# Connectedness of Strong $k$-Colour Graphs

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**Abstract**

For a positive integer $k$ and a graph $G$, we consider proper vertex-colourings of $G$ with $k$ colours in which all $k$ colours are actually used. We call such a colouring a strong $k$-colouring. The strong $k$-colour graph of $G$, $S_k(G)$, is the graph that has all the strong $k$-colourings of $G$ as its vertex set, and two colourings are adjacent in $S_k(G)$ if they differ in colour on only one vertex of $G$. In this paper, we show some results related to the question: For what $G$ and $k$ is $S_k(G)$ connected?

**Keywords**: strong $k$-vertex-colouring, strong $k$-colour graph, strong colour graph.

## 1 Introduction

Throughout this paper a graph $G$ is finite, simple, and loopless, and we also usually assume that $G$ is connected. Most of our terminology and notation will be standard and can be found in any textbook on graph theory such as [5] and [11]. For a positive integer $k$ and a graph $G$, the *$k$-colour graph* of $G$, denoted $C_k(G)$, is the graph that has the proper $k$-vertex-colourings of $G$ as its vertex set, and two such colourings are joined by an edge in $C_k(G)$ if they differ in colour on only one vertex of $G$.

We now introduce a subgraph of $C_k(G)$, called the *strong $k$-colour graph* of $G$, denoted $S_k(G)$. Its vertex set contains only proper $k$-colourings in which all $k$ colours actually appear, and we call such a colouring a *strong $k$-colouring*.

Questions regarding the connectivity of a $k$-colour graph have applications in reassignment problems of the channels used in cellular networks; see, e.g., [1, 7, 8]. For some applications, it is required that all channels in a range are actually used. Such a labelling is sometimes called a “no-hole” or “consecutive” labelling; see, e.g., [6, 9]. In terms of colourings, this corresponds to a strong $k$-colouring. And asking questions about the possibility to reassign channels in a cellular network can be done in such a way that all available channels are actually used. These problems can be expressed in finding paths in the strong $k$-colour graph.

Questions related to the connectivity of a $k$-colour graph have been studied extensively: [2, 3, 4, 10]. In this note we initiate similar research on the connectivity of strong $k$-colour graphs: Given a positive integer $k$ and a graph $G$, is $S_k(G)$ connected? As an example, in Figures 1 and 2, we show the strong 3-colour graph of the paths with 4 and 5 vertices, respectively. One is not connected while the other is connected.

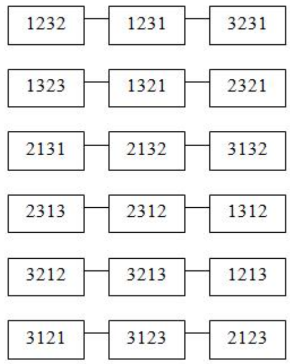



Figure 1: The strong colour graph $S_3(P_4)$.

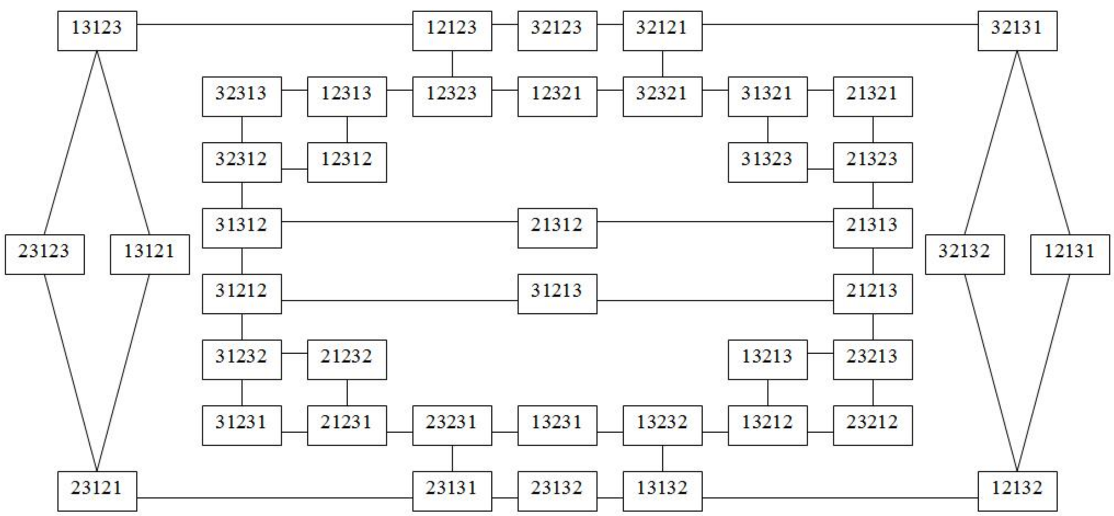



Figure 2: The strong colour graph $S_3(P_5)$.

We usually use lower case Greek letters $\alpha, \beta, \gamma, \dots$ to denote specific colourings, and lower case Latin $a, b, c, \dots$ to denote specific colours.

To avoid trivial cases, we will always assume that $k$ is greater than or equal to the chromatic number of $G$, and the number of vertices of $G$ is at least $k+1$.

## 2 General Results

For all $k \geq 2$ and $m, n \geq 1$, let $\alpha$ be a strong $k$-vertex-colouring of a complete bipartite graph $K_{m,n}$. Each colour that appears in one part of the partition cannot be used in the other part. Now we choose one colour from each part and recolour the graph by swapping these two colours on each vertex coloured with one of these colours. Let $\beta$ be the resulting colouring, so $\beta$ is strong as well. It is easy to see that there is no path in $S_k(K_{m,n})$ from $\alpha$ to $\beta$. Thus

$S_k(K_{m,n})$ is not connected for all $k \geq 2$ and $m, n \geq 1$. This gives the following lemma.

**Lemma 1.** *Let $G$ be a connected, $k$-colourable graph such that $S_k(G)$ is connected, with $k \geq 2$. Then $|V(G)| \geq k+1$, and $G$ does not contain a complete bipartite graph as a spanning subgraph.*

**Theorem 2.** *Let $G$ be a connected, $k$-colourable graph such that $S_k(G)$ is connected, with $k \geq 2$. Suppose the graph $G^*$ is obtained from $G$ by adding a new vertex $v^*$ and joining it to $j$ vertices in $V(G)$, with $1 \leq j \leq k-2$. Then $S_k(G^*)$ is connected.*

We will show in the next section that the strong 3-colour graph of the $n$-vertex path, $S_3(P_n)$, is connected if and only if $n \geq 5$. Now add a new vertex and join it to the first and the last vertex of the path, forming to an $(n+1)$-vertex cycle. We will show in Section 4 that the strong 3-colour graph of the $n$-vertex cycle, $S_3(C_n)$, is not connected for all $n$. This example shows that the restriction $j \leq k-2$ of Theorem 2 is optimal.

***Proof of Theorem 2.*** Let $\alpha^*$ and $\beta^*$ be strong $k$-colourings of $G^*$. We show that there always exists a walk in $S_k(G^*)$ from $\alpha^*$ to $\beta^*$. We say that a colouring of $G^*$ is *good* if all $k$ colours appear on $V(G)$.

First suppose that $\alpha^*$ and $\beta^*$ are good. By ignoring the vertex $v^*$, let $\alpha$ and $\beta$ be the strong $k$-colourings of $G$ obtained from $\alpha^*$ and $\beta^*$, respectively. Since $S_k(G)$ is connected, there is a path from $\alpha$ to $\beta$ in $S_k(G)$. We just follow the recolouring steps of that path to form a walk from $\alpha^*$ to $\beta^*$ in $S_k(G^*)$. The only extra steps happen when we want to recolour a neighbour $u$ of $v^*$ to the same colour as $v^*$. Since $d_{G^*}(v^*) = j \leq k-2$, we can always recolour $v^*$ to a colour different from any of the colours appearing in its neighbourhood and its current colour. After recolouring $v^*$, we can recolour $u$, and continue the walk. This walk in $S_k(G^*)$ finishes in a colouring in which the vertices in $V(G)$ have the same colour as they have in $\beta^*$. If necessary, we can do one recolouring of $v^*$ to its colour in $\beta^*$, completing the walk in $S_k(G^*)$ from $\alpha^*$ to $\beta^*$.

If $\alpha^*$ is not good, then below we show that we can always find a path in $S_k(G^*)$ from $\alpha^*$ to some good colouring ( and if necessary, we do the same for $\beta^*$ ). Together with the method described in the previous paragraph, this completes the proof.

So we now assume that in $\alpha^*$ every vertex of $G$ has received one of $k-1$ colours while $v^*$ has the remaining colour. Let $W$ be the set of vertices in $V(G)$ that do not have a unique colour in $G$ for the colouring $\alpha^*$. Since $|V(G)| \geq k+1$, $W$ is not empty.

**Case 1**: There is a vertex $w \in W$ not adjacent to $v^*$.
By the definition of $W$, there is a vertex $w' \in W$ such that $w$ and $w'$ have the same colour in $\alpha^*$. Then we recolour $w$ to the same colour as $v^*$. The resulting colouring is good.

**Case 2**: All vertices in $W$ are adjacent to $v^*$.
Additionally, define $U = N(v^*) \setminus W$. and $X = V(G) \setminus N(v^*)$. Note that all vertices in $X$ have a unique colour in $\alpha^*$.

**Subcase 2.1**: There is a vertex $x \in X$ that is not adjacent to some vertex $w \in W$.
Again, there is a vertex $w' \in W$ such that $w$ and $w'$ have the same colour in $\alpha^*$. Then we first recolour $w$ to the same colour as $x$, and then recolour $x$ to the same colour as $v^*$. Again, this gives a good colouring.

**Subcase 2.2**: Every vertex in $X$ is adjacent to every vertex in $W$.
Because of Lemma 1, $U$ is not empty (otherwise, the pair $(X, W)$ would form the parts of a spanning complete bipartite subgraph of $G$). Suppose there is some vertex $u \in U$ that is not adjacent to some vertex $w \in W$ and not adjacent to some vertex $x \in X$. Then we can recolour $w$ to the colour of $u$ (this is possible since there is another vertex $w' \in W$ with the same colour as $w$). Then recolour $u$ to the same colour as $x$, and lastly $x$ to the same colour as $v^*$. It is easy to check that the remaining colouring is good.

So we are left with the case that each vertex in $U$ is adjacent to every vertex in $W$ or to every vertex in $X$. Let $U_W$ be the set of vertices in $U$ that are adjacent to every vertex in $W$, and $U_X = U \setminus U_W$. Then the pair $(X \cup U_W, W \cup U_X)$ forms the parts of a spanning complete bipartite subgraph of $G$. Because of Lemma 1, this contradicts that $S_k(G)$ is connected. □

**Theorem 3.** *Let $G$ be a connected $k$-colourable graph so that $S_k(G)$ is connected, with $k \geq 2$. Let $v$ be a vertex of $G$ with neighbourhood $N(v)$. Suppose the graph $G^*$ is obtained from $G$ by adding a new vertex $v^*$ and joining $v^*$ to the vertices in $N^*$ for some $N^* \subseteq N(v)$, $N^* \neq \varnothing$. Then $S_k(G^*)$ is connected.*

*Proof.* Let $\alpha^*$ and $\beta^*$ be strong $k$-colourings of $G^*$. We show that there always exists a walk in $S_k(G^*)$ from $\alpha^*$ to $\beta^*$. We say that a colouring of $G^*$ is *good* if $v$ and $v^*$ are labelled with the same colour.

First suppose that $\alpha^*$ and $\beta^*$ are good. By ignoring the vertex $v^*$, let $\alpha$ and $\beta$ be the strong $k$-colourings of $G$ obtained from $\alpha^*$ and $\beta^*$, respectively. Since $S_k(G)$ is connected, there is a path from $\alpha$ to $\beta$ in $S_k(G)$. We just follow the recolouring steps of that path to form a walk from $\alpha^*$ to $\beta^*$ in $S_k(G^*)$. The only extra step happens when we recolour $v$. In the next step we immediately recolour $v^*$ to the same colour as $v$ just received. It is easy to check that all these recolourings are allowed and give a walk in $S_k(G^*)$ from $\alpha^*$ to $\beta^*$, completing the proof.

Assume that $\alpha^*$ is not good. Below we show that we always can find a path in $S_k(G^*)$ from $\alpha^*$ to some good colouring (and if necessary, we do the same for $\beta^*$). Together with the method described in the previous paragraph, this completes the proof.

If there is a vertex $u \in V(G) \setminus \{v\}$ with the same colour as $v^*$, then we just recolour $v^*$ to the same colour as $v$. This gives a good colouring.

So we now suppose that in $\alpha^*$ every vertex of $G$ has received one of $k-1$ colours while $v^*$ has the remaining colour. Remind that $v$ and $v^*$ received different colours in $\alpha^*$. Let $W$ be the set of vertices in $V(G)$ that did not receive a unique colour in $G$ for the colouring $\alpha^*$. Since $|V(G)| \geq k+1$, $W$ is not empty.

**Case 1**: There is a vertex $w \in W$ not adjacent to $v^*$.
By the definition of $W$, there is a vertex $w' \in W$ such that $w$ and $w'$ have the same colour in $\alpha^*$. Hence we can recolour $w$ to the same colour as $v^*$, and then recolour $v^*$ to the same colour as $v$. The resulting colouring is good.

**Case 2**: All vertices in $W$ are adjacent to $v^*$.
Additionally, define $U = N(v^*) \setminus W$. and $X = V(G) \setminus N(v^*)$. Note that all vertices in $X$ have a unique colour in $\alpha^*$.

**Subcase 2.1**: There is a vertex $x \in X$ that is not adjacent to some vertex $w \in W$.
Again, there is a vertex $w' \in W$ such that $w$ and $w'$ have the same colour in $\alpha^*$. Then we first recolour $w$ to the same colour as $x$. Then recolour $x$ to the same colour as $v^*$, and lastly recolour $v^*$ to the same colour as $v$. Again, this gives a good colouring.

**Subcase 2.2**: Every vertex in $X$ is adjacent to every vertex in $W$.
Because of Lemma 1, $U$ is not empty (otherwise, the pair $(X, W)$ would form the parts of a spanning complete bipartite subgraph of $G$). Suppose there is some vertex $u \in U$ that is not adjacent to some vertex $w \in W$ and not adjacent to some vertex $x \in X$. Then we can recolour $w$ to the colour of $u$ (this is possible since there is another vertex $w' \in W$ with the same colour as $w$). Then recolour $u$ to the same colour as $x$, $x$ to the same colour as $v^*$, and lastly recolour $v^*$ to the same colour as $v$. It is easy to check that the remaining colouring is good.

So we are left with the case that each vertex in $U$ is adjacent to every vertex in $W$ or to every vertex in $X$. Let $U_W$ be the set of vertices in $U$ that are adjacent to every vertex in $W$, and $U_X = U \setminus U_W$. Then the pair $(X \cup U_W, W \cup U_X)$ forms the parts of a spanning complete bipartite subgraph of $G$. Because of Lemma 1, this contradicts that $S_k(G)$ is connected. □

It is easy to see that in the normal colour graph $C_k(G)$ there always is a path from any proper $k$-vertex-colouring to some strong $k$-vertex-colouring. This shows the following.

**Lemma 4.** *If $S_k(G)$ is connected, then $C_k(G)$ is also connected.*

## 3 The Strong $k$-Colour Graph of Paths

In this section, we prove that the strong $k$-colour graph of a path with $n$ vertices, $S_k(P_n)$, is connected if and only if $k \geq 3$, $n \geq 5$, and $n \geq k+1$.

First, suppose we colour a path $P_n$, $n \geq 2$, with two colours. It is easy to see that there are only two strong 2-vertex-colourings of $P_n$, and they are not adjacent in $S_2(P_n)$. Thus $S_2(P_n)$ is not connected for all $n \geq 2$.

For $k = 3$, we have already seen in Figures 1 and 2 that $S_3(P_4)$ is not connected, but $S_3(P_5)$ is connected. It is somewhat more work to show that $S_4(P_5)$ is connected.

**Proposition 5.** *The strong colour graph $S_4(P_5)$ is connected.*

*Proof.* In any strong 4-colouring of $P_5$, there are only two vertices with the same colour. Let $\alpha$ be a strong 4-colouring of $P_5 = v_1 v_2 ... v_5$. We call $\alpha$ an *$a$-standard colouring* if $\alpha(v_1) = \alpha(v_5) = a$.

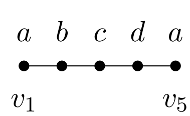
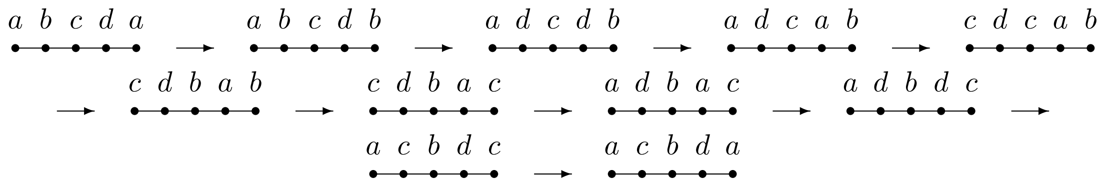



Figure 3: An $a$-standard colouring.

We will prove the proposition by combining one or more of the following three steps.

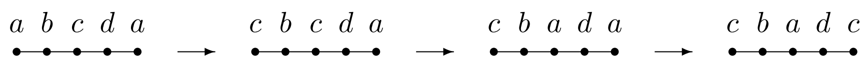
**Step 1**: There is a path from $(a, b, c, d, a)$ to any other $a$-standard colouring.
We will first show that there is a path from $(a, b, c, d, a)$ to $(a, c, b, d, a)$:



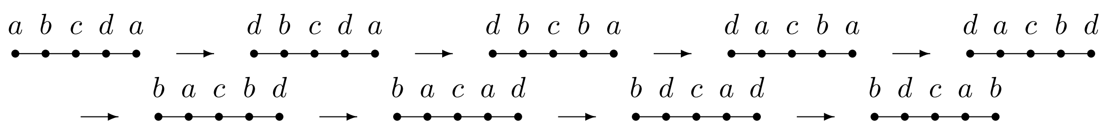

By symmetry, there is a path from $(a, b, c, d, a)$ to $(a, b, d, c, a)$ as well.

Now we consider $a$-standard colourings as permutations of $\{b, c, d\}$. Note that all these permutations can be generated by the transpositions $(b, c)$ and $(c, d)$. Therefore, since we can find a path from $(a, b, c, d, a)$ to $(a, c, b, d, a)$, and from $(a, b, c, d, a)$ to $(a, b, d, c, a)$, there is a path from $(a, b, c, d, a)$ to any other $a$-standard colouring.

**Step 2**: There is a path between any two types of standard colourings.
First, here is a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_3)$-standard colouring:



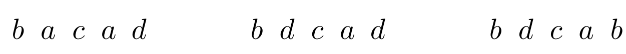

Next, a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_2)$-standard colouring:




By symmetry, there is also a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_4)$-standard colouring.

**Step 3**: Each colouring has a path to some standard colouring.
Let $\alpha$ be a strong 4-colouring of $P_5$. Then $\alpha$ has one of the following forms:


The first form already is an $a$-standard colouring; for the second and the third ones we just recolour the vertex $v_1$ to $d$; while for the fourth and the sixth ones we just recolour the vertex $v_5$ to $b$. Finally, for the fifth form the following is a path to a $b$-standard colouring:




It is straightforward to see that appropriate renaming of the colours and sequence of the paths in Steps $1-3$ will transform any strong 4-colouring of $P_5$ into any other strong 4-colouring. $\square$

We extend the last result by showing that $S_k(P_{k+1})$ is connected, for all $k \geq 4$.

**Proposition 6.** *For all $k \geq 4$, $S_k(P_{k+1})$ is connected.*

*Proof.* We will prove this by induction on $k$. We have already shown the proposition is true for $k = 4$.

Let $\alpha$ and $\beta$ be strong $k$-colourings of $P_{k+1} = v_1v_2...v_{k+1}$, for some $k \geq 5$. We can assume that in $\alpha$, the vertex $v_{k+1}$ has a unique colour. Otherwise, there is another vertex $v_i$ such that $\alpha(v_i) = \alpha(v_{k+1})$. Then just recolour $v_i$ to a colour different from $\alpha(v_{k+1})$. Next, we can assume that this unique colour on $v_{k+1}$ in $\alpha$ is $a$.

If the vertex $v_{k+1}$ is the only vertex coloured $a$ in $\beta$ as well, then we can just remove $v_{k+1}$. Let $\alpha'$ and $\beta'$ be the strong $k$-colourings of $P_k = v_1v_2...v_k$ obtained from $\alpha$ and $\beta$, respectively. Since, by induction, $S_{k-1}(P_k)$ is connected, there is a path from $\alpha'$ to $\beta'$ in $S_{k-1}(P_k)$. Using the same steps on $P_{k+1}$ gives a path from $\alpha$ to $\beta$ in $S_k(P_{k+1})$.

So we can assume that in $\beta$, $v_{k+1}$ is not coloured $a$ or is not the only vertex coloured $a$. We distinguish 4 cases.

**Case 1**: In $\beta$, $v_{k+1}$ is coloured $a$, but there is a second vertex $v_i$ coloured $a$ as well. Then just recolour $v_i$ to some different from $a$, and so $v_{k+1}$ is now the only vertex coloured $a$. We are done by the paragraph above.

**Case 2**: In $\beta$, $v_{k+1}$ and some other vertex $v_i$ have the same colour $b \neq a$, while a third vertex $v_j$ is coloured $a$.

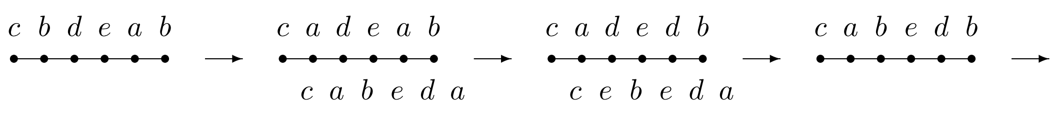
**Subcase 2.1**: $v_{k+1}$ and $v_j$ are not adjacent.
Then just recolour $v_{k+1}$ to $a$, and we are back to Case 1.

**Subcase 2.2**: $v_{k+1}$ and $v_j$ are adjacent, i.e., $j = k$.
Call a colouring of $S_k(P_{k+1})$ *good* if we can recolour $v_i$ to a colour which is not one of $\{\beta(v_{k-1}), \beta(v_k) = a, \beta(v_{k+1}) = b\}$.

Note that $\beta$ is good when $k \geq 6$, or $k = 5$ and $i \neq 2$. If $\beta$ is good, we can recolour $v_i$ to the colour $\beta(v_l)$ for some $l \notin \{i-1, i, i+1, k-1, k, k+1\}$, to obtain the strong $k$-colouring $\gamma$, Let $\delta$ be the strong $k$-colouring of $P_{k+1}$, obtained from $\gamma$ by swapping the colours of $v_i$ and $v_k$. By ignoring the vertex $v_{k+1}$, we can consider $\gamma$ and $\delta$ as strong $(k-1)$-colourings of $P_k$. Since $S_{k-1}(P_k)$ is connected, there is a path between these two colourings. We then apply this path to a path in $S_k(P_{k+1})$ from $\gamma$ to $\delta$.

Next, we will form a path from $\delta$ to a colouring in which vertex $v_{k+1}$ is the only vertex coloured $a$. Therefore, we also have a path from $\beta$ to this colouring. In $\delta$, we first recolour $v_l$ to $\beta(v_{k+1}) = b$ and then recolour $v_{k+1}$ to $a$. Finally, recolour vertex $v_i$, which is previously coloured $a$, to another colour, and we are done.

We now suppose that $\beta$ is not good, i.e., $k = 5$ and $i = 2$. Then there is a path from $\beta$ to a colouring in which $v_{k+1}$ is the only vertex coloured $a$.




**Case 3**: In $\beta$, $v_i$ and $v_j$ are coloured $a$ for some $i, j \neq k+1$.
Without loss of generality, we may assume that $v_i$ is not adjacent to $v_{k+1}$. Then we recolour $v_i$ to $\beta(v_{k+1})$, and we are back to Case 2.

**Case 4**: In $\beta$, $v_i$ and $v_j$ have the same colour $b \neq a$, a third vertex $v_\ell$ is coloured $a$, for some $i, j, \ell \neq k+1$.
Without loss of generality, we may assume that $v_i$ is not adjacent to $v_{k+1}$. Then we recolour $v_i$ to $\beta(v_{k+1})$, and we are back to Case 2. □

Combining it all, we get the promised result on the strong colour graph of paths.

**Theorem 7.** *The strong colour graph $S_k(P_n)$ is connected if and only if $k \geq 3$, $n \geq 5$ and $n \geq k+1$.*

*Proof.* We already have seen that $S_3(P_4)$ and $S_2(P_n)$, $n \geq 3$, are not connected, while $S_3(P_5)$ and $S_k(P_{k+1})$, $k \geq 4$, are connected. Applying Theorem 2 completes the proof. □

# 4 The Strong $k$-Colour Graph of Cycles

In this section we want to show that the strong $k$-colour graph of a cycle with $n$ vertices, $S_k(C_n)$, is connected if and only if $k \geq 4$, $n \geq 6$ and $n \geq k+1$. Before we prove the theorem, we prove some tools used in this proof.

To orient a cycle means to orient each edge on the cycle so that a directed cycle is obtained. If $C$ is a cycle, then by $\overrightarrow{C}$ we denote the cycle with one of the two possible orientations of $d$. Given a 3-colouring $\alpha$ using colours $\{1,2,3\}$, the weight of an edge $e = uv$ oriented from $u$ to $v$ is

$$w(\overrightarrow{uv},\alpha) \;=\; \begin{cases} +1, & \text{if } \alpha(u)\alpha(v) \in \{12,\ 23,\ 31\}; \\ -1, & \text{if } \alpha(u)\alpha(v) \in \{21,\ 32,\ 13\}. \end{cases}$$

The weight $W(\overrightarrow{C},\alpha)$ of an oriented cycle $\overrightarrow{C}$ is the sum of the weights of its oriented edges.

**Lemma 8.** (Cereceda et al. [2]) *Let $\alpha$ be a 3-colouring of a graph $G$ that contains a cycle $C$. If $W(\overrightarrow{C},\alpha) \neq 0$, then $C_k(G)$ is not connected.*

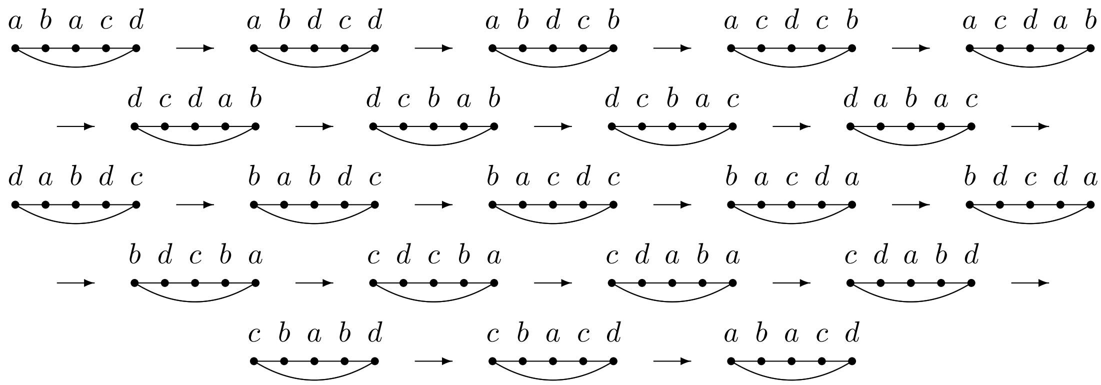
**Proposition 9.** *For all $n \geq 3$, $S_3(C_n)$ is not connected.*

*Proof.* By Lemmas 4 and 8, it is enough to find a strong 3-colouring $\alpha$ with $W(\overrightarrow{C_n},\alpha) \neq 0$. If $n = 3\,\ell$ for some positive integer $\ell$, the pattern 1,2,3,1,2,3,...,1,2,3 provides a 3-colouring $\alpha$ of $C_n$ with $W(\overrightarrow{C_n},\alpha) = n \neq 0$. For $n = 4$, it is easy to see that $S_3(C_4)$ is a graph with 12 isolated vertices. If $n = 3\,\ell + 1 > 4$, then we use the pattern 1,2,3,1,2,3,...,1,2,3,2, which gives $W(\overrightarrow{C_n},\alpha) = n - 4 \neq 0$. Finally, if $n = 3\,\ell + 2 \geq 5$, then we use the pattern 1,2,3,1,2,3,...,1,2,3,1,2, with $W(\overrightarrow{C_n},\alpha) = n - 2 \neq 0$. □

**Proposition 10.** *The strong colour graph $S_4(C_5)$ is not connected.*

*Proof.* For any strong 4-colouring of the 5-cycle $C_5$, there are only two vertices having the same colour. Thus each strong 4-vertex-colouring of $C_5$ can be recoloured only on these two vertices, and each of these two vertices can be recoloured to only one new colour ( since the two different colours of their neighbours are forbidden ). This means each colouring has degree two in $S_4(C_5)$.

Straightforward counting shows that $S_4(C_5)$ has 120 vertices. But each colouring in $S_4(C_5)$ is contained in some cycle of length 20. To see this, we start with some strong 4-colouring of $C_5$ and recolour :




By symmetry, we immediately get that $S_4(C_5)$ is a disjoint union of six copies of $C_{20}$, so it is not connected. □

**Proposition 11.** *The strong colour graph $S_5(C_6)$ is connected.*

*Proof.* In any strong 5-colouring of the 6-cycle $C_6$, there are only two vertices having the same colour. Let $\alpha$ be a strong 5-colouring of $C_6 = v_1v_2...v_5v_6v_1$. We call $\alpha$ an *a-standard colouring* if $\alpha(v_1) = \alpha(v_3) = a$.

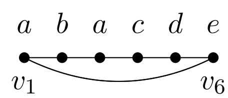
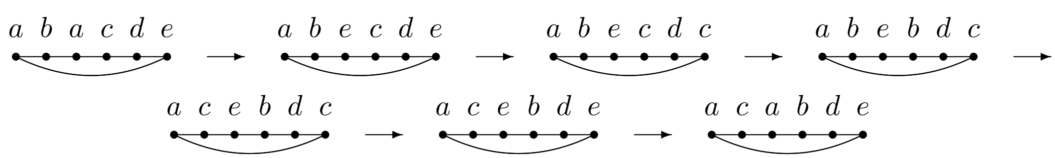


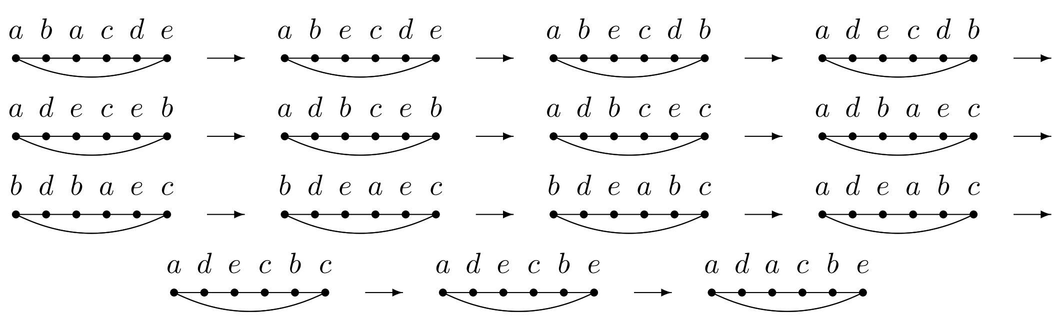

Figure 4: An $a$-standard colouring.

We will prove the proposition by showing the following three steps.

**Step 1**: There is a path from $(a, b, a, c, d, e)$ to any other $a$-standard colourings.
First, we will show that there is a path from $(a, b, a, c, d, e)$ to $(a, c, a, b, d, e)$:

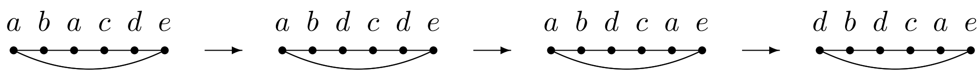



By symmetry, there is also a path from $(a, b, a, c, d, e)$ to $(a, e, a, c, d, b)$.
Next, we show that there is a path from $(a, b, a, c, d, e)$ to $(a, d, a, c, b, e)$:

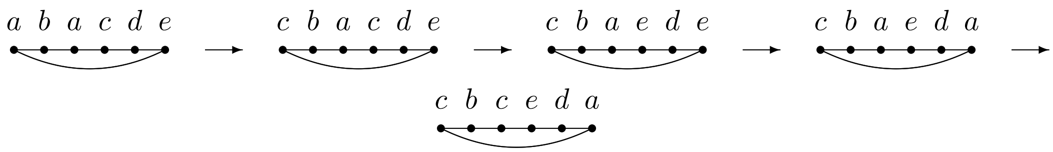


Now we consider $a$-standard colourings as permutations of $\{b, c, d, e\}$. Note that all these permutations can be generated by the transpositions $(b, c)$, $(b, e)$ and $(b, d)$. Therefore, since we can find a path from $(a, b, a, c, d, e)$ to $(a, c, a, b, d, e)$, from $(a, b, a, c, d, e)$ to $(a, e, a, c, d, b)$, and from $(a, b, a, c, d, e)$ to $(a, d, a, c, b, e)$, there is a path from $(e, a, e, b, c, d)$ to any other $a$-standard colourings.

**Step 2**: There is a path between any two types of standard colourings.
First, here is a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_5)$-standard colouring:

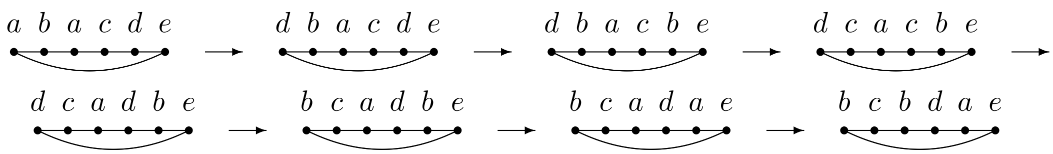



Next, a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_4)$-standard colouring:

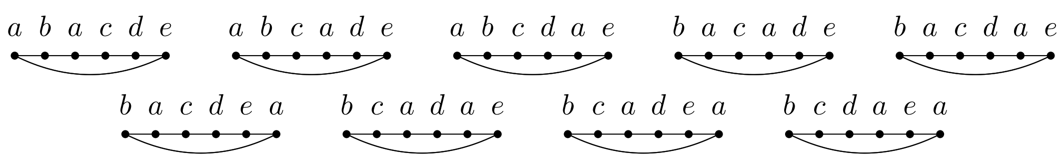



By symmetry, there is also a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_6)$-standard colouring.

And finally, a path from an $a$-standard colouring $\alpha$ to an $\alpha(v_2)$-standard colouring:




**Step 3**: Each colouring has a path to some standard colouring.
Let $\alpha$ be a strong 5-colouring of $C_6$. Then $\alpha$ has one of the following forms:




The first form is already an $a$-standard colouring. For the second and the third forms, we just recolour vertex $v_1$ to $c$, and for the seventh and eight forms, we just recolour vertex $v_3$ to $b$. For all the remaining colourings, we can find a path of length two to some standard colouring. We will leave checking that to the reader.

It is straightforward to see that appropriate renaming of the colours and sequence of the paths in Steps $1-3$ will transform any strong 5-colouring of $P_6$ into any other strong 5-colouring. $\square$

**Theorem 12.** *The strong colour graph $S_k(C_n)$ is connected if and only if $k \geq 4$, $n \geq 6$, and $n \geq k + 1$.*

*Proof.* We already have seen that $S_3(C_n)$, $n \geq 3$ and $S_4(C_5)$ are disconnected. From Theorems 2 and 7 we easily obtain that $S_k(C_n)$ is connected for all $k \geq 4$, $n \geq 6$, and $n \geq k+2$. Since $S_5(C_6)$ is connected, all that is left to prove is that $S_k(C_{k+1})$ is connected for all $k \geq 6$.

Let $k \geq 6$ and let $\alpha$ and $\beta$ be strong $k$-colourings of $C_{k+1} = v_1 v_2 \ldots v_{k+1} v_1$. In $\alpha$, there will be a vertex, say $v_1$, which has an unique colour, say colour $a$. We say that a strong $k$-colouring of $C_{k+1}$ is *good* if $v_1$ is the only vertex in $C_{k+1}$, which is coloured $a$. Thus $\alpha$ is good.

If $\beta$ is good as well, then remove $v_1$, and let $\alpha'$ and $\beta'$ be the strong $(k-1)$-colourings of $P_k$ obtained from $\alpha$ and $\beta$, respectively. Since $k \geq 6$, by Proposition 6 there is a path in $S_{k-1}(P_k)$ from $\alpha'$ to $\beta'$. Using the same steps gives a path from $\alpha$ to $\beta$ in $C_{k+1}$.

So suppose that $\beta$ is not good. As we colour the $k+1$ vertices of $C_{k+1}$ with $k$ colours, there are only two vertices having the same colour. We distinguish five cases.

**Case 1**: In $\beta$, $v_1$ is coloured $a$, but there is a second vertex $v_i$ coloured $a$ as well.
Then just recolour $v_i$ to another colour. The resulting colouring is good, and we are done by the paragraph above.

**Case 2**: In $\beta$, $v_1$ and some other vertex $v_i$ have the same colour $b \neq a$, while a third vertex $v_j$ with $j \neq 2, k+1$ is coloured $a$.
Then we first recolour $v_1$ to $a$, and then recolour $v_j$ to another colour. Again, this gives a good colouring, so we are done.

**Case 3**: In $\beta$, $v_1$ and some other vertex $v_i$ have the same colour $b \neq a$, while a third vertex $v_j$ with $j \in \{2, k+1\}$ is coloured $a$.
Without loss of generality, assume that $j = k+1$.

**Subcase 3.1**: We have $i \geq 5$.
Now first recolour vertex $v_i$ to colour $\beta(v_3)$, then recolour $v_3$ to $\beta(v_{k+1}) = a$, $v_{k+1}$ to $\beta(v_4)$, $v_4$ to $\beta(v_1) = b$, $v_1$ to $a$, and finally recolour $v_3$ to some colour different from $a$. It is easy to check that the remaining colouring is good.

**Subcase 3.2**: We have $i \in \{3, 4\}$.
Recall that $j = k+1 \geq 7$. Now first recolour vertex $v_i$ to colour $\beta(v_6)$, then recolour $v_6$ to $\beta(v_1) = b$. Now $v_1$ and $v_6$ have the same colour $b$, so we are back to Subcase 3.1.

**Case 4**: In $\beta$, $v_1$ has a unique colour $b \neq a$, while there are two vertices $v_i$ and $v_j$ coloured $a$.

**Subcase 4.1**: We have $\{i, j\} = \{2, k+1\}$.
Now first recolour vertex $v_2$ to $\beta(v_4)$, and then recolour $v_4$ to $\beta(v_1) = b$. Then we are back to Subcase 3.2.

**Subcase 4.2**: We have $\{i, j\} \neq \{2, k+1\}$.
Without loss of generality, assume that $i \neq 2, k+1$. Then we can recolour $v_i$ to $\beta(v_1) = b$. This means that $v_1$ and $v_i$ have the same colour $b \neq a$, so we are back in Case 2 or 3.

**Case 5**: In $\beta$, $v_1$ has a unique colour $b \neq a$, there is a unique vertex $v_i$ coloured $a$, and two vertices $v_j$ and $v_\ell$ have the same colour $c \neq a, b$.

**Subcase 5.1**: We have $\{j, \ell\} = \{2, k+1\}$.
Since $k+1 \geq 7$, we must have $i \neq 3$ or $i \neq k$. Without loss of generality, assume that $i \neq 3$.

Then recolour $v_2$ to $\beta(v_i) = a$, and next recolour $v_i$ to $\beta(v_1) = b$. This brings us back to Case 3.

**Subcase 5.2**: We have $\{j, \ell\} \neq \{2, k+1\}$.
Without loss of generality, assume that $j \neq 2, k+1$. Then we can recolour $v_j$ to $\beta(v_1) = b$. This means that $v_1$ and $v_j$ have the same colour $b \neq a$, and we are back in Case 2 or 3. □

## 5 The Strong 3-Colour Graph of Trees

The aim of this section is to classify the trees $T$ for which the strong 3-colour graph $S_3(T)$ is connected.

For this we need to consider some special trees. First, in Section 2 we saw that the strong $k$-colour graph of a complete bipartite graph is not connected, so $S_3(K_{1,n})$ is disconnected for all $n \geq 2$.

For $n \geq 1$ and $p, q \geq 2$, let $I$, $\Psi_n$ and $\Phi_{p,q}$ be the graphs sketched in Figure 5, respectively.

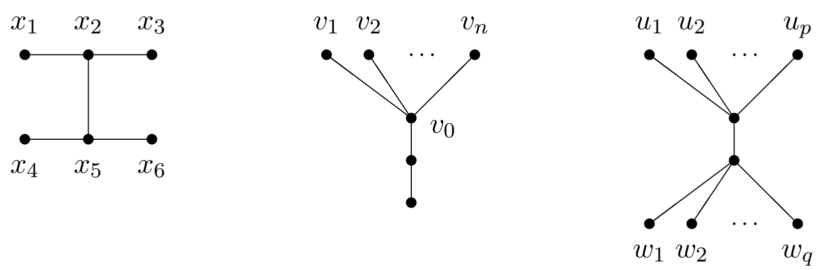



Figure 5: The graphs $I$, $\Psi_n$, and $\Phi_{p,q}$.

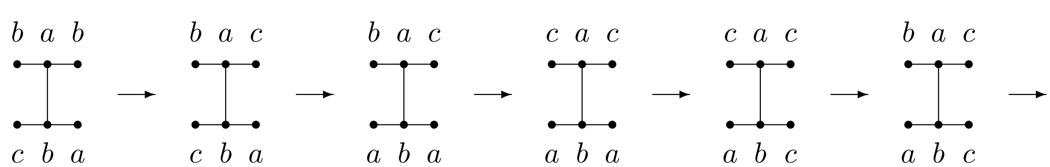
It is straightforward to check that in any strong 3-colouring of $\Psi_n$ we cannot recolour the vertex $v_0$ to another colour so that the resulting 3-colouring is strong again. Hence the strong colour graph $S_3(\Psi_n)$ is disconnected for all $n \geq 1$.

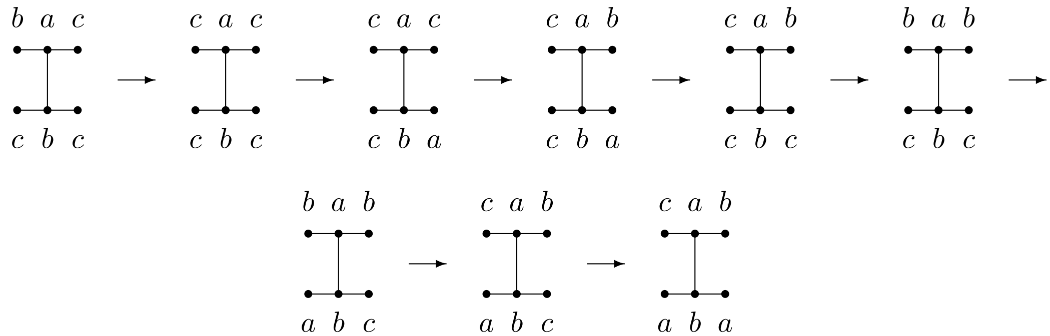
**Proposition 13.** *The strong 3-colour graph $S_3(I)$ is connected.*

*Proof.* Let $\alpha$ be a strong 3-colouring of the graph $I$, with vertex set $\{x_1, x_2, \ldots, x_6\}$ as in Figure 5. We call $\alpha$ an $(ab)$-*standard colouring* if $\alpha(x_2) = a$ and $\alpha(x_5) = b$. Easy counting shows that for fixed $a, b$, there are 15 $(ab)$-standard colourings. ( There are 2 choices for each of the other 4 vertices, but one of the resulting 16 3-colourings is not strong. ) As there are 6 choices for pairs $a, b$ from 3 colours, there are a total of 90 strong 3-colourings of $I$.

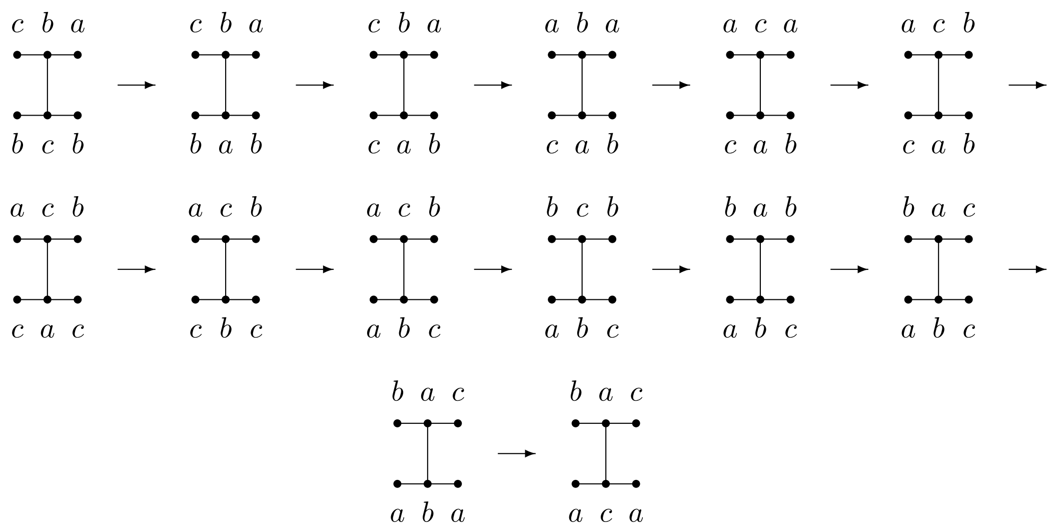
We will prove the proposition by combining the following two steps.

**Step 1**: For given $a, b$, there is a path containing all $(ab)$-standard colourings.






**Step 2**: There is a path containing at least one colouring from each type of standard colourings.




These two steps, together with appropriate renaming of the colours, will give all that is needed to transform any strong 3-colouring of $I$ into any other strong 3-colouring. $\square$

**Theorem 14.** *Let $T$ be a tree. Then $S_3(T)$ is connected if and only if $T$ contains $P_5$ or $I$ as a subgraph.*

*Proof.* Since $S_3(P_5)$ and $S_3(I)$ are connected, one direction is immediately proved by using Theorem 2.

For the other direction, suppose that $T$ does not contain $P_5$, nor $I$. Since $P_5$ is a path with 4 edges, the longest path in $T$ can have length at most 3. Thus $T$ has to be one of the following: $K_1$, $P_2$, $K_{1,m}$, $m \geq 2$, or $\Psi_n$, $n \geq 1$. Note that $T$ cannot be $\Phi_{p,q}$ for all $p, q \geq 2$ since then it would contain $I$ as a subgraph. Since $K_1$ and $P_2$ have fewer than 3 vertices, $T$ cannot be one of these two graphs. We already saw that $S_3(K_{1,m})$, $m \geq 2$, and $S_3(\Psi_n)$, $n \geq 1$, are disconnected. We can conclude that $S_3(T)$ is not connected, which completes the proof. $\square$

## 6 Discussion

We realise that this note contains only some first results on the strong colour graphs. In comparison, there is a growing body of literature on the connectivity of the normal colour graph: [2, 3, 4, 10]. An interesting direction of future research would be to investigate how far the theory of strong colour graphs can be reduced to the theory of normal colour graphs.

We have already seen in Lemma 4 that if the strong colour graph $S_k(G)$ is connected for some $G$ and $k$, then so is the normal colour graph $C_k(G)$. In general, the reverse direction is not true. For instance, for all $m, n \geq 2$, for the complete bipartite graphs $K_{m,n}$ we have that for $k \geq 3$, $C_k(K_{m,n})$ is connected, whereas $S_k(K_{m,n})$ is not connected. In fact, we've already seen that if $G$ has a complete bipartite graph as a spanning subgraph, then $S_k(G)$ is never connected. For $k \geq 3$ it is not hard to construct other graphs apart from complete bipartite graphs that have this property, but all examples we know of have a fairly special structure (see for instance the trees $\Psi_n$ in Figure 5). This makes us raise the following question.

**Question.** *Is it possible to completely describe a class of graphs $\mathcal{H}$ so that if $G$ does not contain a graph from $\mathcal{H}$ as a spanning subgraph, then $C_k(G)$ is connected if and only if $S_k(G)$ is connected?*

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