POWERS OF IDEALS AND THE COHOMOLOGY OF STALKS AND FIBERS OF MORPHISMS

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Abstract. We first provide here a very short proof of a refinement of a theorem of Kodiyalam and Cutkosky, Herzog and Trung on the regularity of powers of ideals. This result implies a conjecture of Hà and generalizes a result of Eisenbud and Harris concerning the case of ideals primary for the graded maximal ideal in a standard graded algebra over a field. It also implies a new result on the regularities of powers of ideal sheaves. We then compare the cohomology of the stalks and the cohomology of the fibers of a projective morphism to the effect of comparing the maximum over fibers and over stalks of the Castelnuovo-Mumford regularities of a family of projective schemes.

1. Introduction

An important result of Kodiyalam and Cutkosky, Herzog and Trung states that the Castelnuovo-Mumford regularity of the power I^t of an ideal over a standard graded algebra is eventually a linear function in t. The leading term of this function has been determined by Kodiyalam in his proof.

This result was first obtained for standard graded algebras over a field, and later extended by Trung and Wang to standard graded algebras over a Noetherian ring.

We first provide here a very short proof of a refinement of this result

Theorem 1.1. *Let A be a positively graded Noetherian algebra, $M \neq 0$ be a finitely generated graded A-module, I be a graded A-ideal, and set*

$$d := \min\{\mu \mid \exists p,\ (I_{\leq\mu})I^pM = I^{p+1}M\}.$$

Then

$$\lim_{t\to\infty}(\mathrm{end}(H^i_{A_+}(I^tM)) + i - td)$$

exists for any i, and is at least equal to the initial degree of M for some i.

The end of a graded module H is $\mathrm{end}(H) := \sup\{\mu \mid H_\mu \neq 0\}$ if $H \neq 0$ and $-\infty$ else.

Very interesting examples showing an hectic behaviour of the value of $a^i(t) := \mathrm{end}(H^i_{A_+}(I^t))$ as t varies were given by Cutkosky in [Cu]. These examples point out that the existence of the limit quoted above do not imply that all of the functions

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$a^i(t)$ are eventually linear functions of t. It only implies that at least one of them is eventually linear in t.

More recently, Eisenbud and Harris proved that in the case of standard graded algebra A over a field, for a graded ideal which is A_+-primary and generated in a single degree, the constant term in the linear function is the maximum of the regularity of the fibers of the morphism defined by a set of minimal generators. In a recent preprint, Huy Tài Hà generalized this result by proving that, if an ideal is generated in a single degree d, a variant of the regularity (the a^*-invariant) satisfies $a^*(I^t) = dt + a$ for $t \gg 0$, where a can be expressed in terms of the maximal of the values of a^* on the stalks of the projection π from the closure of the graph of the map defined by the generators to its image, [Hà, 1.3]. He conjectures that a similar result holds for the regularity.

In Theorem 5.3 we prove this conjecture of Hà. More precisely, we show that the limit in the Theorem above is the maximum of the end degree of the i-th local cohomology of the stalks of π, for ideals generated in a single degree. This holds for graded ideals in a Noetherian positively graded algebra.

An interesting, and perhaps surprising, consequence of this result is the following result on the limit of the regularity of saturation of powers, or equivalently of powers of ideal sheaves, in a positively graded Noetherian algebra :

Corollary 1.2. *Let I be a graded ideal generated in a single degree d. Then,*

$$\lim_{t\to\infty} (\mathrm{reg}((I^t)^{sat}) - dt)$$

exists and the following are equivalent :

(i) the limit is non negative,

(ii) the limit is not $-\infty$,

(iii) the projection π from the closure of the graph of the function defined by minimal generators of I to its image admits a fiber of positive dimension.

This can be applied to ideals generated in degree at most d, replacing I by $I_{\geq d}$.

It gives a simple geometric criterion for an ideal I generated in degree (at most) d to satisfy $\mathrm{reg}((I^t)^{sat}) = dt + b$ for $t \gg 0$: this holds if and only if there exists a subvariety V of the closure of the graph that is contracted in its projection to the closure of the image (*i.e.* $\dim(\pi(V)) < \dim V$). A very simple example is the following : in a polynomial ring in $n+1$ variables any graded ideal generated by n forms of same degree d satisfies $\mathrm{reg}((I^t)^{sat}) = dt + b$ for $t \gg 0$, with $b \geq 0$ – the same result holds if the analytic spread of the ideal is at most n.

The result of Eisenbud and Harris is stated in terms of regularity of fibers. For a finite morphism, there is no difference between the regularity of stalks and the regularity of fibers. This follows from the following result that is likely part of folklore, but that we didn't find in several of the classical references in the field :

Lemma 1.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $S := R[X_1, \dots, X_n]$ be a polynomial ring over R with $\deg X_i > 0$ and $\mathbb{M}$ be a finitely generated graded S-module. Set $d := \dim(\mathbb{M} \otimes_R k)$. Then $H^i_{S_+}(\mathbb{M}) = 0$ for $i > d$ and the natural graded map $H^d_{S_+}(\mathbb{M}) \otimes_R k \longrightarrow H^d_{S_+}(\mathbb{M} \otimes_R k)$ is an isomorphism.*

For morphisms that are not finite or flat, the situation is more subtle – see Proposition 6.3. We show that for families of projective schemes that are close from being flat (if the Hilbert polynomial of any two fibers differ at most by a constant, in the standard graded situation), the maximum of the regularities of stalks and the maximum of the regularities of fibers agree. Also the maximum regularity of stalks bounds above the one for fibers under a weaker hypothesis. Putting this together provides a collection of results that covers the results obtained in [EH] and [Hà]. See Theorem 6.9.

To simplify the statements, we introduce the notion of regularity over a scheme, generalizing the usual notion of regularity with reference to a polynomial extension of a ring. This is natural in our situation : the family of schemes given by closure of the graph over the parameter space given by closure of the image of our map, considered as a projective scheme, is a key ingredient of this study.

This work was inspired by results of Huy Tài Hà in [Hà] and of David Eisenbud and Joe Harris in [EH]. Bernd Ulrich made remarks on a very early version of some of these results and motivated my study of the difference between the regularity of stalks and the regularity of fibers, and Joseph Oesterlé provided references concerning Lemma 6.1. It is my pleasure to thank them for their contribution.

2. Notations and general setup

Let R be a commutative ring and S a polynomial ring over R in finitely many variables.

If S is $\mathbf{Z}$-graded, $R \subset S_0$, and $X_1, \dots, X_n$ are the variables with positive degrees, the Čech complex $\mathcal{C}^\bullet_{S_+}(M)$ with $\mathcal{C}^0_{S_+}(M) = M$ and $\mathcal{C}^i_{S_+}(M) = \oplus_{j_1 < \cdots j_i} M_{X_{j_1} \cdots X_{j_i}}$ for $i > 0$, is graded, whenever M is a graded S-module.

There is an isomorphism $H^i_{S_+}(M) \simeq H^i(\mathcal{C}^\bullet_{S_+}(M))$ for all i, which is graded if M is so. One then defines two invariants attached to such a graded S-module M :

$$a^i(M) := \sup\{\mu \mid H^i_{S_+}(M)_\mu \neq 0\}$$

if $H^i_{S_+}(M) \neq 0$ and $a^i(M) := -\infty$ else, and

$$b_j(M) := \sup\{\mu \mid \operatorname{Tor}^S_j(M, S/S_+)_\mu \neq 0\}$$

if $\operatorname{Tor}^S_j(M, S/S_+) \neq 0$ and $b_j(M) := -\infty$ else. Notice that $a^i(M) = 0$ for $i > n$ and $b_j(M) = 0$ for $j > n$. The Castelnuovo-Mumford regularity of a graded S-module M is then defined as

$$\operatorname{reg}(M) := \max_i\{a^i(M) + i\} = \max_j\{b_j(M) - j\} + n - \sigma$$

where σ is the sum of the degrees of the variables with positive degrees. Other options are possible, in particular when S is not standard graded (when $\sigma \neq n$). Another related invariant is

$$a^*(M) := \max_i \{a^i(M)\} = \max_j \{b_j(M)\} - \sigma.$$

The following classical result is usually stated for positive grading.

Theorem 2.1. *Let S be a Noetherian $\mathbf{Z}$-graded algebra and M be a finitely generated graded S-module. Then, for any i,*
(i) $a^i(M) \in \{-\infty\} \cup \mathbf{Z}$,
(ii) the S_0-module $H^i_{S_+}(M)_\mu$ is finitely generated for any $\mu \in \mathbf{Z}$.

Proof. As S is Noetherian, S is an epimorphic image of a polynomial ring S' over S_0 by a graded morphism. Considering M as S'-module one has $H^i_{S_+}(M) \simeq H^i_{S'_+}(M)$ via the natural induced map, so that we may replace S by S' and assume that

$$S = S_0[Y_1, \dots, Y_m, X_1, \dots, X_n]$$

with $\deg Y_i \leq -1$ and $\deg X_j \geq 1$ for all i and j. We recall that $H^i_{S_+}(S) = 0$ for $i < n$ and $H^n_{S_+}(S) = (X_1 \cdots X_n)^{-1} S_0[Y_1, \dots, Y_m, X_1^{-1}, \dots, X_n^{-1}]$, and notice that $H^n_{S_+}(S)_\mu$ is a finitely generated free S_0-module for any μ.

Let $F_\bullet$ be a graded free S-resolution of M with F_i finitely generated. Both spectral sequences associated to the double complex $\mathcal{C}^\bullet_{S_+} F_\bullet$ degenerate at step 2 and provide graded isomorphisms :

$$H^i_{S_+}(M) \simeq H^{n-i}(H^n_{S_+}(F_\bullet)),$$

which shows that $H^i_{S_+}(M)_\mu$ is a subquotient of $H^n_{S_+}(F_{n-i})_\mu$, hence a finitely generated S_0-module which is zero in degrees $> -n + b_{n-i}$, where b_j is the highest degree of a basis element of F_j over S. $\square$

3. Regularity over a scheme.

Local cohomology and the torsion functor commute with localization on the base R, providing natural graded isomorphisms for a graded S-module M :

$$H^i_{(S \otimes_R R_\mathfrak{p})_+}(M \otimes_R R_\mathfrak{p}) \simeq H^i_{S_+}(M) \otimes_R R_\mathfrak{p}$$

and

$$\mathrm{Tor}_i^{S \otimes_R R_\mathfrak{p}}(M \otimes_R R_\mathfrak{p}, R_\mathfrak{p}) \simeq \mathrm{Tor}_i^S(M, R) \otimes_R R_\mathfrak{p}.$$

Hence $a^i(M) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} a^i(M \otimes_R R_\mathfrak{p})$ and $b_j(M) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} b_j(M \otimes_R R_\mathfrak{p})$. It follows that the regularity is a local notion on R :

$$\mathrm{reg}(M) = \sup_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathrm{reg}(M \otimes_R R_\mathfrak{p}).$$

These supremums are maximums whenever $\mathrm{reg}(M) < +\infty$, for instance if R is Noetherian and M is finitely generated. The same holds for $a^*(M)$.

Extending this definition to the case where the base is a scheme is natural and is given in the following definition.

Definition 3.1. *Let Y be a scheme, $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank, $\mathcal{F}$ be a graded sheaf of* $\mathrm{Sym}_Y(\mathcal{E})$*-modules. Then*

$$a^i(\mathcal{F}) := \sup_{y\in Y} a^i(\mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y,y}),\quad \mathrm{reg}(\mathcal{F}) := \max_i\{a^i(\mathcal{F})+i\}.$$

If $\mathcal{E}$ is free, $\mathrm{Sym}_Y(\mathcal{E}) = \mathcal{O}_Y[X_1,\dots,X_n]$*, and the definition of regularity above makes sense for non standard grading.*

A closed subscheme Z of $\mathrm{Proj}(\mathrm{Sym}_Y(\mathcal{E}))$ *correspond to a unique graded* $\mathrm{Sym}_Y(\mathcal{E})$*-ideal sheaf $\mathcal{I}_Z$ saturated with respect to* $\mathrm{Sym}_Y(\mathcal{E})_+$*. We set*

$$a^i(Z) := \sup_{y\in Y} a^i(\mathcal{O}_{Y,y}[X_0,\dots,X_n]/(\mathcal{I}_Z\otimes_{\mathcal{O}_Y}\mathcal{O}_{Y,y}))$$

(notice that $a^0(Z)=-\infty$) and $\mathrm{reg}(Z) := \max_i\{a^i(Z)+i\}$.

The following Proposition is immediate from the definition and the corresponding results over an affine scheme.

Proposition 3.2. *Assume Y is Noetherian, $\mathcal{E}$ is a locally free coherent sheaf on Y and $\mathcal{F}\neq 0$ is a coherent graded sheaf of* $\mathrm{Sym}_Y(\mathcal{E})$*-modules. Then* $\mathrm{reg}(\mathcal{F})\in\mathbf{Z}$*. If $Z\neq\emptyset$ is a closed subscheme of $\mathbf{P}^{n-1}_Y$, then* $\mathrm{reg}(Z)\geq 0$.

4. First result on cohomology of powers.

Theorem 4.1. *Let A be a positively graded Noetherian algebra, M be a finitely generated graded A-module, I be a graded A-ideal, and set*

$$d := \min\{\mu \mid \exists p,\ (I_{\leq\mu})I^pM = I^{p+1}M\}.$$

Then

$$\lim_{t\to\infty} a^i(I^tM)+i-td$$

exists for any i, and is at least equal to $\mathrm{indeg}(M)$ *for some i.*

Proof. Set $J := I_{\leq d}$ and write $J=(g_1,\dots,g_s)$ with $\deg g_i = d$ for $1\leq i\leq m$ and $\deg g_i < d$ else. Let

$$\mathcal{R}_J := \oplus_{t\geq 0}J(d)^t = \oplus_{t\geq 0}J^t(td)\quad\text{and}\quad \mathcal{R}_I := \oplus_{t\geq 0}I(d)^t = \oplus_{t\geq 0}I^t(td),$$

and $S_0 := A_0[T_1,\dots,T_m]$, $S := S_0[T_{m+1},\dots,T_s,X_1,\dots,X_n]$, with $\deg(T_i) := \deg(g_i)-d$. Setting $\mathrm{bideg}(T_i) := (\deg(T_i),1)$ and $\mathrm{bideg}(X_j) := (\deg(X_j),0)$, one has $J(d) = (\mathcal{R}_J)_{0,1}$, hence a bigraded onto map

$$\begin{aligned} S &\longrightarrow \mathcal{R}_J\\ T_i &\longmapsto g_i\in J(d). \end{aligned}$$

The bigraded into map $\mathcal{R}_J\longrightarrow\mathcal{R}_I$ makes $M\mathcal{R}_I$ a finitely generated bigraded S-module.

The equality of graded A-modules $H^i_{S_+}(M\mathcal{R}_I)_{(*,t)} = H^i_{A_+}(M\mathcal{R}_I)_{(*,t)}$ shows that

$$H^i_{S_+}(M\mathcal{R}_I)_{(\mu,t)} = H^i_{A_+}((M\mathcal{R}_I)_{(*,t)})_\mu = H^i_{A_+}(MI^t)_{\mu+td}.$$

By Theorem 2.1 (i), $a^i(M\mathcal{R}_I) < +\infty$ and the above equalities show that $a^i(MI^t) \leq td + a^i(M\mathcal{R}_I)$, and that equality holds for some t.

Furthermore, Theorem 2.1 (ii) shows that $K_{i,\mu} := H^i_{S_+}(M\mathcal{R}_I)_{(\mu,*)}$ is a finitely generated graded S_0-module (for the standard grading $\deg(T_i) = 1$). It follows that $H^i_{S_+}(M\mathcal{R}_I)_{(\mu,t)} = 0$ for $t \gg 0$ if and only if $K_{i,\mu}$ is annihilated by a power of $\mathfrak{n} := (T_1, \ldots, T_m)$. Hence

$$\lim_{t\to+\infty}(a^i(MI^t) - td) = -\infty$$

if $K_{i,\mu}$ is annihilated by a power of $\mathfrak{n}$ for every $\mu \leq a^i(M\mathcal{R}_I)$, and else

$$\lim_{t\to+\infty}(a^i(MI^t) - td) = \max\{\mu \mid K_{i,\mu} \neq H^0_{\mathfrak{n}}(K_{i,\mu})\}.$$

As $\operatorname{reg}(MI^t) \geq \operatorname{end}(MI^t/R_+MI^t)$, the last claim follows from the next lemma, due to Kodiyalam. □

Lemma 4.2. *With the hypotheses of Theorem 4.1,*

$$\operatorname{end}(MI^t/A_+MI^t) \geq \operatorname{indeg}(M) + td, \quad \forall t.$$

Proof. The proof goes along the same lines as in [Ko, the proof of Proposition 4]. Notice that the needed graded version Nakayama's lemma applies. □

5. Cohomology of powers and cohomology of stalks

The following result is a more elaborated, and more technical, version of Theorem 4.1 that essentially follows from its proof. It implies a conjecture of Hà on the regularity of powers of ideals, and refines the main result in [Hà]. We will see later that it also implies the result of Eisenbud and Harris in [EH].

Proposition 5.1. *Let A be a positively graded Noetherian algebra, M be a finitely generated graded A-module, I be a graded A-ideal and $J \subseteq I$ be a graded ideal such that $JI^pM = I^{p+1}M$ for some p.*

Assume that the ideal J is generated by r forms $f_1, \ldots, f_r$ of respective degrees $d_1 = \cdots = d_m > d_{m+1} \geq \cdots \geq d_r$. Set $d := d_1$, $\deg(T_i) := \deg(f_i) - d$, $\operatorname{bideg}(T_i) := (\deg(T_i), 1)$ and $\operatorname{bideg}(a) := (\deg(a), 0)$ for $a \in A$. Consider the natural bigraded morphism of bigraded A_0-algebras

$$S := A[T_1, \ldots, T_r] \xrightarrow{\psi} \mathcal{R}_I := \oplus_{t\geq 0} I(d)^t = \oplus_{t\geq 0} I^t(dt),$$

sending T_i to f_i and the bigraded map of S-modules :

$$M[T_1, \ldots, T_r] \xrightarrow{1_M \otimes_A \psi} M\mathcal{R}_I := \oplus_{t\geq 0} MI^t(dt).$$

Let $B := A_0[T_1, \ldots, T_m]$ and $B' := B/\operatorname{ann}_B(\ker(1_M \otimes_A \psi))$.

Then,

$$\lim_{t\to+\infty}(a^i(MI^t)-td)=\max_{\mathfrak{q}\in \mathrm{Proj}(B')}\{a^i(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})\}.$$

Proof. First remark that in the proof of Theorem 4.1 we only need the equality $JI^pM=I^{p+1}M$ for some p (as a consequence, for all p big enough). We have shown there that

$$(*)\qquad \lim_{t\to+\infty}(a^i(MI^t)-td)=-\infty,$$

if and only if the finitely generated B-module $H^i_{S_+}(M\mathcal{R}_I)_{(\mu,*)}$ is supported in $V(T_1,\ldots,T_m)$ for any μ. As local cohomology commutes with flat base change and elements in B have degree 0,

$$H^i_{S_+}(M\mathcal{R}_I)_{(\mu,*)}\otimes_{B'}B'_{\mathfrak{q}}=H^i_{S_+}(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})_{(\mu,*)}$$

hence $(*)$ holds if and only if $H^i_{S_+}(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})=0$ for any $\mathfrak{q}\in\mathrm{Proj}(B')$. On the other hand, if this does not hold, there exists μ_0 maximum such that $H^i_{S_+}(M\mathcal{R}_I)_{(\mu_0,*)}$ is not supported in $V(T_1,\ldots,T_m)$, and choosing $\mathfrak{q}\in\mathrm{Proj}(B')\cap\mathrm{Supp}(H^i_{S_+}(M\mathcal{R}_I)_{(\mu_0,*)})$ shows that both members in the asserted equality are equal to μ_0. $\square$

Remark 5.2. *In the above Proposition, as well as in other places in this text, we localize at homogeneous primes $\mathfrak{q}\in\mathrm{Proj}(C)$ for some standard graded algebra C, in other words at graded prime ideals that do not contain C_+. We may as well replace these localization by the degree zero part of the localization at such a prime ideal, usually denoted by $C_{(\mathfrak{q})}$: the multiplication by an element $\ell\in C_1\setminus\mathfrak{q}$ induces an isomorphism $(C_{\mathfrak{q}})_\mu\simeq(C_{\mathfrak{q}})_{\mu+1}$ for any μ. Hence, for any C-module M, $M\otimes_C C_{\mathfrak{q}}=0$ if and only if $M\otimes_C C_{(\mathfrak{q})}=0$.*

In the equal degree case, the following corollary, that we state in a more geometric fashion, implies the conjecture of Hà in [Hà].

Theorem 5.3. *Let $A:=A_0[x_0,\ldots,x_n]$ be a positively graded Noetherian algebra and I be a graded A-ideal generated by $m+1$ forms of degree d. Set $Y:=\mathrm{Spec}(A_0)$ and $X:=\mathrm{Proj}(A/I)\subset\mathrm{Proj}(A)\subseteq\tilde{\mathbf{P}}^n_Y$. Let $\phi:\tilde{\mathbf{P}}^n_Y\setminus X\longrightarrow\mathbf{P}^m_Y$ be the corresponding rational map, W be the closure of the image of ϕ, and*

$$\Gamma\subset\tilde{\mathbf{P}}^n_W\subseteq\tilde{\mathbf{P}}^n_{\mathbf{P}^m_Y}=\tilde{\mathbf{P}}^n_Y\times_Y\mathbf{P}^m_Y$$

be the closure of the graph of ϕ. Let $\pi:\Gamma\longrightarrow W$ be the projection induced by the natural map $\tilde{\mathbf{P}}^n_{\mathbf{P}^m_Y}\longrightarrow\mathbf{P}^m_Y$. Then

$$\lim_{t\to+\infty}(a^i(I^t)-dt)=a^i(\Gamma).$$

Proof. Choose $J:=I$ and $M:=A$ in Proposition 5.1. The equality $\lim_{t\to+\infty}(a^i(I^t)-dt)=a^i(\Gamma)$ directly follows from the conclusion of Proposition 5.1 according the definition of $a^i(\Gamma)$ for $\Gamma\subset\tilde{\mathbf{P}}^n_W$ given in Definition 3.1. $\square$

6. Cohomology of stalks and cohomology of fibers

Lemma 6.1. *Let $R{\to}S$ be a homomorphism of Noetherian rings. Assume $\mathbb{M}$ is finitely generated over S and N is finitely generated over R.*

Then the S-modules $\mathrm{Tor}_q^R(\mathbb{M},N)$ are finitely generated over S and

(i) $\mathrm{Supp}(\mathrm{Tor}_q^R(\mathbb{M},N))\subseteq \mathrm{Supp}(\mathbb{M}\otimes_R N)$ for any q,

(ii) if further $(R,\mathfrak{m})$ is local, $S=R[X_1,\dots,X_n]$, with $\deg X_i>0$ and $\mathbb{M}$ is a graded S-module, then $\mathrm{Supp}(\mathrm{Tor}_q^R(\mathbb{M},R/\mathfrak{m}))\subseteq \mathrm{Supp}(\mathrm{Tor}_1^R(\mathbb{M},R/\mathfrak{m}))$ for any $q\geq 1$.

Proof. First the modules $\mathrm{Tor}_q^R(\mathbb{M},N)$ are finitely generated over S by [BA, X §6 N°4 Corollaire]. Second, $\mathrm{Supp}(\mathbb{M}\otimes_R N)=\mathrm{Supp}(\mathbb{M})\cap\varphi^{-1}(\mathrm{Supp}(N))$, where $\varphi:\mathrm{Spec}(S)\longrightarrow\mathrm{Spec}(R)$ is the natural map induced by $R\longrightarrow S$, by [BAC, II §4 N°4, Proposition 18 & Proposition 19], since $\mathbb{M}\otimes_R N=\mathbb{M}\otimes_S(N\otimes_R S)$. For $\mathfrak{P}\in\mathrm{Spec}(S)$, set $\mathfrak{p}:=\varphi^{-1}(\mathfrak{P})$. Then $\mathrm{Tor}_q^R(\mathbb{M},N)_{\mathfrak{P}}=\mathrm{Tor}_q^{R_{\mathfrak{p}}}(\mathbb{M}_{\mathfrak{P}},N_{\mathfrak{p}})$ vanishes if either $\mathbb{M}_{\mathfrak{P}}=0$ or $N_{\mathfrak{p}}=0$.

For (ii), we can reduce to the case of a local morphism by localizing S and $\mathbb{M}$ at $\mathfrak{m}+S_+$. In this local situation, $\mathrm{Tor}_1^R(\mathbb{M},R/\mathfrak{m})=0$ if and only if $\mathbb{M}$ is A-flat by [An, Lemme 58], which proves our claim by localization at primes $\mathfrak{P}$ such that $\varphi^{-1}(\mathfrak{P})=\mathfrak{m}$. ☐

Let R be a commutative ring, N be a R-module, $S:=R[X_1,\dots,X_n]$ a polynomial ring over R and $\mathbb{M}$ be a graded S-module.

Lemma 6.2. *There are two converging spectral sequences of graded S-modules with same abutment $H^\bullet$ and with respective second terms*

$$'_2E^p_q=H^p_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},N))\Rightarrow H^{p-q}$$

and

$$''_2E^p_q=\mathrm{Tor}_q^R(H^p_{S_+}(\mathbb{M}),N)\Rightarrow H^{p-q}.$$

Let $d_N:=\max\{i\mid H^i_{S_+}(\mathbb{M}\otimes_R N)\neq 0\}$. If R is Noetherian, N is finitely generated over R and $\mathbb{M}$ is finitely generated over S, then

$$H^{d_N}_{S_+}(\mathbb{M}\otimes_R N)\simeq H^{d_N}_{S_+}(\mathbb{M})\otimes_R N$$

and $\mathrm{Tor}_q^R(H^i_{S_+}(\mathbb{M}),N)=H^i_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},N))=0$ *for any q if $i>d_N$.*

Proof. Let $F_\bullet$ be a free R-resolution of N. Consider the double complex $\mathcal{C}^\bullet_{S_+}(\mathbb{M}\otimes_R F_\bullet)=\mathcal{C}^\bullet_{S_+}(\mathbb{M})\otimes_R F_\bullet$, totalizing to $T^\bullet$ with $T^i=\oplus_{p-q=i}\mathcal{C}^p_{S_+}(\mathbb{M})\otimes_R F_q$. It gives rise to two spectral sequences abuting to the homology $H^\bullet$ of $T^\bullet$.

One has first terms $\mathcal{C}^p_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},N))$ and second terms $H^p_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},N))$.

The other spectral sequence has first terms $H^p_{S_+}(\mathbb{M})\otimes_R F_q$ and second terms $\mathrm{Tor}_q^R(H^p_{S_+}(\mathbb{M}),N)$. It provides the quoted spectral sequences.

Recall that if P is a finitely presented S-module, one has $\mathrm{cd}_{S_+}(P')\leq\mathrm{cd}_{S_+}(P)$ whenever $\mathrm{Supp}(P')\subseteq\mathrm{Supp}(P)$. This is proved in [DNT, 2.2] under the assumption that S is Noetherian and P' is finitely generated, which is enough for our purpose.

By Lemma 6.1 (i), $\mathrm{Supp}(\mathrm{Tor}_q^R(\mathbb{M},N)) \subseteq \mathrm{Supp}(\mathbb{M}\otimes_R N)$ for any q, which implies that $H^i_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},N)) = 0$ for any q if $i > d_N$. It follows that $H^{d_N} = H^{d_N}_{S_+}(\mathbb{M}\otimes_R N)$ and $H^i = 0$ for $i > d_N$.

On the other hand, choose i maximal such that $H^i_{S_+}(\mathbb{M}) \otimes_R N \neq 0$. Then $\mathrm{Tor}_q^R(H^p_{S_+}(\mathbb{M}), N) = 0$ for any q if $p > i$, because $H^p_{S_+}(\mathbb{M})_\mu$ is a finitely generated R-module for every μ, and hence $H^i = H^i_{S_+}(\mathbb{M}) \otimes_R N \neq 0$ and $H^j = 0$ for $j > i$. The conclusion follows. □

Proposition 6.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $S := R[X_1,\ldots,X_n]$ be a polynomial ring over R with $\deg X_i > 0$ and $\mathbb{M}$ be a finitely generated graded S-module. Set $\mathrm{M} := \mathbb{M} \otimes_R k$ and $d := \dim \mathrm{M}$. Then one has,*

(i) The natural graded map $H^d_{S_+}(\mathbb{M}) \otimes_R k \longrightarrow H^d_{S_+}(\mathrm{M})$ is an isomorphism and $d = \max\{i \mid H^i_{S_+}(\mathbb{M}) \neq 0\}$. In particular,

$$a^d(\mathbb{M}) = a^d(\mathrm{M}) \in \mathbf{Z}.$$

(ii) For any integers μ and ℓ, if $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M},k)) \leq \ell + 1$ then

$$\{H^i_{S_+}(\mathbb{M})_\mu = 0, \forall i \geq \ell\} \Rightarrow \{H^i_{S_+}(\mathrm{M})_\mu = 0, \forall i \geq \ell\},$$

and both conditions are equivalent if $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M},k)) \leq \ell$. In particular, $\mathrm{reg}(\mathrm{M}) \leq \mathrm{reg}(\mathbb{M})$ if $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M},k)) \leq 1$ and equality holds if $\mathrm{depth}_{S_+}(\mathbb{M}) > 0$.

Proof. We consider the two spectral sequences in Lemma 6.2,

$$'_2E^p_q = H^p_{S_+}(\mathrm{Tor}_q^R(\mathbb{M},k)) \Rightarrow H^{p-q}$$

and

$$''_2E^p_q = \mathrm{Tor}_q^R(H^p_{S_+}(\mathbb{M}),k) \Rightarrow H^{p-q}.$$

Let $A := k[X_1,\ldots,X_n]$. The module $\mathrm{Tor}_q^R(\mathbb{M},k)$ is a $R[X_1,\ldots,X_n]$-module of finite type, annihilated by $\mathfrak{m}$ and $\mathrm{ann}_S(\mathbb{M})$. Hence M is a graded A-module of finite type and $\mathrm{Tor}_q^R(\mathbb{M},k)$ is a graded $(A/\mathrm{ann}_A(\mathrm{M}))$-module of finite type, for any q.

Notice that $d = \mathrm{cd}_{S_+}(\mathrm{M}) = \mathrm{cd}_{A_+}(\mathrm{M})$. It follows that $'_2E^p_q = 0$ if $p > d$, and $'_2E^d_0 \neq 0$.

By Lemma 6.2, $''_2E^p_q = 0$ for all q if $p > d$, in particular $H^p_{S_+}(\mathbb{M})_\mu \otimes_R k = 0$ for any μ if $p > d$. Hence $H^p_{S_+}(\mathbb{M})_\mu = 0$ for any μ if $p > d$. In other words, $H^p_{S_+}(\mathbb{M}) = 0$ for any $p > d$.

The same lemma shows that $H^d_{S_+}(\mathrm{M}) = H^d_{S_+}(\mathbb{M}) \otimes_R k$, and finishes the proof of (i).

For (ii) let μ be an integer. We prove the result by descending recursion on ℓ from the case $\ell = d$, which we already proved.

Assume the results hold for $\ell+1$. Recall that, for any p, he map ${}'_r d^{p-r}_{1-r}$: ${}'_r E^{p-r}_{1-r} \longrightarrow {}'_r E^p_0$ is zero for $r \geq 2$ and the map ${}''_r d^p_0 : {}''_r E^p_0 \longrightarrow {}''_r E^{p+1-r}_{-r}$ is the zero map for $r \geq 1$.

If $H^i_{S_+}(\mathbb{M})_\mu = 0, \forall i \geq \ell$, then $({}''_2 E^p_q)_\mu = 0$ for $p \geq \ell$ and all q. As ${}''_2 E^p_q = 0$ for $q < 0$, it follows that $({}''_2 E^p_q)_\mu = 0$ if $p - q \geq \ell$.

If $\mathrm{cd}_{S_+}(\mathrm{Tor}^R_1(\mathbb{M}, k)) \leq \ell + 1$ then ${}'_2 E^p_q = 0$ for $p \geq \ell + 2$ and $q > 0$ by Lemma 6.1 (ii), in particular the map

$$({}'_r d^\ell_0)_\mu : ({}'_r E^\ell_0)_\mu \longrightarrow ({}'_r E^{\ell+r}_{r-1})_\mu$$

is the zero map for any $r \geq 2$, hence $H^\ell_{S_+}(\mathbb{M})_\mu = ({}'_2 E^\ell_0)_\mu = ({}'_\infty E^\ell_0)_\mu = 0$ as claimed.

For the reverse implication, the hypothesis implies that ${}'_2 E^p_q = 0$ if $q \geq 1$ and $p \geq \ell + 1$ by Lemma 6.1 (ii). Hence $({}'_2 E^p_q)_\mu = 0$ for $p - q \geq \ell$ if $H^\ell_{S_+}(\mathbb{M})_\mu = 0$. By recursion hypothesis, $H^p_{S_+}(\mathbb{M})_\mu \otimes_R k = 0$ for $p \geq \ell + 1$. Hence $({}'_2 E^p_q)_\mu = \mathrm{Tor}^R_q(H^p_{S_+}(\mathbb{M})_\mu, k) = 0$ for $p \geq \ell + 1$ and all q. It implies that $H^\ell_{S_+}(\mathbb{M})_\mu \otimes_R k = ({}''_\infty E^\ell_0)_\mu = 0$, and proves the claimed equivalence.

Finally, recall that $H^i_{S_+}(\mathbb{M}) = 0$ for $i < \mathrm{depth}_{S_+}(\mathbb{M})$.

□

Lemma 6.4. *Let p be an integer. In the setting of Proposition 6.3, assume that R is reduced and S is standard graded. Then the following are equivalent :*

(i) $\dim(\mathrm{Tor}^R_1(\mathbb{M}, k)) \leq p$*,*

(ii) The Hilbert polynomials of $\mathbb{M} \otimes_R k$ *and* $\mathbb{M} \otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ *differ at most by a polynomial of degree* $< p$*, for any* $\mathfrak{p} \in \mathrm{Spec}(R)$*.*

Proof. We induct on p. The result is standard when $p = 0$, see for instance [Ha, III 9.9] or [Ei, Ex. 20.14].

Assume (i) and (ii) are equivalent for $p - 1 \geq 0$, for any Noetherian local domain, standard graded polynomial ring over it and graded module of finite type.

Set $K := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, $M_K := \mathbb{M} \otimes_R K$, $A := k[X_1, \ldots, X_n]$ and $B := K[X_1, \ldots, X_n]$. Consider variables $U_1, \ldots, U_n$ (of degree 0) and let $\ell := U_1 X_1 + \cdots + U_n X_n$. By the Dedekind-Mertens lemma,

(a) $\quad \ker(\mathbb{M}[U] \xrightarrow{\times \ell} \mathbb{M}[U](1)) \subseteq H^0_{S_+}(\mathbb{M})[U]$,

(b) $\quad \ker(M[U] \xrightarrow{\times \ell} M[U](1)) \subseteq H^0_{A_+}(M)[U]$,

(c) $\quad \ker(M_K[U] \xrightarrow{\times \ell} M_K[U](1)) \subseteq H^0_{B_+}(M_K)[U]$,

(d) $\quad \ker(\mathrm{Tor}^{R'}_1(\mathbb{M}, k)[U] \xrightarrow{\times \ell} \mathrm{Tor}^{R'}_1(\mathbb{M}, k)[U](1)) \subseteq H^0_{A_+}(\mathrm{Tor}^R_1(\mathbb{M}, k))[U]$.

Let $R' := R(U)$ be obtained from $R[U]$ by inverting all polynomials whose coefficient ideal is the unit ideal, and denote by N' the extension of scalars from R to R' for the module N. Recall that $R(U)$ is local with maximal ideal $\mathfrak{m}R(U)$, residue field $k' = k(U)$ and $K' = K(U)$ –see for instance [Na, p. 17]. As the zero local cohomology modules above vanish in high degrees, (b) and (c) show that $\mathbb{M}'/\ell\mathbb{M}'$

satisfies condition (ii) of the Lemma for $p-1$, R' and $R'[X_1,\dots,X_n]$. Now (a) and (d) provide an exact sequence for $\mu \gg 0$:

$$0 \longrightarrow \mathrm{Tor}_1^R(\mathbb{M}',k')_{\mu-1} \xrightarrow{\times \ell} \mathrm{Tor}_1^{R'}(\mathbb{M}',k')_\mu \longrightarrow \mathrm{Tor}_1^{R'}(\mathbb{M}'/\ell\mathbb{M}',k')_\mu \longrightarrow 0$$

which shows in particular that

$$\dim \mathrm{Tor}_1^{R'}(\mathbb{M}'/\ell\mathbb{M}',k') = \dim \mathrm{Tor}_1^{R'}(\mathbb{M}',k') - 1 = \dim \mathrm{Tor}_1^{R}(\mathbb{M},k) - 1,$$

and proves our claim by induction. □

Remark 6.5. *If the grading is not standard, a quasi-polynomial is attached to any finitely generated graded module, and in Lemma 6.4 property (ii) should be replaced by the following :*

(ii) the difference between the quasi-polynomials of $\mathbb{M}\otimes_R k$ *and* $\mathbb{M}\otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ *is a quasi-polynomial of degree* $< p$ *for any* $\mathfrak{p}\in \mathrm{Spec}(R)$.

The degree of a quasi-polynomial is the highest degree of the polynomials that defines it. The proof of [Ha, III 9.9] *extends to this case when* $p = 0$, *and our proof extends after a slight modification : in the proof of (ii)⇒(i), one should take* $\ell := U_1X_1^{w/w_1} + \cdots + U_nX_n^{w/w_n}$, *where* $w_i := \deg(X_i)$ *and* $w := lcm(w_1,\dots,w_n)$.

The local statement of Lemma 6.4 implies a global statement, by comparing Hilbert functions at generic points of the components and at closed points. We state it below in a ring theoretic form.

Proposition 6.6. *Let p be an integer, R be a reduced commutative ring, S be a Noetherian positively graded polynomial ring over R and $\mathbb{M}$ be a finitely generated graded S-module. Then the following are equivalent :*

(i) $H^i_{S_+}(\mathrm{Tor}_1^S(\mathbb{M},R)) = 0$, *for* $i > p$,

(ii) for any two ideals $\mathfrak{p}\subset\mathfrak{q}$ *in* $\mathrm{Spec}(R)$, *the quasi-polynomials of* $\mathbb{M}\otimes_R R/\mathfrak{p}$ *and* $\mathbb{M}\otimes_R R/\mathfrak{q}$ *differ by a quasi-polynomial of degree* $< p$,

(iii) over a connected component of $\mathrm{Spec}(R)$ *the quasi-polynomials of two fibers differ by a quasi-polynomial of degree* $< p$.

In parallel to the definition of the regularity over a scheme, we define the fiber-regularity $\mathfrak{freg}$ as the maximum over the fibers of their regularity.

Definition 6.7. *In the setting of Definition 3.1,*

$$\tilde{a}_i(\mathcal{F}) := \sup_{y\in Y} a^i(\mathcal{F}\otimes_{\mathcal{O}_Y} k(y)), \quad \mathfrak{freg}(\mathcal{F}) := \max_i\{\tilde{a}_i(\mathcal{F})+i\},$$

and $\mathfrak{freg}(Z) := \max_{i\geq 1}\{\tilde{a}_i(\mathrm{Sym}_Y(\mathcal{E})/\mathcal{I}_Z)+i\}$.

Notice that $\mathfrak{freg}(\mathcal{F})$ is finite if Y is covered by finitely many affine charts and $\mathcal{F}$ is coherent. This holds since the regularity of a graded module over a polynomial ring over a field is bounded in terms of the number of generators and the degrees of generators and relations (see *e.g.* [CFN, 3.5]).

Remark 6.8. *It follows from Theorem 5.3 that the dimension of the fibers of π are bounded above by the cohomological dimension of A/I.*

From the comparison of cohomology of stalks and cohomology of fibers, we get from Theorem 5.3 the following result,

Theorem 6.9. *Let $A := A_0[x_0, \dots, x_n]$ be a positively graded Noetherian algebra and I be a graded A-ideal generated by $m+1$ forms of degree d. Set $Y := \mathrm{Spec}(A_0)$ and $X := \mathrm{Proj}(A/I) \subset \mathrm{Proj}(A) \subseteq \tilde{\mathbf{P}}^n_Y$. Let $\phi : \tilde{\mathbf{P}}^n_Y \setminus X \longrightarrow \mathbf{P}^m_Y$ be the corresponding rational map, W be the closure of the image of ϕ, and*

$$\Gamma \subset \tilde{\mathbf{P}}^n_W \subseteq \tilde{\mathbf{P}}^n_{\mathbf{P}^m_Y} = \tilde{\mathbf{P}}^n_Y \times_Y \mathbf{P}^m_Y$$

be the closure of the graph of ϕ. Let $\pi : \Gamma \longrightarrow W$ be the projection induced by the natural map $\tilde{\mathbf{P}}^n_{\mathbf{P}^m_Y} \longrightarrow \mathbf{P}^m_Y$. Then,

(i) $\lim_{t\to+\infty}(\mathrm{reg}((I^t)^{sat}) - dt) = \max_{i\geq 2}\{a^i(\Gamma) + i\}$.

(ii) If π admits a fiber $Z \subseteq \tilde{\mathbf{P}}^n_{\mathrm{Spec}\,\mathfrak{K}}$ of dimension $i-1$, then

$$\lim_{t\to\infty}(a^i(I^t) + i - td) \geq a^i(Z) + i = \tilde{a}^i(Z) + i \geq 0.$$

(iii) Let δ be the maximal dimension of a fiber of π. Then,

$$a^{\delta+1}(I^t) - td = a^{\delta+1}(\Gamma) = \tilde{a}^{\delta+1}(\Gamma),\ \forall t \gg 0.$$

(iv) If $\pi : \Gamma \to \mathbf{P}^m_Y$ is finite, then

$$\mathrm{reg}(I^t) = a^1(I^t) + 1 = \mathfrak{f}\mathrm{reg}(\Gamma) + td, \quad \forall t \gg 0$$

and $\lim_{t\to\infty}(a^i(I^t) - td) = -\infty$ *for $i \neq 1$.*

(v) If A is standard graded and reduced, $\pi : \Gamma \to \pi(\Gamma)$ has fibers of dimension one, all of same degree, then

$$\mathrm{reg}(I^t) = \mathfrak{f}\mathrm{reg}(\Gamma) + td,\ \forall \mu \gg 0,$$

and $\lim_{t\to\infty}(a^i(I^t) - td) = -\infty$ *for $i \geq 2$. Furthermore*

$$\lim_{t\to\infty}(a^1(I^t) - td) \geq \tilde{a}^1(\Gamma)$$

and equality holds if $\mathrm{reg}(I^t) = a^1(I^t) + 1$ *for $t \gg 0$.*

(vi) If A is reduced and, for every connected component T of $\pi(\Gamma)$, the Hilbert quasi-polynomials of fibers of π over any two points in $\mathrm{Spec}(T)$ differ by a periodic function, then

$$\mathrm{reg}(I^t) = \mathfrak{f}\mathrm{reg}(\Gamma) + td,\ \forall \mu \gg 0.$$

Proof. (i) is a direct corollary of Theorem 5.3. Statements (ii), (iii) and (iv) follow from Theorem 5.3 and Proposition 6.3 (i).

Statements (v) and (vi) follow from Theorem 5.3, Proposition 6.3 (ii) – notice that $\operatorname{depth}_{S_+}(\mathcal{R}_I) \geq 1$ – and the equivalence $(i) \Leftrightarrow (iii)$ in Proposition 6.6 applied on the affine charts covering $\pi(\Gamma)$. $\square$

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