A Bourgain type bilinear estimate for a class of water-wave models

Qifan Li

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We consider the general form of the equation of water-wave models on torus $\mathbb{T}$

$$\partial_t u + \sum_{k=1}^{N} b_k \partial_x^{2k+1} u + Q(u, \partial_x u, \cdots \partial_x^{2N+1} u) = 0 \tag{1}$$

where Q denotes nonlinear term of the equation and b_ks are real constants. This equation was first introduced and studied by [3] on the real line. The symbol of the linear partial differential operator $\mathcal{L} = \sum_{k=1}^{N} b_k \partial_x^{2k+1}$ is

$$m(\xi) = \sum_{k=1}^{N} b_k (2\pi i \xi)^{2k+1} = P(\xi) i$$

where $P(\xi) = \sum_{k=1}^{N} c_k \xi^{2k+1}$ and $c_k = (-1)^k (2\pi)^{2k+1} b_k$. We assume $c_k \leq 0$ for all the $k \geq 1$. In particular, generalized Kawahara equations [2] and fifth-order KdV equations [4] are the special cases satisfying this condition. We introduce the Bourgain space-time space $X^{s,b}$ with the norm

$$\|u\|_{X^{s,b}}^2 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |\lambda + P(n)|)^{2b} (1 + |n|)^{2s} |\hat{u}(n, \lambda)|^2 d\lambda. \tag{2}$$

We are going to establish a bilinear estimate which extends the proposition 7.15 in [1] to the higher order derivatives.

Lemma 0.1. *Let functions $u, \ v: \ \mathbb{T} \times [0, T] \to \mathbb{R}$, we have the bilinear estimate*

$$\|uv\|_{L_x^2 L_t^2} \lesssim \|u\|_{X^{0, \frac{N+1}{4N+2}}} \|v\|_{X^{0, \frac{N+1}{4N+2}}}. \tag{3}$$

Proof. Let

$$A_m = \left(\int_{\mathbb{R}} (1 + |\xi|)^{\frac{N+1}{2N+1}} |\widehat{u_m}(\xi)|^2 d\xi \right)^{1/2}$$

$$B_m = \left(\int_{\mathbb{R}} (1 + |\xi|)^{\frac{N+1}{2N+1}} |\widehat{v_m}(\xi)|^2 d\xi \right)^{1/2}$$

and

$$u(x,t) = \sum_{m \in \mathbb{Z}} e^{2\pi i (mx - P(m)t)} u_m(t), \quad v(x,t) = \sum_{m \in \mathbb{Z}} e^{2\pi i (mx - P(m)t)} v_m(t). \tag{4}$$

The proof of (3) is reduced to show $\|uv\|_{L_x^2 L_t^2} \lesssim \|A_m\|_{l_n^2} \|B_m\|_{l_n^2}$.

Let quadratic polynomial $Q(m,l) = P(m+l) - P(m) - P(l)$. We have

$$(u\bar{v})(x,t) = \sum_{l\in\mathbb{Z}} e^{2\pi i(lx - P(l)t)} \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (u_m \overline{v_{m+l}})(t)$$

and

$$\|u\bar{v}\|_{L_x^2 L_t^2}^2 = \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \left| \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (u_m \overline{v_{m+l}})(t) \right|^2 dt. \tag{5}$$

For a integer $j > 0$, define Paley-Littlewood operator

$$\Delta_j f(t) = (\mathbf{1}_{2^{j-1}\le|\xi|\le 2^j} \hat{f}(\xi))^{\vee}(t)$$

where $\mathbf{1}_{2^{j-1}\le|\xi|\le 2^j}$ denotes the characteristic function on the set $[2^{j-1}, 2^j] \cup [-2^j, -2^{j-1}]$, and set

$$\Delta_0 f(t) = (\mathbf{1}_{|\xi|\le 1} \hat{f}(\xi))^{\vee}(t).$$

We have the Paley-Littlewood decomposition for u_m and v_m

$$u_m = \sum_{p\ge 0} \Delta_p u_m, \qquad v_m = \sum_{q\ge 0} \Delta_q v_m.$$

We assert that (5) can be estimated by

$$\begin{aligned} \|u\bar{v}\|_{L_x^2 L_t^2}^2 \lesssim &\sum_{l\in\mathbb{Z}} \left(\sum_{q\ge p} \left\| \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L_t^2} \right)^2 \\ &+ \sum_{l\in\mathbb{Z}} \left(\sum_{q\ge p} \left\| \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (\Delta_p v_m \overline{\Delta_q u_{m+l}})(t) \right\|_{L_t^2} \right)^2. \end{aligned} \tag{6}$$

Since it is easy to see that

$$\|u\bar{v}\|_{L_x^2 L_t^2}^2 \le \sum_{l\in\mathbb{Z}} \left(\sum_{q\in\mathbb{Z}} \sum_{p\in\mathbb{Z}} \left\| \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L_t^2} \right)^2.$$

For the summation over $q \le p$ inside the brackets, take $m' = m + l$ and $l' = -l$ we have $Q(m' + l', -l') = P(m') - P(m' + l') - P(-l') = -Q(m', l')$. By this reason, we can write

$$\begin{aligned} \sum_{l\in\mathbb{Z}} \left(\sum_{q\le p} \left\| \sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L_t^2} \right)^2 &= \sum_{l'\in\mathbb{Z}} \left(\sum_{q\le p} \left\| \sum_{m'\in\mathbb{Z}} e^{2\pi i Q(m'+l',-l')t} (\Delta_p u_{m'+l'} \overline{\Delta_q v_{m'}})(t) \right\|_{L_t^2} \right)^2 \\ &= \sum_{l'\in\mathbb{Z}} \left(\sum_{q\le p} \left\| \sum_{m'\in\mathbb{Z}} \overline{e^{2\pi i Q(m',l')t} (\overline{\Delta_p u_{m'+l'}} \Delta_q v_{m'})(t)} \right\|_{L_t^2} \right)^2. \end{aligned} \tag{7}$$

This proves the estimate (6).

We will need the pointwise estimate

$$\sum_m |\Delta_i u_m|^2 \leq 2^j \left(\sum_m \|\Delta_i u_m\|^2_{L^2_t} \right). \tag{8}$$

To prove (8), we shall use Jensen inequality and Plancherel theorem.

$$\begin{aligned} \sum_m |\Delta_i u_m|^2 &= \sum_m 2^{2j} \left| \frac{1}{2^j} \int_{2^{j-1} \leq |\xi| \leq 2^j} \widehat{u_m}(\xi) e^{2\pi i \xi t} d\xi \right|^2 \\ &\leq \sum_m 2^j \int_{2^{j-1} \leq |\xi| \leq 2^j} |\widehat{u_m}(\xi)|^2 d\xi = 2^j \left(\sum_m \|\Delta_i u_m\|^2_{L^2_t} \right). \end{aligned}$$

We distinguish three cases

$$\text{(i)} \;\; |l|^{2N+1} \leq 2^q \qquad \text{(ii)} \;\; |l|^{2N} \leq 2^q < |l|^{2N+1} \qquad \text{(iii)} \;\; 2^q < |l|^{2N}.$$

Contribution of (i). By (8), we have the estimate

$$\begin{aligned} \left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L^2_t} &\leq \left\| (\sum_m |\Delta_p u_m|^2)^{1/2} (\sum_m |\Delta_p v_{m+l}|^2)^{1/2} \right\|_{L^2_t} \\ &\leq 2^{p/2} \left(\sum_m \|\Delta_p u_m\|^2_{L^2_t} \right)^{1/2} \left(\sum_m \|\Delta_q v_{m+l}\|^2_{L^2_t} \right)^{1/2}. \end{aligned}$$

By Plancherel theorem we get

$$\begin{aligned} \left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L^2_t} \lesssim & 2^{p \frac{N}{4N+2}} 2^{-q \frac{N+1}{4N+2}} \left(\sum_m \int_{2^{p-1} \leq |\xi| \leq 2^p} (1+|\xi|)^{\frac{N+1}{2N+1}} |\widehat{u_m}(\xi)|^2 d\xi \right)^{1/2} \\ & \times \left(\sum_m \int_{2^{q-1} \leq |\xi| \leq 2^q} (1+|\xi|)^{\frac{N+1}{2N+1}} |\widehat{v_m}(\xi)|^2 d\xi \right)^{1/2}. \end{aligned} \tag{9}$$

Estimate (9) implies

$$\begin{aligned} \sum_{q \geq p} \left\| \sum_m e^{2\pi i Q(m,l)t} (\Delta_p u_m \overline{\Delta_q v_{m+l}})(t) \right\|_{L^2_t} \lesssim & \sum_{2^p \geq |l|^{2N+1}} 2^{-p \frac{1}{4N+2}} \left(\sum_m \int_{2^{p-1} \leq |\xi| \leq 2^p} (1+|\xi|)^{\frac{N+1}{2N+1}} |\widehat{u_m}(\xi)|^2 d\xi \right)^{1/2} \\ & \times \left(\sum_m B_m^2 \right)^{1/2} \\ \lesssim & |l|^{1/4} \left(\sum_{2^p \geq |l|^{2N+1}} 2^{-p \frac{1}{4N+2}} \sum_m \int_{2^{p-1} \leq |\xi| \leq 2^p} (1+|\xi|)^{\frac{N+1}{2N+1}} |\widehat{u_m}(\xi)|^2 d\xi \right)^{1/2} \\ & \times \left(\sum_m B_m^2 \right)^{1/2}. \end{aligned}$$

The last step is followed by Hölder inequality. For $j \geq 0$, let positive integer $|l| = 2^j$ and $p' = p - j(2N+1)$, we can obtain the estimate

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\left(\sum_{q\geq p}\left\|\sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t}(\Delta_p u_m \overline{\Delta_q v_{m+l}})(t)\right\|_{L_t^2}\right)^2 &\lesssim \sum_{l\in\mathbb{Z}}\sum_{2^p\geq|l|^{2N+1}}\sum_m \int_{2^{p-1}\leq|\xi|\leq 2^p}(1+|\xi|)^{\frac{N+1}{2N+1}}|\widehat{u_m}(\xi)|^2 d\xi \\ &\quad \times |l|^{1/2} 2^{-p\frac{1}{4N+2}}\sum_m B_m^2 \\ &\approx \sum_{j\in\mathbb{Z}}\sum_{p'\geq 0}\sum_m \int_{2^{p'+j(2N+1)-1}\leq|\xi|\leq 2^{p'+j(2N+1)}}(1+|\xi|)^{\frac{N+1}{2N+1}} \\ &\quad \times |\widehat{u_m}(\xi)|^2 d\xi 2^{-p'\frac{1}{4N+2}}\sum_m B_m^2 \\ &= \sum_{p'\geq 0} 2^{-p'\frac{1}{4N+2}}\sum_m A_m^2 \sum_m B_m^2 \lesssim \sum_m A_m^2\sum_m B_m^2.\end{aligned}$$

Similarly, we have

$$\sum_{l\in\mathbb{Z}}\left(\sum_{q\geq p}\left\|\sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t}(\Delta_p v_m \overline{\Delta_q u_{m+l}})(t)\right\|_{L_t^2}\right)^2 \lesssim \sum_m A_m^2 \sum_m B_m^2.$$

Contribution of (ii). Consider the quantity

$$\left\|\sum_{m\in\mathbb{Z}} e^{2\pi i Q(m,l)t}(\Delta_p u_m \overline{\Delta_q v_{m+l}})(t)\right\|_{L_t^2}.$$

We first assume $l \geq 0$. For a large positive $K > 0$, by Plancherel theorem, we write

$$\left\|\sum_{|m|\leq K} e^{2\pi i Q(m,l)t}(\Delta_p u_m \overline{\Delta_q v_{m+l}})(t)\right\|_{L_t^2} = \left\|\sum_{|m|\leq K}(\widehat{\Delta_p u_m} * \widehat{\overline{\Delta_q v_{m+l}}})(\xi - Q(m,l))\right\|_{L_\xi^2}. \tag{10}$$

Since supp $(\widehat{\Delta_p u_m} * \widehat{\overline{\Delta_q v_{m+l}}})(\xi) \subset [-2^{q+1}, 2^{q+1}]$, splitting the summation into $62^q/|l|^{2N}$ summations over arithmetic progressions of increment $62^q/|l|^{2N}$, say $\mathcal{M}_s$ for $s = 1, \cdots, 62^q/|l|^{2N}$. if $m_1,\ m_2 \in \mathcal{M}_s$ then

$$m_1 = n_1 62^q/|l|^{2N} + d, \qquad m_2 = n_2 62^q/|l|^{2N} + d.$$

where d, n_1 and n_2 denote be integers and $n_1 > n_2$, $d < 62^q/|l|^{2N}$. We write

$$(m+l)^k - m^k = \sum_{\alpha+\beta=k,\ \beta\geq 1} a_{\alpha,\beta} m^\alpha l^\beta$$

and obviously $a_{\alpha,\beta} \geq 0$ for all indices α, β. In this case we have

$$\begin{aligned}|Q(m_1,l) - Q(m_2,l)| &= \left|\sum_{k=1}^{2N+1}\sum_{\alpha+\beta=k,\ \beta\geq 1} c_k a_{\alpha,\beta}(m_1^\alpha - m_2^\alpha) l^\beta\right| \\ &\geq (m_1 - m_2) l^{2N} \geq \max\{62^q, l^{2N}\}.\end{aligned} \tag{11}$$

By a orthogonality consideration and (10) we write

$$\begin{aligned}\left\|\sum_{m\in\mathcal{M}_s}(\widehat{\Delta_p u_m}*\widehat{\overline{\Delta_q v_{m+l}}})(\xi-Q(m,l))\right\|_{L^2_\xi}^2 &= \sum_{m\in\mathcal{M}_s}\left\|(\widehat{\Delta_p u_m}*\widehat{\overline{\Delta_q v_{m+l}}})(\xi-Q(m,l))\right\|_{L^2_\xi}^2\\ &= \sum_{m\in\mathcal{M}_s}\left\|(\widehat{\Delta_p u_m}*\widehat{\overline{\Delta_q v_{m+l}}})(\xi)\right\|_{L^2_\xi}^2 \qquad (12)\\ &= \sum_{m\in\mathcal{M}_s}\left\|\Delta_p u_m\overline{\Delta_q v_{m+l}}\right\|_{L^2_t}^2.\end{aligned}$$

From Young inequality, (12) and (8) we get

$$\begin{aligned}\left\|\sum_{|m|\le K}e^{2\pi iQ(m,l)t}(\Delta_p u_m\overline{\Delta_q v_{m+l}})(t)\right\|_{L^2_t}^2 &\le 6\frac{2^q}{|l|^{2N}}\sum_{s=1}^{62^q/|l|^{2N}}\left\|\sum_{m\in\mathcal{M}_s}e^{2\pi iQ(m,l)t}(\Delta_p u_m\overline{\Delta_q v_{m+l}})(t)\right\|_{L^2_t}^2\\ &= 6\frac{2^q}{|l|^{2N}}\sum_{s=1}^{62^q/|l|^{2N}}\sum_{m\in\mathcal{M}_s}\left\|(\Delta_p u_m\overline{\Delta_q v_{m+l}})(t)\right\|_{L^2_t}^2\\ &\le 6\frac{2^q}{|l|^{2N}}2^p\sum_m\|\Delta_p u_m\|_{L^2_t}^2\|\Delta_p v_m\|_{L^2_t}^2\\ &\le 6|l|^{-2N}2^{2Np/(2N+1)}2^{2Nq/(2N+1)}\sum_m\int_{2^{p-1}\le|\xi|\le 2^p}(1+|\xi|)^{\frac{N+1}{2N+1}}\\ &\quad\times|\widehat{u_m}(\xi)|^2d\xi\int_{2^{q-1}\le|\xi|\le 2^q}(1+|\xi|)^{\frac{N+1}{2N+1}}|\widehat{v_{m+l}}(\xi)|^2d\xi.\end{aligned}$$

Let $K\to\infty$, the estimate holds for the whole integer set. The same estimate can be obtained if we interchange the role of u_m and v_m. As for the case $l<0$, by (7) we can reduce this case to $l\ge 0$.

Therefore, we have

$$\begin{aligned}\sum_{l\in\mathbb{Z}}\left(\sum_{q\ge p,\ 2^q<|l|^{2N+1}}\left\|\sum_{m\in\mathbb{Z}}e^{2\pi iQ(m,l)t}(\Delta_p u_m\overline{\Delta_q v_{m+l}})(t)\right\|_{L^2_t}\right)^2 &\le \sum_m\sum_l A_m^2B_{m+l}^2\\ &= \sum_m A_m^2\sum_m B_m^2.\end{aligned}$$

Contribution of (iii). We have known from (11) that the orthogonality seems more natural in this case. The arguments are similar as the case of (ii) and even simpler. □

The result is sharp. For example if we take

$$u_K(x,t)=\sum_{|n|\le K}\int_{|\lambda|\le K^{2N+1}}e^{2\pi i(nx+\lambda t)}d\lambda.$$

Then $\|u_K\|_{L^2_xL^2_t}\approx K^{N+1}$ and

$$\|u_K\|_{L^4_xL^4_t}\approx K^{3/4}(K^{2N+1})^{3/4}\approx K^{3(N+1)/2}.$$

On the other hand,

$$\|u_K\|_{X^{0,\frac{N+1}{4N+2}}} = \left(\sum_{|n|\le K}\int_{|\lambda|\le K^{2N+1}} (1+|\lambda+P(n)|)^{\frac{N+1}{2N+1}} |\widehat{u_K}(n,\lambda)|^2 d\lambda\right)^{1/2}$$
$$\approx K^{3(N+1)/2} \approx \|u_K\|_{L_x^4 L_t^4}.$$

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