arXiv:1009.1258v1 [math.AP] 7 Sep 2010

Vanishing viscosity limit for viscous magnetohydrodynamic equations with a slip boundary condition

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Abstract

We consider the evolutionary MHD systems, and study the the regularity and vanishing viscosity limit of the 3-D viscous system in a class of bounded domains with a slip boundary condition. We derive the convergence is in H^{2k+1}, for $k \geq 1$, if the initial date holds some sufficient conditions.

Key words: Magnetohydrodynamic system; slip boundary condition; vanishing viscosity limit.

1 Introduction and results

Let Ω be an open bounded domain in R^3. We consider the initial and boundary value problem for the system of viscous MHD equations

$$\begin{cases} \partial_t \mathbf{u} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{H} \cdot \nabla)\mathbf{H} + \nabla \mathbf{p} = \mathbf{0} \text{ in } \mathbf{\Omega}, \\ \nabla \cdot \mathbf{u} = \mathbf{0} \text{ in } \mathbf{\Omega}, \\ \partial_t \mathbf{H} - \mu \triangle \mathbf{H} + (\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u} = \mathbf{0} \text{ in } \mathbf{\Omega}, \\ \nabla \cdot \mathbf{H} = \mathbf{0} \text{ in } \mathbf{\Omega}, \\ \mathbf{u} = \mathbf{u_0}, \mathbf{H} = \mathbf{H_0}, \text{ at } \mathbf{t} = \mathbf{0}, \end{cases} \tag{1}$$

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with the following slip without friction boundary conditions

$$\mathbf{u}\cdot\mathbf{n}=\mathbf{0},\nabla\times\mathbf{u}\cdot\tau=\mathbf{0},\mathbf{H}\cdot\mathbf{n}=\mathbf{0},\nabla\times\mathbf{H}\cdot\tau=\mathbf{0}\text{ on }\partial\boldsymbol{\Omega} \tag{2}$$

where $\nabla\cdot$ and $\nabla\times$ denote the div and curl operators, $\mathbf{n}$ the outward normal vector and τ any unit tangential vector of $\partial\Omega$.

The corresponding ideal MHD system is usually equipped with the slip boundary condition, namely

$$\begin{cases}\partial_t\mathbf{u^0}+(\mathbf{u^0}\cdot\nabla)\mathbf{u^0}-(\mathbf{H^0}\cdot\nabla)\mathbf{H^0}+\nabla\mathbf{p^0}=\mathbf{0}\text{ in }\boldsymbol{\Omega},\\ \nabla\cdot\mathbf{u^0}=\mathbf{0}\text{ in }\boldsymbol{\Omega},\\ \partial_t\mathbf{H^0}+(\mathbf{u^0}\cdot\nabla)\mathbf{H^0}-(\mathbf{H^0}\cdot\nabla)\mathbf{u^0}=\mathbf{0}\text{ in }\boldsymbol{\Omega},\\ \nabla\cdot\mathbf{u^0}=\mathbf{0}\text{ in }\boldsymbol{\Omega},\\ \mathbf{u^0}=\mathbf{u_0},\mathbf{H^0}=\mathbf{H_0},\text{ at }\mathbf{t}=\mathbf{0},\end{cases} \tag{3}$$

$$\mathbf{u^0}\cdot\mathbf{n}=\mathbf{0},\mathbf{H^0}\cdot\mathbf{n}=\mathbf{0}\text{ on }\boldsymbol{\Omega}. \tag{4}$$

Our aim is to investigate strong convergence, up to the boundary, of the solution $(\mathbf{u},\mathbf{H})$ of the MHD (1) to the solution $(\mathbf{u^0},\mathbf{H^0})$ of the ideal MHD system (3), as $(\nu,\mu)\to 0$.

The boundary conditions (2) are a special Navier-type slip boundary conditions, which allow the fluid to slip at a slip velocity proportional to the shear stress introduced by Navier [1]. This type of boundary conditions has been used in many fluid problems (see e.g. [2], [3], [4], [5]).

The viscous MHD system in the whole space or with non-slip boundary conditions has been studied extensively (see e.g. [6], [7], [8], [9], [10]). The solvability, regularity of the 3-D viscous MHD system with a slip boundary condition, we refer to [11].

The issue of vanishing viscosity limits of the Navier-Stokes equations is classical and fundamental importance in fluid dynamics and turbulence theory (see e.g. [12] [13], [14], [15], [16], [17], [18]).

In flat boundary case, the 3-D inviscid limit for solution $(\mathbf{u},\mathbf{H})$ to the slip boundary problem (1) and (2) has been considered in [11]. In [11], they state the following result. Assume $\nabla\cdot\mathbf{u_0}=\mathbf{0}$, $\nabla\cdot\mathbf{H_0}=\mathbf{0}$, and $(\mathbf{u_0},\mathbf{H_0})\in\mathbf{H^3}$ satisfy the boundary conditions (2). Then, as $(\nu,\mu)\to 0$,

$$(\mathbf{u},\mathbf{H})\to(\mathbf{u^0},\mathbf{H^0})\text{ in }\mathbf{L^p}(\mathbf{0},\mathbf{T};\mathbf{H^3}(\boldsymbol{\Omega}))\cap\mathbf{C}([\mathbf{0},\mathbf{T}];\mathbf{H^2}(\boldsymbol{\Omega})), \tag{5}$$

for some $T>0$ and any $p\in[1,+\infty)$, where $(\mathbf{u^0},\mathbf{H^0})$ is the solution to the ideal MHD equations (3) and (4).

It should be noted that the approach encounters great difficulties for general domains as pointed out by [16]. Thus, following [16], we restrict the problem to a cubic domain $Q = [0,1]^2_{per} \times (0,1)$ with the boundary conditions on two opposite faces $z = 0$ and $z = 1$, and others be assumed periodic, which was called flat boundary case.

Our approach here is motivated by the idea introduced in [19] to study the same problems for the Navier-Stokes equations. We prove the following result.

Theorem 1.1 *Let the initial data* $\mathbf{u_0} \in \mathbf{V^{2k-1}} \cap \mathbf{H^{2k+1}}$, $\mathbf{H_0} \in \mathbf{V^{2k-1}} \cap \mathbf{H^{2k+1}}$, $k \geq 1$. *Then there exist strong solution of the MHD equation (1) and (2) in the "cubic domain" (flat boundary case) on some time interval* $[0,T]$, *s.t.*

$$\begin{aligned} &\|\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}} + \|\mathbf{H}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}} \leq \mathbf{C}, \\ &\|\partial_t \mathbf{u}\|_{\mathbf{L^2(0,T;H^{2k})}} + \|\partial_\mathbf{t} \mathbf{H}\|_{\mathbf{L^2(0,T;H^{2k})}} \leq \mathbf{C}. \end{aligned} \tag{6}$$

And

$$(\mathbf{u}, \mathbf{H}) \to (\mathbf{u^0}, \mathbf{H^0}) \text{ in } \mathbf{C([0,T]; H^{2k})}, \text{ as } (\nu, \mu) \to \mathbf{0}, \tag{7}$$

where $(\mathbf{u^0}, \mathbf{H^0})$ *is the unique solution of the ideal MHD equations (3) and (4).*

Further, denoting $\omega^0 = \nabla \mathbf{u^0}$ *and* $\zeta^0 = \nabla \mathbf{H^0}$, *if* $\|\partial_n^{2k} \omega_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))} \leq C$, $\|\partial_n^{2k} \zeta_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))} \leq C$, $\|\partial_t \partial_n^{2k} \omega_\tau^0\|_{L^2(0,T;C^1(\partial\Omega))} \leq C$, $\|\partial_t \partial_n^{2k} \zeta_\tau^0\|_{L^2(0,T;C^1(\partial\Omega))} \leq C$, *then*

$$(\mathbf{u}, \mathbf{H}) \to (\mathbf{u^0}, \mathbf{H^0}) \text{ in } \mathbf{C([0,T]; H^{2k+1})}, \text{ as } (\nu, \mu) \to \mathbf{0}. \tag{8}$$

The paper is organized as follows. Some tools are drawn in section 2. A priori estimates to the MHD systems are given in section 3. The results of vanishing viscosity limit and the convergence rate are presented in section 4.

2 Notations and preliminaries

Throughout the rest of this paper, denote by $\mathbf{v}_\tau = \mathbf{v} \cdot \tau$ and $\mathbf{v_n} = \mathbf{v} \cdot \mathbf{n}$ on the boundary $\partial\Omega$. For the flat boundary case, $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ and $\nabla \times \mathbf{v} = \mathbf{0}$ are equivalent to $\mathbf{v_n} = \mathbf{0}$ and $\partial_n \mathbf{v}_\tau = \mathbf{0}$ on ∂Q. And $\partial Q = \{(x,y,z); z = 0, z = 1\} \cap \overline{Q}$. For convenience, Ω and Q may be omitted when we write these spaces without confusion.

We begin our analysis with a formula of integration by parts.

Let Ω be a regular open, bounded set in R^3. Then, for sufficiently regular vector fields $\mathbf{v}$,

$$-\int_{\Omega} \triangle \mathbf{v} \cdot \mathbf{v} \mathbf{dx} = \|\nabla \mathbf{v}\|_{\mathbf{L^2}}^{\mathbf{2}} - \int_{\partial \mathbf{\Omega}} \partial_{\mathbf{n}} \mathbf{v} \cdot \mathbf{v} \mathbf{d}\sigma. \tag{9}$$

It is easily shown that if $\mathbf{v}$ is, sufficiently regular, vector fields in a flat boundary domain then

$$\partial_{\mathbf{n}} \mathbf{v} \cdot \mathbf{v} = \partial_{\mathbf{n}} \mathbf{v}_{\tau} \cdot \mathbf{v}_{\tau} + \partial_{\mathbf{n}} \mathbf{v}_{\mathbf{n}} \cdot \mathbf{v}_{\mathbf{n}}. \tag{10}$$

It follows that $\partial_{\mathbf{n}} \mathbf{v} \cdot \mathbf{v}$ vanishes on the boundary if either of the following conditions is satisfied,

$$\begin{aligned} &(a) \quad \mathbf{v} \cdot \mathbf{n} = \mathbf{0}, \nabla \times \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial \mathbf{\Omega}, \\ &(b) \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial \mathbf{\Omega}, \nabla \cdot \mathbf{u} = \mathbf{0} \text{ in } \mathbf{\Omega}. \end{aligned} \tag{11}$$

To study functions with either of above boundary conditions, we introduce series of function sets.

Let

$$H = \{\mathbf{v} \in \mathbf{H^1}; \nabla \cdot \mathbf{v} = \mathbf{0} \text{ in } \mathbf{\Omega}\},$$

$$V^{-1} = \{\mathbf{v} \in \mathbf{H}; \mathbf{v_n} = \mathbf{0} \text{ on } \partial \mathbf{\Omega}\},$$

$$V^{0} = \{\mathbf{v} \in \mathbf{H}; \mathbf{v}_{\tau} = \mathbf{0} \text{ on } \partial \mathbf{\Omega}\},$$

$$V^{2k} = \{\mathbf{v} \in \mathbf{H^{2k+1}}; \partial_{\mathbf{n}}^{\mathbf{2j}} \mathbf{v} \in \mathbf{V^0}, \mathbf{j} = \mathbf{0}, \mathbf{1}, \cdots, \mathbf{k}\},$$

$$V^{2k+1} = \{\mathbf{v} \in \mathbf{H^{2k+2}}; \mathbf{v} \in \mathbf{V^{-1}}, \partial_{\mathbf{n}}^{\mathbf{2j+1}} \mathbf{v} \in \mathbf{V^0}, \mathbf{j} = \mathbf{0}, \mathbf{1}, \cdots, \mathbf{k}\}.$$

Then, the following propositions are easily obtained

Proposition 2.1 *Let $k \geq 1$. Then $V^k \subseteq V^{k-2}$.*

Proposition 2.2 *Let $k \geq -1$, and $\mathbf{v} \in \mathbf{V^k}$. Then $\partial_n^j \mathbf{v} \in \mathbf{V^{k-j}}$, $j = 0, 1, \cdots, k+1$.*

Rewrite (11) with the new notations,

Lemma 2.1 *Let $k \geq 0$ and $\mathbf{v} \in \mathbf{V^k}$. Then $\partial_n \mathbf{v} \cdot \mathbf{v} = \mathbf{0}$ on $\partial\Omega$.*

It should be considered that when $\mathbf{v}$ is not in V^0 or V^1. For energy estimates, we construct a boundary layer to fill the gap.

Lemma 2.2 *In the flat boundary case, assume* $\|\mathbf{h}_\tau\|_{\mathbf{C^1(\partial\Omega)}} \le \mathbf{C}$ *for* $k \ge 1$. *Then, for any* $\varepsilon << 1$, *there is a* $\mathbf{v}^\varepsilon \in \mathbf{V^{2k-1}}$, $\chi^\varepsilon = \nabla \times \mathbf{v}^\varepsilon$, *such that* $\nabla \times \mathbf{v}^\varepsilon \equiv \mathbf{0}$ *as* $\varepsilon^{\frac{1}{2}} \le z \le 1 - \varepsilon^{\frac{1}{2}}$, *furthermore,*

$$\begin{aligned}
&\chi_\tau^\varepsilon \in \mathbf{C^{2k+1}}(\overline{\mathbf{\Omega}}), \chi_{\mathbf{n}}^\varepsilon \in \mathbf{C^{2k}}(\overline{\mathbf{\Omega}}),\\
&\partial_n^{2k}\chi_\tau^\varepsilon = \mathbf{h}_\tau, \partial_{\mathbf{n}}^{\mathbf{2k}}\chi_{\mathbf{n}}^\varepsilon = \mathbf{0} \text{ on } \partial\mathbf{\Omega},\\
&\|z^i(1-z)^i\partial_n^{2k+1}\chi_\tau^\varepsilon\|_{\mathbf{L^p}} \le \mathbf{C}\varepsilon^{\frac{1}{2p}+\frac{i-1}{2}},\\
&\|z^i(1-z)^i\partial_n^{2k}\chi_{\mathbf{n}}^\varepsilon\|_{\mathbf{L^p}} \le \mathbf{C}\varepsilon^{\frac{1}{2p}+\frac{i+1}{2}},\\
&\|\partial_n^{2k}\chi^\varepsilon\|_{\mathbf{L^p}} \le \mathbf{C}\varepsilon^{\frac{1}{2p}+\frac{i}{2}},
\end{aligned} \tag{12}$$

for $i \in R^+$, $1 \le p < +\infty$.

Proof. It's trivial to find a function $\varphi(z) \in C^1[0,\infty)$, s.t.

$$\begin{cases} \varphi(z) = 1 \text{ at } z = 0,\\ \varphi(z) = 0 \text{ at } z \ge 1,\\ \displaystyle\int_0^1 F^j(\varphi)(s)ds = 0, j = 0, 1, \cdots, 2k-1, \end{cases} \tag{13}$$

where F is an integrate operator from $C[0,\infty)$ to $C^1[0,\infty)$, and $F(f)(z) = \int_0^z f(s)ds$, $F^0(f) = f$, $F^j = F(F^{j-1})$, $j \ge 1$.

Denote by $\varphi^\varepsilon(z) = \varphi(\frac{z}{\varepsilon^{\frac{1}{2}}})$. Then,

$$\|z^i\partial_z^j\varphi^\varepsilon\|_{L^p} \le C\varepsilon^{\frac{1}{2p}+\frac{i-j}{2}} \text{ for } i \in R^+, j \le 2, 1 \le p \le +\infty.$$

Set $\psi^\varepsilon(z)_\tau = \mathbf{h}_\tau(\mathbf{0})\varphi^\varepsilon(\mathbf{z}) + \mathbf{h}_\tau(\mathbf{1})\varphi^\varepsilon(\mathbf{1}-\mathbf{z})$, and $\psi_3^\varepsilon = -\int_0^z \nabla_\tau \cdot \psi_\tau(x,y,s)ds$. It follows that

$$\nabla \cdot \psi^\varepsilon = 0 \text{ in } \Omega, \psi_\tau^\varepsilon = \mathbf{h}_\tau \text{ on } \partial\mathbf{\Omega}.$$

Next, set

$$\chi^\varepsilon = F^{2k}(\psi^\varepsilon).$$

Since $F^j(\varphi^\varepsilon) = 0$ on $\partial\Omega$, for $j = 1, 2, \cdots, 2k$, it follows that $\partial_z^j\chi_\tau^\varepsilon = 0$, $\partial_z^{j+1}\chi_3 = 0$ on $\partial\Omega$, for $j = 0, 1, \cdots, 2k-1$. Furthermore, $\nabla \cdot \chi^\varepsilon = 0$ in Ω. In other words, $\chi^\varepsilon \in V^{2k-2}$. Therefore, $\int \chi_3 = 0$.

Finally, let ζ^ε satisfy the following equations

$$\begin{cases} -\triangle\zeta^\varepsilon = \chi^\varepsilon \text{ in } \Omega,\\ \zeta_\tau^\varepsilon = 0, \partial_z\zeta_3^\varepsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{14}$$

The necessary condition $\int \chi_3 = 0$ of existence holds by classical elliptic theories. Applying div to equation (14), together with div$\zeta^\varepsilon = 0$ on $\partial\Omega$, then $\nabla \cdot \zeta^\varepsilon = 0$ in Ω.

Set $\mathbf{v}^\varepsilon = \nabla \times \zeta^\varepsilon$ and notice that $\nabla \times \nabla \zeta^\varepsilon = -\triangle \chi^\varepsilon$, then the proof is completed after a simple calculation. ■

Now, we derive some results of nonlinearities.

Lemma 2.3 $(\mathbf{u} \cdot \nabla)\mathbf{v}$ *is normal to the boundary, if either of the following conditions holds*

$$\begin{aligned} &(a) \qquad \mathbf{u} \in \mathbf{V^0}, \mathbf{v} \in \mathbf{V^1}, \\ &(b) \qquad \mathbf{u} \in \mathbf{V^{-1}}, \mathbf{v} \in \mathbf{V^0}. \end{aligned} \tag{15}$$

Lemma 2.4 *Let* $j \geq 0$, $\mathbf{u}, \mathbf{v} \in \mathbf{V^j}$. *Then,* $(\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ *on* $\partial\Omega$.

The proof is left to the reader.

Theorem 2.1 *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{V^{2k+1}}$, $k \geq 1$. *Then, for* $0 \leq j \leq 2k+1$,

$$\begin{cases} \partial_n^j (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{n} = \mathbf{0} \ \ \text{on } \partial\mathbf{\Omega}, \quad \text{if } \mathbf{j} \text{ is even}, \\ \partial_n^j (\mathbf{u} \cdot \nabla)\mathbf{v} \times \mathbf{n} = \mathbf{0} \ \text{on } \partial\mathbf{\Omega}, \quad \text{if } \mathbf{j} \text{ is odd}. \end{cases} \tag{16}$$

Proof. It's easily derived,

$$\partial_n^j (\mathbf{u} \cdot \nabla)\mathbf{v} = \sum_{\mathbf{i=0}}^{\mathbf{j}} \partial_{\mathbf{n}}^{\mathbf{i}} (\mathbf{u} \cdot \nabla) \partial_{\mathbf{n}}^{\mathbf{j-i}} \mathbf{v}.$$

If j is odd. i and $j-i$, or $j-i$ and i, are odd and even, respectively. Recalling Proposition 2.1, $\partial_n^i \mathbf{u} \in \mathbf{V^1}$ and $\partial_n^{j-i} \mathbf{v} \in \mathbf{V^0}$, or $\partial_n^i \mathbf{u} \in \mathbf{V^0}$ and $\partial_n^{j-i} \mathbf{v} \in \mathbf{V^1}$. It follows Lemma 2.3, $\partial_n^i (\mathbf{u} \cdot \nabla) \partial_{\mathbf{n}}^{\mathbf{j-i}} \mathbf{v} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and the desired result is obtained.

If j is even. i and $j-i$ are all odd or even. Recalling proposition 2.1 and Lemma 2.4, $\partial_n^i (\mathbf{u} \cdot \nabla) \partial_{\mathbf{n}}^{\mathbf{j-i}} \mathbf{v} \cdot \mathbf{n} = \mathbf{0}$. And the desired result is obtained. ■

Finally, denote $\nabla \times (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{v})$ by $F(Du, Dv)$. By appealing to Theorem 2.1, the following results can be obtained.

Corollary 2.1 *Let* $\mathbf{u}, \mathbf{v} \in \mathbf{V^{2k+1}}$, $k \geq 1$. *Then, for* $0 \leq j \leq k$,

$$\partial_n^{2j} (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{v}) \times \mathbf{n} = \mathbf{0}, \partial_{\mathbf{n}}^{\mathbf{2j}} \mathbf{F}(\mathbf{Du}, \mathbf{Dv}) \times \mathbf{n} = \mathbf{0} \ \text{on } \partial\mathbf{\Omega}.$$

3 A priori estimates

Now, we derive formal energy estimates assuming that $\mathbf{u_0}$, $\mathbf{u}$, $\mathbf{H_0}$, $\mathbf{H}$ are sufficiently regular. As pointed out in [11] and [16], the key in studying the vanishing viscosity limit is to control the vorticity created on the boundary.

Set

$$\omega = \nabla \times \mathbf{u}, \zeta = \nabla \times \mathbf{H}.$$

Recalling the boundary conditions (2) together with the notations introduced in section 2,

$$\mathbf{u} \in \mathbf{V^1}, \quad \mathbf{H} \in \mathbf{V^1}, \quad \omega \in \mathbf{V^0}, \quad \zeta \in \mathbf{V^0}. \tag{17}$$

By applying the operator curl to both sides of the equation (1) one gets,

$$\partial_t \omega - \nu \triangle \omega + (\mathbf{u} \cdot \nabla)\omega - (\mathbf{H} \cdot \nabla)\zeta + \mathbf{F}(\mathbf{Du}, \mathbf{Du}) - \mathbf{F}(\mathbf{DH}, \mathbf{DH}) = \mathbf{0} \text{ in } \mathbf{\Omega}. \tag{18}$$

$$\partial_t \zeta - \mu \triangle \zeta + (\mathbf{u} \cdot \nabla)\zeta - (\mathbf{H} \cdot \nabla)\omega + \mathbf{F}(\mathbf{Du}, \mathbf{DH}) - \mathbf{F}(\mathbf{DH}, \mathbf{Du}) = \mathbf{0} \text{ in } \mathbf{\Omega}. \tag{19}$$

By appealing to Corollary 2.1 and (17), one obtains $\partial_t \omega, \partial_t \zeta, \triangle_\tau \omega, \triangle_\tau \zeta,$ $(\mathbf{u} \cdot \nabla)\omega, (\mathbf{H} \cdot \nabla)\zeta, (\mathbf{u} \cdot \nabla)\zeta, (\mathbf{H} \cdot \nabla)\omega, \mathbf{F}(\mathbf{Du}, \mathbf{Du}), \mathbf{F}(\mathbf{DH}, \mathbf{DH}), \mathbf{F}(\mathbf{Du}, \mathbf{DH})$ and $\mathbf{F}(\mathbf{DH}, \mathbf{Du})$ are all normal to boundary. Then, by equations (18) and (19), $\partial_n^2 \omega$ and $\partial_n^2 \zeta$ are normal to $\partial\Omega$. It follows that $\omega \in V^2$, $\zeta \in V^2$, $\mathbf{u} \in \mathbf{V^3}$ and $\mathbf{H} \in \mathbf{V^3}$.

Similarly, applying operator ∂_n^2 to both sides of equations (18) and (19). Step by step, the following result is obtained.

Lemma 3.1 *Let* $\mathbf{u}$ *and* $\mathbf{H}$ *be sufficient regularity. Then, for* $k \in N$*,*

$$\mathbf{u} \in \mathbf{V^{2k+1}}, \mathbf{H} \in \mathbf{V^{2k+1}}, \omega \in \mathbf{V^{2k}}, \zeta \in \mathbf{V^{2k}}. \tag{20}$$

Applying the operator $\partial^\alpha_{x,y,z}$, (α is a multi-index and $|\alpha| \le 2k$) to both sides of equations (18) and (19), one gets

$$\begin{aligned}
&\partial_t \partial^\alpha_{x,y,z}\omega - \nu \triangle \partial^\alpha_{x,y,z}\omega + (\mathbf{u} \cdot \nabla)\partial^\alpha_{\mathbf{x,y,z}}\omega - (\mathbf{H} \cdot \nabla)\partial^\alpha_{\mathbf{x,y,z}}\zeta \\
&+ \sum_{|\beta|=1, \beta+\gamma=\alpha} (\partial^\beta_{x,y,z}\mathbf{u} \cdot \nabla)\partial^\gamma_{\mathbf{x,y,z}}\omega - \sum_{|\beta|=\mathbf{1}, \beta+\gamma=\alpha} (\partial^\beta_{\mathbf{x,y,z}}\mathbf{H} \cdot \nabla)\partial^\gamma_{\mathbf{x,y,z}}\zeta \\
&+ \sum_{|\beta|\ge 1, \beta+\gamma=\alpha} (\partial^\beta_{x,y,z}\mathbf{u} \cdot \nabla)\partial^\gamma_{\mathbf{x,y,z}}\omega - \sum_{|\beta|\ge \mathbf{1}, \beta+\gamma=\alpha} (\partial^\beta_{\mathbf{x,y,z}}\mathbf{H} \cdot \nabla)\partial^\gamma_{\mathbf{x,y,z}}\zeta \\
&+ F(D\partial^\alpha_{x,y,z}u, Du) + F(Du, D\partial^\alpha_{x,y,z}u) - F(D\partial^\alpha_{x,y,z}H, DH) - F(DH, D\partial^\alpha_{x,y,z}H) \\
&+ \sum_{1\le|\beta|\le|\alpha|-1, \beta+\gamma=\alpha} F(D\partial^\beta_{x,y,z}u, D\partial^\beta_{x,y,z}u) - \sum_{1\le|\beta|\le|\alpha|-1, \beta+\gamma=\alpha} F(D\partial^\beta_{x,y,z}u, D\partial^\beta_{x,y,z}u) = 0,
\end{aligned}$$

$$
\begin{aligned}
&\partial_t\partial^\alpha_{x,y,z}\zeta-\mu\triangle\partial^\alpha_{x,y,z}\zeta+(\mathbf{u}\cdot\nabla)\partial^\alpha_{\mathbf{x,y,z}}\zeta-(\mathbf{H}\cdot\nabla)\partial^\alpha_{\mathbf{x,y,z}}\omega\\
&+\sum_{|\beta|=1,\beta+\gamma=\alpha}(\partial^\beta_{x,y,z}\mathbf{u}\cdot\nabla)\partial^\gamma_{\mathbf{x,y,z}}\zeta-\sum_{|\beta|=\mathbf{1},\beta+\gamma=\alpha}(\partial^\beta_{\mathbf{x,y,z}}\mathbf{H}\cdot\nabla)\partial^\gamma_{\mathbf{x,y,z}}\omega\\
&+\sum_{|\beta|\geq 1,\beta+\gamma=\alpha}(\partial^\beta_{x,y,z}\mathbf{u}\cdot\nabla)\partial^\gamma_{\mathbf{x,y,z}}\zeta-\sum_{|\beta|\geq\mathbf{1},\beta+\gamma=\alpha}(\partial^\beta_{\mathbf{x,y,z}}\mathbf{H}\cdot\nabla)\partial^\gamma_{\mathbf{x,y,z}}\omega\\
&+F(D\partial^\alpha_{x,y,z}u,DH)+F(Du,D\partial^\alpha_{x,y,z}H)-F(D\partial^\alpha_{x,y,z}H,Du)-F(DH,D\partial^\alpha_{x,y,z}u)\\
&+\sum_{1\leq|\beta|\leq|\alpha|-1,\beta+\gamma=\alpha}F(D\partial^\beta_{x,y,z}u,D\partial^\beta_{x,y,z}H)-\sum_{1\leq|\beta|\leq|\alpha|-1,\beta+\gamma=\alpha}F(D\partial^\beta_{x,y,z}H,D\partial^\beta_{x,y,z}u)=0.
\end{aligned}
$$

Next, multiplying both sides of the above equations by $\partial^\alpha_{x,y,z}\omega$ and $\partial^\alpha_{x,y,z}\zeta$, respectively, integrating in Ω, and summing them up. Note that $\partial^\alpha_{x,y,z}\omega^\nu\in V^0$ and $\partial^\alpha_{x,y,z}\zeta^\nu\in V^0$ or $\partial^\alpha_{x,y,z}\omega^\nu\in V^1$ and $\partial^\alpha_{x,y,z}\zeta^\nu\in V^1$ for $|\alpha|\leq 2k$,

$$
\int(\mathbf{H}\cdot\nabla)\partial^\alpha_{\mathbf{x,y,z}}\zeta\cdot\partial^\alpha_{\mathbf{x,y,z}}\omega+\int(\mathbf{H}\cdot\nabla)\partial^\alpha_{\mathbf{x,y,z}}\omega\cdot\partial^\alpha_{\mathbf{x,y,z}}\zeta=\mathbf{0},
$$

$$
\|\nabla\mathbf{u}\|_{\mathbf{L}^\infty}+\|\nabla\mathbf{H}\|_{\mathbf{L}^\infty}\leq\mathbf{C}\|\mathbf{u}\|_{\mathbf{H^3}}+\mathbf{C}\|\mathbf{H}\|_{\mathbf{H^3}}\leq\mathbf{C}\|\omega\|_{\mathbf{H^{2k}}}+\mathbf{C}\|\zeta\|_{\mathbf{H^{2k}}},
$$

and

$$
\|\mathbf{u}\|_{\mathbf{W^{j,4}}}+\|\mathbf{H}\|_{\mathbf{W^{j,4}}}\leq\mathbf{C}\|\omega\|_{\mathbf{W^{j-1,4}}}+\mathbf{C}\|\zeta\|_{\mathbf{W^{j-1,4}}}\leq\mathbf{C}\|\omega\|_{\mathbf{H^j}}+\mathbf{C}\|\zeta\|_{\mathbf{H^j}},\mathbf{2}\leq\mathbf{j}\leq\mathbf{2k}.
$$

By Lemma 2.1 and summing up for all $|\alpha|\leq 2k$, one obtains

$$
\frac{1}{2}\frac{d}{dt}(\|\omega\|^2_{H^{2k}}+\|\zeta\|^2_{H^{2k}})+\nu\|\nabla\omega\|^2_{H^{2k}}+\nu\|\nabla\zeta\|^2_{H^{2k}}\leq C\|\omega\|^3_{H^{2k}}+C\|\zeta\|^3_{H^{2k}}.
$$

Comparing with the ordinary differential equation

$$
\begin{cases}
y'(t)=Cy^{\frac{3}{2}},\\
y(0)=\|\omega_0\|^2_{H^{2k}}+\|\zeta_0\|^2_{H^{2k}},
\end{cases}
\tag{21}
$$

where $\omega_0=\nabla\times\mathbf{u_0}$ and $\zeta_0=\nabla\times\mathbf{H_0}$, then denoting by T^* the blow up time, it follows that a priori estimates hold, for $T<T^*$,

$$
\|\omega\|_{L^\infty(0,T;H^{2k})}+\|\zeta\|_{L^\infty(0,T;H^{2k})}\leq C. \tag{22}
$$

Thus, we have the following result.

Theorem 3.1 *Let* $\mathbf{u_0}\in\mathbf{V^{2k-1}}\cap\mathbf{H^{2k+1}}$ *and* $\mathbf{H_0}\in\mathbf{V^{2k-1}}\cap\mathbf{H^{2k+1}}$, $k\geq 1$. *Then there exist* T *and* $C(\|\mathbf{u_0}\|_{\mathbf{H^{2k+1}}},\mathbf{T})$, *s.t.*

$$
\|\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}}+\|\mathbf{H}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}}\leq\mathbf{C}. \tag{23}
$$

Taking the inner product $((18), \partial_t\omega)_{H^{2k-1}} + ((19), \partial_t\zeta)_{H^{2k-1}}$, one obtains that $\|\partial_t\omega\|_{L^2(0,T;H^{2k-1})} + \|\partial_t\zeta\|_{L^2(0,T;H^{2k-1})} \le C$. It follows that $\|\partial_t\mathbf{u}\|_{\mathbf{L^2(0,T;H^{2k})}} + \|\partial_\mathbf{t}\mathbf{H}\|_{\mathbf{L^2(0,T;H^{2k})}} \le \mathbf{C}$.

According to $\|\omega_0\|_{H^{2k}} \le C$ and $\|\zeta_0\|_{H^{2k}} \le C$, then by equations (18) and (19), $\|\partial_t\omega|_{t=0}\|_{H^{2k-2}} + \|\partial_t\zeta|_{t=0}\|_{H^{2k-2}} \le C$

Similarly, applying operator $\partial_t\partial^\alpha_{x,y,z}$ to both sides of equations (18) and (19), for $|\alpha| \le 2k-2$, and multiplying $\partial_t\partial^\alpha_{x,y,z}\omega$ and $\partial_t\partial^\alpha_{x,y,z}\zeta$, respectively, we have,

$$\|\partial_t\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k-1})}} + \|\partial_\mathbf{t}\mathbf{H}\|_{\mathbf{L^\infty(0,T;H^{2k-1})}} \le \mathbf{C}. \tag{24}$$

Thus, we can conclude

Theorem 3.2 *Let the conditions of Theorem 3.1 be satisfied, then for $s \le k$*

$$\begin{aligned}
&\|\partial_t^s\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} + \|\partial_\mathbf{t}^\mathbf{s}\mathbf{H}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} \le \mathbf{C},\\
&\|\partial_t^{s+1}\mathbf{u}\|_{\mathbf{L^2(0,T;H^{2k-2s})}} + \|\partial_\mathbf{t}^\mathbf{s+1}\mathbf{H}\|_{\mathbf{L^2(0,T;H^{2k-2s})}} \le \mathbf{C}.
\end{aligned} \tag{25}$$

where $C = C(\|\mathbf{u_0}\|_{\mathbf{H^{2k+1}}}, \mathbf{T})$.

Then, the regularity of the solution of MHD equations (1) and (2) is investigated,

Theorem 3.3 *Let the conditions of Theorem 3.1 be satisfied. Then for $s \le k$, there exist a time T depending on the initial date and unique classical solution of MHD equations (1) with boundary condition (2). In addition,*

$$\begin{aligned}
&\|\partial_t^s\mathbf{u}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} + \|\partial_\mathbf{t}^\mathbf{s}\mathbf{H}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} \le \mathbf{C},\\
&\|\partial_t^{s+1}\mathbf{u}\|_{\mathbf{L^2(0,T;H^{2k-2s})}} + \|\partial_\mathbf{t}^\mathbf{s+1}\mathbf{H}\|_{\mathbf{L^2(0,T;H^{2k-2s})}} \le \mathbf{C},
\end{aligned} \tag{26}$$

where $C = C(\|\mathbf{u_0}\|_{\mathbf{H^{2k+1}}}, \|\mathbf{H_0}\|_{\mathbf{H^{2k+1}}}, \mathbf{T})$.

4 The vanishing viscosity limit

This section focuses on the vanishing viscosity limit of the MHD system for the flat boundary case.

Theorem 4.1 *Let the conditions of Theorem 3.1 be satisfied for $k \ge 1$. Then as $(\nu, \mu) \to 0$, $(\mathbf{u}, \mathbf{H})$ converge to the unique solution $(\mathbf{u^0}, \mathbf{H^0})$ of the ideal MHD system with the same initial date in the sense*

$$(\mathbf{u}, \mathbf{H}) \to (\mathbf{u^0}, \mathbf{H^0}) \text{ in } \mathbf{C(0, T; H^{2k})}. \tag{27}$$

Proof. It follows from Theorem 3.3 that

$$\mathbf{u}(\nu,\mu),\mathbf{H}(\nu,\mu) \text{ is uniformly bounded in } \mathbf{L}^{\infty}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k+1}}), \tag{28}$$

and

$$\partial_t\mathbf{u}(\nu,\mu),\partial_\mathbf{t}\mathbf{H}(\nu,\mu) \text{ is uniformly bounded in } \mathbf{L}^{\mathbf{2}}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k}}), \tag{29}$$

for all $\nu > 0$ and $\mu > 0$. By the standard compactness result, there exist a subsequence ν_k of ν, μ_k of μ and vector functions $\mathbf{u^0}$ and $\mathbf{H^0}$, such that

$$(\mathbf{u}(\nu_\mathbf{k},\mu_\mathbf{k}),\mathbf{H}(\nu_\mathbf{k},\mu_\mathbf{k})) \to (\mathbf{u^0},\mathbf{H^0}) \text{ in } \mathbf{C}(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k}}), \tag{30}$$

as $(\nu,\mu)\to 0$. Passing to the limit, we can find $(\mathbf{u^0},\mathbf{H^0})$ solves the ideal MHD equations (3) and (4). Together with the uniqueness of the strong solution of the ideal MHD systems, we then show the convergence of whole sequence. ■

Now, we present the convergence rate.

Theorem 4.2 *Let the conditions of Theorem 3.1 be satisfied for $k \geq 1$. Then,*

$$\|\mathbf{u}-\mathbf{u^0}\|_{\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k-1}})} + \|\mathbf{H}-\mathbf{H^0}\|_{\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{H}^{\mathbf{2k-1}})} \leq \mathbf{C}\nu + \mathbf{C}\mu. \tag{31}$$

Proof. Set $\omega^0 = \nabla\times\mathbf{u^0}$ and $\zeta^0 = \nabla\times\mathbf{H^0}$. Recalling Lemma 3.1 and Theorem 4.1, one obtains

$$\begin{aligned}&\mathbf{u^0}(\mathbf{t})\in\mathbf{X}^{\mathbf{2k-1}}\cap\mathbf{H}^{\mathbf{2k+1}},\mathbf{H^0}(\mathbf{t})\in\mathbf{X}^{\mathbf{2k-1}}\cap\mathbf{H}^{\mathbf{2k+1}},\\ &\omega^0(t)\in X^{2k-2}\cap H^{2k},\zeta^0(t)\in X^{2k-2}\cap H^{2k}.\end{aligned} \tag{32}$$

Set $\widetilde{\mathbf{u}}=\mathbf{u}-\mathbf{u^0}$, $\widetilde{\omega}=\nabla\times\widetilde{\mathbf{u}}$, $\widetilde{\mathbf{H}}=\mathbf{H}-\mathbf{H^0}$, $\widetilde{\zeta}=\nabla\times\widetilde{\mathbf{H}}$. We can find $\widetilde{\mathbf{u}},\widetilde{\omega}$ solve

$$\begin{aligned}&\partial_t\widetilde{\omega}-\nu\triangle\widetilde{\omega}+(\mathbf{u}\cdot\nabla)\widetilde{\omega}+(\widetilde{\mathbf{u}}\cdot\nabla)\omega^\mathbf{0}-(\mathbf{H}\cdot\nabla)\widetilde{\zeta}-(\widetilde{\mathbf{H}}\cdot\nabla)\zeta^\mathbf{0}\\ &+F(D\mathbf{u},\mathbf{D}\widetilde{\mathbf{u}})+\mathbf{F}(\mathbf{D}\widetilde{\mathbf{u}},\mathbf{D}\mathbf{u^0})-\mathbf{F}(\mathbf{DH},\mathbf{D}\widetilde{\mathbf{H}})-\mathbf{F}(\mathbf{D}\widetilde{\mathbf{H}},\mathbf{DH^0})=\nu\triangle\omega^\mathbf{0},\end{aligned} \tag{33}$$

$$\begin{aligned}&\partial_t\widetilde{\zeta}-\mu\triangle\widetilde{\zeta}+(\mathbf{u}\cdot\nabla)\widetilde{\zeta}+(\widetilde{\mathbf{u}}\cdot\nabla)\zeta^\mathbf{0}-(\mathbf{H}\cdot\nabla)\widetilde{\omega}-(\widetilde{\mathbf{H}}\cdot\nabla)\zeta^\mathbf{0}\\ &+F(D\mathbf{u},\mathbf{D}\widetilde{\mathbf{H}})+\mathbf{F}(\mathbf{D}\widetilde{\mathbf{u}},\mathbf{DH^0})-\mathbf{F}(\mathbf{DH},\mathbf{D}\widetilde{\mathbf{u}})-\mathbf{F}(\mathbf{D}\widetilde{\mathbf{H}},\mathbf{Du^0})=\mu\triangle\omega^\mathbf{0},\end{aligned} \tag{34}$$

Noting that $\widetilde{\omega}^0\in V^{2k-2}$, $\widetilde{\zeta}^0\in V^{2k-2}$, $\|\triangle\omega\|_{H^{2k-2}}\leq C$ and $\|\triangle\zeta\|_{H^{2k-2}}\leq C$, the same argument in proof of Theorem 3.1 can be followed. Taking the inner products $((33),\widetilde{\omega})_{H^{2k-2}}+((34),\widetilde{\zeta})_{H^{2k-2}}$, and the desired result can be obtained. ■

There is a gap between $(\partial_n^{2k}\omega_\tau^0,\partial_n^{2k}\zeta_\tau^0)$ and 0. In other words, ω^0 and ζ^0 are not in V^{2k}. Assuming $\|\partial_n^{2k}\omega_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))}\leq C$, $\|\partial_n^{2k}\zeta_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))}\leq C$,

$\|\partial_t\partial_n^{2k}\omega_\tau^0\|_{L^2(0,T;C^2(\partial\Omega))} \le C$, $\|\partial_t\partial_n^{2k}\zeta_\tau^0\|_{L^2(0,T;C^2(\partial\Omega))} \le C$, by Lemma 2.2, there exist $\mathbf{v^i} \in \mathbf{V^{2k-1}}$, $\chi^i = \nabla\times\mathbf{v}^\nu \in \mathbf{V^{2k-2}}$ $(i=1,2)$, s.t. $\partial_n^{2k}\chi_\tau^1 = -\partial_n^{2k}\omega_\tau^0$, $\partial_n^{2k}\chi_\tau^2 = -\partial_n^{2k}\zeta_\tau^0$ on $\partial\Omega$, $\|z^j(1-z)^j\partial_n^{2k+1}\chi^i\|_{L^\infty(0,T;L^2)} \le C\nu^{\frac{2j-1}{4}}$, and further $\|\partial_t\chi^i\|_{L^2(0,T;H^{2k})} \le C\nu^{\frac14}$, $\|\partial_\tau\chi^i\|_{L^\infty(0,T;H^{2k})} \le C\nu^{\frac14}$, $i=1,2$, $j=0,1$.

Set $\widehat{\mathbf{u}} = \mathbf{u}-\mathbf{u^0}-\mathbf{v^1}$, $\widehat{\mathbf{H}} = \mathbf{H}-\mathbf{H^0}-\mathbf{v^2}$ $\widehat{\omega} = \nabla\times\widehat{\mathbf{u}} = \omega-\omega^0-\chi^1$, $\widehat{\zeta} = \nabla\times\widehat{\mathbf{H}} = \zeta-\zeta^0-\chi^2$.

From equations (33) and (34), one obtains,

$$\begin{aligned}
&\partial_t\widehat{\omega}-\nu\triangle\widehat{\omega}+(\mathbf{u}\cdot\nabla)\widehat{\omega}+(\widehat{\mathbf{u}}\cdot\nabla)\omega^{\mathbf{0}}-(\mathbf{H}\cdot\nabla)\widehat{\zeta}-(\widehat{\mathbf{H}}\cdot\nabla)\zeta^{\mathbf{0}}\\
&+F(D\mathbf{u},\mathbf{D}\widehat{\mathbf{u}})+\mathbf{F}(\mathbf{D}\widehat{\mathbf{u}},\mathbf{Du^0})-\mathbf{F}(\mathbf{DH},\mathbf{D}\widehat{\mathbf{H}})+\mathbf{F}(\mathbf{D}\widehat{\mathbf{H}},\mathbf{DH^0})\\
&=\nu\triangle\omega^0-\partial_t\chi^1+\nu\triangle\chi^0-(\mathbf{u}\cdot\nabla)\chi^{\mathbf{1}}-(\mathbf{v^1}\cdot\nabla)\omega^0+(\mathbf{H}\cdot\nabla)\chi^{\mathbf{2}}+(\mathbf{v^2}\cdot\nabla)\zeta^{\mathbf{0}}\\
&-F(D\mathbf{u},\mathbf{Dv^1})-\mathbf{F}(\mathbf{Dv^1},\mathbf{Du^0})+\mathbf{F}(\mathbf{DH},\mathbf{Dv^2})+\mathbf{F}(\mathbf{Dv^2},\mathbf{DH^0}),
\end{aligned}\tag{35}$$

$$\begin{aligned}
&\partial_t\widehat{\zeta}-\mu\triangle\widehat{\zeta}+(\mathbf{u}\cdot\nabla)\widehat{\zeta}+(\widehat{\mathbf{u}}\cdot\nabla)\zeta^{\mathbf{0}}-(\mathbf{H}\cdot\nabla)\widehat{\omega}-(\widehat{\mathbf{H}}\cdot\nabla)\zeta^{\mathbf{0}}\\
&+F(D\mathbf{u},\mathbf{D}\widehat{\mathbf{H}})+\mathbf{F}(\mathbf{D}\widehat{\mathbf{u}},\mathbf{DH^0})-\mathbf{F}(\mathbf{DH},\mathbf{D}\widehat{\mathbf{u}})-\mathbf{F}(\mathbf{D}\widehat{\mathbf{H}},\mathbf{Du^0})\\
&=\mu\triangle\omega^0-\partial_t\chi^2+\mu\triangle\chi^2-(\mathbf{u}\cdot\nabla)\chi^{\mathbf{2}}-(\mathbf{v^1}\cdot\nabla)\zeta^{\mathbf{0}}+(\mathbf{H}\cdot\nabla)\chi^{\mathbf{1}}+(\mathbf{v^2}\cdot\nabla)\zeta^{\mathbf{0}}\\
&-F(D\mathbf{u},\mathbf{Dv^2})-\mathbf{F}(\mathbf{Dv^1},\mathbf{DH^0})+\mathbf{F}(\mathbf{DH},\mathbf{Dv^1})+\mathbf{F}(\mathbf{Dv^2},\mathbf{Du^0}),
\end{aligned}\tag{36}$$

Then, taking the inner products $((35),\widehat{\omega})_{H^{2k}}+((36),\widehat{\zeta})_{H^{2k}}$,

Note that

$$\partial_n^{2k-1}\widehat{\omega}_n=0,\partial_n^{2k}\widehat{\omega}_\tau=0,\partial_n^{2k-1}\widehat{\zeta}_n=0,\partial_n^{2k}\widehat{\zeta}_\tau=0 \text{ on } \partial\Omega,$$

$$|\nu\int\triangle\partial_n^{2k}\chi^1\cdot\partial_n^{2k}\widehat{\omega}^\nu|=|\nu\int\nabla\partial_n^{2k}\chi^1\cdot\nabla\partial_n^{2k}\widehat{\omega}^\nu|\le\frac{\nu}{8}\|\nabla\partial_n^{2k}\widehat{\omega}^\nu\|_{L^2}^2+C\nu^{\frac12},$$

$$|\nu\int\triangle\partial_n^{2k}\omega^0\cdot\partial_n^{2k}\widehat{\omega}^\nu|=|\nu\int\nabla\partial_n^{2k}\omega^0\cdot\nabla\partial_n^{2k}\widehat{\omega}^\nu|\le\frac{\nu}{8}\|\nabla\partial_n^{2k}\widehat{\omega}^\nu\|_{L^2}^2+C\nu,$$

$$\|(\mathbf{u}\cdot\nabla)\partial_\mathbf{n}^{\mathbf{2k}}\chi^\mathbf{1}\|_{\mathbf{L^2}}\le\|(\mathbf{u}_\tau\cdot\nabla_\tau)\partial_\mathbf{n}^{\mathbf{2k}}\chi^\mathbf{1}\|_{\mathbf{L^2}}+\|\frac{\mathbf{u_3}}{\mathbf{z(1-z)}}\|_{\mathbf{L^\infty}}\|\mathbf{z(1-z)}\partial_\mathbf{n}^{\mathbf{2k+1}}\chi^\mathbf{1}\|_{\mathbf{L^2}}\le\mathbf{C}\nu^{\frac14}.$$

And it follows in the same manner, that

$$\|\widehat{\omega}\|_{L^\infty(0,T;H^{2k})}+\|\widehat{\zeta}\|_{L^\infty(0,T;H^{2k})}\le C\nu^{\frac14}+C\mu^{\frac14}.$$

The following result is concluded,

Theorem 4.3 *Let the conditions of Theorem 3.1 be satisfied for* $k\ge1$. *Assume the solution* $(\mathbf{u^0},\mathbf{H^0})$ *of the ideal MHD equations (3) and (4) satisfy* $\|\partial_n^{2k}\omega_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))}\le C$, $\|\partial_n^{2k}\zeta_\tau^0\|_{L^\infty(0,T;C^2(\partial\Omega))}\le C$, $\|\partial_t\partial_n^{2k}\omega_\tau^0\|_{L^2(0,T;C^2(\partial\Omega))}\le C$, $\|\partial_t\partial_n^{2k}\zeta_\tau^0\|_{L^2(0,T;C^2(\partial\Omega))}\le C$, *then,*

$$\|\mathbf{u}-\mathbf{u^0}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}}+\|\mathbf{H}-\mathbf{H^0}\|_{\mathbf{L^\infty(0,T;H^{2k+1})}}\le\mathbf{C}\nu^{\frac14}+\mathbf{C}\mu^{\frac14}.\tag{37}$$

Finally, we give two remarks.

Remark 4.1 *If the conditions of Theorem 4.2 are all satisfied, then for* $s \leq k-1$,

$$\|\partial_t^s \mathbf{u} - \partial_\mathbf{s}^\mathbf{t} \mathbf{u^0}\|_{\mathbf{L^\infty(0,T;H^{2k-1-2s})}} + \|\partial_\mathbf{t}^\mathbf{s} \mathbf{H} - \partial_\mathbf{s}^\mathbf{t} \mathbf{H^0}\|_{\mathbf{L^\infty(0,T;H^{2k-1-2s})}} \leq \mathbf{C}\nu + \mathbf{C}\mu. \quad (38)$$

Remark 4.2 *If the conditions of Theorem 4.3 are all satisfied, then for* $s \leq k$,

$$\|\partial_t^s \mathbf{u} - \partial_\mathbf{t}^\mathbf{s} \mathbf{u^0}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} + \|\partial_\mathbf{t}^\mathbf{s} \mathbf{H} - \partial_\mathbf{t}^\mathbf{s} \mathbf{H^0}\|_{\mathbf{L^\infty(0,T;H^{2k+1-2s})}} \leq \mathbf{C}\nu^{\frac{1}{4}} + \mathbf{C}\mu^{\frac{1}{4}}. \quad (39)$$

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