DIFFERENTIAL HARNACK INEQUALITIES FOR NONLINEAR HEAT EQUATIONS WITH POTENTIALS UNDER THE RICCI FLOW

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Abstract. In this paper, we prove several differential Harnack inequalities for positive solutions of nonlinear (backward) heat equations with different potentials under the Ricci flow. In particular, for a closed surface, we derive a nonlinear version of the interpolated Harnack inequality. These results include many well known differential Harnack inequalities for linear heat equations with potentials coupled with the Ricci flow.

1. Introduction

The study of differential Harnack estimates for parabolic equations originated with the work of P. Li and S.-T. Yau [27], who first proved a gradient estimate for the heat equation via the maximum principle and then derived a sharp Harnack inequality by integrating the gradient estimate along space-time paths. Later S.-T. Yau generalized this result to the Harnack inequalities for some nonlinear heat-type equations [38] and non-self-adjoint evolution equations [39]. Surprisingly, R. Hamilton employed similar techniques to obtain Harnack inequalities for the Ricci flow [21], and the mean curvature flow [23]. In dimension two, a differential Harnack estimate for the positive scalar curvature was proved by R. Hamilton [20], which was then extended by B. Chow [39] when the scalar curvature changes sign. Moreover, B. Chow applied similar techniques to obtain the Harnack inequalities for the Gauss curvature flow [11] and the Yamabe flow [12]. H.-D. Cao [3] proved a Harnack inequality for the Kähler-Ricci flow. B. Andrews [1] obtained several Harnack inequalities for general curvature flows of hypersurfaces. In [15], B. Chow and R. Hamilton give an extension of Li-Yau's Harnack inequality, which they called constrained and linear Harnack inequalities. For more detail discussions above, we refer the interested reader to Chapter 10 of [17].

On the other hand, R. Hamilton [22] also generalized Li-Yau's Harnack inequality to a matrix Harnack form on a class of Riemannian manifolds with nonnegative sectional curvature. This result was then further extended to the constrained matrix Harnack inequalities by B. Chow and R. Hamilton [15]. H.-D. Cao and L. Ni [4] proved a matrix Harnack estimate for the heat equation on Kähler manifolds. B. Chow and L. Ni [32] proved a matrix Harnack estimate for Kähler-Ricci flow by the interpolation consideration originated by B. Chow [13].

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In another direction, differential Harnack inequalities for (backward) heat-type equations coupled with the Ricci flow have become an important object, which can be traced back to the work of R. Hamilton [20]. Later this aspect was furthermore exploited by B. Chow [13], B. Chow and R. Hamilton [15], B. Chow and D. Knopf [16], and H.-B. Cheng [9], etc. Perhaps the most spectacular one is the differential Harnack inequality for the fundamental solution of conjugate heat equation coupled with the Ricci flow without any curvature assumption discovered by G. Perelman [33]. Perelman's Harnack inequality has many important applications. For example it is essential in proving the pseudolocality theorems. Recently, X. Cao [5], and S.-L. Kuang and Qi S. Zhang [26] both extended Perelman's Harnack inequality. They independently proved a differential Harnack inequality for all positive solutions of the conjugate heat equation under the Ricci flow on closed manifolds with nonnegative scalar curvature.

For differential Harnack inequalities for the linear heat equation coupled with the Ricci flow, there were also many research activities on this direction; see for example [2], [6], [8], [14], [19], [28], [36] and [40].

In the past few years there has been an increasing interest in the study of the case of nonlinear heat equations coupled with the Ricci flow. A nice example of nonlinear heat equations is the following form:

$$\frac{\partial}{\partial t}f = \Delta f - af\ln f - bf, \tag{1.1}$$

where a and b are real constants, which was introduced by L. Ma in [29]. However L. Ma only proved a local gradient estimate for positive solutions to the corresponding elliptic version

$$\Delta f - af\ln f - bf = 0 \tag{1.2}$$

on a complete manifold with a fixed metric. Indeed, F. R. K. Chung and S.-T. Yau [18] observed that the elliptic equation (1.2) is linked with the Gross Logarithmic Sobolev inequality. They also established a Logarithmic Harnack inequality for this equation when $a < 0$. In [37], Y. Yang derived local gradient estimates for positive solutions to (1.1) on a complete manifold with a fixed metric (see also [10], [25], [34], [35]). Recently, Yang's result has been developed by L. Ma in [30, 31]. In [24], S.-Y. Hsu proved a local gradient estimate for the nonlinear heat equation (1.1) under the Ricci flow. We need to emphasize that the above equations (1.1) and (1.2) often appear in geometric evolution equation, and are also closely related to the gradient Ricci solitons. See, for example, [7] and [29] for nice explanations on this subject.

Very recently, X. Cao and Z. Zhang [7] employed X. Cao and R. Hamilton's argument in [6] and proved an interesting differential Harnack inequality for positive solutions of the nonlinear heat equation

$$\frac{\partial}{\partial t}f = \Delta f - f\ln f + Rf \tag{1.3}$$

coupled with the Ricci flow equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{1.4}$$

on a closed manifold. Here the symbol Δ, R and R_{ij} are the Laplacian, scalar curvature and Ricci curvature of the metric $g(t)$ moving under the Ricci flow.

In this paper, we will be concerned with general time-dependent nonlinear (backward) heat equations of the type (1.1) with different potentials on closed Riemannian manifolds under the Ricci flow.

Firstly, suppose $(M, g(t))$, $t \in [0, T)$, is a solution to the Ricci flow (1.4) on a closed manifold. Let f be a positive solution of the nonlinear heat equation with potential R, i.e.,

$$\frac{\partial}{\partial t} f = \Delta f - af \ln f + Rf, \tag{1.5}$$

where $a \geq 0$ is a real constant. In this case, we have the following differential Harnack inequality.

Theorem 1.1. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the nonlinear heat equation (1.5), $u = -\ln f$ and*

$$H = 2\Delta u - |\nabla u|^2 - 3R - 2\frac{n}{t}.$$

Then for all time $t \in (0, T)$,

$$H \leq \frac{n}{4} a.$$

Remark 1.2. If we choose $a = 1$, this result belongs to X. Cao and Z. Zhang (see Theorem 1.1 in [7]). On the other hand, we observe that $H \leq \frac{n}{4} a$ is equivalent to

$$\frac{|\nabla f|^2}{f^2} - 2\left(\frac{f_t}{f} + a \ln f + \frac{R}{2}\right) \leq 2\frac{n}{t} + \frac{n}{4} a.$$

However, in [37] (see also [35]), the classical Li-Yau gradient estimate for positive solutions to the nonlinear heat equation (1.1) is

$$\frac{|\nabla f|^2}{f^2} - 2\left(\frac{f_t}{f} + a \ln f + b\right) \leq 2\frac{n}{t} + na$$

on manifolds with a fixed metric satisfying nonnegative Ricci curvature. Hence our Harnack inequality is similar to the classical Li-Yau gradient estimate for the nonlinear heat equation (1.1).

In particular, if we restrict ourselves on surfaces, we can derive a new differential interpolated Harnack inequality, which is originated by B. Chow [13].

Theorem 1.3. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the ε-Ricci flow ($\varepsilon \geq 0$)*

$$\frac{\partial}{\partial t} g_{ij} = -\varepsilon R g_{ij} \tag{1.6}$$

on a closed surface with $R > 0$. Let f be a positive solution to the nonlinear heat equation

$$\frac{\partial}{\partial t} f = \Delta f - af \ln f + \varepsilon Rf, \tag{1.7}$$

$u = -\ln f$ and $H_\varepsilon = \Delta u - \varepsilon R$. Then for all time $t \in (0, T)$,

$$H_\varepsilon \leq \frac{1}{t},$$

i.e.,

$$\frac{\partial}{\partial t} \ln f - |\nabla \ln f|^2 + a \ln f + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \geq 0.$$

Secondly, we consider differential Harnack inequalities for positive solutions of the nonlinear backward heat equation with potential $2R$, i.e.,

$$\frac{\partial}{\partial t}f = -\Delta f + af\ln f + 2Rf, \tag{1.8}$$

where $a \geq 0$ is a real constant, under the Ricci flow. For this system, we have

Theorem 1.4. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold. Let f be a positive solution to the nonlinear backward heat equation* (1.8)*, $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq \frac{n}{2}a.$$

If we further assume that our solution to the Ricci flow is of type I, i.e.,

$$|Rm| \leq \frac{d_0}{T-t}$$

for some constant d_0, where T is the blow-up time, then we can show that

Theorem 1.5. *Let $(M, g(t))$, $t \in [0, T]$, be a type I solution to the Ricci flow on a closed manifold. Let f be a positive solution to the nonlinear backward heat equation* (1.8)*, $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + 2R - d\frac{n}{\tau},$$

where $d = d(d_0, n) \geq 2$ is some constant such that $H(\tau) < 0$ for small τ. Then for all time $t \in [0, T)$,

$$H \leq \frac{n}{2}a.$$

Thirdly, let us now consider the nonlinear conjugate heat equation

$$\frac{\partial}{\partial t}f = -\Delta f + af\ln f + Rf, \tag{1.9}$$

where $a \geq 0$ is a real constant, under the Ricci flow. For this system, we prove that

Theorem 1.6. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold with nonnegative scalar curvature. Let f be a positive solution to the nonlinear conjugate heat equation* (1.9)*, $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + R - 2\frac{n}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq \frac{n}{4}a.$$

Alternately, by modifying the Harnack quantity of Theorem 1.6, we can prove the following form of differential Harnack inequalities.

Theorem 1.7. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold. Let f be a positive solution to the nonlinear conjugate heat equation* (1.9)*, $v = -\ln f - \frac{n}{2}\ln(4\pi\tau)$, $\tau = T - t$,*

$$P = 2\Delta v - |\nabla v|^2 + R - 2\frac{n}{\tau}$$

and

$$\tilde{P} = 2\Delta v - |\nabla v|^2 + R - 3\frac{n}{\tau}.$$

Then the following differential Harnack inequalities are true:

(1) *if the scaler curvature $R \geq 0$, then for all time $t \in [0, T)$,*

$$P \leq \frac{n}{4}a;$$

(2) *without assuming the nonnegativity of R, then for all time $t \in [\frac{T}{2}, T)$,*

$$\tilde{P} \leq \frac{n}{4}a.$$

Remark 1.8. The above mentioned theorems include the recent results of X. Cao (see Theorems 1.1, 1.2, 1.3 and 3.6 in [5]), X. Cao and R. Hamilton (see Theorem 1.1 in [6]), B. Chow (see Theorem 1 in [13]), and S.-L. Kuang and Qi S. Zhang (see Theorem 2.1 in [26]). In fact if $a = 0$, we recover their theorems.

The proof of our all theorems nearly follows the arguments of X. Cao [5], X. Cao and R. Hamilton [6], X. Cao and Z. Zhang [7], and S.-L. Kuang and Qi S. Zhang [26], where computations of evolution equations and the maximum principle for parabolic equations are employed. The only difference is that our Harnack estimates are for the nonlinear heat equations.

An interesting feature of this paper is that our differential Harnack inequalities are not only like the Perelman's Harnack inequalities, but also similar to the classical Li-Yau Harnack inequalities for the corresponding nonlinear heat equation (see Remark 1.2 above). Due to the soliton potential functions are linked with our nonlinear heat equations, we expect that our differential Harnack inequalities will be useful in understanding the Ricci solitons.

The rest of this paper is organized as follows: In Sect. 2, we first prove Theorem 1.1. Then we prove an integral version of the Harnack inequality (see Theorem 2.2). For a closed surface, we will prove a new differential interpolated Harnack inequality, i.e., Theorem 1.3. In Sect. 3, we shall prove Theorems 1.4 and 1.5. Meanwhile, a classical Harnack inequality of Theorem 1.4 will be derived (see Theorem 3.2). In Sect. 4, we will prove Theorem 1.6 as well as its classical Harnack version (Theorem 4.2). By modifying the Harnack quantity of Theorem 1.6, we will prove Theorem 1.7. In Sect. 5, we will prove gradient estimates for the nonlinear (backward) heat equation (without the potential term), i.e., Theorems 5.1 and 5.3.

2. On the nonlinear heat equation with potentials

In this section, we will prove the differential Harnack inequalities for positive solutions of nonlinear heat equations with potentials coupled with the Ricci flow (Theorems 1.1 and 1.3). An analogous result was obtained by X. Cao and Z. Zhang (see Theorem 1.1 in [7]).

Let f be a positive solution of the nonlinear heat equation (1.5). By maximum principle, we conclude that the solution for (1.5) will remain positive along the Ricci flow when scalar curvature is positive. If $u = -\ln f$, we can easily compute that

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial t}\ln f,$$

and

$$\nabla u = -\nabla \ln f, \quad \Delta u = -\Delta \ln f.$$

Hence u satisfies the following equation,

$$\frac{\partial}{\partial t}u = \Delta u - |\nabla u|^2 - R - au. \tag{2.1}$$

We shall follow X. Cao and R. Hamilton's argument in [6] (see also [5]) and shall prove Theorem 1.1. We observe that the evolution equation of u, is similar to the equation (2.1) ($c = 1$) in [6]. So we can borrow their computation for the very general setting there to simplify our calculation. The only difference is that we have extra terms coming from time derivative $\frac{\partial}{\partial t}u$.

Proof of Theorem 1.1. Let f and u be defined as above. Set

$$H = 2\Delta u - |\nabla u|^2 - 3R - 2\frac{n}{t}.$$

Comparing with the equation (2.3) in [6], using (2.1), we have

$$\begin{aligned}\frac{\partial}{\partial t}H =&\Delta H - 2\nabla H\cdot\nabla u - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 - 2\left|\nabla_i\nabla_j u - R_{ij} - \frac{1}{t}g_{ij}\right|^2\\ &- 2\left(\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R\cdot\nabla u + 2R_{ij}u_iu_j\right) - 2a(\Delta u - |\nabla u|^2)\\ \le&\Delta H - 2\nabla H\cdot\nabla u - \frac{2}{t}H - \frac{2}{t}|\nabla u|^2 - \frac{2}{n}\left(\Delta u - R - \frac{n}{t}\right)^2\\ &- 2a(\Delta u - |\nabla u|^2),\end{aligned} \tag{2.2}$$

where in the last step above we used

$$\frac{\partial}{\partial t}R + \frac{R}{t} + 2\nabla R\cdot\nabla u + 2R_{ij}u_iu_j \ge 0,$$

which is called the trace Harnack inequality for the Ricci flow proved by R. Hamilton in [21], since $g(t)$ has weakly positive curvature operator. Meanwhile, we also used the elementary inequality

$$\left|\nabla_i\nabla_j u - R_{ij} - \frac{1}{t}g_{ij}\right|^2 \ge \frac{1}{n}\left(\Delta u - R - \frac{n}{t}\right)^2.$$

By the definition of H, we also note that

$$-2a(\Delta u - |\nabla u|^2) = -2aH + 2a\left(\Delta u - R - \frac{n}{t}\right) - 4aR - \frac{2n}{t}a.$$

Plugging this into (2.2), we have

$$\begin{aligned}\frac{\partial}{\partial t}H \le&\ \Delta H - 2\nabla H\cdot\nabla u - \left(\frac{2}{t} + 2a\right)H - \frac{2}{t}|\nabla u|^2\\ &- \frac{2}{n}\left(\Delta u - R - \frac{n}{t} - \frac{n}{2}a\right)^2 + \frac{n}{2}a^2 - 4aR - \frac{2n}{t}a.\end{aligned} \tag{2.3}$$

Adding $-\frac{n}{4}a$ to H yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(H - \frac{n}{4}a\right) \le&\ \Delta\left(H - \frac{n}{4}a\right) - 2\nabla\left(H - \frac{n}{4}a\right)\cdot\nabla u - \left(\frac{2}{t} + 2a\right)\left(H - \frac{n}{4}a\right)\\ &- \frac{2}{t}|\nabla u|^2 - \frac{2}{n}\left(\Delta u + R - \frac{n}{t} - \frac{n}{4}a\right)^2 - 4aR - \frac{5n}{2t}a.\end{aligned}$$

Since $R \geq 0$ and $a \geq 0$, it is easy to see that for t small enough then $H - \frac{n}{4}a < 0$. Applying the maximum principle to the above evolution formula yields

$$H - \frac{n}{4}a \leq 0$$

for all time t. This completes the proof of Theorem 1.1. □

Remark 2.1. Theorem 1.1 is also true on complete noncompact manifolds when we can apply maximum principle. For example, we can assume that the Ricci flow solution $g(t)$ is complete with the curvature tensor and all the covariant derivatives being uniformly bounded, and Δu has a upper bound for all time t.

We now integrate the inequality

$$H - \frac{n}{4}a \leq 0$$

along a space-time path and derive a classical Harnack inequality. More precisely, since

$$\frac{\partial}{\partial t}u = \Delta u - |\nabla u|^2 - R - au,$$

combining this with

$$H - \frac{n}{4}a = 2\Delta u - |\nabla u|^2 - 3R - 2\frac{n}{t} - \frac{n}{4}a \leq 0$$

we have

$$2\frac{\partial}{\partial t}u + |\nabla u|^2 - R - 2\frac{n}{t} + 2au - \frac{n}{4}a.$$

Pick a space-time path $\gamma(x,t)$ joining (x_1,t_1) and (x_2,t_2) with $0 < t_1 < t_2$. Along the path γ, we have

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + \nabla u \cdot \gamma \\ &\leq -\frac{1}{2}|\nabla u|^2 + \frac{1}{2}R + \frac{n}{t} - au + \frac{n}{8}a + \nabla u \cdot \gamma \\ &\leq \frac{1}{2}\left(|\dot\gamma|^2 + R + \frac{n}{4}a\right) + \frac{n}{t} - au,\end{aligned}$$

where in the last step above we used the inequality

$$-\frac{1}{2}|\nabla u|^2 + \nabla u \cdot \gamma - \frac{1}{2}|\dot\gamma|^2 \leq 0.$$

Rearranging terms yields

$$\frac{d}{dt}\left(e^{at} \cdot u\right) \leq \frac{e^{at}}{2}\left(|\dot\gamma|^2 + R + \frac{n}{4}a + \frac{2n}{t}\right).$$

Integrating this inequality we obtain

$$e^{at_2} \cdot u(x_2,t_2) - e^{at_1} \cdot u(x_1,t_1) \leq \frac{1}{2}\inf_{\gamma}\int_{t_1}^{t_2} e^{at}\left(|\dot\gamma|^2 + R + \frac{n}{4}a + \frac{2n}{t}\right) dt.$$

Note that $u = -\ln f$. Hence we have the following classical Harnack inequality.

Theorem 2.2. *Let $(M, g(t))$, $t \in [0,T)$, be a solution to the Ricci flow on a closed manifold, and suppose that $g(0)$ (and so $g(t)$) has weakly positive curvature operator. Let f be a positive solution to the nonlinear heat equation* (1.5). *Assume* *that* (x_1,t_1) *and* (x_2,t_2), $0<t_1<t_2<T$, *are two points in* $M\times(0,T)$. *Then we have*

$$e^{at_1}\cdot\ln f(x_1,t_1)-e^{at_2}\cdot\ln f(x_2,t_2)\leq\frac{1}{2}\inf_{\gamma}\int_{t_1}^{t_2}e^{at}\left(|\dot{\gamma}|^2+R+\frac{n}{4}a+\frac{2n}{t}\right)dt,$$

where γ *is any space-time path joining* (x_1,t_1) *and* (x_2,t_2).

Similar to the trick of proof of Theorem 1.1, now we can finish the proof of Theorems 1.3.

Proof of Theorem 1.3. The proof follows from a direct computation and the parabolic maximum principle. We can compute that u satisfies

$$\frac{\partial}{\partial t}u=\Delta u-|\nabla u|^2-\varepsilon R-au. \tag{2.4}$$

Let

$$H_\varepsilon=\Delta u-\varepsilon R.$$

A direct computation yields

$$\begin{aligned}\frac{\partial}{\partial t}H_\varepsilon=&\Delta H_\varepsilon-2\left|\nabla_i\nabla_j u-\frac{\varepsilon}{2}Rg_{ij}\right|^2-2\nabla H_\varepsilon\cdot\nabla u-\varepsilon RH_\varepsilon\\&-R\left|\nabla u+\varepsilon\nabla\ln R\right|^2-\varepsilon R\left(\frac{\partial\ln R}{\partial t}-\varepsilon|\nabla\ln R|^2\right)-a\Delta u\\\leq&\Delta H_\varepsilon-H_\varepsilon^2-2\nabla H_\varepsilon\cdot\nabla u-(\varepsilon R+a)H_\varepsilon+\frac{\varepsilon}{t}R-\varepsilon aR.\end{aligned}\tag{2.5}$$

The reason for the last inequality above is that the trace Harnack inequality for the ε-Ricci flow on a closed surface proved by B. Chow in [13] (see also Lemma 2.1 in [36]) implies

$$\frac{\partial\ln R}{\partial t}-\varepsilon|\nabla\ln R|^2=\varepsilon(\Delta\ln R+R)\geq-\frac{1}{t},$$

since $g(t)$ has positive scalar curvature. Besides this, we also used the elementary inequality

$$\left|\nabla_i\nabla_j u-\frac{\varepsilon}{2}Rg_{ij}\right|^2\geq\frac{1}{2}(\Delta u-\varepsilon R)^2=\frac{1}{2}H_\varepsilon^2$$

and the relation

$$-a\Delta u=-aH_\varepsilon-\varepsilon aR.$$

Adding $-\frac{1}{t}$ to H_ε in (2.5) yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(H_\varepsilon-\frac{1}{t}\right)\leq&\Delta\left(H_\varepsilon-\frac{1}{t}\right)-2\nabla\left(H_\varepsilon-\frac{1}{t}\right)\cdot\nabla u-\left(H_\varepsilon+\frac{1}{t}\right)\left(H_\varepsilon-\frac{1}{t}\right)\\&-(\varepsilon R+a)\left(H_\varepsilon-\frac{1}{t}\right)-\frac{a}{t}-\varepsilon aR.\end{aligned}\tag{2.6}$$

It is easy to see that for t small enough then $H_\varepsilon-\frac{1}{t}<0$. Since $R>0$ and $a\geq0$, applying the maximum principle to the evolution formula (2.6) we conclude

$$H_\varepsilon-\frac{1}{t}\leq0$$

for all time t, and the proof of the theorem is completed. □

In Theorem 1.3, let us take $a = 0$ and then we get the following interpolation proved by B. Chow [13] between the classical Li-Yau Harnack inequality for the heat equation and the Chow-Hamilton linear trace Harnack inequality for the Ricci flow on a closed surface:

Corollary 2.3 (B. Chow [13])**.** *Let $(M, g(t))$ be a solution to the ε-Ricci flow* (1.6) *on a closed surface with $R > 0$. If f is a positive solution to*

$$\frac{\partial}{\partial t}f = \Delta f + \varepsilon Rf,$$

then

$$\frac{\partial}{\partial t}\ln f - |\nabla \ln f|^2 + \frac{1}{t} = \Delta \ln f + \varepsilon R + \frac{1}{t} \geq 0. \tag{2.7}$$

On the other hand, if we take $\varepsilon = 0$ in Theorem 1.3, we can get the following differential Harnack inequality for a nonlinear heat equation on closed surfaces with a fixed metric:

Corollary 2.4. *If $f : M\times \in [0, T) \to \mathbb{R}$, is a positive solution to the nonlinear heat equation*

$$\frac{\partial}{\partial t}f = \Delta f - af \ln f, \tag{2.8}$$

where $a \geq 0$ is a real constant, on a closed surface (M, g) with $R > 0$, then for all time $t \in (0, T)$,

$$\frac{\partial}{\partial t}\ln f - |\nabla \ln f|^2 + a \ln f + \frac{1}{t} = \Delta \ln f + \frac{1}{t} \geq 0.$$

If we take $\varepsilon = 1$ in Theorem 1.3, then

Corollary 2.5. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed surface with $R > 0$. If f is a positive solution to the nonlinear heat equation* (1.5)*, then for all time $t \in (0, T)$,*

$$\frac{\partial}{\partial t}\ln f - |\nabla \ln f|^2 + a \ln f + \frac{1}{t} = \Delta \ln f + R + \frac{1}{t} \geq 0.$$

Remark 2.6. Interestingly, Theorem 1.3 is a nonlinear interpolated Harnack inequality which links Corollary 2.4 to Corollary 2.5.

3. On the nonlinear backward heat equation with potentials

In this section, we start to study several differential Harnack inequalities for positive solutions of nonlinear backward heat equations with potentials:

$$\frac{\partial}{\partial t}f = -\Delta f + af \ln f + 2Rf,$$

where $a \geq 0$ is a real constant, under the Ricci flow.

To prove Theorems 1.4 and 1.5, we shall apply a similar trick as in the proof of Theorem 1.1. We will see that evolution equations of u, is very similar to what is considered in [5]. Just as the case of the proof of Theorem 1.1, the only difference is we have more terms coming from time derivative $\frac{\partial}{\partial t}u$.

Proof of Theorems 1.4 and 1.5. As before, it is easy to compute that u satisfies the following equation,

$$\frac{\partial}{\partial \tau}u = \Delta u - |\nabla u|^2 + 2R - au. \tag{3.1}$$

Let

$$H = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau}. \tag{3.2}$$

Comparing with the equation (2.4) in [5], we have

$$\begin{aligned}\frac{\partial}{\partial \tau}H =&\Delta H - 2\nabla H \cdot \nabla u - \frac{2}{\tau}H - \frac{2}{\tau}|\nabla u|^2 - 2|Rc|^2\\ &- 2\left|\nabla_i\nabla_j u + R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 - 2a(\Delta u - |\nabla u|^2)\\ \leq&\Delta H - 2\nabla H \cdot \nabla u - \frac{2}{\tau}H - \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}R^2\\ &- \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau}\right)^2 - 2a(\Delta u - |\nabla u|^2).\end{aligned} \tag{3.3}$$

From (3.2), we know that

$$-2a(\Delta u - |\nabla u|^2) = -2aH + 2a\left(\Delta u + R - \frac{n}{\tau}\right) + 2aR - \frac{2n}{\tau}a.$$

Plugging this into (3.3) yields

$$\begin{aligned}\frac{\partial}{\partial \tau}H \leq& \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2a\right)H - \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}R^2\\ &- \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 + \frac{n}{2}a^2 + 2aR - \frac{2n}{\tau}a\\ =& \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{2}{\tau} + 2a\right)H - \frac{2}{\tau}|\nabla u|^2\\ &- \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 - \frac{2}{n}\left(R - \frac{n}{2}a\right)^2 - \frac{2n}{\tau}a + na^2.\end{aligned}$$

Therefore adding $-\frac{n}{2}a$ to H, we have

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(H - \frac{n}{2}a\right) \leq& \Delta\left(H - \frac{n}{2}a\right) - 2\nabla\left(H - \frac{n}{2}a\right)\cdot\nabla u - \left(\frac{2}{\tau} + 2a\right)\left(H - \frac{n}{2}a\right)\\ &- \frac{2}{\tau}|\nabla u|^2 - \frac{2}{n}\left(\Delta u + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 - \frac{2}{n}\left(R - \frac{n}{2}a\right)^2 - \frac{3n}{\tau}a.\end{aligned} \tag{3.4}$$

If τ small enough, we can easily see that $H - \frac{n}{2}a < 0$. Then applying the maximum principle to the evolution equation (3.4) yields

$$H - \frac{n}{2}a \leq 0$$

for all time τ, hence for all $t \in [0, T)$. This finishes the proof of Theorem 1.4.

As for the proof of Theorem 1.5, from the above proof, we can easily see that Theorem 1.5 is true for all time $t < T$. □

Remark 3.1. Theorem 1.4 is also true on complete non-compact Riemannian manifolds as long as we can apply maximum principle.

From Theorem 1.4, we can derive a classical Harnack inequality by integrating along a space-time path.

Theorem 3.2. *Let $(M, g(t))$, $t \in [0,T]$, be a solution to the Ricci flow on a closed manifold. Let f be a positive solution to the nonlinear backward heat equation* (1.8)*. Assume that (x_1,t_1) and (x_2,t_2), $0 \le t_1 < t_2 < T$, are two points in $M \times [0,T)$. Then we have*

$$e^{at_2}\cdot \ln f(x_2,t_2) - e^{at_1}\cdot \ln f(x_1,t_1) \le \frac{1}{2}\inf_{\gamma}\int_{t_1}^{t_2} e^{a(T-t)}\left(|\dot\gamma|^2 + 2R + \frac{n}{2}a + \frac{2n}{T-t}\right)dt,$$

where γ is any space-time path joining (x_1,t_1) and (x_2,t_2).

Proof. Our strategy in this proof will be similar to that in the proof of Theorem 2.2. We include it here because this is a Harnack inequality for the nonlinear backward heat equation. Let us consider the solutions of

$$\frac{\partial}{\partial \tau}u = \Delta u - |\nabla u|^2 + 2R - au.$$

Combining this with

$$H - \frac{n}{2}a = 2\Delta u - |\nabla u|^2 + 2R - 2\frac{n}{\tau} - \frac{n}{2}a \le 0,$$

we have

$$2\frac{\partial}{\partial \tau}u + |\nabla u|^2 - 2R - 2\frac{n}{\tau} + 2au - \frac{n}{2}a.$$

Pick a space-time path $\gamma(x,t)$ joining (x_2,τ_2) and (x_1,τ_1) with $\tau_1 > \tau_2 > 0$. Along the space-time path γ, we have

$$\begin{aligned}\frac{du}{d\tau} &= \frac{\partial u}{\partial \tau} + \nabla u\cdot\gamma\\ &\le -\frac{1}{2}|\nabla u|^2 + R + \frac{n}{\tau} - au + \frac{n}{4}a + \nabla u\cdot \gamma\\ &\le \frac{1}{2}\left(|\dot\gamma|^2 + 2R + \frac{n}{2}a\right) + \frac{n}{\tau} - au.\end{aligned}$$

Rearranging terms yields

$$\frac{d}{d\tau}\left(e^{a\tau}\cdot u\right) \le \frac{e^{a\tau}}{2}\left(|\dot\gamma|^2 + 2R + \frac{n}{2}a + \frac{2n}{\tau}\right).$$

Integrating this inequality we get

$$e^{a\tau_1}\cdot u(x_1,\tau_1) - e^{a\tau_2}\cdot u(x_2,\tau_2) \le \frac{1}{2}\inf_{\gamma}\int_{\tau_2}^{\tau_1} e^{a\tau}\left(|\dot\gamma|^2 + 2R + \frac{n}{2}a + \frac{2n}{\tau}\right)d\tau.$$

Namely,

$$e^{at_1}\cdot u(x_1,t_1) - e^{at_2}\cdot u(x_2,t_2) \le \frac{1}{2}\inf_{\gamma}\int_{t_1}^{t_2} e^{a(T-t)}\left(|\dot\gamma|^2 + 2R + \frac{n}{2}a + \frac{2n}{T-t}\right)dt.$$

Therefore the desired classical Harnack inequality follows. ∎

4. On the nonlinear conjugate heat equation with potentials

In this section, we will study several differential Harnack inequalities for positive solutions of the nonlinear conjugate heat equation

$$\frac{\partial}{\partial t}f=-\Delta f+af\ln f+Rf,$$

where $a\geq 0$ is a real constant, under the Ricci flow. In this case, in general we need to assume that the initial metric $g(0)$ has nonnegative scalar curvature, it is well known that this property is preserved by the Ricci flow.

Proof of Theorem 1.6. We go on applying a similar trick in last sections to prove Theorem 1.6. It is easy to compute that u satisfies the following equation,

$$\frac{\partial}{\partial \tau}u=\Delta u-|\nabla u|^2+R-au. \tag{4.1}$$

Comparing with the equation (3.2) in [5], we have

$$\begin{aligned}\frac{\partial}{\partial \tau}H=&\Delta H-2\nabla H\cdot\nabla u-\frac{2}{\tau}H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}\\&-2\left|\nabla_i\nabla_j u+R_{ij}-\frac{1}{\tau}g_{ij}\right|^2-2a(\Delta u-|\nabla u|^2).\end{aligned}\tag{4.2}$$

Note that

$$H=2\Delta u-|\nabla u|^2+R-2\frac{n}{\tau},$$

which implies

$$-2a(\Delta u-|\nabla u|^2)=-2aH+2a\left(\Delta u+R-\frac{n}{\tau}\right)-\frac{2n}{\tau}a.$$

Plugging this into (4.2), we obtain

$$\begin{aligned}\frac{\partial}{\partial \tau}H\leq&\ \Delta H-2\nabla H\cdot\nabla u-\left(\frac{2}{\tau}+2a\right)H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}\\&-\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}\right)^2+2a\left(\Delta u+R-\frac{n}{\tau}\right)-\frac{2n}{\tau}a\\=&\ \Delta H-2\nabla H\cdot\nabla u-\left(\frac{2}{\tau}+2a\right)H-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}\\&-\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}-\frac{n}{2}a\right)^2-\frac{2n}{\tau}a+\frac{n}{2}a^2.\end{aligned}$$

Therefore adding $-\frac{n}{4}a$ to H yields

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(H-\frac{n}{4}a\right)\leq&\ \Delta\left(H-\frac{n}{4}a\right)-2\nabla\left(H-\frac{n}{4}a\right)\cdot\nabla u-\left(\frac{2}{\tau}+2a\right)\left(H-\frac{n}{4}a\right)\\&-\frac{2}{\tau}|\nabla u|^2-2\frac{R}{\tau}-\frac{2}{n}\left(\Delta u+R-\frac{n}{\tau}-\frac{n}{2}a\right)^2-\frac{5n}{2\tau}a.\end{aligned}\tag{4.3}$$

Since $R\geq 0$ and $a\geq 0$, it is easy to see that for τ small enough then $H-\frac{n}{4}a<0$. Applying the maximum principle to the evolution formula (4.3), we have

$$H-\frac{n}{4}a\leq 0$$

for all time τ, hence for all t. This finishes the proof of Theorem 1.6. □

As in Section 3, we can see that if the solution to the Ricci flow is of type I, i.e.,

$$|Rm| \leq \frac{d_0}{T-t}$$

for some constant d_0, where T is the blow-up time, then the following Harnack estimate holds for all time $t < T$.

Theorem 4.1. *Let $(M, g(t))$, $t \in [0, T]$, be a type I solution to the Ricci flow on a closed manifold with nonnegative scalar curvature. Let f be a positive solution to the nonlinear backward heat equation (1.9), $u = -\ln f$, $\tau = T - t$ and*

$$H = 2\Delta u - |\nabla u|^2 + R - d\frac{n}{\tau},$$

here $d = d(d_0, n) \geq 1$ is some constant such that $H(\tau) < 0$ for small τ. Then for all time $t \in [0, T)$,

$$H \leq \frac{n}{4}a.$$

In the same way, from Theorem 1.6 we can derive a classical Harnack inequality by integrating along a space-time path.

Theorem 4.2. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold with nonnegative scalar curvature. Let f be a positive solution to the nonlinear conjugate heat equation (1.9). Assume that (x_1, t_1) and (x_2, t_2), $0 \leq t_1 < t_2 < T$, are two points in $M \times [0, T)$. Then we have*

$$e^{at_2} \cdot \ln f(x_2, t_2) - e^{at_1} \cdot \ln f(x_1, t_1) \leq \frac{1}{2} \inf_{\gamma} \int_{t_1}^{t_2} e^{a(T-t)} \left(|\dot{\gamma}|^2 + R + \frac{n}{4}a + \frac{2n}{T-t} \right) dt,$$

where γ is any space-time path joining (x_1, t_1) and (x_2, t_2).

In the rest of this section, we will finish the proof of Theorem 1.7.

Proof of Theorem 1.7. We first prove Theorem 1.7 in the case $R \geq 0$. Compute that u satisfies the following equation,

$$\frac{\partial}{\partial \tau} v = \Delta v - |\nabla v|^2 + R - \frac{n}{2\tau} - a\left(v + \frac{n}{2}\ln(4\pi\tau)\right). \tag{4.4}$$

Comparing with the equation (3.7) in [5], we have

$$\begin{aligned} \frac{\partial}{\partial \tau} P =& \Delta P - 2\nabla P \cdot \nabla v - \frac{2}{\tau}P - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau} \\ & - 2\left|\nabla_i \nabla_j v + R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 - 2a(\Delta v - |\nabla v|^2). \end{aligned} \tag{4.5}$$

By the definition of P, we have

$$-2a(\Delta v - |\nabla v|^2) = -2aP + 2a\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{2n}{\tau}a.$$

Plugging this into (4.5) yields

$$\begin{aligned}\frac{\partial}{\partial\tau}P \leq& \Delta P - 2\nabla P\cdot\nabla v - \left(\frac{2}{\tau}+2a\right)P - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau}\\ &- \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau}\right)^2 + 2a\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{2n}{\tau}a\\ =& \Delta P - 2\nabla P\cdot\nabla v - \left(\frac{2}{\tau}+2a\right)P - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau}\\ &- \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 - \frac{2n}{\tau}a + \frac{n}{2}a^2.\end{aligned}$$

Therefore adding $-\frac{n}{4}a$ to P gives

(4.6)

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(P-\frac{n}{4}a\right) \leq& \Delta\left(P-\frac{n}{4}a\right) - 2\nabla\left(P-\frac{n}{4}a\right)\cdot\nabla v - \left(\frac{2}{\tau}+2a\right)\left(P-\frac{n}{4}a\right)\\ &- \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau} - \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 - \frac{5n}{2\tau}a.\end{aligned}$$

Since $R\geq 0$ and $a\geq 0$, it is easy to see that for τ small enough then $P-\frac{n}{4}a<0$. Applying the maximum principle to the evolution formula (4.6) yields

$$P-\frac{n}{4}a\leq 0$$

for all time τ, hence for all t. Hence the theorem in the case $R\geq 0$ is proved.

Next we prove the Harnack inequality without the nonnegativity assumption for the scalar curvature R. Similarly, we have

$$\begin{aligned}\frac{\partial}{\partial\tau}\tilde{P} =& \Delta\tilde{P} - 2\nabla\tilde{P}\cdot\nabla v - \frac{2}{\tau}\tilde{P} - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau} - \frac{n}{\tau^2}\\ &- 2\left|\nabla_i\nabla_j v + R_{ij} - \frac{1}{\tau}g_{ij}\right|^2 - 2a(\Delta v - |\nabla v|^2).\end{aligned}\tag{4.7}$$

According to the definition of $\tilde{P}$, we have

$$-2a(\Delta v - |\nabla v|^2) = -2a\tilde{P} + 2a\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{4n}{\tau}a.$$

Substituting this into (4.7), we get

$$\begin{aligned}\frac{\partial}{\partial\tau}\tilde{P} \leq& \Delta\tilde{P} - 2\nabla\tilde{P}\cdot\nabla v - \left(\frac{2}{\tau}+2a\right)\tilde{P} - \frac{2}{\tau}|\nabla v|^2 - 2\frac{R}{\tau} - \frac{n}{\tau^2}\\ &- \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau}\right)^2 + 2a\left(\Delta v + R - \frac{n}{\tau}\right) - \frac{4n}{\tau}a\\ =& \Delta\tilde{P} - 2\nabla\tilde{P}\cdot\nabla v - \left(\frac{2}{\tau}+2a\right)\tilde{P} - \frac{2}{\tau}|\nabla v|^2 - \frac{2}{\tau}\left(R + \frac{n}{2\tau}\right)\\ &- \frac{2}{n}\left(\Delta v + R - \frac{n}{\tau} - \frac{n}{2}a\right)^2 - \frac{4n}{\tau}a + \frac{n}{2}a^2.\end{aligned}\tag{4.8}$$

Note that the evolution of scalar curvature under the Ricci flow is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2 \geq \Delta R + \frac{2}{n}R^2.$$

Applying the maximum principle to this inequality yields

$$R \geq -\frac{n}{2t}.$$

Since $t \geq T/2$, then $1/t \leq 1/\tau$. Hence

$$R \geq -\frac{n}{2t} \geq -\frac{n}{2\tau},$$

i.e.,

$$R + \frac{n}{2\tau} \geq 0.$$

Combining this with (4.8), we have

$$\frac{\partial}{\partial \tau}\tilde{P} \leq \Delta \tilde{P} - 2\nabla \tilde{P} \cdot \nabla v - \left(\frac{2}{\tau} + 2a\right)\tilde{P} - \frac{4n}{\tau}a + \frac{n}{2}a^2.$$

Adding $-\frac{n}{4}a$ to $\tilde{P}$, we get

(4.9)

$$\frac{\partial}{\partial \tau}\left(\tilde{P} - \frac{n}{4}a\right) \leq \Delta\left(\tilde{P} - \frac{n}{4}a\right) - 2\nabla\left(\tilde{P} - \frac{n}{4}a\right) \cdot \nabla v - \left(\frac{2}{\tau} + 2a\right)\left(\tilde{P} - \frac{n}{4}a\right) - \frac{9n}{2\tau}a.$$

Since $a \geq 0$, it is easy to see that for τ small enough then $\tilde{P} - \frac{n}{4}a < 0$. Applying the maximum principle to the evolution formula (4.9) yields

$$\tilde{P} - \frac{n}{4}a \leq 0$$

for all time $t \geq T/2$. Hence the theorem is completely proved. □

Remark 4.3. Note that Theorem 1.7 in the case $R \geq 0$ is equivalent to Theorem 1.6. Similar to Theorems 4.1 and 4.2, we can prove similar theorems by the standard argument from Theorem 1.7.

5. Gradient estimates for nonlinear (backward) heat equations

In this section we first consider the positive solution $f(x,t) < 1$ to the nonlinear heat equation without any potentials

$$\frac{\partial}{\partial t}f = \Delta f - af\ln f, \tag{5.1}$$

where $a \geq 0$ is a real constant, with the metric evolved by the Ricci flow (1.4). If $u = -\ln f$, then

$$\frac{\partial}{\partial t}u = \Delta u - |\nabla u|^2 - au$$

and $u > 0$.

Following the arguments of X. Cao and R. Hamilton's paper [6], we let

$$H = |\nabla u|^2 - \frac{u}{t}.$$

Comparing with the equation (5.3) in [6], we have

$$\begin{aligned}\frac{\partial}{\partial t}H &= \Delta H - 2\nabla H \cdot \nabla u - \frac{1}{t}H - 2|\nabla\nabla u|^2 - 2a|\nabla u|^2 + \frac{a}{t}u \\ &= \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{t} + a\right)H - 2|\nabla\nabla u|^2 - a|\nabla u|^2.\end{aligned} \tag{5.2}$$

Notice that if t small enough, then $H < 0$. Then applying the maximum principle to (5.2), we have

Theorem 5.1. *Let $(M, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on a closed manifold. Let $f < 1$ be a positive solution to the nonlinear heat equation* (5.1), *$u = -\ln f$ and*

$$H = |\nabla u|^2 - \frac{u}{t}.$$

Then for all time $t \in (0, T)$,

$$H \leq 0.$$

Remark 5.2. Theorem 5.1 implies X. Cao and R. Hamilton's result (see Theorem 5.1 in [6]). If $a = 0$, then we recover their result. In this case, we do not need any curvature assumption.

On the other hand, we can also consider the positive solution $f(x,t) < 1$ to the nonlinear backward heat equation without any potentials

$$\frac{\partial}{\partial t} f = -\Delta f + af \ln f, \tag{5.3}$$

where $a \geq 0$ is a real constant, with the metric evolved by the Ricci flow (1.4). Let $u = -\ln f$. Then we have

$$\frac{\partial}{\partial \tau} u = \Delta u - |\nabla u|^2 - au$$

and $u > 0$.

Following the arguments of X. Cao's paper [5], let

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Comparing with the equation (5.3) in [5], we have

$$\begin{aligned} \frac{\partial}{\partial \tau} H &= \Delta H - 2\nabla H \cdot \nabla u - \frac{1}{\tau} H - 2|\nabla\nabla u|^2 - 4R_{ij}u_i u_j - 2a|\nabla u|^2 + \frac{a}{\tau} u \\ &= \Delta H - 2\nabla H \cdot \nabla u - \left(\frac{1}{\tau} + a\right) H - 2|\nabla\nabla u|^2 - 4R_{ij}u_i u_j - a|\nabla u|^2. \end{aligned} \tag{5.4}$$

If we assume $R_{ij}(g(t)) \geq -K$, where $0 \leq K \leq \frac{a}{4}$, then

$$-4R_{ij}u_i u_j - a|\nabla u|^2 \leq (4K - a)|\nabla u|^2 \leq 0.$$

Hence if τ small enough, then $H < 0$. Then applying the maximum principle to (5.4), we have

Theorem 5.3. *Let $(M, g(t))$, $t \in [0, T]$, be a solution to the Ricci flow on a closed manifold with the Ricci curvature satisfying $R_{ij}(g(t)) \geq -K$, where $0 \leq K \leq \frac{a}{4}$. Let $f < 1$ be a positive solution to the nonlinear backward heat equation* (5.3), *$u = -\ln f$, $\tau = T - t$ and*

$$H = |\nabla u|^2 - \frac{u}{\tau}.$$

Then for all time $t \in [0, T)$,

$$H \leq 0.$$

Remark 5.4. Theorem 5.3 generalizes X. Cao's result (see Theorem 5.1 in [5]). If we choose $a = 0$, our theorem reduces to his.

Remark 5.5. In Theorems 5.1 and 5.3, we give the assumption $f < 1$ to guarantee the maximum principle can be used. However this assumption, not like the case of linear equation, does not seem natural for this nonlinear equation.

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