{}_i$ from (5.102), and use the functions $t^m = t^m(x, y)$ specified by (5.106), we obtain the vanishing $d_i\lambda(x, y_1, y_2) = 0$, when $h = const$. To verify the statement, it is worth deriving the equality

$$\frac{\partial\lambda}{\partial y_1^k} = h^2\frac{B_1 v_{2k} + q_1^2 b_k A_2 - b_1 A_2 v_{1k} - v_{12}\left(h^2 v_{1k} + \left(b_k + \frac{1}{2}g\frac{1}{q_1}v_{1k}\right)A_1\right)}{B_1\sqrt{B_1}\sqrt{B_2}}, \tag{5.109}$$

together with the counterpart

$$\frac{\partial\lambda}{\partial y_2^k} = h^2\frac{B_2 v_{1k} + q_2^2 b_k A_1 - b_2 A_1 v_{2k} - v_{12}\left(h^2 v_{2k} + \left(b_k + \frac{1}{2}g\frac{1}{q_2}v_{2k}\right)A_2\right)}{B_2\sqrt{B_2}\sqrt{B_1}}, \tag{5.110}$$

where $A_1 = A(x, y_1)$, $A_2 = A(x, y_2)$, $B_1 = B(x, y_1)$, $B_2 = B(x, y_2)$, $q_1 = q(x, y_1)$, $q_2 = q(x, y_2)$, $b_1 = b(x, y_1)$, $b_2 = b(x, y_2)$, together with $v_{1i} = r_{in}(x)y_1^n$ and $v_{2i} = r_{in}(x)y_2^n$. Plugging these derivatives in $d_i\lambda(x, y_1, y_2)$ results in the claimed vanishing $d_i\lambda(x, y_1, y_2) = 0$ after attentive couplepage reductions.

It will be noted that

$$b^k\frac{\partial\lambda}{\partial y_1^k} = h^2\frac{q_1^2 A_2 - v_{12}A_1}{B_1\sqrt{B_1}\sqrt{B_2}}, \qquad b^k\frac{\partial\lambda}{\partial y_2^k} = h^2\frac{q_2^2 A_1 - v_{12}A_2}{B_2\sqrt{B_1}\sqrt{B_2}}.$$

We have also

$$\frac{\partial\lambda}{\partial g} = -\frac{1}{2}\left(\frac{b_1 q_1}{B_1} + \frac{b_2 q_2}{B_2}\right)\lambda + \frac{q_1 A_2 + q_2 A_1 - g v_{12}}{2\sqrt{B_1}\sqrt{B_2}},$$

or

$$\frac{\partial\lambda}{\partial g}=\frac{1}{2\sqrt{B_1}\sqrt{B_2}}\left[\frac{q_1^2A_2}{B_1}\sigma_1+\frac{q_2^2A_1}{B_2}\sigma_2-v_{12}\left(\frac{A_1}{B_1}\sigma_1+\frac{A_2}{B_2}\sigma_2\right)\right],$$

where

$$\sigma_1=\frac{g}{2}A_1+h^2q_1\equiv q_1+\frac{g}{2}b_1,\qquad \sigma_2=\frac{g}{2}A_2+h^2q_2\equiv q_2+\frac{g}{2}b_2. \tag{5.111}$$

There arises the equality

$$\frac{\partial\lambda}{\partial g}=\frac{1}{2h^2}\left[\sigma_1b^k\frac{\partial\lambda}{\partial y_1^k}+\sigma_2b^k\frac{\partial\lambda}{\partial y_2^k}\right]\equiv\frac{1}{h^2}\left[z_1C_1^k\frac{\partial\lambda}{\partial y_1^k}+z_2C_2^k\frac{\partial\lambda}{\partial y_2^k}\right], \tag{5.112}$$

where

$$z_1=\frac{q_1K_1^2}{NgB_1}\sigma_1,\qquad z_2=\frac{q_2K_2^2}{NgB_2}\sigma_2.$$

6. Conclusions

In the two-dimensional approach, $N=2$, the general representation for the coefficients $N^m{}_i=N^m{}_i(x,y)$ entailing the property of preservation of two-vector angle can be indicated locally for arbitrary sufficiently smooth Finsler metric function [8,9]. Such a general possibility can doubtfully be meet in the dimensions $N\geq 3$, for in these dimensions the two-vector is of complicated nature except for rare particular cases. Such lucky cases are just proposed by the Finsler spaces which are conformally automorphic to the Riemannian spaces. The respective two-vector angle is explicit, namely is given by the simple formulas (1.7) and (2.31)-(2.32). Such Finsler spaces can be characterized by the constancy of the indicatrix curvature. In each tangent space, the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}=H^2$ is obtained and the relevant conformal multiplier is given by p^2 with $p=(1/H)F^{1-H}$. This p is constructed from the Finsler metric function F. The H is the degree of conformal automorphism. In the case $H=1$ the Finsler space under consideration reduces to the Riemannian space proper.

In indicatrix-homogeneous case, the required connection coefficients are presented by the pair $\{N^j{}_i,\,D^j{}_{ik}\}$, where $D^j{}_{ik}=-\partial N^j{}_i/\partial y^k$. The equality $N^j{}_i=-D^j{}_{ik}y^k$ holds.

In the Riemannian geometry the two-vector angle is $\alpha^{\text{Riem}}_{\{x\}}(y_1,y_2)=a_{mn}(x)y_1^my_2^n/S_1S_2$, where $S_1=\sqrt{a_{mn}(x)y_1^my_1^n}$ and $S_2=\sqrt{a_{mn}(x)y_2^my_2^n}$. Starting with the fundamental property of the metrical linear Riemannian connection that the Riemannian angle is preserving under the parallel displacements of the involved vectors, which in terms of our notation can be written as

$$d_i^{\text{Riem}}\alpha^{\text{Riem}}_{\{x\}}(y_1,y_2)=0,\qquad y_1,y_2\in T_xM,$$

with

$$d_i^{\text{Riem}}=\frac{\partial}{\partial x^i}+L^k{}_i(x,y_1)\frac{\partial}{\partial y_1^k}+L^k{}_i(x,y_2)\frac{\partial}{\partial y_2^k},$$

where $L^k{}_i(x,y_1)=-a^k{}_{ij}(x)y_1^j$, $L^k{}_i(x,y_2)=-a^k{}_{ij}(x)y_2^j$, and $a^k{}_{ij}$ are the Riemannian Christoffel symbols fulfilling the Riemannian Levi-Civita connection, the important question can be set forth: Can we have the similar vanishing in the Finsler space? It proves that the respective extension of the Riemannian equation $d_i^{\text{Riem}}\alpha^{\text{Riem}}=0$ to the equation

$d_i\alpha = 0$ applicable to the Finsler space under consideration can straightforwardly be solved giving the required coefficients $N^j{}_i$ indicated in (2.36). They admit the remarkable alternative representation $N^n{}_i = d_i^{\rm Riem} y^n$ (see (1.24)). In this way we obtain the connection $\{N^j{}_i,\, D^j{}_{ik}\}$ which is metrical and simultaneously angle-preserving. The key vanishing $y_k N^k{}_{mnj} = 0$ holds fine.

Remarkably, the Finsler connection presented by this pair $\{N^j{}_i,\, D^j{}_{ik}\}$ is the image of the metrical linear Riemannian connection under conformally-automorphic transformations. When going from the considered Finsler space to the underlined Riemannian space, the covariant derivative behaves transitively and the non-linear deformation which materializes the conformal automorphism is parallel. In particular, the Riemannian vanishing $d_m^{\rm Riem} S = 0$ just entails the Finslerian counterpart $d_m F = 0$.

Also, the involved coefficients $N^m{}_i$ fulfill the representation $N^k{}_{mnj} = -\mathcal{D}_m C^k{}_{nj}$ (see Proposition 3.2). Just the same representation is valid in the two-dimensional Finsler spaces (see (2.14) in [8,9]). Is the equation

$$\frac{\partial^2 N^k{}_m}{\partial y^n \partial y^j} = -\mathcal{D}_m C^k{}_{nj}$$

meaningful in other (in any?) Finsler spaces to find the coefficients $N^k{}_m$ required to preserve the two-vector angle? The question is addressed to readers.

The curvature tensor $\rho_k{}^n{}_{ij}$ has been explicated from commutators of arisen covariant derivatives which is attractive to develop in future the theory of curvature for the Finsler space $\mathcal{F}^N$.

For the $\mathcal{FS}$-space specialized by (1.25) we have got at our disposal the simple example of the parallel deformation transformation, namely proposing by (5.27), which entails the coefficients $N^m{}_i$ possessing the property of angle preservation. The coefficients are given explicitly by the representation (5.72)–(5.75), which admits the alternative form (5.101)-(5.102). The space proves to be of the Finsleroid type, with the Finsleroid characteristic parameter g manifesting the meaning: $h = \sqrt{1-(g^2/4)}$ is the homogeneity degree (denoted above by H) of the conformal automorphism. The Finsleroid metric function K when considered on the b-slit tangent bundle $\mathcal{T}_bM \ := TM \setminus 0 \setminus b \setminus -b$ is smooth of the class C^∞ regarding the global y-dependence. The same regularity property is valid for the coefficients $N^m{}_i$ given by (5.102).

Appendix A: Proof of Proposition 2.1

Let us verify the validity of Proposition 2.1, starting with the conformal tensor

$$u_{ij} = F^{2a} g_{ij}, \qquad a = a(x),$$

and denoting $u_{ijk} = \partial u_{ij}/\partial y^k$. We get $u_{ijk} = 2(a/F)F^{2a} l_k g_{ij} + 2F^{2a} C_{ijk}$, where $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$. Constructing the coefficients

$$Z_{ijk} \ := \frac{1}{2}(u_{kji} + u_{iki} - u_{ijk})$$

leads to

$$Z_{ijk} = \frac{a}{F} F^{2a}(l_i g_{kj} + l_j g_{ik} - l_k g_{ij}) + F^{2a} C_{ijk}.$$

Since the components u^{ij} reciprocal to u_{ij} are of the form $u^{ij} = F^{-2a} g^{ij}$, the coefficients $Z^m{}_{ij} = u^{mh} Z_{ijh}$ read merely

$$Z^m{}_{ij} = \frac{a}{F}(l_i \delta^m_j + l_j \delta^m_i - l^m g_{ij}) + C^m{}_{ij}.$$

We obtain

$$\frac{\partial Z^m{}_{ni}}{\partial y^j} = -\frac{a}{F^2} l_j (l_n \delta^m_i + l_i \delta^m_n - l^m g_{ni}) + \frac{a}{F^2}(h_{ij}\delta^m_n + h_{nj}\delta^m_i - h^m_j g_{in} - 2l^m F C_{inj}) + \frac{\partial C^m{}_{ni}}{\partial y^j}$$

and

$$\frac{\partial Z^m{}_{ni}}{\partial y^j} - \frac{\partial Z^m{}_{nj}}{\partial y^i} = -\frac{a}{F^2}[l_n(l_j \delta^m_i - l_i \delta^m_j) - l^m(l_j g_{ni} - l_i g_{nj})]$$

$$+\frac{a}{F^2}[(h_{nj}\delta^m_i - h_{ni}\delta^m_j) - (h^m_j g_{in} - h^m_i g_{jn})] + \frac{\partial C^m{}_{ni}}{\partial y^j} - \frac{\partial C^m{}_{nj}}{\partial y^i},$$

so that

$$\frac{\partial Z^m{}_{ni}}{\partial y^j} - \frac{\partial Z^m{}_{nj}}{\partial y^i} = \frac{2a}{F^2}(h_{nj} h^m_i - h_{ni} h^m_j) + \frac{\partial C^m{}_{ni}}{\partial y^j} - \frac{\partial C^m{}_{nj}}{\partial y^i}.$$

Also,

$$Z^h{}_{ni} Z^m{}_{hj} - Z^h{}_{nj} Z^m{}_{hi} = \frac{a}{F}\left[\frac{a}{F}[l_n(l_i \delta^m_j - l_j \delta^m_i) + l^m(l_j g_{ni} - l_i g_{nj})] + (l_i C^m{}_{nj} - l_j C^m{}_{ni})\right]$$

$$-\left(\frac{a}{F}\right)^2 (g_{in}\delta^m_j - g_{jn}\delta^m_i) + \frac{a}{F}(l_j C^m{}_{in} - l_i C^m{}_{jn}) + C^h{}_{ni} C^m{}_{hj} - C^h{}_{nj} C^m{}_{hi},$$

or

$$Z^h{}_{ni} Z^m{}_{hj} - Z^h{}_{nj} Z^m{}_{hi} = -\left(\frac{a}{F}\right)^2 (h_{in} h^m_j - h_{jn} h^m_i) + C^h{}_{ni} C^m{}_{hj} - C^h{}_{nj} C^m{}_{hi}.$$

The curvature tensor

$$\widetilde{R}_n{}^m{}_{ij} := \frac{\partial Z^m{}_{ni}}{\partial y^j} - \frac{\partial Z^m{}_{nj}}{\partial y^i} + Z^h{}_{ni} Z^m{}_{hj} - Z^h{}_{nj} Z^m{}_{hi}$$

is found as follows: $F^2 \widetilde{R}_n{}^m{}_{ij} = a(a+2)(h_{nj} h^m_i - h_{ni} h^m_j) + S_n{}^m{}_{ij}$, where

$$S_n{}^m{}_{ij} = \left(\frac{\partial C^m{}_{ni}}{\partial y^j} - \frac{\partial C^m{}_{nj}}{\partial y^i} + C^h{}_{ni} C^m{}_{hj} - C^h{}_{nj} C^m{}_{hi}\right) F^2.$$

In term of the covariant components $\widetilde{R}_{nmij} = u_{mh} \widetilde{R}_n{}^h{}_{ij}$ and $S_{nmij} = g_{mh} S_n{}^h{}_{ij}$, we obtain

$$F^2 \widetilde{R}_{nmij} = S_{nmij} + a(a+2)(h_{nj} h_{mi} - h_{ni} h_{mj}).$$

Therefore, if $\widetilde{R}_{nmij} = 0$, then

$$S_{nmij} = C(h_{nj}h_{mi} - h_{ni}h_{mj}), \qquad C = -a(a+2). \tag{A.1}$$

Since $\mathcal{C}_{\text{Ind.}} = 1 - C$ (see Section 5.8 in [1]), we get $\mathcal{C}_{\text{Ind.}} = H^2$, where $H = a+1$. The proposition is valid.

Appendix B: Proof of Proposition 2.2

Let us verify the validity of Proposition 2.2. From the equation

$$\frac{\partial \lambda}{\partial x^i} + N^k{}_{1i}\frac{\partial \lambda}{\partial y_1^k} + N^k{}_{2i}\frac{\partial \lambda}{\partial y_2^k} = 0$$

we want to find the tensors

$$n_{1i}^m = t_{1k}^m N^k{}_{1i}, \qquad n_{2i}^m = t_{2k}^m N^k{}_{2i}. \tag{B.1}$$

Using (2.33) and (2.34), we obtain

$$\frac{a_{mn,i}t_1^m t_2^n}{S_1S_2} - \frac{1}{2}\lambda\left[\frac{1}{S_1S_1}a_{mn,i}t_1^m t_1^n + \frac{1}{S_2S_2}a_{mn,i}t_2^m t_2^n\right]$$

$$+\left[\frac{a_{mn}t_2^n}{S_1S_2} - \frac{a_{mn}t_1^n}{S_1S_1}\lambda\right]\left(n_{1i}^m + \frac{\partial t_1^m}{\partial x^i}\right) + \left[\frac{a_{mn}t_1^n}{S_2S_1} - \frac{a_{mn}t_2^n}{S_2S_2}\lambda\right]\left(n_{2i}^m + \frac{\partial t_2^m}{\partial x^i}\right) = 0,$$

which can be written in the concise form

$$\frac{1}{S_1}\left[\frac{a_{mn}t_2^n}{S_2} - \frac{a_{mn}t_1^n}{S_1}\lambda\right]\nu_{1i}^m + \frac{1}{S_2}\left[\frac{a_{mn}t_1^n}{S_1} - \frac{a_{mn}t_2^n}{S_2}\lambda\right]\nu_{2i}^m = 0,$$

where

$$\nu_{1i}^m = n_{1i}^m + \frac{\partial t_1^m}{\partial x^i} + a^m{}_{ik}t_1^k, \qquad \nu_{2i}^m = n_{2i}^m + \frac{\partial t_2^m}{\partial x^i} + a^m{}_{ik}t_2^k,$$

and $a^m{}_{ik}$ are the Riemannian Christoffel symbols (2.37).

In this way we come to the equation

$$(S_1S_2a_{mn}t_2^n - S_2S_2a_{mn}t_1^n)\,\nu_{1i}^m + (S_1S_2a_{mn}t_1^n - S_1S_1a_{mn}t_2^n)\,\nu_{2i}^m = 0. \tag{B.2}$$

Use

$$d_iF = \frac{\partial F}{\partial x^i} + l_kN^k{}_i = \frac{\partial F}{\partial x^i} + \frac{1}{H}F^{2(1-H)}t_mn^m{}_i,$$

so that

$$t_mn^m{}_i = HF^{2(H-1)}\left(d_iF - \frac{\partial F}{\partial x^i}\right).$$

From $S^2 = F^{2H}$ it follows that

$$t_m\left(\frac{\partial t^m}{\partial x^i} + a^m{}_{ik}t^k\right) = H\frac{1}{F}F^{2H}\frac{\partial F}{\partial x^i}$$

($H = const$ is implied). We obtain

$$t_m \nu^m{}_i = HF^{2(H-1)} d_i F, \qquad \nu^m{}_i = n^m{}_i + \frac{\partial t^m}{\partial x^i} + a^m{}_{ik} t^k, \qquad n^m_i = t^m_k N^k{}_i, \tag{B.3}$$

where the equality $t_h t^h_n = HF^{2(H-1)} y_n$ (see (2.16)) has been used. When $d_i F = 0$, we have unambiguously $t_m \nu^m{}_i = 0$ and the equation (B.2) reduces to

$$a_{mn} t^n_2 \nu^m_{1i} + a_{mn} t^n_1 \nu^m_{2i} = 0. \tag{B.4}$$

Thus we may conclude that when $H = const$ and $d_i F = 0$ is fulfilled, the started equation $d_i \lambda = 0$ is equivalent to the equation (B.4).

The case $\nu^m{}_i = 0$ reads

$$n^m{}_i = -\left(\frac{\partial t^m}{\partial x^i} + a^m{}_{ik} t^k \right), \tag{B.5}$$

which is equivalent to (2.36). The examined proposition is valid.

Appendix C: Validity of Proposition 3.1

Let us consider the term

$$a_{sh} t^h_{ni} T^s_m + t_s \left(\frac{\partial t^s_{ni}}{\partial x^m} + a^s{}_{mh} t^h_{ni} \right) = \frac{\partial t_h t^h_{ni}}{\partial x^m} - t^j t^h_{ni} \frac{\partial a_{jh}}{\partial x^m} + a_{sj} t^j_{ni} a^s{}_{mh} t^h + t_s a^s{}_{mh} t^h_{ni}$$

$$= \frac{\partial t_h t^h_{ni}}{\partial x^m} - t^j t^h_{ni} \frac{\partial a_{jh}}{\partial x^m} + \frac{1}{2} t^j_{ni} \left(\frac{\partial a_{jh}}{\partial x^m} + \frac{\partial a_{jm}}{\partial x^h} - \frac{\partial a_{mh}}{\partial x^j} \right) t^h + \frac{1}{2} t^j \left(\frac{\partial a_{jh}}{\partial x^m} + \frac{\partial a_{jm}}{\partial x^h} - \frac{\partial a_{mh}}{\partial x^j} \right) t^h_{ni}$$

$$= \frac{\partial t_h t^h_{ni}}{\partial x^m}.$$

We can take $t_h t^h_{ni}$ from (2.18). By doing so and introducing the notation $P = 1 - H$, we transform the representation (3.3) to

$$y_k N^k{}_{mni} + 2C_{lni} N^l{}_m = g_{ki} (y^k_{sl} t^l_n T^s_m + y^k_s T^s_{n,m}) + g_{kn} (y^k_{sl} t^l_i T^s_m + y^k_s T^s_{i,m})$$

$$- \frac{2P}{H} F^{-2H} \left[(g_{ni} - 2H l_n l_i) t_s + (y_n a_{sl} t^l_i + y_i a_{sl} t^l_n) \right] T^s_m$$

$$- \left(2 \frac{P}{H} F^{-2H} y_n t_s + \frac{1}{H} F^{2(1-H)} a_{sh} t^h_n \right) T^s_{i,m} - \left(2 \frac{P}{H} F^{-2H} y_i t_s + \frac{1}{H} F^{2(1-H)} a_{sh} t^h_i \right) T^s_{n,m}$$

$$- P F^{2(1-H)} \frac{\partial F^{2(H-1)} (g_{ni} - 2 l_n l_i)}{\partial x^m}$$

and take the tensor g_{ki} from (2.9), obtaining

$$y_k {N^k}_{mni} + 2C_{lni}{N^l}_m = g_{ki} y^k_{sl} t^l_n T^s_m + g_{kn} y^k_{sl} t^l_i T^s_m$$

$$-\frac{2P}{H} F^{-2H} \left[(g_{ni} - 2H l_n l_i) t_s + (y_n a_{sl} t^l_i + y_i a_{sl} t^l_n) \right] T^s_m$$

$$-\left(2\frac{P}{H} F^{-2H} y_n t_s - \frac{P}{H^2} F^{2(1-H)} a_{sh} t^h_n \right) T^s_{i,m} - \left(2\frac{P}{H} F^{-2H} y_i t_s - \frac{P}{H^2} F^{2(1-H)} a_{sh} t^h_i \right) T^s_{n,m}$$

$$-PF^{2(1-H)} \frac{\partial \left(\frac{1}{H^2} a_{su} t^u_n t^s_i - 2F^{2(H-1)} l_n l_i \right)}{\partial x^m}.$$

After that, we take into account the formula (3.31) which specifies the object $T^s_{n,m}$. This yields

$$y_k {N^k}_{mni} + 2C_{lni}{N^l}_m = g_{ki} y^k_{sl} t^l_n T^s_m + g_{kn} y^k_{sl} t^l_i T^s_m$$

$$-\frac{2P}{H} F^{-2H} \left[(g_{ni} - 2H l_n l_i) t_s + (y_n a_{sl} t^l_i + y_i a_{sl} t^l_n) \right] T^s_m$$

$$-2\frac{P}{H} F^{-2H} t_s \left[y_n \left(\frac{\partial t^s_i}{\partial x^m} + {a^s}_{mh} t^h_i \right) + y_i \left(\frac{\partial t^s_n}{\partial x^m} + {a^s}_{mh} t^h_n \right) \right] + \frac{1}{H^2} F^{2(1-H)} a_{sl} {a^s}_{mh} (t^l_n t^h_i + t^l_i t^h_n)$$

$$-\frac{P}{H^2} F^{2(1-H)} t^u_n t^s_i \frac{\partial a_{su}}{\partial x^m} + 2\frac{P}{H^2} F^{2(1-H)} \frac{\partial F^{-2H} t_h t_s t^h_n t^s_i}{\partial x^m}.$$

Here, $t_h t^h_n = HF^{2(H-1)} y_n$ (see (2.16)).

Noting that

$$g_{ki} y^k_{sl} t^l_n = p^2 a_{uv} t^u_k t^v_i \frac{\partial y^k_s}{\partial y^n} = -p^2 a_{uv} t^v_i t^u_{kn} y^k_s,$$

taking C_{lni} from (2.19), and using the vanishing $H y_k y^k_s - F^{2(1-H)} t_s = 0$ (see (2.15)), we perform simplifications and remain with

$$y_k {N^k}_{mni} = 4PF^{-2H} l_n l_i t_s T^s_m - \frac{2P}{H} F^{-2H} (y_n a_{sl} t^l_i + y_i a_{sl} t^l_n) T^s_m - 2\frac{P}{H} F^{-2H} t_s {a^s}_{mh} (y_n t^h_i + y_i t^h_n)$$

$$+\frac{1}{H^2} F^{2(1-H)} a_{sl} {a^s}_{mh} (t^l_n t^h_i + t^l_i t^h_n) - \frac{P}{H^2} F^{2(1-H)} t^u_n t^s_i \frac{\partial a_{su}}{\partial x^m} + 2\frac{P}{H^2} F^{2(1-H)} t^h_n t^s_i \frac{\partial F^{-2H} t_h t_s}{\partial x^m}.$$

Finally, we apply (3.4)

$$\frac{\partial F}{\partial x^m} = \frac{F}{HS^2} t_s T^s_m,$$

take T^s_m from (3.1), and notice the vanishing

$$-\frac{2P}{H} F^{-2H} (y_n t^l_i + y_i t^l_n) \frac{\partial t_l}{\partial x^m} + 2\frac{P}{H^2} F^{2(1-H)} t^h_n t^s_i F^{-2H} \frac{\partial t_h t_s}{\partial x^m} = 0.$$

We arrive at

$$y_k N^k{}_{mni} = -\frac{2P}{H} F^{-2H} (y_n t^l_i + y_i t^l_n) \left(-t^s \frac{\partial a_{sl}}{\partial x^m} + a_{sl} a^s{}_{mh} t^h \right) - \frac{2P}{H} F^{-2H} t_s a^s{}_{ml} (y_n t^l_i + y_i t^l_n)$$

$$+ \frac{P}{H^2} F^{2(1-H)} a_{su} a^s{}_{mh} (t^u_n t^h_i + t^u_i t^h_n) - \frac{P}{H^2} F^{2(1-H)} t^u_n t^h_i \frac{\partial a_{hu}}{\partial x^m} = 0.$$

We have verified the validity of Proposition 3.1.

Appendix D: Verifying Proposition 3.2

With the convenient notation

$$X^k{}_{mn} = y^k_{sl} t^l_n a^s{}_{mh} t^h + y^k_s a^s{}_{mh} t^h_n - y^k_s a^{su} t^l_n \frac{\partial a_{ul}}{\partial x^m}$$

we can write (3.1) in the form

$$N^k{}_{mn} + X^k{}_{mn} = -y^k_{sl} t^l_n \frac{\partial t^s}{\partial x^m} - y^k_s a^{su} \frac{\partial a_{ul} t^l_n}{\partial x^m}.$$

Differentiating this equality with respect to y^j and using the notation $X^k{}_{mnj} = \partial X^k{}_{mn}/\partial y^j$, we get

$$N^k{}_{mnj} + X^k{}_{mnj} = -(y^k_{slj} t^l_n + y^k_{sl} t^l_{nj}) \frac{\partial t^s}{\partial x^m} - y^k_{sl} t^l_n \frac{\partial t^s_j}{\partial x^m} - y^k_{sl} t^l_j a^{su} \frac{\partial a_{ul} t^l_n}{\partial x^m} - y^k_s a^{su} \frac{\partial y^h_u Z^l_{nj}}{\partial x^m},$$

where $Z^l{}_{nj} = a_{vl} t^v_h t^l_{nj}$ and the identity $y^h_u t^v_h = \delta^v_u$ has been taken into account. Here, the equality $y^k_s a^{su} y^h_u = p^2 g^{kh}$ should be used.

This method results in

$$N^k{}_{mnj} + X^k{}_{mnj} = -(y^k_{slj} t^l_n + y^k_{sl} t^l_{nj}) \frac{\partial t^s}{\partial x^m} - y^k_{sl} t^l_n \frac{\partial t^s_j}{\partial x^m} - y^k_{sl} t^l_j a^{su} \frac{\partial a_{ul} t^l_n}{\partial x^m}$$

$$-y^k_s a^{su} a_{vl} t^v_h t^l_{nj} \frac{\partial y^h_u}{\partial x^m} - p^2 g^{kh} \frac{\partial Z^l_{nj}}{\partial x^m}.$$

Since

$$C_{hnj}=(1-H)\frac{1}{F}(l_jg_{hn}+l_ng_{hj}-l_hg_{nj})+p^2Z^l{}_{nj}$$

(see (2.22)), we can write

$$N^k{}_{mnj}+X^k{}_{mnj}=-\left(\frac{\partial}{\partial y^j}\frac{\partial y^k_l}{\partial y^n}\right)t^l_uy^u_s\frac{\partial t^s}{\partial x^m}-y^k_{sl}t^l_n\frac{\partial t^s_j}{\partial x^m}-y^k_{sl}t^l_ja^{su}\frac{\partial a_{ul}t^l_n}{\partial x^m}$$

$$-y^k_sa^{su}a_{vl}t^v_ht^l_{nj}\frac{\partial y^h_u}{\partial x^m}+g^{kh}a_{vl}t^v_ht^l_{nj}\frac{\partial p^2}{\partial x^m}-g^{kh}\frac{\partial C_{hnj}}{\partial x^m}$$

$$-(1-H)g^{kh}\frac{1}{F^2}(l_jg_{hn}+l_ng_{hj}-l_hg_{nj})\frac{\partial F}{\partial x^m}$$

$$+(1-H)g^{kh}\frac{1}{F}\left[\left(\frac{\partial l_j}{\partial x^m}g_{hn}+\frac{\partial l_n}{\partial x^m}g_{hj}-\frac{\partial l_h}{\partial x^m}g_{nj}\right)+\left(l_j\frac{\partial g_{hn}}{\partial x^m}+l_n\frac{\partial g_{hj}}{\partial x^m}-l_h\frac{\partial g_{nj}}{\partial x^m}\right)\right].$$

Considering the relation

$$\left(\frac{\partial}{\partial y^j}\frac{\partial y^k_l}{\partial y^n}\right)t^l_u=\frac{\partial}{\partial y^j}\left(t^l_u\frac{\partial y^k_l}{\partial y^n}\right)-\frac{\partial y^k_l}{\partial y^n}t^l_{uj}=-\frac{\partial}{\partial y^j}\left(y^k_lt^l_{un}\right)-\frac{\partial y^k_l}{\partial y^n}t^l_{uj}$$

and noting that $y^k_l=g^{kh}p^2a_{lv}t^v_h$, we obtain the useful equality

$$\left(\frac{\partial}{\partial y^j}\frac{\partial y^k_l}{\partial y^n}\right)t^l_u=-\frac{\partial}{\partial y^j}\left(g^{kh}p^2a_{lv}t^v_ht^l_{un}\right)-\frac{\partial y^k_l}{\partial y^n}t^l_{uj}.$$

Along this way we come to

$$N^k{}_{mnj}+X^k{}_{mnj}+g^{kh}\frac{\partial C_{hnj}}{\partial y^u}y^u_sa^s{}_{mh}t^h$$

$$=\left(\frac{\partial}{\partial y^j}\left(g^{kh}p^2a_{lv}t^v_ht^l_{un}\right)+\frac{\partial y^k_l}{\partial y^n}t^l_{uj}\right)y^u_s\frac{\partial t^s}{\partial x^m}-y^k_{sl}t^l_n\frac{\partial t^s_j}{\partial x^m}-y^k_{sl}t^l_ja^{su}\frac{\partial a_{ul}t^l_n}{\partial x^m}$$

$$+p^2g^{kh}a_{sl}t^l_{nj}\frac{\partial t^s_h}{\partial x^m}+g^{kh}a_{vl}t^v_ht^l_{nj}\frac{\partial p^2}{\partial x^m}$$

$$-(1-H)g^{kh}\left[\frac{1}{F^2}(l_jg_{hn}+l_ng_{hj}-l_hg_{nj})\frac{\partial F}{\partial x^m}-\frac{1}{F}\left(\frac{\partial l_j}{\partial x^m}g_{hn}+\frac{\partial l_n}{\partial x^m}g_{hj}-\frac{\partial l_h}{\partial x^m}g_{nj}\right)\right]$$

$$+(1-H)g^{kh}\frac{1}{F}\left(l_j\frac{\partial g_{hn}}{\partial x^m}+l_n\frac{\partial g_{hj}}{\partial x^m}-l_h\frac{\partial g_{nj}}{\partial x^m}\right)-g^{kh}\frac{\partial C_{hnj}}{\partial y^u}y^u_s\frac{\partial t^s}{\partial x^m}-g^{kh}d_mC_{hnj},$$

where

$$d_m C_{hnj} = \frac{\partial C_{hnj}}{\partial x^m} + N^u{}_m \frac{\partial C_{hnj}}{\partial y^u}.$$

Reducing similar terms yields

$$N^k{}_{mnj} = -g^{kh} d_m C_{hnj} + J^k{}_{mnj}, \tag{D.1}$$

where

$$J^k{}_{mnj} = -X^k{}_{mnj} - g^{kh} \frac{\partial C_{hnj}}{\partial y^u} y^u_s a^s{}_{mh} t^h - y^k_{sl} t^l_j a^{su} t^l_n \frac{\partial a_{ul}}{\partial x^m}$$

$$+ \left(t^l_{un} \frac{\partial}{\partial y^j} \left(g^{kh} p^2 a_{lv} t^v_h \right) + \frac{\partial y^k_l}{\partial y^n} t^l_{uj} \right) y^u_s \frac{\partial t^s}{\partial x^m} - y^k_{sl} t^l_n \frac{\partial t^s_j}{\partial x^m} - y^k_{sl} t^l_j \frac{\partial t^s_n}{\partial x^m}$$

$$+ p^2 g^{kh} a_{sl} t^l_{nj} \frac{\partial t^s_h}{\partial x^m} + g^{kh} a_{vl} t^v_h t^l_{nj} \frac{\partial p^2}{\partial x^m} - (1-H) g^{kh} \frac{1}{F^2} (l_j g_{hn} + l_n g_{hj} - l_h g_{nj}) \frac{\partial F}{\partial x^m}$$

$$+ (1-H) g^{kh} \frac{1}{F} \left[\left(\frac{\partial l_j}{\partial x^m} g_{hn} + \frac{\partial l_n}{\partial x^m} g_{hj} - \frac{\partial l_h}{\partial x^m} g_{nj} \right) + \left(l_j \frac{\partial g_{hn}}{\partial x^m} + l_n \frac{\partial g_{hj}}{\partial x^m} - l_h \frac{\partial g_{nj}}{\partial x^m} \right) \right]$$

$$+ (1-H) g^{kh} \frac{1}{F^2} \left[(l_j g_{hn} + l_n g_{hj} - l_h g_{nj}) l_u y^u_s \frac{\partial t^s}{\partial x^m} - (h_{ju} g_{hn} + h_{nu} g_{hj} - h_{hu} g_{nj}) y^u_s \frac{\partial t^s}{\partial x^m} \right]$$

$$- 2(1-H) g^{kh} \frac{1}{F} (l_j C_{hnu} + l_n C_{hju} - l_h C_{nju}) y^u_s \frac{\partial t^s}{\partial x^m} - g^{kh} t^l_{jn} \frac{\partial}{\partial y^u} \left(p^2 a_{lv} t^v_h \right) y^u_s \frac{\partial t^s}{\partial x^m}. \tag{D.2}$$

We also find the contraction

$$N^r{}_{mh} C_{rnj} = - \left[y^r_{sl} t^l_h \left(\frac{\partial t^s}{\partial x^m} + a^s{}_{mh} t^h \right) + y^r_s \left(\frac{\partial t^s_h}{\partial x^m} + a^s{}_{mv} t^v_h \right) \right] C_{rnj}$$

(see (3.1)), getting

$$N^r{}_{mh} C_{rnj} = -C_{rnj} y^r_{sl} t^l_h \frac{\partial t^s}{\partial x^m} - \left[(1-H) \frac{1}{F} (l_j g_{rn} + l_n g_{rj} - l_r g_{nj}) + p^2 t^v_r t^l_{nj} a_{vl} \right] y^r_s \frac{\partial t^s_h}{\partial x^m}$$

$$- \left(y^r_{sl} t^l_h a^s{}_{mh} t^h + y^r_s a^s{}_{mv} t^v_h \right) C_{rnj}. \tag{D.3}$$

The indicated relations are sufficient to obtain the representation

$$N^k{}_{mnj} = -\mathcal{D}_m C^k{}_{nj} \equiv -d_m C^k{}_{nj} + N^k{}_{mt} C^t{}_{nj} - N^t{}_{mn} C^k{}_{tj} - N^t{}_{mj} C^k{}_{nt}$$

after performing required substitutions.

Appendix E: Validity of Proposition 5.1

With (5.39), we get the expression

$$\frac{\mu}{H^2}a_{mn}t_k^m t_h^n = \left[1 - \sin^2\varrho + \frac{1}{4\breve{k}^2}(1+\breve{k}^2)^2\sin^2\varrho\right](\varrho')^2\frac{1}{b^2}w^2 e_k e_h F^{2H}$$

$$+\sin^2\varrho F^{2H}\left(a_{hk} - \frac{1}{c^2}b_k b_h - \frac{1}{\widetilde{w}^2}[w^2 e_k(w^2 e_h + \widetilde{w}^2 b_h) + \widetilde{w}^2 b_k(w^2 e_h + \widetilde{w}^2 b_h)]\right)\left(\frac{1}{b\widetilde{w}}\right)^2$$

$$+\sqrt{\mu}\frac{1}{F}l_k\left[\sqrt{\mu}\frac{1}{F}l_h + \frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_h\right]F^{2H}$$

$$+\frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_k\left[\sqrt{\mu}\frac{1}{F}l_h + \frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}e_h\right]F^{2H}$$

$$+\sin\varrho\left[\cos\varrho - \frac{1+\breve{k}^2}{4\breve{k}^2}[1-\breve{k}^2+(1+\breve{k}^2)\cos\varrho]\right]e_k\left[\sqrt{\mu}\frac{1}{F}l_h + \frac{1-\breve{k}^2}{2\breve{k}}\sin\varrho\,\varrho'\frac{w}{b}e_h\right]\varrho'\frac{w}{b}\frac{H}{\sqrt{\mu}}F^{2H}$$

$$+\sin\varrho\left[\cos\varrho - \frac{1+\breve{k}^2}{4\breve{k}^2}[1-\breve{k}^2+(1+\breve{k}^2)\cos\varrho]\right]e_h\left[\sqrt{\mu}\frac{1}{F}l_k + \frac{1-\breve{k}^2}{2\breve{k}}\sin\varrho\,\varrho'\frac{w}{b}e_k\right]\varrho'\frac{w}{b}\frac{H}{\sqrt{\mu}}F^{2H}$$

which can readily be simplified to read

$$\frac{\mu}{H^2}a_{mn}t_k^m t_h^n = \left[1 + \frac{1}{2\breve{k}^2}(1-\breve{k}^2)^2\sin^2\varrho\right](\varrho')^2\frac{1}{b^2}w^2 e_k e_h F^{2H}$$

$$+\sin^2\varrho F^{2H}\left(a_{kh} - b_k b_h - w^2(e_h+b_h)(e_k+b_k) + \left(w^2 - \frac{w^4}{\widetilde{w}^2}\right)e_k e_h\right)\left(\frac{1}{b\widetilde{w}}\right)^2$$

$$+\mu\frac{1}{F^2}l_k l_h + \sqrt{\mu}\frac{1}{F}\frac{1}{2\breve{k}}(1-\breve{k}^2)\sin\varrho\,\varrho'\frac{w}{b}(l_k e_h + l_h e_k)F^{2H}$$

$$-\frac{1-\breve{k}^2}{4\breve{k}^2}\sin\varrho\left[1+\breve{k}^2+(1-\breve{k}^2)\cos\varrho\right]\left[\frac{\sqrt{\mu}}{F}(e_k l_h + e_h l_k) + \frac{1-\breve{k}^2}{\breve{k}}\sin\varrho\,\varrho'\frac{w}{b}e_k e_h\right]\frac{w}{b}\frac{\varrho' H}{\sqrt{\mu}}F^{2H}$$

$$= (\varrho')^2 \frac{w^2}{b^2} e_k e_h F^{2H} + \mu \frac{1}{F^2} l_k l_h$$

$$+ \sin^2 \varrho F^{2H} \left(a_{kh} - b_k b_h - w^2(e_h + b_h)(e_k + b_k) + \left(w^2 - \frac{w^4}{\widetilde{w}^2} \right) e_k e_h \right) \left(\frac{1}{b\widetilde{w}} \right)^2 .$$

We use here the expansion

$$g_{kh} = l_k l_h + \frac{F^2}{b^2 \tau} \left(a_{kh} - b_k b_h - w^2(e_h + b_h)(e_k + b_k) + \frac{\tau - w(\tau' - w)}{\tau} w^2 e_h e_k \right)$$

of the involved Finslerian metric tensor and take μ from (5.40) and $\widetilde{w}^2/\tau$ from (5.41).

We obtain

$$\frac{1}{\widetilde{w}^2} \tau \sin^2 \varrho \frac{1}{H^2} a_{mn} t^m_k t^n_h = (\varrho')^2 \frac{w^2}{b^2} e_k e_h F^{2H} + \frac{1}{\widetilde{w}^2} \tau \sin^2 \varrho F^{2(H-1)} g_{kh}$$

$$+ \sin^2 \varrho F^{2H} \left(-\frac{\tau - w(\tau' - w)}{\tau} w^2 e_h e_k + \left(w^2 - \frac{w^4}{\widetilde{w}^2} \right) e_k e_h \right) \left(\frac{1}{b\widetilde{w}} \right)^2 . \qquad \text{(E.1)}$$

This representation is obviously equivalent to (5.42).

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