Upper bound for the generalized repetition threshold.

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Abstract

Let A be an a-letter alphabet. We consider fractional powers of A-strings: if x is a n-letter string, x^r is a prefix of $xxxx\ldots$ having length nr.

Let l be a positive integer. Ilie, Ochem and Shallit defined $R(a,l)$ as the infimum of reals $r>1$ such that there exist a sequence of A-letters without factors (substrings) that are fractional powers $x^{r'}$ where x has length at least l and $r' \geq r$.

We prove that $1+\frac{1}{la} \leq R(a,l) \leq 1+\frac{c}{la}$ for some constant c.

1 Introduction

A fractional power x^r of a string x is defined as $x^r = xxx\ldots xxy$ where y is a prefix of x and $|x^r| = r|x|$. (We assume that $r>1$ is a fraction with denominator $|x|$.)

One may ask whether there exists an infinite sequence of letters that does not contain fractional powers x^r with large r and long x. More precisely, for a given alphabet size a, a given integer l and a given real α one may ask whether there exists an infinite sequence of letters that does not contain fractional powers x^r with $r>\alpha$ and $|x| \geq l$.

For $\alpha = 1$ the answer is evidently negative (each string x is a fractional power x^1). On the other hand, it is easy to see that for any $a \geq 2$ and $l \geq 1$ the answer is positive if α is large enough (there exists a binary sequence that does not contain factors x^3). The threshold value that separates negative and positive answers is denoted by $R(a,l)$ in [7]; the authors note that $1 < R(a,l) \leq 2$ and compute exact values of $R(a,l)$ for some pairs (a,l). Evidently, $R(a,l)$ decreases when a or l increase.

To get a lower bound for $R(a,l)$, let us apply the pigeonhole principle to $a+1$ letters at positions $0, l, 2l, \ldots, al$. Two of them should be equal and

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this creates a fractional power x^r where $|x| \geq l$ and $r \leq 1 + 1/la$ (this power starts and ends with a letter that appears twice). Therefore,

$$R(a, l) \geq 1 + \frac{1}{la}.$$

Francesca Fiorenzi, Pascal Ochem and Elise Vaslet in [8] gave stronger lower bounds and also some upper bounds for $R(a, l)$. In particular, they proved that

$$1 + \frac{1}{1 + \lfloor \frac{3l+2}{4}(a-1) \rfloor} \leq R(a, l) \leq 1 + \frac{2 \ln l}{l \ln \lambda} + O\left(\frac{1}{l}\right),$$

where $\lambda = \frac{(a-1)+\sqrt{(a-1)(a+3)}}{2}$ and a constant in O may depend on a but not on l.

In this paper we use Lovász local lemma to prove a stronger upper bound for $R(a, l)$. Our upper bound differs from the lower bound only by a constant:

$$R(a, l) \leq 1 + \frac{c}{la}$$

for some c and for all $a \geq 2$, $l \geq 1$.

2 Kolmogorov complexity of subsequences

We present the proof using the notion of *Kolmogorov complexity* (also called *algorithmic complexity* or *description complexity*). We refer the reader to [1] or [10] for the definition and basic properties of Kolmogorov complexity.

For an infinite sequence ω and finite set $X \subset \mathbb{N}$ let $\omega(X)$ be a string of length $\#X$ formed by ω_i with $i \in X$ (in the same order as in ω).

We use the following result from [9] that guarantees the existence of a sequence ω such that strings $\omega(X)$ have high Kolmogorov complexity for all simple X:

Theorem 1. *Let α be a positive real number less than 1. There exists a binary sequence ω and an integer N such that for any finite set X of cardinality at least N the inequality*

$$\mathrm{K}(X, \omega(X)|t) \geq \alpha \# X$$

holds for some $t \in A$.

Here $\mathrm{K}(X, \omega(X)|t)$ is conditional Kolmogorov complexity of a pair $(X, \omega(X))$ relative to t.

We need a slightly more general version of this result (for any alphabet size):

Theorem 2. *Let $a \geq 2$ be an integer. Let α be a positive real less than 1. There exists a sequence ω in a-letters alphabet and an integer N such that for any finite set X of cardinality at least N the inequality*

$$\mathrm{K}(X, \omega(X)|t) \geq \alpha \# X \log a$$

holds for some $t \in X$.

Proof. Theorem 2 can be proven using exactly the same argument as in [9] (Lovasz local lemma technique). It can also be formally derived from Theorem 1 as follows: we encode a letters of the alphabet by bit blocks of some length t (large enough). This encoding is not bijective (several blocks encode the same letter) but is chosen in such a way that all letters have almost the same number of encodings (about $2^t/a$). Then we take a sequence from Theorem 1, split it into t-bit blocks and replace these blocks by corresponding letters. If some subsequence formed by the letters is simple, then the corresponding bit subsequence is simple, too. (Technically we should change α slightly to compensate for "boundary effects".) □

3 Weak upper bound

To illustrate the technique, we first prove a simple generalization of a result obtained by Berk [6] and provide an upper bound for $R(a, l)$ that is weaker that our final bound:

Theorem 3. *For every $a \geq 2$ and every real number $b \in (1, a)$ there exists a number N and a sequence ω in a-letters alphabet such that for every $n \geq N$ the distance between any two different occurrences of the same substring of length n in ω is at least b^n.*

Proof. Construct a sequence ω using Theorem 2 with α close enough to 1.

Let I and J ($|I| = |J| = n$) be different intervals where the same substring of length n occurs in ω. Let $X = I \cup J$. Then $n < \#X \leq 2n$ (intervals I and J are not necessarily disjoint) and the first n letters of $\omega(X)$ are equal to the last n letters of $\omega(X)$. It is easy to see that the string $\omega(X)$ is determined by its first $\#X - n$ letters, n and $\#X$, so $\mathrm{K}(\omega(X)) \leq (\#X - n) \log a + O(\log n)$.

Assume $t \in X$. Then X is determined by t, the number n, the distance between I and J and the ordinal number of t in X. So if the distance between I and J is less than b^n then $\mathrm{K}(\omega(X), X|t) \leq (|X| - n) \log a + n \log b + O(\log n) \leq \alpha |X| \log n$ for large enough n and α that is close enough to 1 (because $\log b < \log a$). This contradicts the inequality of Theorem 2. Therefore sequence ω does not contain a pair of different occurrences of the same substring of sufficiently large length n with distance between them less than b^n. □

In particular, for every integer $a \geq 2$, every real number $b \in (1, a)$ and for large enough l the following inequality holds:

$$R(a, l) < 1 + \frac{\log_b l}{l}.$$

4 The final upper bound

In the weak upper bound we used the same sequence for all values of l. And now we need different sequences for different values of l but we want the constant c to be the same. To achieve this goal we use the following "l-uniform" version of Theorem 1.

Theorem 4. *Let α be a positive real number less than 1. There exists an integer N such that for every integer l there exists a binary sequence ω that has the following property: for every finite set X of cardinality at least N the inequality*

$$\mathrm{K}(X, \omega(X) | t, l) \geq \alpha \# X$$

holds for some $t \in A$.

Note that ω may depend on l while N is the same for all values of l. (If we allowed N to be dependent on l, this would be a standard relativization of Theorem 1.)

Proof. Theorem 4 can be proven in the same way as Theorem 1. And it can also be formally derived from it: if a sequence τ and a number N satisfy the requirements of Theorem 1 and $z : \mathbb{N}^2 \to \mathbb{N}$ is a computable bijection, then the sequence $i \mapsto \omega_i = \tau_{z(i,l)}$ and the same number N satisfy the requirements of Theorem 4 for the integer l. (The bijection adds $O(1)$-term, but this can be compensated by a small change in α: the statement is true for every $\alpha < 1$.) □

Now we can start proving the upper bound.

Theorem 5. *There exists a constant c such that for any $a \geq 2$ and $l \geq 1$ the following inequality holds:*

$$1 + \frac{1}{al} \leq R(a, l) \leq 1 + \frac{c}{al}.$$

Proof. The lower bound is easy (as shown in the introduction). Let us prove the upper bound. Let as assume first that $a = 2$ (the general case can be reduced to this special one).

Consider a sequence ω satisfying the requirements of Theorem 4 for some $\alpha > \frac{1}{2}$. Then the required sequence with long fractional powers will be constructed as

$$\tau_i = \omega_{f(i)}$$

for some mapping $f : \mathbb{N} \to \mathbb{N}$.

At first let us define f at the first l integers (the value of integer constant m will be chosen later):

1. $f(i) = i \bmod m$ for $i < l$ and $(i \bmod m) \neq m-1$ (we say that these indexes have rank 1).

2. $f(mi+m-1) = (m-1)+(i \bmod m)$ for $mi+m-1 < l$ and $(i \bmod m) \neq m-1$ (we say that these indexes have rank 2).

3. $f(m^2 i + m^2 - 1) = 2(m-1) + (i \bmod m)$ for $m^2 i + m^2 - 1 < l$ and $(i \bmod m) \neq m-1$ (we say that these indexes have rank 3).

   (And so on until f is defined at all first l integers.)

Then we define f on other blocks of l integers in the same way but using fresh bits each time. So if $f(\{0, 1, \dots, l-1\}) = \{0, 1, \dots, L-1\}$ then $f(i + jl) = f(i) + jL$.

Suppose the sequence $\tau_i = \omega_{f(i)}$ contains some fractional power xyx with $|xy| \geq l$ and the exponent $\dfrac{|xyx|}{|xy|} \geq 1 + \dfrac{c}{2l}$. Without loss of generality we can assume that the exponent $1 + \frac{c}{2l}$ is not greater than 2 (otherwise the statement of the theorem follows from the existence of a binary sequence, called Thue-Morse sequence, that does not contain any fractional power with exponent greater than 2, see [2], [3]). Also we can assume that $c > 2m$ (increasing c, we make our task easier). So $l \geq \frac{c}{2} > m$ and $|x| \geq \frac{c}{2l}|xy| > m$.

First we consider the case when both occurrences of x in xyx lie entirely in some blocks of size l (in two different blocks, because $|xy| \geq l$). Denote by n the number of l-sized blocks between these two occurrences of x and denote by k the integer number that satisfies the inequality $m^{k-1} \leq |x| < m^k$. Then $m^k > \frac{c}{2}n$ and $k \geq 2$ (because $|x| \geq \frac{c}{2l}|xy| > m$).

Let us denote by I and J the sets of values of f for the first and second occurrences of x (respectively) whose rank is not greater than k (obviously there is at most 1 index in each of these occurrences of x whose rank is greater than k). The sets I and J are disjoint because these occurrences of x lies in the different l-sized blocks. Assume $Z = I \cup J$, then for some $t \in Z$ we have $\mathrm{K}(Z, \omega(Z)|t, l) \geq \alpha \# Z$ by the statement of Theorem 4 (we need here that $m > N + 1$ since $\# Z$ should be greater than N).

Obviously,

$$\frac{1}{2}\# Z = \# I + O(1) = \# J + O(1) = (k-1)(m-1) + \frac{|x|}{m^{k-1}} + O(1).$$

The set Z is determined by t, l, m, n, k, $|x|$ and the start/end positions for the two occurrences of the word x modulo m^k (and one bit saying whether t belongs to the first occurrence of x or to the second one). So $\mathrm{K}(Z \mid t, l) \leq \log n + O(\log(m^k)) = O(k \log m)$ (since $m^k > \frac{c}{2}n$). We can also calculate

$\omega(Z)$ if $\omega(I)$ is given (we need at most one extra bit for calculating the entire string x). Therefore

$$O(k \log m) + \frac{1}{2}\#Z \geq \alpha \#Z,$$

but $\alpha > \frac{1}{2}$ and $\#Z \geq 2(k-1)(m-1) + O(1) \geq k(m-1) + O(1)$. So $k(m-1) < O(k \log m)$ that is a contradiction if m is large enough. (Recall that the choice of m was postponed.)

Consider now the general case for the position of the two occurrences of x. If length of x is not large, i.e. $|x| \leq l$, we can reduce this case to the previous one by splitting x into parts and choosing the largest part (we must multiply the constant c by 3). Now let x be longer than the block size ($|x| > l$). We can assume that there is no l-sized block that intersects both occurrences of x (in the other case we also split the word x in parts).

Let us denote by I and J the sets of values of f in the first and second occurrences of x respectively. The sets I and J are disjoint. Assume $Z = I \cup J$. Then for some $t \in Z$ we have $\mathrm{K}(Z, \omega(Z)|t, l) \geq \alpha \#Z$.

The set Z is determined by t, l, m and the relative start/end positions of the two occurrence of the word x with respect to the one of the preimages of t (for example, the first one). So $\mathrm{K}(Z \mid t, l) \leq \log |xy| + O(\log l) = O(\log |x|)$ (since $|x| \geq l$ and $|x| \geq \frac{c}{2l}|xy|$). To compute $\omega(Z)$, it is enough to know at most a half of it ($\omega(I)$ or $\omega(J)$, whichever is smaller). Therefore

$$O(\log |x|) + \frac{1}{2}\#Z \geq \alpha \#Z,$$

but $\alpha > \frac{1}{2}$ and $\#Z = \Omega\left(\frac{|x|}{l}(m-1)\log_m l\right) = \Omega\left((\log |x|)\frac{m-1}{\log m}\right)$ (here we use that $|x| > l > m$ and $\frac{|x|}{\log |x|} \geq \frac{l}{\log l}$). That is a contradiction if m is large enough.

This finishes the proof for $a = 2$.

Assume now that $a \geq 6$ and a is even. Let ω be the sequence constructed for binary alphabet and $l' = \frac{a-2}{2}l$. To get the required sequence ν we will color the terms of ω into $\frac{a}{2}$ colors: the i-th block of size l gets color $i \bmod \frac{a}{2}$. Then the size of the alphabet of sequence ν (whose terms are now $\langle$bit, color$\rangle$ pairs) equals to a and ν does not contain fractional powers z^p with $|z| \geq \frac{a-2}{2}l$ and $p \geq 1 + \frac{c}{(a-2)l}$. And obviously ν does not contain any fractional powers z^p with $l \leq |z| \leq \frac{a-2}{2}l$ (because it does not contain pairs of equal letters at these distances).

Therefore $R(a, l) \leq 1 + \frac{c}{(a-2)l}$ if $a \geq 6$ and a is even, and $R(2, l) \leq 1 + \frac{c}{2l}$.

To prove the theorem for arbitrary a it remains to note that that $R(a, l)$ is decreasing in a, so $R(a, l) \leq 1 + \frac{3c}{al}$ for every $a \geq 2$, $l \geq 1$. □

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