Boxicity of Line Graphs

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Abstract. Boxicity of a graph H, denoted by $box(H)$, is the minimum integer k such that H is an intersection graph of axis-parallel k-dimensional boxes in $\mathbb{R}^k$. In this paper, we show that for a line graph G of a multigraph, $box(G) \leq 2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1$, where Δ denotes the maximum degree of G. Since $\Delta \leq 2(\chi - 1)$, for any line graph G with chromatic number χ, $box(G) = O(\chi \log_2 \log_2(\chi))$. For the d-dimensional hypercube H_d, we prove that $box(H_d) \geq \frac{1}{2}(\lceil \log_2 \log_2 d \rceil + 1)$. The question of finding a non-trivial lower bound for $box(H_d)$ was left open by Chandran and Sivadasan in [L. Sunil Chandran and Naveen Sivadasan. The cubicity of Hypercube Graphs. *Discrete Mathematics*, 308(23):5795-5800, 2008].
The above results are consequences of bounds that we obtain for the boxicity of fully subdivided graphs (a graph which can be obtained by subdividing every edge of a graph exactly once).

Key words: Intersection graph, Interval graph, Boxicity, Line graph, Edge graph, Hypercube, Subdivision

1 Introduction

Given a family $\mathcal{F}$ of sets, a graph $G = (V, E)$ is called an *intersection graph* of sets from $\mathcal{F}$, if there exists a map $f : V(G) \to \mathcal{F}$ such that $(u, v) \in E(G) \Leftrightarrow f(u) \cap f(v) \neq \emptyset$. If the sets in $\mathcal{F}$ are intervals on a real line, then we call G an *interval graph*. In other words, interval graphs are intersection graphs of intervals on the real line. In $\mathbb{R}^k$, an axis parallel k-dimensional box or a k-*box* is a cartesian product $R_1 \times R_2 \times \cdots \times R_k$, where each R_i is a closed interval $[a_i, b_i]$ on the real line. A graph G is said to have a k-box representation if there exists a mapping from the vertices of G to k-boxes in the k-dimensional eucledian space such that two vertices in G are adjacent if and only if their corresponding k-boxes have a non-empty intersection. *Boxicity* of G, denoted by $box(G)$, is the minimum positive integer k such that G has a k-box representation. As each interval can also be viewed as an axis parallel 1-dimensional box, interval graphs are precisely the class of graphs with boxicity 1. We take the boxicity of a complete graph to be 1.

1.1 Background

The concept of boxicity was introduced by F.S. Roberts in 1969 [17]. Cozzens [11] showed that computing the boxicity of a graph is NP-hard. Yannakakis in [21] improved this result. Finally, Kratochvil [16] showed that deciding whether the boxicity of a graph is at most 2 itself is NP-complete.

Box representation of graphs finds application in niche overlap (competition) in ecology and to problems of fleet maintenance in operations research (see [12]). Given a low dimensional box representation, some well known NP-hard problems become polynomial time solvable. For instance, the max-clique problem is polynomial time solvable for graphs with boxicity k because the number of maximal cliques in such graphs is only $O((2n)^k)$.

Roberts proved that for every graph G on n vertices, $box(G) \leq \lfloor \frac{n}{2} \rfloor$. He gave a tight example to this by showing that a complete $\frac{n}{2}$-partite graph with 2 vertices in each part has its boxicity equal to $\frac{n}{2}$. In [4], it was shown that if t denotes the size of a minimum vertex cover of G, then $box(G) \leq \lfloor \frac{t}{2} \rfloor + 1$. Chandran, Francis and Sivadasan showed in [8] that, for any graph G on n vertices having maximum degree Δ, $box(G) \leq (\Delta + 2) \ln n$. An upper bound solely in terms of the maximum degree Δ, which says $box(G) \leq 2\Delta^2$, is proved in [7]. Esperet [15] improved this bound to $\Delta^2 + 2$. Recently Adiga, Bhowmick and Chandran [1] showed that $box(G) = O(\Delta \log^2 \Delta)$. Chandran and Sivadasan in [9] found a relation between treewidth and boxicity which says $box(G) \leq \mathrm{tw}(G) + 2$, where $\mathrm{tw}(G)$ denotes the treewidth of graph G.

Attempts on finding better bounds for boxictiy of special graph classes can also be seen in the literature. Scheinerman [18] showed that outerplanar graphs have boxicity at most 2. Thomassen [20] proved that the boxicity of planar graphs is not greater than 3. Cozzens and Roberts [12] have done a study on the boxicity of split graphs. Results on the boxicity of Chordal graphs, AT-free graphs, permutation graphs etc. can be seen in [9]. Better bounds for the boxicity of Circular Arc graphs and AT-free graphs can be seen in [2, 3]. In [5] it was shown that, there exist chordal bipartite graphs with arbitrarily high boxicity.

1.2 An Equivalent Definition for Boxicity

Let $G, G_1, G_2, \ldots, G_b$ be a collection of graphs with $V(G) = V(G_i)$, for any $i \leq b$. We say $G = \bigcap_{i=1}^{b} G_i$ when $E(G) = \bigcap_{i=1}^{b} E(G_i)$. The following lemma gives the relationship between interval graphs and intersection graphs of k-boxes.

Lemma 1 (Roberts[17]). *For any graph G, $box(G) \leq k$ if and only if there exist k interval graphs $I_1, I_2, \ldots, I_k$ such that $G = \bigcap_{i=1}^{k} I_i$.*

From the above lemma, we can say that boxicity of a graph G is the minimum positive integer k for which there exist k interval graphs $I_1, I_2 \ldots, I_k$ such that $G = \bigcap_{i=1}^{k} I_i$.

We have seen that intervals graphs are intersection graphs of intervals on the real line. Hence for any interval graph I, there exists a map $f : V(I) \rightarrow \{X \subseteq$

$\mathbb{R} \mid X$ is a closed interval$\}$ such that, for any $u, v \in V(I)$, $(u, v) \in E(I)$ if and only if $f(u) \cap f(v) \neq \emptyset$. Such a map f is called an *interval representation* of I. An interval graph can have more than one interval representation. It is known that given an interval graph I, we can find an interval representation for I in which no two intervals share any endpoints.

1.3 Preliminaries

Except in Theorem 3, Section 4, we consider only finite, undirected, and simple graphs. In Theorem 3, we consider finite, undirected multigraphs. For any finite positive integer n, let $[n]$ denote the set $\{1, 2, \ldots n\}$. For a graph G, we use $V(G)$ and $E(G)$ to denote the set of its vertices and edges respectively. For any $v \in V(G)$, $N_G(v) := \{u \mid (v, u) \in E(G)\}$ and $d_G(v) := |N_G(v)|$. The maximum degree of G is denoted by $\Delta(G)$. $\chi(G)$ represents the chromatic number of G. We say that an edge e_i is a neighbour of another edge e_j in G, if they share an endpoint. Given two graphs G and H, we say $G = H$ when G is isomorphic to H.

We say that a graph G is obtained by *fully subdividing* H, if G is obtained as a result of subdividing every edge of H exactly once. Given a multigraph H, we define a graph $L(H)$ in the following way: $V(L(H)) = E(H)$ and $E(L(H)) = \{(e_1, e_2) \mid e_1, e_2 \in E(H),\ e_1 \text{ and } e_2 \text{ share an endpoint in } H\}$. A graph G is a *line graph* if and only if there exists a multigraph H such that G is isomorphic to $L(H)$. Let I be an interval graph and f an interval representation of I. Then, $\forall x \in V(I)$, we use $l(f(x))$ and $r(f(x))$ to denote the left and right endpoint respectively of the interval $f(x)$.

1.4 Our Results

In this paper, we show that for a line graph G with maximum degree Δ,

$$box(G) \leq 2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1.$$

From the above result, we also infer that if chromatic number of G is χ, then $box(G) = O(\chi \log_2 \log_2(\chi))$. Recall that, in [1] it was shown that for any graph G, $box(G) \leq c \cdot \Delta \log^2 \Delta$, where c is a large constant. Hence, for the class of line graphs, our result is an improvement over the best bound known for general graphs. Moreover, in contrast with the result in [1], the proof here is constructive and easily gives an efficient algorithm to get a box representation for the given line graph. We leave the tightness of our result open.

The main supporting result that we have used to prove the above result is the following (this itself may be independently interesting): For a graph G obtained by fully subdividing another graph H, $box(G) \leq \lceil \log_2 \log_2(\Delta) \rceil + 3$, where Δ is the maximum degree of G. At the end of the paper, we point out another consequence of this supporting result. For the d-dimensional hypercube H_d,

$$box(H_d) \geq \frac{\lceil \log_2 \log_2 d \rceil + 1}{2}.$$

It was shown by Chandran and Sivadasan in [10] that $box(H_d) \leq \frac{cd}{\log d}$, where c is a constant. They had raised the question of finding a non-trivial lower bound for $box(H_d)$.

2 Boxicity of a Fully Subdivided Complete Graph

Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$ be a set of permutations of $[n]$, where n is any finite positive integer. S is called *k-suitable* for $[n]$ if for any k-element subset $X \subseteq [n]$ and for any $x \in X$, there exists a permutation $\sigma \in S$ with the following property:

$$\sigma^{-1}(x) \geq \sigma^{-1}(y), \forall y \in X.$$

The minimum cardinality of a k-suitable set for $[n]$ is denoted by $N'(n,k)$. Spencer [19] proved that

$$N'(n,3) < \log_2 \log_2 n + \frac{1}{2} \log_2 \log_2 \log_2 n + \log_2(\sqrt{2}\pi) + o(1).$$

In this paper, we are interested in a slightly relaxed version of the notion of 3-suitability. Given a permutation σ of $[n]$ and $s, t \in [n]$, let

$$\begin{aligned} \beta(s,t,\sigma) = \{x \mid\ & \sigma^{-1}(s) < \sigma^{-1}(x) < \sigma^{-1}(t) \\ & \text{or } \sigma^{-1}(t) < \sigma^{-1}(x) < \sigma^{-1}(s)\}. \end{aligned} \tag{1}$$

A set $S = \{\sigma_1, \sigma_2, \ldots, \sigma_p\}$ is called *simply 3-suitable* for $[n]$, if for each pair $s, t \in [n]$, $\bigcap_{i=1}^{p} \beta(s,t,\sigma_i) = \emptyset$. In other words, for every triple $x, s, t \in [n]$ there exists a permutation $\sigma \in S$ such that either $\sigma^{-1}(x) < \min\left(\sigma^{-1}(s), \sigma^{-1}(t)\right)$ or $\sigma^{-1}(x) > \max\left(\sigma^{-1}(s), \sigma^{-1}(t)\right)$. It is easy to see that any 3-suitable set is also a simply 3-suitable set while the converse is clearly not true. Let $N(n)$ be the minimum possible cardinality of a simply 3-suitable set for $[n]$. From Spencer's bound on $N'(n,3)$, we have $N(n) \leq N'(n,3) < \log_2 \log_2 n + \frac{1}{2} \log_2 \log_2 \log_2 n + \log_2(\sqrt{2}\pi) + o(1)$. But since simply 3-suitability is a more relaxed notion than 3-suitability, we can get the following exact formula for $N(n)$:

Lemma 2. $N(n) = \lceil \log_2 \log_2 n \rceil + 1$.

Proof. Erdős and Szekeres [14] proved that if σ_1 and σ_2 are two permutations of $[n^2+1]$, then there exists some $X \subset [n^2+1]$ with $|X| = n+1$ such that the permutation of X obtained by restricting σ_1 to X is the same as the permutation obtained by restricting σ_2 to X. By an easy inductive argument (as Spencer points out in [19]) we can show that if $\sigma_1, \sigma_2, \ldots \sigma_{s+1}$ are permutations of $[2^{2^s}+1]$, then there exists some triple $\{x, y, z\}$ such that the order of these 3 elements with respect to each permutation $\sigma_1, \sigma_2, \ldots \sigma_{s+1}$ is the same. This implies that $N(n) \geq \lceil \log_2 \log_2 n \rceil + 1$.

We need to show that when $n \leq 2^{2^i}$, $N(n) \leq i+1$. Note that when the permutations in a simply 3-suitable set S for $[n]$ are restricted to $[n_1]$ (where $n_1 < n$), S becomes a simply 3-suitable set for $[n_1]$. Hence it is enough to prove that, when $n = 2^{2^i}$, $N(n) \leq i+1$. We prove this by induction on i. The base case, when $i = 0$ and $n = 2$, is trivially true. For any $i < i_1$, assume $N(n) \leq i+1$. Let $i = i_1$, $n = 2^{2^{i_1}}$ and $n_1 = 2^{2^{i_1-1}}$. Then $n = n_1 \cdot n_1$. So set $[n]$ can be partitioned into n_1 sets $A_1, A_2, \ldots A_{n_1}$, where for any $p \in [n_1]$, $A_p = \{(p-1)n_1+1, (p-1)n_1+2, \ldots, (p-1)n_1+n_1\}$. Clearly for any $a \in [n]$, there exist $k, p \in [n_1]$ such that $a = (p-1)n_1 + k$. By induction hypothesis, there exists a simply 3-suitable set $S' = \{\eta_1, \eta_2, \ldots \eta_{i_1}\}$ of $[n_1]$. Then we define i_1+1 permutations $S = \{\sigma_1, \ldots, \sigma_{i_1+1}\}$ for $[n]$ as follows:

$$\sigma_j^{-1}(a) = (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k), \text{ where } 1 \leq j \leq i_1.$$
$$\sigma_{i_1+1}^{-1}(a) = (n_1 - \eta_{i_1}^{-1}(p))n_1 + \eta_{i_1}^{-1}(k).$$

We claim that S is a simply 3-suitable set for $[n]$ i.e., for any $s, t \in [n]$, $\bigcap_{i=1}^{i_1+1} \beta(s,t,\sigma_i) = \emptyset$. Let $s \in A_p$ and $t \in A_q$. Consider the 2 cases below:
case 1: If $p = q$, then there exist $k_1, k_2 \in [n_1]$ with $k_1 \neq k_2$ such that, $s = (p-1)n_1 + k_1$ and $t = (p-1)n_1 + k_2$. Consider a permutation σ_j, where $j \in [i_1]$.

$$\begin{aligned}
\beta(s,t,\sigma_j) &= \{x \mid \sigma_j^{-1}(s) < \sigma_j^{-1}(x) < \sigma_j^{-1}(t) \\
&\quad \text{ or } \sigma_j^{-1}(t) < \sigma_j^{-1}(x) < \sigma_j^{-1}(s)\} \\
&= \{x \mid (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_1) < \sigma_j^{-1}(x) < (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_2) \\
&\quad \text{ or } (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_2) < \sigma_j^{-1}(x) < (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_1)\}.
\end{aligned}$$

If $\beta(s,t,\sigma_j) \neq \emptyset$, then consider any $x \in \beta(s,t,\sigma_j)$. Clearly $x \in A_p$. Let $x = (p-1)n_1+k_3$. From the above, it is clear that either $\eta_j^{-1}(k_1) < \eta_j^{-1}(k_3) < \eta_j^{-1}(k_2)$ or $\eta_j^{-1}(k_2) < \eta_j^{-1}(k_3) < \eta_j^{-1}(k_1)$. This means that $x \in \beta(s,t,\sigma_j) \implies k_3 \in \beta(k_1,k_2,\eta_j)$. Therefore, $\bigcap_{j=1}^{i_1} \beta(s,t,\sigma_j) \neq \emptyset \implies \bigcap_{j=1}^{i_1} \beta(k_1,k_2,\eta_j) \neq \emptyset$. By induction hypothesis, we know that $\bigcap_{j=1}^{i_1} \beta(k_1,k_2,\eta_j) = \emptyset$. Hence $\bigcap_{j=1}^{i_1} \beta(s,t,\sigma_j) = \emptyset$.
case 2: If $p \neq q$, then $\exists k_1, k_2 \in [n_1]$ such that $s = (p-1)n_1 + k_1$ and $t = (q-1)n_1 + k_2$. Let $x = (r-1)n_1 + k_3$. Now $x \in \bigcap_{j=1}^{i_1} \beta(s,t,\sigma_j)$ implies, for any $j \in [n_1]$, $(\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_1) < (\eta_j^{-1}(r)-1)n_1 + \eta_j^{-1}(k_3) < (\eta_j^{-1}(q)-1)n_1 + \eta_j^{-1}(k_2)$ or $(\eta_j^{-1}(q)-1)n_1 + \eta_j^{-1}(k_2) < (\eta_j^{-1}(r)-1)n_1 + \eta_j^{-1}(k_3) < (\eta_j^{-1}(p)-1)n_1 + \eta_j^{-1}(k_1)$. It follows that $\eta_j^{-1}(p) \leq \eta_j^{-1}(r) \leq \eta_j^{-1}(q)$ or $\eta_j^{-1}(q) \leq \eta_j^{-1}(r) \leq \eta_j^{-1}(p)$. If $r \notin \{p,q\}$, then $\eta_j^{-1}(p) < \eta_j^{-1}(r) < \eta_j^{-1}(q)$ or $\eta_j^{-1}(q) < \eta_j^{-1}(r) < \eta_j^{-1}(p)$ i.e., $r \in \bigcap_{j=1}^{i_1} \beta(p,q,\eta_j)$ which contradicts the induction hypothesis that $\bigcap_{j=1}^{i_1} \beta(p,q,\eta_j) = \emptyset$.

Therefore we infer that $r = p$ or $r = q$. Let $r = p$ (proof is similar when $r = q$). If $x \in \bigcap_{j=1}^{i_1+1} \beta(s,t,\sigma_j)$ then we have $x \in \beta(s,t,\sigma_{i_1})$ and therefore $\sigma_{i_1}^{-1}(s) < \sigma_{i_1}^{-1}(x) < \sigma_{i_1}^{-1}(t)$ or $\sigma_{i_1}^{-1}(t) < \sigma_{i_1}^{-1}(x) < \sigma_{i_1}^{-1}(s)$. Without loss of generality, let $\sigma_{i_1}^{-1}(s) < \sigma_{i_1}^{-1}(x) < \sigma_{i_1}^{-1}(t)$. Then $(\eta_{i_1}^{-1}(p)-1)n_1 + \eta_{i_1}^{-1}(k_1) < (\eta_{i_1}^{-1}(r)-1)n_1 + \eta_{i_1}^{-1}(k_3) < (\eta_{i_1}^{-1}(q)-1)n_1 + \eta_{i_1}^{-1}(k_2)$. Since $p = r$, we have $\eta_{i_1}^{-1}(p) = \eta_{i_1}^{-1}(r)$ and therefore $\eta_{i_1}^{-1}(k_1) < \eta_{i_1}^{-1}(k_3)$. This also allows us to infer that $(n_1 -$

$\eta_{i_1}^{-1}(p))n_1 + \eta_{i_1}^{-1}(k_1) < (n_1 - \eta_{i_1}^{-1}(r))n_1 + \eta_{i_1}^{-1}(k_3)$. That is $\sigma_{i_1+1}^{-1}(s) < \sigma_{i_1+1}^{-1}(x)$. On the other hand, $(n_1 - \eta_{i_1}^{-1}(q))n_1 + \eta_{i_1}^{-1}(k_2) < (n_1 - \eta_{i_1}^{-1}(p))n_1 + \eta_{i_1}^{-1}(k_1)$ (since $\eta_{i_1}^{-1}(p) < \eta_{i_1}^{-1}(q)$). Therefore, $\sigma_{i_1+1}^{-1}(t) < \sigma_{i_1+1}^{-1}(s)$. So we have, $\sigma_{i_1+1}^{-1}(t) < \sigma_{i_1+1}^{-1}(s) < \sigma_{i_1+1}^{-1}(x)$. Hence $x \notin \beta(s, t, \sigma_{i_1+1})$ contradicting our assumption that $x \in \bigcap_{j=1}^{i_1+1} \beta(s, t, \sigma_j)$. □

Theorem 1. *Let G be the graph obtained by fully subdividing the complete graph K_n. Then $\frac{\lceil \log_2 \log_2 n \rceil + 1}{2} \leq box(G) \leq \lceil \log_2 \log_2 n \rceil + 2$.*

Proof. Let $v_1, v_2, \ldots v_n$ be the vertices of K_n and $e_1, e_2, \ldots e_m$ its edges, where $m = \binom{n}{2}$. Let $u_{p \cdot q}$ denote the vertex introduced when subdividing the edge $(v_p, v_q) \in E(K_n)$, where $p < q$. Thus the graph G obtained by fully subdividing K_n has the vertex set $V(G) = \{v_1, v_2, \ldots v_n\} \cup \{u_{p \cdot q} \mid 1 \leq p < q \leq n\}$ and $E(G) = \{(v_p, u_{p \cdot q}) \mid 1 \leq p < q \leq n\} \cup \{(v_q, u_{p \cdot q} \mid 1 \leq p < q \leq n)\}$.

We first show that $box(G) \leq \lceil \log_2 \log_2 n \rceil + 2$. Let $k = \lceil \log_2 \log_2 n \rceil + 1$. By Lemma 2, there exists a simply 3-suitable set $S = \{\sigma_1, \ldots, \sigma_k\}$ for $[n]$. Using S, we construct a $(k+1)$-dimensional box representation for G. Corresponding to each permutation σ_i of $[n]$ in S, we construct an interval graph I_i as follows. Let f_i denote the interval representation of I_i.

for every $v_p \in V(G)$, $f_i(v_p) = [\sigma_i^{-1}(p), \sigma_i^{-1}(p)]$.
for every $u_{p \cdot q} \in V(G)$, $f_i(u_{p \cdot q}) = [\sigma_i^{-1}(p), \sigma_i^{-1}(q)]$, if $\sigma_i^{-1}(p)] < \sigma_i^{-1}(q)$.
for every $u_{p \cdot q} \in V(G)$, $f_i(u_{p \cdot q}) = [\sigma_i^{-1}(q), \sigma_i^{-1}(p)]$, if $\sigma_i^{-1}(q) < \sigma_i^{-1}(p)]$.

The interval representation f_{k+1} of the $(k+1)$th interval graph I_{k+1} is as follows:

for every $v_p \in V(G)$, $f_{k+1}(v_p) = [1, m]$.
for every $u_{p \cdot q} \in V(G)$, $f_{k+1}(u_{p \cdot q}) = [j, j]$, where $u_{p \cdot q}$ was obtained by subdividing edge $e_j = (v_p, v_q)$ of K_n.

By Lemma 1, in order to prove that $box(G) \leq k + 1$ it is sufficient to show that $\bigcap_{i=1}^{k+1} I_i = G$, i.e.,
(i) each I_j is a supergraph of G.
(ii) for any $(x, y) \notin E(G)$, there exists some interval graph I_i such that $(x, y) \notin E(I_i)$.

Recall that any edge of G is of the form (v_p, u_{pq}) or (v_q, u_{pq}), where $v_p, v_q \in V(K_n)$. It is easy to verify that, for any $i \in [k+1]$, $f_i(u_{pq}) \cap f_i(v_p) \neq \emptyset$ and $f_i(u_{pq}) \cap f_i(v_q) \neq \emptyset$. Therefore (i) is true.

Let $(x, y) \notin E(G)$. In order to prove (ii), we consider the following cases:
case 1: $x = v_p, y = v_q$, for some $1 \leq p < q \leq n$.
It is easy to see that $f_1(v_p) \cap f_1(v_q) = \emptyset$ and therefore $(v_p, v_q) \notin E(I_1)$.
case 2: $x = u_{p \cdot q}, y = u_{r \cdot s}$ and $u_{p \cdot q} \neq u_{r \cdot s}$.
Clearly, $f_{k+1}(u_{p \cdot q}) \cap f_{k+1}(u_{r \cdot s}) = \emptyset$ and therefore $(u_{p \cdot q}, u_{r \cdot s}) \notin E(I_{k+1})$.
case 3: $x = v_p, y = u_{r \cdot s}$, for any $p, r, s \in [n]$, $p \notin \{r, s\}$ and $r < s$.
Since S is a simply 3-suitable set for $[n]$ there exists a permutation σ_j such

that $p \notin \beta(r,s,\sigma_j)$ i.e., either $\sigma_j^{-1}(p) < \min(\sigma_j^{-1}(r),\sigma_j^{-1}(s))$ or $\sigma_j^{-1}(p) > \max(\sigma_j^{-1}(r),\sigma_j^{-1}(s))$. Now it is easy to see that, $f_j(v_p) \cap f_j(u_{r\cdot s}) = \emptyset$ and therefore $(v_p, u_{r\cdot s}) \notin E(I_j)$. We thus prove (ii) and thereby prove that $box(G) \leq \lceil \log_2 \log_2 n \rceil + 2$.

We now show that $box(G) \geq \frac{\lceil \log_2 \log_2 n \rceil + 1}{2}$. Let $box(G) = b$. By Lemma 1 there exist b interval graphs, say $I_1, I_2, \ldots, I_b$, such that $G = \bigcap_{i=1}^{b} I_i$. For any $i \in [b]$, let f_i be an interval representation of I_i such that no two intervals share any endpoints. From each f_i, generate two permutations L_i and R_i of $[n]$ in the following way. For $p, q \in [n]$, $p \neq q$, $L_i^{-1}(p) < L_i^{-1}(q) \Leftrightarrow l(f_i(v_p)) < l(f_i(v_q))$. Similarly, $R_i^{-1}(p) < R_i^{-1}(q) \Leftrightarrow r(f_i(v_p)) < r(f_i(v_q))$

Consider the set $S = \{L_1, R_1, L_2, R_2, \ldots L_b, R_b\}$ of permutations of $[n]$. We claim that S is a simply 3-suitable set for $[n]$. Let $s, t \in [n]$. Then for any $i \in [b]$,

$$\begin{aligned} x \in \beta(s,t,L_i) &\implies \left(L_i^{-1}(s) < L_i^{-1}(x) < L_i^{-1}(t)\right) \text{ or} \\ &\qquad \left(L_i^{-1}(t) < L_i^{-1}(x) < L_i^{-1}(s)\right) \\ &\implies (l(f_i(v_s)) < l(f_i(v_x)) < l(f_i(v_t))) \text{ or} \\ &\qquad (l(f_i(v_t)) < l(f_i(v_x)) < l(f_i(v_s)))\,. \end{aligned} \tag{2}$$

$$\begin{aligned} x \in \beta(s,t,R_i) &\implies \left(R_i^{-1}(s) < R_i^{-1}(x) < R_i^{-1}(t)\right) \text{ or} \\ &\qquad \left(R_i^{-1}(t) < R_i^{-1}(x) < R_i^{-1}(s)\right) \\ &\implies (r(f_i(v_s)) < r(f_i(v_x)) < r(f_i(v_t))) \text{ or} \\ &\qquad (r(f_i(v_t)) < r(f_i(v_x)) < r(f_i(v_s)))\,. \end{aligned} \tag{3}$$

Suppose, for contradiction, $x \in \bigcap_{j=1}^{b} (\beta(s,t,L_j) \cap \beta(s,t,R_j))$. Consider any $i \in [b]$. Let $y = \max(l(f_i(v_s)), l(f_i(v_t)))$ and $z = \min(r(f_i(v_s)), r(f_i(v_t)))$. Consider the two cases below:

case 1: $y < z$. Then by implications (2) and (3) it is clear that $l(f_i(v_x)) < y = \max(l(f_i(v_s)), l(f_i(v_t)))$ and $r(f_i(v_x)) > z = \min(r(f_i(v_s)), r(f_i(v_t)))$. Therefore, $[y,z] \subseteq f_i(v_x)$. Now we will show that $f_i(u_{s\cdot t}) \cap [y,z] \neq \emptyset$ which will immediately imply that $f_i(u_{s\cdot t}) \cap f_i(v_x) \neq \emptyset$. If $f_i(u_{s\cdot t}) \cap [y,z] = \emptyset$, then either $r(f_i(u_{s\cdot t})) < y$ or $l(f_i(u_{s\cdot t})) > z$. In both these cases, it is easy to see that either $(u_{s\cdot t}, v_s) \notin E(I_i)$ or $(u_{s\cdot t}, v_t) \notin E(I_i)$. This contradicts the fact that I_i is a supergraph of G. Hence $f_i(u_{s\cdot t}) \cap [y,z] \neq \emptyset$ and therefore $(u_{s\cdot t}, v_x) \in E(I_i)$.
case 2: $y > z$. Since $(u_{s\cdot t}, v_s) \in E(I_i)$ and $(u_{s\cdot t}, v_t) \in E(I_i)$, we have $r(f_i(u_{s\cdot t})) > y$ and $l(f_i(u_{s\cdot t})) < z$. Therefore, $[z,y] \subseteq f_i(u_{s\cdot t})$. Now we will show that $f_i(v_x) \cap [z,y] \neq \emptyset$ which will immediately imply that $f_i(u_{s\cdot t}) \cap f_i(v_x) \neq \emptyset$. If $f_i(v_x) \cap [z,y] = \emptyset$, then either $r(f_i(v_x)) < z$ or $l(f_i(v_x))) > y$. In both these cases, we contradict implications (2) and (3) which state that $r(f_i(v_x))$ is sandwiched between $r(f_i(v_s))$ and $r(f_i(v_t))$, and $l(f_i(v_x))$ is sandwiched between $l(f_i(v_s))$ and $l(f_i(v_t))$. Hence $f_i(v_x) \cap [z,y] \neq \emptyset$ and therefore $(u_{s\cdot t}, v_x) \in E(I_i)$.

Thus we conclude that if there exists an $x \notin \{s,t\}$ such that $x \in \bigcap_{j=1}^{2b} \beta(s,t,\sigma_j)$, then $(u_{s\cdot t}, v_x) \in E(\bigcap_{i=1}^{b} I_i)$ which implies that $(u_{s\cdot t}, v_x) \in E(G)$. But this contradicts the fact that $(u_{s\cdot t}, v_x) \notin E(G)$ and hence $\bigcap_{j=1}^{2b} \beta(s,t,\sigma_j) = \emptyset$ i.e.,

S is a simply 3-suitable set. Then by Lemma 2, $|S| = 2b \geq \lceil \log_2 \log_2 n \rceil + 1$ or $box(G) \geq \frac{\lceil \log_2 \log_2 n \rceil + 1}{2}$. $\square$

Remark 1. Louis Esperet informed us that he had independently observed Theorem 1. But he has not published it. We thank him for personal communication. In [15], he also conjectures that for any graph G, (i) $box(G) \leq a(G) + \kappa$, (ii) $box(G) \leq \lambda \cdot a(G)$, where κ, λ are constants and $a(G)$ refers to the arboricity of G. As arboricity of any graph is upper bounded by its degeneracy and since fully subdivided complete graphs are 2-degenerate, Theorem 1 disproves Esperet's both conjectures.

3 Boxicity of a Fully Subdivided Graph of Chromatic Number χ

Theorem 2. *Let H be a graph with chromatic number χ and let G be the graph obtained by fully subdividing H. Then, $box(G) \leq \lceil \log_2 \log_2 \chi \rceil + 3$.*

Proof. Given a colouring of H using χ colours, let $C_1, C_2 \ldots C_\chi$ represent the χ colour classes. Let $|C_i| = c_i$ and $c_{max} = \max_i(c_i)$. Give an arbitrary order to the vertices in each colour class. Let v_{ij} denote the j-th vertex in the i-th colour class, where $i \in [\chi]$ and $j \in [c_i]$. Let $E(H) = \{e_1, e_2, \ldots, e_m\}$ be the edge set of H. Let $u_{pq \cdot rs}$ denote the vertex introduced while subdividing the edge (v_{pq}, v_{rs}), where $p < r$. Let $k = \lceil \log_2 \log_2 \chi \rceil + 1$. By Lemma 2, there exists a simply 3-suitable set $S = \{\sigma_1, \ldots \sigma_k\}$ for $[\chi]$. We use S to construct a $(k+2)$-dimensional box representation for G. Corresponding to each permutation $\sigma_i \in S$, we construct an interval graph I_i as follows. Let f_i denote the interval representation of I_i. When $i \leq k$,

$$\begin{aligned}
\text{for every } v_{pq} \in E(G), \quad f_i(v_{pq}) &= [g_i(p,q), g_i(p,q)], \\
&\text{where } g_i(p,q) = \sigma_i^{-1}(p) + \frac{q-1}{c_{max}}. \\
\text{for every } u_{pq \cdot rs} \in E(G), \; f_i(u_{pq \cdot rs}) &= [g_i(p,q), g_i(r,s)], \text{ if } g_i(p,q) < g_i(r,s). \\
\text{for every } u_{pq \cdot rs} \in E(G), \; f_i(u_{pq \cdot rs}) &= [g_i(r,s), g_i(p,q)], \text{ if } g_i(r,s) < g_i(p,q), \\
&\text{where } g_i(p,q) = \sigma_i^{-1}(p) + \frac{q-1}{c_{max}} \\
&\text{and } g_i(r,s) = \sigma_i^{-1}(r) + \frac{s-1}{c_{max}}.
\end{aligned}$$

The interval representations of the remaining 2 interval graphs namely I_{k+1} and I_{k+2} are as follows:-

$$
\begin{aligned}
\text{for every } v_{pq} \in E(G), \quad f_{k+1}(v_{pq}) &= [1, m]. \\
\text{for every } u_{pq\cdot rs} \in E(G), \ f_{k+1}(u_{pq\cdot rs}) &= [j, j], \\
&\quad \text{where } u_{pq\cdot rs} \text{ was obtained by} \\
&\quad \text{subdividing edge } e_j = (v_{pq}, v_{rs}) \text{ of } H. \\
\text{for every } v_{pq} \in E(G), \quad f_{k+2}(v_{pq}) &= [h_k(p,q), h_k(p,q)], \\
&\quad \text{where } h_k(p,q) = (\chi+1) - \sigma_k^{-1}(p) + \frac{q-1}{c_{max}}. \\
\text{for every } u_{pq\cdot rs} \in E(G), \ f_{k+2}(u_{pq\cdot rs}) &= [h_k(p,q), h_k(r,s)], \text{ if } h_k(p,q) < h_k(r,s). \\
\text{for every } u_{pq\cdot rs} \in E(G), \ f_{k+2}(u_{pq\cdot rs}) &= [h_k(r,s), h_k(p,q)], \text{ if } h_k(r,s) < h_k(p,q), \\
&\quad \text{where } h_k(p,q) = (\chi+1) - \sigma_k^{-1}(p) + \frac{q-1}{c_{max}} \\
&\quad \text{and } h_k(r,s) = (\chi+1) - \sigma_k^{-1}(r) + \frac{s-1}{c_{max}}.
\end{aligned}
$$

Observe that every edge in G is of the form $(u_{pq\cdot rs}, v_{pq})$ or $(u_{pq\cdot rs}, v_{rs})$ where v_{pq} and v_{rs} are vertices of H and $u_{pq\cdot rs}$ is the vertex introduced while subdividing edge (v_{pq}, v_{rs}). Any interval graph I_i, where $1 \leq i \leq k$, is clearly a supergraph of G because in f_i the interval corresponding to $u_{pq\cdot rs}$ has its endpoints on the point intervals assigned to v_{pq} and v_{rs}. The same is true with interval graph I_{k+2}. In the interval representation f_{k+1} of I_{k+1} , any vertex v_{pq} is assigned an interval $[1, m]$ which overlaps with the interval of every other vertex. Hence all interval graphs $I_1, I_2, \ldots, I_{k+2}$ are supergraphs of G.

In order to show that for every $(x, y) \notin E(G)$ there exists some interval graph I_i in our collection such that $(x, y) \notin E(I_i)$, we consider the following cases:

case 1: $x = v_{pq}$, $y = v_{rs}$, where $v_{pq} \neq v_{rs}$.
As $f_1(v_{pq}) \cap f_1(v_{rs}) = \emptyset$, $(v_{pq}, v_{rs}) \notin E(I_1)$.

case 2: $x = u_{pq\cdot rs}, y = u_{wx\cdot yz}$, where $u_{pq\cdot rs} \neq u_{wx\cdot yz}$.
It is easy to verify that $f_{k+1}(u_{pq\cdot rs}) \cap f_{k+1}(u_{wx\cdot yz}) = \emptyset$ and hence $(u_{pq\cdot rs}, u_{wx\cdot yz}) \notin E(I_{k+1})$.

case 3: $x = u_{pq\cdot rs}, y = v_{ab}$ and $a \notin \{p, r\}$.
Note that $p, r, a \in [\chi]$ and since S is a simply 3-suitable set for $[\chi]$, there exists a $\sigma_i \in S$ such that $a \notin \beta(p, r, \sigma_i)$ i.e., $\sigma_i^{-1}(a) < \min(\sigma_i^{-1}(p), \sigma_i^{-1}(r))$ or $\sigma_i^{-1}(a) > \max(\sigma_i^{-1}(p), \sigma_i^{-1}(r))$. $f_i(v_{ab}) = [g_i(a,b), g_i(a,b)]$ and $f_i(u_{pq\cdot rs}) = [g_i(p,q), g_i(r,s)]$. Recalling that, for any $x_1 \in [\chi]$ and $x_2 \in [c_i]$, $g_i(x_1, x_2) = \sigma_i^{-1}(x_1) + \frac{x_2-1}{c_{max}}$ it is easy to verify that $f_i(v_{ab}) \cap f_i(u_{pq\cdot rs}) = \emptyset$.

case 4: $x = u_{pq\cdot rs}, y = v_{ab}$ and $a \in \{p, r\}$.
Assume $a = p$ (proof is similar when $a = r$). Assume $(v_{pb}, u_{pq\cdot rs}) \in E(I_i), \forall i \in \{1, 2, \ldots, k+2\}$. It means $(v_{pb}, u_{pq\cdot rs}) \in E(I_k) \implies \sigma_k^{-1}(p) + \frac{q-1}{c_{max}} < \sigma_k^{-1}(p) + \frac{b-1}{c_{max}} < \sigma_k^{-1}(r) + \frac{s-1}{c_{max}} \implies q < b$ (here we assume that $\sigma_i^{-1}(p) < \sigma_i^{-1}(r)$. Proof is similar when $\sigma_i^{-1}(p) > \sigma_i^{-1}(r)$). In f_{k+2}, note that $u_{pq\cdot rs}$ is assigned the interval $[(\chi+1) - \sigma_k^{-1}(r) + \frac{s-1}{c_{max}}, (\chi+1) - \sigma_k^{-1}(p) + \frac{q-1}{c_{max}}]$ and v_{ab} $(= v_{pb})$ is

assigned the interval $[(\chi+1)-\sigma_k^{-1}(p)+\frac{b-1}{c_{max}},(\chi+1)-\sigma_k^{-1}(p)+\frac{b-1}{c_{max}}]$. Therefore, $(v_{pb},u_{pq\cdot rs})\in E(I_{k+2}) \implies b<q$. But this contradicts our earlier inference that $q<b$. Therefore, either $(v_{ab},u_{pq\cdot rs})\notin E(I_k)$ or $(v_{ab},u_{pq\cdot rs})\notin E(I_{k+2})$.

We have thus shown that for any $(x,y)\notin E(G)$, $\exists i\in[k+2]$ such that $(x,y)\notin E(I_i)$. As each I_i is a supergraph of G, we have $G=\bigcap_{i=1}^{k+2} I_i$. Applying Lemma 1, we get $box(G)\le\lceil\log_2\log_2\chi\rceil+3$. □

Corollary 1. *Given a graph H, let G be the graph obtained by fully subdividing H. Then, $box(G)\le\lceil\log_2\log_2(\Delta(H))\rceil+3\le\lceil\log_2\log_2(\Delta(G))\rceil+3$*

Proof. By Brooks' theorem (see chapter 5 in [13]), $\chi\le\Delta(H)$ unless the graph H is isomorphic to a complete graph $K_{\Delta(H)+1}$ or to an odd cycle. If H is isomorphic to $K_{\Delta(H)+1}$, then by Theorem 1, $box(G)\le\lceil\log_2\log_2(\Delta(H)+1)\rceil+2\le\lceil\log_2\log_2(\Delta(H))\rceil+3$. If H is an odd cycle, then G will be a cycle and hence $box(G)\le 2<\lceil\log_2\log_2(\Delta(H))\rceil+3$. Therefore applying Theorem 2, we have $box(G)\le\lceil\log_2\log_2(\Delta(H))\rceil+3$. As $\Delta(H)\le\Delta(G)$, the corollary follows. □

4 Line Graphs

For any bipartite graph G with bipartition $\{A,B\}$, we use $C_A(G)$ to denote the graph with $V(C_A(G))=V(G)$ and $E(C_A(G))=E(G)\cup\{(x,y)\mid x,y\in A\}$. Thus $C_A(G)$ is the graph obtained from G by making A a clique. Similarly one can define $C_B(G)$.

Lemma 3. *For any bipartite graph G with bipartition $\{A,B\}$, $box(C_A(G))\le 2\cdot box(G)$.*

Proof. Proof of this lemma is similar to the proof of Lemma 7 in [5]. In [5] it is proved that $box(C_{AB}(G))\le 2\cdot box(G)$, where $C_{AB}(G)$ refers to the graph obtained by making both A and B cliques. For the sake of completeness, we give a proof to our lemma below.

Let $box(G)=b$. Then by Lemma 1, there exist b interval graphs, say $I_1,I_2,\ldots,I_b$, such that $G=\bigcap_{i=1}^{b} I_i$. Let f_i denote an interval representation of I_i, where $i\in[b]$. Let $s_i=\min_{x\in A}(l(f_i(x)))$ and $t_i=\max_{x\in A}(r(f_i(x)))$. From these b interval graphs we construct $2b$ interval graphs namely $I'_1,I'_2,\ldots I'_b,I''_1,I''_2,\ldots I''_b$ as follows. Let f'_i, f''_i denote interval representations of I'_i and I''_i respectively, where $i\in[b]$.

Construction of f'_i :

$$\forall x\in A,\ f'_i(x)=[s_i,r(f_i(x))].$$
$$\forall x\in B,\ f'_i(x)=f_i(x).$$

Construction of f''_i :

$$\forall x\in A,\ f''_i(x)=[l(f_i(x)),t_i].$$
$$\forall x\in B,\ f''_i(x)=f_i(x).$$

We claim that $C_A(G) = \bigcap_{i=1}^{b}(I'_i \cap I''_i)$. Consider any $(x, y) \in E(C_A(G))$. To show that $(x, y) \in E(I'_i)$ and $(x, y) \in E(I''_i)$, $\forall i \in [b]$, we consider the following 2 cases. If $(x, y) \in E(G)$, clearly $(x, y) \in E(I_i)$. From the construction of f'_i and f''_i, it is easy to see that I'_i and I''_i are supergraphs of I_i. Otherwise if $(x, y) \notin E(G)$, then $x, y \in A$ and therefore $[s_i, s_i] \subseteq f'_i(x) \cap f'_i(y)$ and $[t_i, t_i] \subseteq f''_i(x) \cap f''_i(y)$.

Now, consider any $(x, y) \notin E(C_A(G)$. We know that $(x, y) \notin E(C_A(G)) \implies (x, y) \notin E(G) \implies (x, y) \notin E(I_i)$, for some $i \in [b]$. It is then easy to verify that,

(a) if $x \in A$, $y \in B$, then $(f'_i(x) \cap f'_i(y) = \emptyset)$ or $(f''_i(x) \cap f''_i(y) = \emptyset)$.

(b) if $x, y \in B$, then $(f'_i(x) \cap f'_i(y) = \emptyset)$ and $(f''_i(x) \cap f''_i(y) = \emptyset)$.

Thus we prove the claim that $C_A(G) = \bigcap_{i=1}^{b}(I'_i \cap I''_i)$.Therefore by Lemma 1, $box(C_A(G)) \leq 2 \cdot box(G)$. □

Lemma 4. *Let G be a bipartite graph with bipartition $\{X, Y\}$ having the following two properties: (i) for any $y \in Y$, $d_G(y) \leq 2$ and (ii) for any $y_1, y_2 \in Y$, if $y_1 \neq y_2$ then $N_G(y_1) \neq N_G(y_2)$. Then, $box(G) \leq \lceil \log_2 \log_2(\Delta(G)) \rceil + 3$.*

Proof. If $\Delta(G) = 1$, then G is a collection of isolated edges and therefore $box(G) = 1 \leq \lceil \log_2 \log_2(\Delta(G)) \rceil + 3$. Let $\Delta(G) \geq 2$. From G, we construct a bipartite graph G' with bipartition $\{X', Y'\}$ in the following way: To start with, let $G' = G$. For each vertex $u \in Y'$ with $d_{G'}(u) = 1$, we add a new vertex n_u to X' such that u is the only neighbour of n_u. For each $v \in Y'$ with $d_{G'}(v) = 0$, delete v from Y'. So $X' = X \cup \{n_u \mid u \in Y \text{ and } d_G(u) = 1\}$ and $Y' = Y \setminus \{v \in Y \mid v \text{ is an isolated vertex}\}$. We claim that $box(G) \leq box(G')$. This is because the graph obtained by removing isolated vertices from G is an induced subgraph of G' and therefore its boxicity is at most that of G'. As adding isolated vertices to any graph does not increase its boxicity, our claim follows.

From the construction of G' we can say that, for every $y \in Y'$, $d_{G'}(y) = 2$. Let G'' be the subgraph induced on vertices of X' in ${G'}^2$, where ${G'}^2$ denotes the square of graph G'. It is easy to see that G' can be obtained by fully subdividing G'' (Here note that if G and thereby G' had not satisfied property (ii), then the graph obtained by fully subdividing G'' would have just been a subgraph of G'). Therefore by our above claim and applying Corollary 1, we get

$$box(G) \leq box(G') \leq \lceil \log_2 \log_2(\Delta(G')) \rceil + 3.$$

From the construction of G' and recalling that $\Delta(G) \geq 2$, we infer that $\Delta(G') \leq \Delta(G)$. Therefore,

$$box(G) \leq \lceil \log_2 \log_2(\Delta(G)) \rceil + 3.$$

□

A *critical clique* of a graph G is a clique K where the vertices of K all have the same set of neighbours in $G \setminus K$, and K is maximal under this property. Let $\mathcal{K}$ denote the collection of critical cliques in G. The *critical clique graph* of a graph G, denoted by $CC(G)$, has $V(CC(G)) = \mathcal{K}$ and $E(CC(G)) = \{(K_1, K_2) \mid K_1, K_2 \in \mathcal{K} \text{ and } V(K_1) \cup V(K_2) \text{ induces a clique in } G\}$. Notice that $CC(G)$ is isomorphic to some induced subgraph of G. For example, we can take a representative vertex from each critical clique and the induced subgraph on this set of vertices is isomorphic to $CC(G)$. The following lemma is due to Chandran, Francis and Mathew [6] :

Lemma 5. *For any graph G, $box(G) = box(CC(G))$.*

We now prove the main result of the paper. Recall that, given a multigraph H, we define its line graph $L(H)$ in the following way: $V(L(H)) := E(H)$ and $E(L(H)) := \{(e_1, e_2) \mid e_1, e_2 \in E(H),\ e_1 \text{ and } e_2 \text{ share an endpoint in } H\}$. A graph G is a line graph if and only if there exists a multigraph H such that G is isomorphic to $L(H)$.

Theorem 3. *Given a multigraph H, let G be a graph isomorphic to $L(H)$. Let Δ denote $\Delta(G)$ and χ represent $\chi(G)$. Then, $box(G) \leq 2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1$.*

Proof. Given a vertex colouring of G using χ colours, let $D_1, D_2, \ldots, D_\chi$ be the colour classes. For any $1 \leq i \leq (\chi - 1)$, let G_i, with $V(G_i) = V(G)$ and $E(G_i) = E(G) \cup \{(x, y) \mid x, y \in \overline{D_i}\}$, be the split graph where D_i is an independent set and $\overline{D_i}$ a clique (here $\overline{D_i} = \{x \in V(G) \mid x \notin D_i\}$). Let G_χ^+ be the graph having $V(G_\chi^+) = V(G)$ and $E(G_\chi^+) = \{(x, y) \mid x \in \overline{D_\chi}, y \in V(G)\}$. It is easy to see that

$$G = G_1 \cap G_2 \cap \cdots \cap G_{(\chi-1)} \cap G_\chi^+.$$

Therefore by Lemma 1,

$$box(G) \leq \Sigma_{i=1}^{(\chi-1)} box(G_i) + box(G_\chi^+).$$

By Lemma 5, we know that $box(G_i) = box(CC(G_i))$. Also, observe that G_χ^+ is an interval graph and hence its boxicity is 1. Therefore,

$$box(G) \leq \Sigma_{i=1}^{(\chi-1)} box(CC(G_i)) + 1. \tag{4}$$

We know that, $\forall i \in [(\chi - 1)]$, G_i is a split graph, where D_i is an independent set and $\overline{D_i}$ a clique. As $CC(G_i)$ is isomorphic to some subgraph of G_i, it is also a split graph with $V(CC(G_i)) = X_i \uplus Y_i$, where $X_i \subseteq D_i$ is an independent set and $Y_i \subseteq \overline{D_i}$ a clique. Let H_i be the bipartite graph obtained from $CC(G_i)$ by making Y_i an independent set. By Lemma 3, we have $box(CC(G_i)) \leq 2 \cdot box(H_i)$. Applying this to inequality (4), we get

$$box(G) \leq 2\Sigma_{i=1}^{(\chi-1)} box(H_i) + 1. \tag{5}$$

Claim 1. For any $i \in [(\chi - 1)]$ and $y \in Y_i$, $d_{H_i}(y) \leq 2$.

Proof. Recall that $G = L(H)$ and therefore a proper vertex colouring of G is equivalent to a proper edge colouring of H. Since in any edge colouring of H a given edge e cannot have more than 2 monochromatic neighbours, for any $y \in \overline{D_i}$, $|N_G(y) \cap D_i| \leq 2$. Observe that the bipartite graph H_i is a subgraph of G. Therefore, for any $y \in Y_i \subseteq \overline{D_i}$, we get $|N_{H_i}(y) \cap X_i| = |N_{H_i}(y)| = d_{H_i}(y) \leq 2$.

For any $i \in [(\chi - 1)]$, H_i is a bipartite graph with bipartition $\{X_i, Y_i\}$ satisfying the following two properties:
(i) by Claim 1, for any $y \in Y_i$, $d_{H_i}(y) \leq 2$.
(ii) for any $y_1, y_2 \in Y_i$, if $y_1 \neq y_2$ then $N_{H_i}(y_1) \neq N_{H_i}(y_2)$. Assume for contradiction that there exist some $y_1, y_2 \in Y_i$ with $y_1 \neq y_2$ and $N_{H_i}(y_1) = N_{H_i}(y_2)$. Then we have $N_{CC(G_i)}(y_1) = N_{CC(G_i)}(y_2)$ which contradicts the fact that $CC(G_i)$ is the critical clique graph of G_i.

Therefore by Lemma 4, we get $box(H_i) \leq \lceil \log_2 \log_2(\Delta(H_i)) \rceil + 3$. Since H_i is a subgraph of G, $\Delta(H_i) \leq \Delta$. Hence,

$$box(H_i) \leq \lceil \log_2 \log_2 \Delta \rceil + 3.$$

We thus rewrite inequality (5) as,

$$box(G) \leq 2(\chi - 1)(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1 \leq 2\Delta(\lceil \log_2 \log_2 \Delta \rceil + 3) + 1.$$

As $G = L(H)$, $\Delta \leq 2(\Delta(H) - 1) \leq 2(\chi - 1)$. Therefore,

$$box(G) \leq 2(\chi - 1)(\lceil \log_2 \log_2(2(\chi - 1)) \rceil + 3) + 1.$$

□

5 Lower Bound for Boxicity of a Hypercube

For any non-negative integer d, a d-dimensional hypercube H_d has its vertices corresponding to the 2^d binary strings each of length d. Two vertices are adjacent if and only if their binary strings differ from each other in exactly one bit position.

Theorem 4. $box(H_d) \geq \frac{\lceil \log_2 \log_2 d \rceil + 1}{2}$

Proof. For any vertex $v \in V(H_d)$, let $g(v)$ denote the number of ones in the bit string associated with v. Let $X = \{v \in V(H) \mid g(v) = 1 \text{ or } g(v) = 2\}$. Let H' be the subgraph of H induced on the vertex set X. We can see that H' is a bipartite graph with bipartition $\{A, B\}$, where $A = \{v \in V(H') \mid g(v) = 1\}$ and $B = \{v \in V(H') \mid g(v) = 2\}$.

It is easy to observe that H' is a graph obtained by fully subdividing $K_{|A|}$, where $K_{|A|}$ refers to a complete graph on $|A| = d$ vertices. Then by Theorem 1, we can say that

$$box(H') \geq \frac{\lceil \log_2 \log_2 d \rceil + 1}{2}.$$

As H' is an induced subgraph of H,

$$box(H) \geq box(H') \geq \frac{\lceil \log_2 \log_2 d \rceil + 1}{2}.$$

□

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