

# $L \log L$ condition for supercritical branching Hunt processes

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## Abstract

In this paper we use the spine decomposition and martingale change of measure to establish a Kesten-Stigum  $L \log L$  theorem for branching Hunt processes. This result is a generalization of the results in [1] and [9] for branching diffusions.

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## 1 Introduction

Suppose that  $\{Z_n; n \geq 1\}$  is a Galton-Watson process with each particle having probability  $p_n$  of giving birth to  $n$  offspring. Let  $L$  stand for a random variable with this offspring distribution. Let  $m := \sum_{n=1}^{\infty} np_n$  be the mean number of offspring per particle. Then  $Z_n/m^n$  is a non-negative martingale. Let  $W$  be the limit of  $Z_n/m^n$  as  $n \rightarrow \infty$ . Kesten and Stigum proved in [10] that if  $1 < m < \infty$  (that is, in the supercritical case) then  $W$  is non-degenerate (i.e., not almost surely zero) if and only if

$$E(L \log^+ L) = \sum_{n=1}^{\infty} p_n n \log n < \infty, \quad (1.1)$$

here, and in the rest of this paper, we use the notation that  $\log^+ r = 0 \vee \log r$  for all  $r > 0$ . This result is usually referred to the Kesten-Stigum  $L \log L$  theorem.

In 1995, Lyons, Pemantle and Peres developed a martingale change of measure method in [20] to give a new proof for the Kesten-Stigum  $L \log L$  theorem for single type branching processes. Later this method was extended to prove the  $L \log L$  theorem for multiple and general multiple type branching processes in [2], [14] and [19].

In this paper we will extend this method to supercritical branching Hunt processes and establish an  $L \log L$  criterion for branching Hunt processes. To review the known results and state our main result, we need to introduce the setup we are going to work with first.

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In this paper  $E$  always stands for a locally compact separable metric space. We will use  $E_\Delta := E \cup \{\Delta\}$  to denote the one-point compactification of  $E$ . We will use  $\mathcal{B}(E)$  and  $\mathcal{B}(E_\Delta)$  to denote the Borel  $\sigma$ -fields on  $E$  and  $E_\Delta$ , respectively.  $\mathcal{B}_b(E)$  (respectively,  $\mathcal{B}^+(E)$ ) will denote the set of all bounded (respectively, non-negative)  $\mathcal{B}(E)$ -measurable functions on  $E$ . All functions  $f$  on  $E$  will be automatically extended to  $E_\Delta$  by setting  $f(\Delta) = 0$ . Let  $\mathbf{M}_p(E)$  be the space of finite point measures on  $E$ , that is, measures of the form  $\mu = \sum_{i=1}^n \delta_{x_i}$  where  $n = 0, 1, 2, \dots$  and  $x_i \in E, i = 1, \dots, n$ . (When  $n = 0$ ,  $\mu$  is the trivial zero measure.) For any function  $f$  on  $E$  and any measure  $\mu \in \mathbf{M}_p(E)$  we use  $\langle f, \mu \rangle$  to denote the integral of  $f$  with respect to  $\mu$ .

We will always assume that  $Y = \{Y_t, \Pi_x, \zeta\}$  is a Hunt process on  $E$ , where  $\zeta = \inf\{t > 0 : Y_t = \Delta\}$  is the lifetime of  $Y$ . Let  $\{P_t, t \geq 0\}$  be the transition semigroup of  $Y$ :

$$P_t f(x) = \Pi_x[f(Y_t)] \quad \text{for } f \in \mathcal{B}^+(E).$$

Let  $m$  be a positive Radon measure on  $E$  with full support.  $\{P_t, t \geq 0\}$  can be extended to a strongly continuous semigroup on  $L^2(E, m)$ . Let  $\{\hat{P}_t, t \geq 0\}$  be the semigroup on  $L^2(E, m)$  such that

$$\int_E f(x) P_t g(x) m(dx) = \int_E g(x) \hat{P}_t f(x) m(dx), \quad f, g \in L^2(E, m).$$

We will use  $\mathbf{A}$  and  $\hat{\mathbf{A}}$  to denote the generators of the semigroups  $\{P_t\}$  and  $\{\hat{P}_t\}$  on  $L^2(E, m)$  respectively.

Throughout this paper we assume that

**Assumption 1.1** (i) *There exists a family of continuous strictly positive functions  $\{p(t, \cdot, \cdot); t > 0\}$  on  $E \times E$  such that for any  $(t, x) \in (0, \infty) \times E$ , we have*

$$P_t f(x) = \int_E p(t, x, y) f(y) m(dy), \quad \hat{P}_t f(x) = \int_E p(t, y, x) f(y) m(dy).$$

(ii) *The semigroups  $\{P_t\}$  and  $\{\hat{P}_t\}$  are ultracontractive, that is, for any  $t > 0$ , there exists a constant  $c_t > 0$  such that*

$$p(t, x, y) \leq c_t \quad \text{for any } (x, y) \in E \times E.$$

Consider a branching system determined by the following three parameters:

- (a) a Hunt process  $Y = \{Y_t, \Pi_x, \zeta\}$  with state space  $E$ ;
- (b) a nonnegative bounded measurable function  $\beta$  on  $E$ ;
- (c) an offspring distribution  $\{(p_n(x))_{n=0}^\infty; x \in E\}$ .

Put

$$\psi(x, z) = \sum_{n=0}^{\infty} p_n(x) z^n, \quad z \geq 0. \tag{1.2}$$

$\psi$  is the generating function for the number of offspring generated at point  $x$ .

This branching system is characterized by the following properties:

- (i) Each particle has a random birth and a random death time.
- (ii) Given that a particle is born at  $x \in E$ , the conditional distribution of its path after birth is determined by  $\Pi_x$ .
- (iii) Given the path  $Y$  of a particle and given that the particle is alive at time  $t$ , its probability of dying in the interval  $[t, t + dt)$  is  $\beta(Y_t)dt + o(dt)$ .
- (iv) When a particle dies at  $x \in E$ , it splits into  $n$  particles at  $x$  with probability  $p_n(x)$ .
- (v) The point  $\Delta$  is a cemetery. When a particle reaches  $\Delta$ , it stays at  $\Delta$  for ever and there is no branching at  $\Delta$ .

We assume that the functions  $p_n(x)$ ,  $n = 0, 1, \dots$ , and  $A(x) := \psi'(x, 1) = \sum_{n=0}^{\infty} np_n(x)$  are bounded  $\mathcal{B}(E)$ -measurable and that  $p_0(x) + p_1(x) = 0$  on  $E$ . The last condition implies  $A(x) \geq 2$  on  $E$ . The assumption  $p_0(x) = 0$  on  $E$  is essential for the probabilistic proof of this paper since we need the spine to be defined for all  $t \geq 0$ . The assumption  $p_1(x) = 0$  on  $E$  is just for convenience as the case  $p_1(x) > 0$  can be reduced to the case  $p_1(x) = 0$  by changing the parameters  $\beta$  and  $\psi$  of the branching Hunt process.

For any  $c \in \mathcal{B}_b(E)$ , we define

$$e_c(t) = \exp\left(-\int_0^t c(Y_s)ds\right).$$

Let  $X_t(B)$  be the number of particles located in  $B \in \mathcal{B}(E)$  at time  $t$ . Then  $\{X_t, t \geq 0\}$  is a Markov process in  $\mathbf{M}_p(E)$ . This process is called a  $(Y, \beta, \psi)$ -branching Hunt process. For any  $\mu \in \mathbf{M}_p(E)$ , let  $P_\mu$  be the law of  $\{X_t, t \geq 0\}$  when  $X_0 = \mu$ . Then we have

$$P_\mu \exp\langle -f, X_t \rangle = \exp\langle \log u_t(\cdot), \mu \rangle \quad (1.3)$$

where  $u_t(x)$  satisfies the equation

$$u_t(x) = \Pi_x \left[ e_\beta(t) \exp(-f(Y_t)) + \int_0^t e_\beta(s) \beta(Y_s) \psi(Y_s, u_{t-s}(Y_s)) ds \right] \quad \text{for } t \geq 0. \quad (1.4)$$

The formula (1.4) deals with a process started at time 0 with one particle located at  $x$ , and it has a clear heuristic meaning: the first term in the brackets corresponds to the case when the particle is still alive at time  $t$ ; the second term corresponds to the case when it dies before  $t$ . The formula (1.4) implies that

$$u_t(x) = \Pi_x \int_0^t [\psi(Y_s, u_{t-s}(Y_s)) - u_{t-s}(Y_s)] \beta(Y_s) ds + \Pi_x \exp(-f(Y_t)) \quad \text{for } t \geq 0 \quad (1.5)$$

(see [7, Section 2.3]). For any  $\mu \in \mathbf{M}_p(E)$ ,  $f \in \mathcal{B}_b^+(E)$  and  $t \geq 0$ , we have

$$P_\mu [\langle f, X_t \rangle] = \Pi_\mu [e_{(1-A)\beta}(t) f(Y_t)]. \quad (1.6)$$

Let  $\{P_t^{(1-A)\beta}, t \geq 0\}$  be the Feynman-Kac semigroup defined by

$$P_t^{(1-A)\beta} f(x) := \Pi_x [f(Y_t) e_{(1-A)\beta}(t) f(Y_t)], \quad f \in \mathcal{B}(E).$$

Let  $\{\widehat{P}_t^{(1-A)\beta}, t \geq 0\}$  be the dual semigroup of  $\{P_t^{(1-A)\beta}, t \geq 0\}$  on  $L^2(E, m)$ .

Under Assumption 1.1, we can easily show that the semigroups  $\{P_t^{(1-A)\beta}\}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  are strongly continuous on  $L^2(E, m)$ . Moreover,  $P_t^{(1-A)\beta}$  admits a density  $p^{(1-A)\beta}(t, x, y)$  that is jointly continuous in  $(x, y)$  for each  $t > 0$ :

$$P_t^{(1-A)\beta} f(x) = \int_E p^{(1-A)\beta}(t, x, y) f(y) m(dy), \quad \text{for every } f \in \mathcal{B}^+(E).$$

The generators of  $\{P_t^{(1-A)\beta}\}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  can be formally written as  $\mathbf{A} + (A - 1)\beta$  and  $\widehat{\mathbf{A}} + (A - 1)\beta$  respectively.

Let  $\sigma(\mathbf{A} + (A - 1)\beta)$  and  $\sigma(\widehat{\mathbf{A}} + (A - 1)\beta)$  denote the spectrum of operator  $\mathbf{A} + (A - 1)\beta$  and  $\widehat{\mathbf{A}} + (A - 1)\beta$ , respectively. It follows from Jentzsch's Theorem (Theorem V.6.6 on page 333 of [23]) and the strong continuity of  $\{P_t^{(1-A)\beta}\}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  that the common value  $\lambda_1 := \sup \operatorname{Re}(\sigma(\mathbf{A} + (A - 1)\beta)) = \sup \operatorname{Re}(\sigma(\widehat{\mathbf{A}} + (A - 1)\beta))$  is an eigenvalue of multiplicity 1 for both  $\mathbf{A} + (A - 1)\beta$  and  $\widehat{\mathbf{A}} + (A - 1)\beta$ , and that an eigenfunction  $\phi$  of  $\mathbf{A} + (A - 1)\beta$  associated with  $\lambda_1$  can be chosen to be strictly positive a.e. on  $E$  and an eigenfunction  $\widetilde{\phi}$  of  $\widehat{\mathbf{A}} + (A - 1)\beta$  associated with  $\lambda_1$  can be chosen to be strictly positive a.e. on  $E$ . By [11, Proposition 2.3] we know that  $\phi$  and  $\widetilde{\phi}$  are strictly positive and continuous on  $E$ . We choose  $\phi$  and  $\widetilde{\phi}$  so that  $\int_E \phi(x) \widetilde{\phi}(x) m(dx) = 1$ . Then

$$\phi(x) = e^{-\lambda_1 t} P_t^{(1-A)\beta} \phi(x), \quad \widetilde{\phi}(x) = e^{-\lambda_1 t} \widehat{P}_t^{(1-A)\beta} \widetilde{\phi}, \quad x \in E. \quad (1.7)$$

Throughout this paper we assume that

**Assumption 1.2**  $\lambda_1 > 0$ .

The above assumption is the condition for the supercriticality of the branching Hunt process. Indeed, if  $\lambda_1 < 0$ , it is easy to see that extinction occurs almost surely from the martingale  $M_t(\phi)$  defined below.

Let  $\mathcal{E}_t = \sigma(Y_s; s \leq t)$ . Note that

$$\frac{\phi(Y_t)}{\phi(x)} e^{-\lambda_1 t} e_{(1-A)\beta}(t)$$

is a martingale under  $\Pi_x$ , and so we can define a martingale change of measure by

$$\frac{d\Pi_x^\phi}{d\Pi_x} \Big|_{\mathcal{E}_t} = \frac{\phi(Y_t)}{\phi(x)} e^{-\lambda_1 t} e_{(1-A)\beta}(t).$$

Then  $\{Y, \Pi_x^\phi\}$  is a conservative Markov process, and  $\phi \widetilde{\phi}$  is the unique invariant probability density for the semigroup  $P_t^{(1-A)\beta}$ , that is, for any  $f \in \mathcal{B}^+(E)$ ,

$$\int_E \phi(x) \widetilde{\phi}(x) P_t^{(1-A)\beta} f(x) m(dx) = \int_E f(x) \phi(x) \widetilde{\phi}(x) m(dx).$$

Let  $p^\phi(t, x, y)$  be the transition density of  $Y$  in  $E$  under  $\Pi_x^\phi$ . Then

$$p^\phi(t, x, y) = \frac{e^{-\lambda_1 t}}{\phi(x)} p^{(1-A)\beta}(t, x, y) \phi(y).$$

Throughout this paper, we assume the following

**Assumption 1.3** *The semigroups  $\{P_t^{(1-A)\beta}\}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  are intrinsic ultracontractive, that is, for any  $t > 0$  there exists a constant  $c_t$  such that*

$$p^{(1-A)\beta}(t, x, y) \leq c_t \phi(x) \widetilde{\phi}(y), \quad x, y \in E.$$

**Remark 1.4** *Here are some examples of Hunt processes satisfying Assumptions 1.1 and 1.3.*

(1) *Suppose  $E = D$ , a domain in  $\mathbb{R}^d$ , and  $m$  is the Lebesgue measure on  $D$ . If  $\{Y, \Pi_x, x \in D\}$  is a diffusion killed upon leaving  $D$  with generator*

$$\mathbf{A} = \frac{1}{2} \nabla \cdot a \nabla + b \cdot \nabla$$

*where  $(a_{ij}(x))_{ij}$  is uniformly elliptic and bounded with  $a_{ij}, i, j = 1, \dots, d$ , being bounded functions in  $C^1(\mathbb{R}^d)$  such that all their first partial derivatives are bounded, and  $b_i, i = 1, \dots, d$ , are bounded Borel functions on  $\mathbb{R}^d$ . It was proven in [11] and [12] that the semigroups  $\{P_t^{(1-A)\beta}\}_{t \geq 0}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  are intrinsic ultracontractive when  $D$  is a bounded Lipschitz domain. For more general conditions on  $D$  and the coefficients for  $\{P_t^{(1-A)\beta}\}_{t \geq 0}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}$  to be intrinsic ultracontractive, see [12].*

(2) *Suppose  $E = D$ , a bounded open set in  $\mathbb{R}^d$ , and  $m$  is the Lebesgue measure on  $D$ . If  $\{Y, \Pi_x, x \in D\}$  is a symmetric  $\alpha$ -stable process killed upon exiting  $D$ , where  $0 < \alpha < 2$ , then it follows from [5] and [15] that the semigroups  $\{P_t^{(1-A)\beta}\}_{t \geq 0}$  and  $\{\widehat{P}_t^{(1-A)\beta}\}_{t \geq 0}$  are intrinsic ultracontractive.*

(3) *For examples in which  $E$  is unbounded, see [16] and [17].*

(4) *For more examples of discontinuous Markov processes satisfying Assumptions 1.1 and 1.3, we refer our readers to [13] and the references therein.*

It follows from [11, Theorem 2.8] that

$$\left| \frac{e^{-\lambda_1 t} p^{(1-A)\beta}(t, x, y)}{\phi(x) \widetilde{\phi}(y)} - 1 \right| \leq c e^{-\nu t}, \quad x \in E, \quad (1.8)$$

for some positive constants  $c$  and  $\nu$ , which is equivalent to

$$\sup_{x \in E} \left| \frac{p^\phi(t, x, y)}{\phi(y) \widetilde{\phi}(y)} - 1 \right| \leq c e^{-\nu t}. \quad (1.9)$$

Thus for any  $f \in \mathcal{B}_b^+(E)$  we have

$$\sup_{x \in E} \left| \int_E p^\phi(t, x, y) f(y) m(dy) - \int_E \phi(y) \widetilde{\phi}(y) f(y) m(dy) \right| \leq c e^{-\nu t} \int_E \phi(y) \widetilde{\phi}(y) f(y) m(dy).$$

Consequently we have

$$\lim_{t \rightarrow \infty} \frac{\int_E p^\phi(t, x, y) f(y) m(dy)}{\int_E \phi(y) \tilde{\phi}(y) f(y) m(dy)} = 1, \quad \text{uniformly for } f \in \mathcal{B}_b^+(E) \text{ and } x \in E. \quad (1.10)$$

For any nonzero measure  $\mu \in \mathbf{M}_p(E)$ , we define

$$M_t(\phi) := e^{-\lambda_1 t} \frac{\langle \phi, X_t \rangle}{\langle \phi, \mu \rangle} \quad t \geq 0.$$

**Lemma 1.5** *For any nonzero measure  $\mu \in \mathbf{M}_p(E)$ ,  $M_t(\phi)$  is a nonnegative martingale under  $P_\mu$ , and therefore there exists a limit  $M_\infty(\phi) \in [0, \infty)$ ,  $P_\mu$ -a.s.*

**Proof.** By the Markov property of  $X$ , (1.6) and (1.7),

$$\begin{aligned} P_\mu [M_{t+s}(\phi) | \mathcal{F}_t] &= \frac{1}{\langle \phi, \mu \rangle} e^{-\lambda_1 t} P_{X_t} [e^{-\lambda_1 s} \langle \phi, X_s \rangle] \\ &= \frac{1}{\langle \phi, \mu \rangle} e^{-\lambda_1 t} \left\langle e^{-\lambda_1 s} \Pi. [e_{(1-A)\beta}(s) \phi(Y_s)], X_t \right\rangle \\ &= \frac{1}{\langle \phi, \mu \rangle} e^{-\lambda_1 t} \langle \phi, X_t \rangle = M_t(\phi). \end{aligned}$$

This proves that  $\{M_t(\phi), t \geq 0\}$  is a non-negative  $P_\mu$ -martingale and so it has an almost sure limit  $M_\infty(\phi) \in [0, \infty)$  as  $t \rightarrow \infty$ .  $\square$

In this paper we are concerned with the following classical question: under what condition is the limit  $M_\infty(\phi)$  non-degenerate (that is,  $P_\mu(M_\infty(\phi) > 0) > 0$ )? In [1], Asmussen and Hering gave a criterion for  $M_\infty(\phi)$  to be non-degenerate for a general class of branching Markov processes under regularity conditions. More precisely, it was proved in [1] that if the underlying Markov process  $Y$  is positive regular (see [1] for the precise definition),  $M_\infty(\phi)$  is non-degenerate if and only if

$$\int_E m(dy) \tilde{\phi}(y) P_{\delta_y} [\langle \phi, X_t \rangle \log^+ \langle \phi, X_t \rangle] < \infty \quad \text{for some } t > 0. \quad (1.11)$$

This condition is not easy to verify since it involves the branching process  $X$  itself. It is more desirable to have a criterion in terms of the natural model parameters  $\mathbf{A}$ ,  $\beta$  and  $\{p_n(x)\}$  of the branching process. Such a criterion is found in [1] and [9] for branching diffusions and it was proved that, in the case of branching diffusions on a bounded open set  $E \subset \mathbb{R}^d$  with  $E$  being the union of finite number of bounded  $C^3$ -domains,  $M_\infty(\phi)$  is non-degenerate if and only if

$$\int_E \tilde{\phi}(y) \beta(y) l(y) m(dy) < \infty. \quad (1.12)$$

where

$$l(x) = \sum_{k=2}^{\infty} k \phi(x) \log^+(k \phi(x)) p_k(x), \quad x \in E. \quad (1.13)$$

The arguments of [1] and [9] are mainly analytic.

The purpose of this paper is two-folds. First, we generalize the result above to general branching Hunt processes. Even in the case of branching diffusions, our main result is more general than the corresponding result in [9] in some aspect since our requirement on the regularity of the domain is very weak. Secondly, we give a more probabilistic proof of the result, using the spine decomposition and martingale change of measure. Our probabilistic proof is similar to the probabilistic proofs of [8], [18] and [20].

The main result of this paper can be stated as follows.

**Theorem 1.6** *Suppose that  $\{X_t; t \geq 0\}$  is a  $(Y, \beta, \psi)$ -branching Hunt process and that Assumptions 1.1–1.3 are satisfied. Then  $M_\infty(\phi)$  is non-degenerate under  $P_\mu$  for any nonzero measure  $\mu \in \mathbf{M}_p(E)$  if and only if*

$$\int_E \tilde{\phi}(x)\beta(x)l(x)m(dx) < \infty, \quad (1.14)$$

where  $l$  is defined in (1.13).

It follows from the branching property that when  $\mu \in \mathbf{M}_p(E)$  is given by  $\mu = \sum_{i=1}^n \delta_{x_i}$ ,  $n = 1, 2, \dots$ ,  $\{x_i; i = 1, \dots, n\} \subset E$ , we have

$$M_t(\phi) = \sum_{i=1}^n e^{-\lambda_1 t} \frac{\langle \phi^t, X_t^i \rangle}{\phi(x_i)} \cdot \frac{\phi(x_i)}{\langle \phi, \mu \rangle},$$

where  $X_t^i$  is a branching Hunt process starting from  $\delta_{x_i}$ ,  $i = 1, \dots, n$ . If the conclusions hold for the cases that  $\mu = \delta_x$ , for any  $x \in E$ , then the conclusions also hold for the general cases. So in the remainder of this paper, we assume that the initial measure is of the form  $\mu = \delta_x$ ,  $x \in E$ , and  $P_{\delta_x}$  will be denoted as  $P^x$ .

This paper is organized as follows. In the next section we will discuss the spine decomposition of branching Markov processes. The main result, Theorem 1.6, is proved in the last section.

## 2 Spine decomposition

The materials of this section are mainly taken from [8]. We also refer to [18] for some materials. The main reason we present the details here is to clarify some of the points in [8].

Let  $\mathbb{N} = \{1, 2, \dots\}$ . We will use

$$\Gamma := \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

(where  $\mathbb{N}^0 = \{\emptyset\}$ ) to describe the genealogical structure of our branching Hunt process. The length (or generation)  $|u|$  of each  $u \in \mathbb{N}^n$  is defined to be  $n$ . When  $n \geq 1$  and  $u = (u_1, \dots, u_n)$ , we denote  $(u_1, \dots, u_{n-1})$  by  $u-1$  and call it the parent of  $u$ . For each  $i \in \mathbb{N}$  and  $u = (u_1, \dots, u_n)$ , we write  $ui = (u_1, \dots, u_n, i)$  for the  $i$ -th child of  $u$ . More generally, for  $u = (u_1, \dots, u_n), v = (v_1, \dots, v_m) \in \Gamma$ , we will use  $uv$  to stand for the concatenation  $(u_1, \dots, u_n, v_1, \dots, v_m)$  of  $u$  and

$v$ . We will use the notation  $v < u$  to mean that  $v$  is an ancestor of  $u$ . The set of all ancestors of  $u$  is given by  $\{v \in \Gamma : v < u\} = \{v \in \Gamma : \exists w \in \Gamma \setminus \{\emptyset\} \text{ such that } vw = u\}$ . The notation  $v \leq u$  has the obvious meaning that either  $v < u$  or  $v = u$ .

A subset  $\tau \subset \Gamma$  is called a Galton-Watson tree if a)  $\emptyset \in \tau$ ; b) if  $u, v \in \Gamma$ , then  $uv \in \tau$  implies  $u \in \tau$ ; c) for all  $u \in \tau$ , there exists  $r_u \in \mathbb{N}$  such that when  $j \in \mathbb{N}$ ,  $uj \in \tau$  if and only if  $1 \leq j \leq r_u$ . We will denote the collection of Galton-Watson trees by  $\mathbb{T}$ . Each  $u \in \tau$  is called a node of  $\tau$  or an individual in  $\tau$  or just a particle.

To fully describe the branching Hunt process  $X$ , we need to introduce the concept of marked Galton-Watson trees. We suppose that each individual  $u \in \tau$  has a mark  $(Y_u, \sigma_u, r_u)$  where:

- (i)  $\sigma_u$  is the lifetime of  $u$ , which determines the fission time or the death time of particle  $u$  as  $\zeta_u = \sum_{v \leq u} \sigma_v$  ( $\zeta_\emptyset = \sigma_\emptyset$ ), and the birth time of  $u$  as  $b_u = \sum_{v < u} \sigma_v$  ( $b_\emptyset = 0$ );
- (ii)  $Y_u : [b_u, \zeta_u) \rightarrow E_\Delta$  gives the location of  $u$ . Given  $Y_{u-1}(\zeta_{u-1}-)$  and  $b_u$ ,  $(Y_u, u \in [b_u, \zeta_u))$  is the restriction to  $[b_u, \zeta_u)$  of a copy of a Hunt process starting from  $Y_{u-1}(\zeta_{u-1}-)$  at time  $b_u$ , i.e., a process with law  $\Pi_{Y_{u-1}(\zeta_{u-1}-)}$  shifted by  $b_u$ .
- (iii)  $r_u$  gives the number of the offspring born by  $u$  when it dies. It is distributed as  $P(Y_u(\zeta_u-)) = (p_k(Y_u(\zeta_u-)))_{k \in \mathbb{N}}$  which is as defined in Section 1.

We will use  $(\tau, Y, \sigma, r)$  (or simply  $(\tau, M)$ ) to denote a marked Galton-Watson tree. We denote the set of all marked Galton-Watson trees by  $\mathcal{T} = \{(\tau, M) : \tau \in \mathbb{T}\}$ .

For any  $\tau \in \mathbb{T}$ , we can select a line of decent  $\xi = \{\xi_0 = \emptyset, \xi_1, \xi_2, \dots\}$ , where  $\xi_{n+1} \in \tau$  is an offspring of  $\xi_n \in \tau$ ,  $n = 0, 1, \dots$ . Such a genealogical line is called a spine. We will write  $(M, \xi)$  for a marked spine. We will write  $u \in \xi$  to mean that  $u = \xi_i$  for some  $i \geq 0$ . We will use

$$\tilde{\mathcal{T}} = \{(\tau, Y, \sigma, r, \xi) : \xi \subset \tau \in \mathbb{T}\}$$

denote the set of marked trees with distinguished spines.  $L_t = \{u \in \tau : b_u \leq t < \zeta_u\}$  is the set of particles that are alive at time  $t$ .

We will use  $\tilde{Y} = (\tilde{Y}_t)_{t \geq 0}$  to denote the spatial path followed by a spine and  $n = (n_t : t \geq 0)$  to denote the counting process of fission times along the spine. More precisely,  $\tilde{Y}_t = Y_u(t)$  and  $n_t = |u|$ , if  $u \in L_t \cap \xi$ . We use  $\text{node}_t((\tau, M, \xi))$ , or simply  $\text{node}_t(\xi)$ , to denote the node in the spine that is alive at time  $t$ :

$$\text{node}_t(\xi) := \text{node}_t((\tau, M, \xi)) := u \quad \text{if } u \in \xi \cap L_t.$$

It is clear that  $\text{node}_t(\xi) = \xi_{n_t}$ .

If  $v \in \xi$ , then at the fission time  $\zeta_v$ , it gives birth to  $r_v$  offspring, one of which continues the spine whilst the others go off to create sub-trees which are copies of the original branching Hunt process. Let  $O_v$  be the set of offspring of  $v$  except the one belonging to the spine, then for any  $j = 1, \dots, r_v$  such that  $vj \in O_v$ , we will use  $(\tau, M)_j^v$  to denote the marked tree rooted at  $vj$ .



Now we introduce five filtrations on  $\tilde{\mathcal{T}}$  that we shall use. Define

$$\begin{aligned}\mathcal{F}_t &:= \sigma \{[u, r_u, \sigma_u, (Y_u(s), s \in [b_u, \zeta_u]) : u \in \tau \in \mathbb{T} \text{ with } \zeta_u \leq t] \text{ and} \\ &\quad [u, (Y_u(s), s \in [b_u, t]) : u \in \tau \in \mathbb{T} \text{ with } t \in [b_u, \zeta_u)]\}; \\ \tilde{\mathcal{F}}_t &:= \sigma(\mathcal{F}_t, (\text{node}_s(\xi), s \leq t)); \\ \mathcal{G}_t &:= \sigma(\tilde{Y}_s : 0 \leq s \leq t); \\ \hat{\mathcal{G}}_t &:= \sigma(\mathcal{G}_t, (\text{node}_s(\xi) : s \leq t), (\zeta_u, u < \text{node}_t(\xi))); \\ \tilde{\mathcal{G}}_t &:= \sigma(\mathcal{G}_t, (\text{node}_s(\xi) : s \leq t), (\zeta_u, u < \text{node}_t(\xi)), (r_u : u < \text{node}_t(\xi))).\end{aligned}$$

The filtrations  $\mathcal{F}_t$ ,  $\tilde{\mathcal{F}}_t$ ,  $\mathcal{G}_t$ , and  $\tilde{\mathcal{G}}_t$  were introduced in [8], while the filtration  $\hat{\mathcal{G}}_t$  is newly defined. It is obvious that  $\mathcal{G}_t \subset \hat{\mathcal{G}}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$ . Set  $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$ ,  $\tilde{\mathcal{F}} = \bigcup_{t \geq 0} \tilde{\mathcal{F}}_t$ ,  $\mathcal{G} = \bigcup_{t \geq 0} \mathcal{G}_t$ ,  $\hat{\mathcal{G}} = \bigcup_{t \geq 0} \hat{\mathcal{G}}_t$  and  $\tilde{\mathcal{G}} = \bigcup_{t \geq 0} \tilde{\mathcal{G}}_t$ .

For each  $x \in E$ , let  $P^x$  be the measure on  $(\tilde{\mathcal{T}}, \mathcal{F})$  such that the filtered probability space  $(\tilde{\mathcal{T}}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (P^x)_{x \in E})$  is the canonical model for  $X$ , the branching Hunt process in  $E$ . For detailed constructions of  $P^x$ , we refer our readers to [3], [4] and [21]. As noted by Hardy and Harris [8], it is convenient to consider  $P^x$  as a measure on the enlarged space  $\tilde{\mathcal{T}}$ , rather than on  $\mathcal{T}$ . We shall use  $P_t^x$  for the restriction of  $P^x$  to  $\mathcal{F}_t$ .

We need to extend the probability measures  $P^x$  to probability measures  $\tilde{P}^x$  on  $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$  so that the spine is a single genealogical line of descent chosen from the underlying tree. We will assume that at each fission time we make a uniform choice amongst the offspring to decide which line of descent continues the spine  $\xi$ . Then for  $u \in \tau$  we have

$$\text{Prob}(u \in \xi) = \prod_{v < u} \frac{1}{r_v}.$$

It is easy to see that

$$\sum_{u \in L_t} \prod_{v < u} \frac{1}{r_v} = 1.$$

To define  $\tilde{P}^x$  we recall the following representation from [19].

**Theorem 2.1** *Every  $\tilde{\mathcal{F}}_t$ -measurable function  $f$  can be written as*

$$f = \sum_{u \in L_t} f_u(\tau, M) I_{\{u \in \xi\}}, \quad (2.15)$$

where  $f_u$  is  $\mathcal{F}_t$ -measurable.

We define the measure  $\tilde{P}^x$  on  $\tilde{\mathcal{F}}_t$  by

$$d\tilde{P}^x(\tau, M, \xi) \Big|_{\tilde{\mathcal{F}}_t} = d\Pi_x(\tilde{Y}) dL^{\beta(\tilde{Y})}(\mathbf{n}) \prod_{v < \xi_{n_t}} p_{r_v}(\tilde{Y}_{\zeta_v}) \prod_{v < \xi_{n_t}} \frac{1}{r_v} \prod_{j: vj \in O_v} dP_{t-\zeta_v}^{\tilde{Y}_{\zeta_v}}((\tau, M)_j^v), \quad (2.16)$$

where  $L^{\beta(\tilde{Y})}(\mathbf{n})$  is the law of the Poisson random measure  $\mathbf{n} = \{\{\sigma_i : i = 1, \dots, n_t\} : t \geq 0\}$  with intensity  $\beta(\tilde{Y}_t)dt$  along the path of  $\tilde{Y}$ ,  $\Pi_x(\tilde{Y})$  is the law of the diffusion  $\tilde{Y}$  starting from  $x \in E$ , and  $p_{r_v}(y) = \sum_{k \geq 2} p_k(y)I_{(r_v=k)}$  is the probability that individual  $v$ , located at  $y \in E$ , has  $r_v$  offspring.

It follows from Theorem 2.1 that for any bounded  $f \in \tilde{\mathcal{F}}_t$ ,

$$\begin{aligned} \tilde{P}^x(f|\mathcal{F}_t) &= \tilde{P}^x \left( \sum_{u \in L_t} f_u(\tau, M) I_{\{u \in \xi\}} \middle| \mathcal{F}_t \right) \\ &= \sum_{u \in L_t} f_u(\tau, M) \tilde{P}^x (I_{\{u \in \xi\}} | \mathcal{F}_t) \\ &= \sum_{u \in L_t} f_u(\tau, M) \prod_{v < u} \frac{1}{r_v}. \end{aligned}$$

Then we have

$$\tilde{P}^x(f) = P^x \left( \sum_{u \in L_t} f_u(\tau, M) \prod_{v < u} \frac{1}{r_v} \right), \quad \text{for any bounded } f \in \tilde{\mathcal{F}}_t, t \geq 0. \quad (2.17)$$

In particular,

$$\tilde{P}^x(\tilde{\mathcal{T}}) = P^x \left( \sum_{u \in L_t} \prod_{v < u} \frac{1}{r_v} \right) = P^x(1) = 1,$$

which implies  $\tilde{P}^x$  is a probability measure.  $\tilde{P}^x$  is an extension of  $P^x$  onto  $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$  and for any bounded  $f \in \tilde{\mathcal{F}}_t$  we have

$$\int_{\tilde{\mathcal{T}}} f \, d\tilde{P}^x = \int_{\tilde{\mathcal{T}}} \sum_{u \in L_t} f_u \prod_{v < u} \frac{1}{r_v} \, dP^x. \quad (2.18)$$

The decomposition (2.16) of  $\tilde{P}^x$  suggests the following intuitive construction of the system under  $\tilde{P}^x$ :

- (i) the root of  $\tau$  is at  $x$ , and the spine process  $\tilde{Y}_t$  moves according to the measure  $\Pi_x$ ;
- (ii) given the trajectory  $\tilde{Y}$  of the spine, the fission time  $\zeta_v$  of node  $v$  on the spine is distributed according to  $L^{\beta(\tilde{Y})}$ , where  $L^{\beta(\tilde{Y})}$  is the law of the Poisson random measure with intensity  $\beta(\tilde{Y}_t)dt$ ;
- (iii) at the fission time of node  $v$  on the spine, the single spine particle is replaced by a random number  $r_v$  of offspring with  $r_v$  being distributed according to the law  $P(\tilde{Y}_{\zeta_v}) = (p_k(\tilde{Y}_{\zeta_v}))_{k \geq 1}$ ;
- (vi) the spine is chosen uniformly from the  $r_v$  offspring of  $v$  at the fission time of  $v$ ;
- (v) each of the remaining  $r_v - 1$  particles  $v_j \in O_v$  gives rise to the independent subtrees  $(\tau, M)_j^v$ , which evolve as independent subtrees determined by the probability measure  $P^{\tilde{Y}_{\zeta_v}}$  shifted to the time of creation.

**Definition 2.2** Suppose that  $(\Omega, \mathcal{H}, P)$  is a probability space,  $\{\mathcal{H}_t, t \geq 0\}$  is a filtration on  $(\Omega, \mathcal{H})$  and that  $\mathcal{K}$  is a sub- $\sigma$ -field of  $\mathcal{H}$ . A real-valued process  $U_t$  on  $(\Omega, \mathcal{F}, P)$  is called a  $P(\cdot | \mathcal{K})$ -martingale with respect to  $\{\mathcal{H}_t, t \geq 0\}$  if (i) it is adapted to  $\{\mathcal{H}_t \vee \mathcal{K}, t \geq 0\}$ ; (ii) for any  $t \geq 0$ ,  $E(|U_t|) < \infty$  and (iii) for any  $t > s$ ,

$$E(U_t | \mathcal{H}_s \vee \mathcal{K}) = U_s, \quad \text{a.s.}$$

We also say that  $U_t$  is a martingale with respect to  $\{\mathcal{H}_t, t \geq 0\}$ , given  $\mathcal{K}$ .

**Lemma 2.3** Suppose that  $(\Omega, \mathcal{H}, P)$  is a probability space,  $\{\mathcal{H}_t, t \geq 0\}$  is a filtration on  $(\Omega, \mathcal{H})$  and that  $\mathcal{K}_1, \mathcal{K}_2$  are two sub- $\sigma$ -fields of  $\mathcal{H}$  such that  $\mathcal{K}_1 \subset \mathcal{K}_2$ . Assume that  $U_t^1$  is a  $P(\cdot | \mathcal{K}_1)$ -martingale with respect to  $\{\mathcal{H}_t, t \geq 0\}$ ,  $U_t^2$  is a  $P(\cdot | \mathcal{K}_2)$ -martingale with respect to  $\{\mathcal{H}_t, t \geq 0\}$ . If  $U_t^1 \in \mathcal{K}_2$ ,  $U_t^2 \in \mathcal{H}_t$ , and  $E(|U_t^1 U_t^2|) < \infty$  for any  $t \geq 0$ , then the product  $U_t^1 U_t^2$  is a  $P(\cdot | \mathcal{K}_1)$ -martingale with respect to  $\{\mathcal{H}_t, t \geq 0\}$ .

**Proof.** Suppose that  $t \geq s \geq 0$ . The assumption that  $U_t^1 \in \mathcal{K}_2$  implies that  $U_t^1 \in \mathcal{H}_s \vee \mathcal{K}_2$ . Then

$$\begin{aligned} P(U_t^1 U_t^2 | \mathcal{H}_s \vee \mathcal{K}_1) &= P[P(U_t^1 U_t^2 | \mathcal{H}_s \vee \mathcal{K}_2) | \mathcal{H}_s \vee \mathcal{K}_1] \\ &= P[U_t^1 P(U_t^2 | \mathcal{H}_s \vee \mathcal{K}_2) | \mathcal{H}_s \vee \mathcal{K}_1] \\ &= P[U_t^1 U_s^2 | \mathcal{H}_s \vee \mathcal{K}_1] \\ &= U_s^2 P[U_t^1 | \mathcal{H}_s \vee \mathcal{K}_1] \\ &= U_s^1 U_s^2, \end{aligned}$$

where in the last second equality we used the assumption that  $U_s^2 \in \mathcal{H}_s$ .  $\square$

**Lemma 2.4** Suppose that  $\mathbf{n} = \{\{\zeta_i : i = 1, \dots, n_t\} : t \geq 0\}$  is a Poisson random measure with intensity  $\beta(\tilde{Y}_t) dt$  along the path of  $\tilde{Y}$ . Then

$$\eta_t^{(1)} := \prod_{i \leq n_t} A(\tilde{Y}_{\zeta_i}) \cdot \exp\left(-\int_0^t ((A-1)\beta)(\tilde{Y}_s) ds\right)$$

is an  $L^{\beta(\tilde{Y})}$ -martingale with respect to the natural filtration  $\{\mathcal{L}_t\}$  of  $\mathbf{n}$ .

**Proof.** First note that

$$L^{\beta(\tilde{Y})} \left[ \prod_{i \leq n_t} A(\tilde{Y}_{\zeta_i}) \right] = \exp\left(\int_0^t ((A-1)\beta)(\tilde{Y}_s) ds\right), \quad (2.19)$$

which implies that  $L^{\beta(\tilde{Y})}(\eta_t^{(1)}) = 1$ . It is easy to check that  $\eta_t^{(1)}$  is a martingale under  $L^{\beta(\tilde{Y})}$  by using the Markov property of  $\mathbf{n}$ . We omit the details.  $\square$

It follows from the lemma above that we can define a measure  $L^{(A\beta)(\tilde{Y})}$  by

$$\frac{dL^{(A\beta)(\tilde{Y})}}{dL^{\beta(\tilde{Y})}} \Big|_{\mathcal{L}_t} = \prod_{i \leq n_t} A(\tilde{Y}_{\zeta_i}) \cdot \exp \left( - \int_0^t ((A-1)\beta)(\tilde{Y}_s) ds \right).$$

**Lemma 2.5** *For any  $x \in E$  and  $t \geq 0$ , we have*

$$\tilde{P}^x \left[ \prod_{v < \xi_{n_t}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})} \Big| \hat{\mathcal{G}} \right] = 1. \quad (2.20)$$

**Proof.** It follows from (2.16) that, given  $\hat{\mathcal{G}}$ , for each  $v < \xi_{n_t}$ ,

$$\tilde{P}^x(r(\tilde{Y}_{\zeta_v}) | \hat{\mathcal{G}}) = A(\tilde{Y}_{\zeta_v}).$$

Since, given  $\hat{\mathcal{G}}$ ,  $\{r_v, v < \xi_{n_t}\}$  are independent, we have

$$\tilde{P}^x \left( \prod_{v < \xi_{n_t}} \frac{r(\tilde{Y}_{\zeta_i})}{A(\tilde{Y}_{\zeta_i})} \Big| \hat{\mathcal{G}} \right) = 1.$$

□

The following lemma corresponds to Theorems 5.4 and 5.5 in [8] which were not proved there. Our results are somewhat different from those stated in Theorems 5.4 and 5.5 in [8].

**Lemma 2.6** (1) *The process*

$$\tilde{\eta}_t^{(1)} := \prod_{v < \xi_{n_t}} A(\tilde{Y}_{\zeta_v}) \cdot \exp \left( - \int_0^t ((A-1)\beta)(\tilde{Y}_s) ds \right)$$

is a  $\tilde{P}^x(\cdot | \mathcal{G})$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ .

(2) *The process*

$$\tilde{\eta}_t^{(2)} := \prod_{v < \xi_{n_t}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})}$$

is a  $\tilde{P}^x(\cdot | \hat{\mathcal{G}})$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ .

**Proof.** (1) For  $s, t \geq 0$ , by the Markov property, we have

$$\begin{aligned} \tilde{P}^x \left[ \tilde{\eta}_{t+s}^{(1)} \Big| \tilde{\mathcal{F}}_t \vee \mathcal{G} \right] &= \tilde{P}^x \left[ \prod_{v < \xi_{n_{t+s}}} A(\tilde{Y}_{\zeta_v}) \cdot \exp \left( - \int_0^{t+s} ((A-1)\beta)(\tilde{Y}_r) dr \right) \Big| \tilde{\mathcal{F}}_t \vee \mathcal{G} \right] \\ &= \prod_{v < \xi_{n_t}} A(\tilde{Y}_{\zeta_v}) \cdot \exp \left( - \int_0^t ((A-1)\beta)(\tilde{Y}_r) dr \right) \cdot \\ &\quad \tilde{P}^x \left[ \prod_{\xi_{n_t} \leq v < \xi_{n_{t+s}}} A(\tilde{Y}_{\zeta_v}) \cdot \exp \left( - \int_0^s ((A-1)\beta)(\tilde{Y}_{r+t}) dr \right) \Big| \tilde{\mathcal{F}}_t \vee \mathcal{G} \right] \\ &= \tilde{\eta}_t^{(1)} \exp \left( - \int_0^s ((A-1)\beta)(\tilde{Y}_{r+t}) dr \right) \tilde{P}^x \left[ \prod_{\xi_{n_t} \leq v < \xi_{n_{t+s}}} A(\tilde{Y}_{\zeta_v}) \Big| \mathcal{G} \right]. \end{aligned}$$

For fixed  $t > 0$ , given the path of  $\tilde{Y}$ , the collection of fission times  $\{\{\zeta_v : \xi_{n_t} \leq v < \xi_{n_t+s}\} : s \geq 0\}$  is a Poisson random measure with intensity  $\beta(\tilde{Y}_{t+s})ds$ , and has law  $L^{\beta(\tilde{Y}_{t+\cdot})}$ . It follows from (2.19) that

$$P^x \left[ \prod_{\xi_{n_t} \leq v < \xi_{n_t+s}} A(\tilde{Y}_{\zeta_v}) \middle| \mathcal{G} \right] = \exp \left( \int_0^s ((A-1)\beta)(\tilde{Y}_{r+t}) dr \right).$$

Then we get

$$\tilde{P}^x \left[ \tilde{\eta}_{t+s}^{(1)} \middle| \tilde{\mathcal{F}}_t \vee \mathcal{G} \right] = \tilde{\eta}_t^{(1)}.$$

(2) For  $s, t \geq 0$ , by the Markov property, we have

$$\begin{aligned} P^x \left[ \tilde{\eta}_{t+s}^{(2)} \middle| \tilde{\mathcal{F}}_t \vee \hat{\mathcal{G}} \right] &= \tilde{P}^x \left[ \prod_{v < \xi_{n_t+s}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})} \middle| \tilde{\mathcal{F}}_t \vee \hat{\mathcal{G}} \right] \\ &= \prod_{v < \xi_{n_t}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})} \cdot \tilde{P}^x \left[ \prod_{\xi_{n_t} \leq v < \xi_{n_t+s}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})} \middle| \hat{\mathcal{G}} \right] \\ &= \tilde{\eta}_t^{(2)}, \end{aligned}$$

where in the last equality we used (2.20). Then we have

$$\tilde{P}^x \left[ \tilde{\eta}_{t+s}^{(2)} \middle| \tilde{\mathcal{F}}_t \vee \hat{\mathcal{G}} \right] = \tilde{\eta}_t^{(2)}.$$

□

The effect of a change of measure using the martingale  $\tilde{\eta}_t^{(1)}$  will increase the fission rate along the spine from  $\beta(\tilde{Y}_t)$  to  $(A\beta)(\tilde{Y}_t)$ . The effect of a change of measure using the martingale  $\tilde{\eta}_t^{(2)}$  will change offspring distribution from  $P(\tilde{Y}_{\zeta_i}) = (p_k(\tilde{Y}_{\zeta_i}))_{k \geq 1}$  to the size-biased distribution  $\hat{P}(\tilde{Y}_{\zeta_i}) = (\hat{p}_k(\tilde{Y}_{\zeta_i}))_{k \geq 1}$ , where  $\hat{p}_k(y)$  is defined by

$$\hat{p}_k(y) = \frac{kp_k(y)}{A(y)}, \quad k \geq 1, y \in E.$$

Define

$$\tilde{\eta}_t^{(3)}(\phi) := \frac{\phi(\tilde{Y}_t)}{\phi(x)} \exp \left( - \int_0^t (\lambda_1 - (A-1)\beta)(\tilde{Y}_s) ds \right).$$

$\tilde{\eta}_t^{(3)}(\phi)$  is a  $\tilde{P}^x$ -martingale with respect to  $\{\mathcal{G}_t, t \geq 0\}$ , and it is also a  $\tilde{P}^x$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ , since  $\tilde{\eta}_t^{(3)}(\phi)$  can be expressed as

$$\tilde{\eta}_t^{(3)}(\phi) = \sum_{u \in L_t} \phi(x)^{-1} \phi(\tilde{Y}_u(t)) \exp \left( - \int_0^t (\lambda_1 - (A-1)\beta)(\tilde{Y}_s) ds \right) I_{\{u \in \xi\}}. \quad (2.21)$$

And then we define

$$\begin{aligned} \tilde{\eta}_t(\phi) &= \prod_{v < \xi_{n_t}} r_v \exp \left( - \int_0^t ((A-1)\beta)(\tilde{Y}_s) ds \right) \times \tilde{\eta}_t^{(3)}(\phi) \\ &= \prod_{v < \xi_{n_t}} \frac{r_v}{A(\tilde{Y}_{\zeta_v})} \prod_{v < \xi_{n_t}} A(\tilde{Y}_{\zeta_v}) \exp \left( - \int_0^t ((A-1)\beta)(\tilde{Y}_s) ds \right) \times \tilde{\eta}_t^{(3)}(\phi) \\ &= \tilde{\eta}_t^{(1)} \times \tilde{\eta}_t^{(2)} \times \tilde{\eta}_t^{(3)}(\phi). \end{aligned}$$

The following result corresponds to Definition 5.6 in [8].

**Lemma 2.7**  $\tilde{\eta}_t(\phi)$  is a  $\tilde{P}^x$ -martingale with respect to  $\tilde{\mathcal{F}}_t$ .

**Proof.**  $\tilde{\eta}_t^{(1)}$  is a  $\tilde{P}^x(\cdot|\mathcal{G})$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ , and  $\tilde{\eta}_t^{(2)}$  is a  $\tilde{P}^x(\cdot|\hat{\mathcal{G}})$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ . Note that  $\mathcal{G} \subset \hat{\mathcal{G}}$ , and  $\tilde{\eta}_t^{(1)} \in \hat{\mathcal{G}}, \tilde{\eta}_t^{(2)} \in \tilde{\mathcal{F}}_t$  for any  $t \geq 0$ . Using Lemma 2.3,  $\tilde{\eta}_t^{(1)}\tilde{\eta}_t^{(2)}$  is a  $\tilde{P}^x(\cdot|\mathcal{G})$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ . Note that  $\tilde{\eta}_t^{(3)}(\phi) \in \mathcal{G}$  and  $\tilde{\eta}_t^{(1)}\tilde{\eta}_t^{(2)} \in \tilde{\mathcal{F}}_t$  for any  $t \geq 0$ . Using Lemma 2.3 again, we see that  $\tilde{\eta}_t(\phi) = \tilde{\eta}_t^{(1)}\tilde{\eta}_t^{(2)}\tilde{\eta}_t^{(3)}(\phi)$  is a  $\tilde{P}^x$ -martingale with respect to  $\{\tilde{\mathcal{F}}_t, t \geq 0\}$ .  $\square$

**Lemma 2.8**  $M_t(\phi)$  is the projection of  $\tilde{\eta}_t(\phi)$  onto  $\mathcal{F}_t$ , i.e.,

$$M_t(\phi) = \tilde{P}^x(\tilde{\eta}_t(\phi)|\mathcal{F}_t).$$

**Proof.** By (2.21), we have

$$\tilde{\eta}_t(\phi) = \sum_{u \in L_t} \prod_{v < u} r_v e^{-\lambda_1 t} \phi(x)^{-1} \phi(Y_u(t)) I_{(u \in \xi)}.$$

Then

$$\begin{aligned} \tilde{P}^x(\tilde{\eta}_t(\phi)|\mathcal{F}_t) &= \sum_{u \in L_t} e^{-\lambda_1 t} \phi(x)^{-1} \phi(Y_u(t)) \prod_{v < u} r_v \tilde{P}^x(I_{(u \in \xi)}|\mathcal{F}_t) \\ &= \sum_{u \in L_t} e^{-\lambda_1 t} \phi(x)^{-1} \phi(Y_u(t)) = M_t(\phi), \end{aligned}$$

where in the second equality we used the fact that  $\tilde{P}^x(I_{(u \in L_t \cap \xi)}|\mathcal{F}_t) = I_{(u \in L_t)} \times \prod_{v < u} r_v^{-1}$ .  $\square$

Now we define a probability measure  $\tilde{Q}^x$  on  $(\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$  by

$$\frac{d\tilde{Q}^x}{d\tilde{P}^x} \Big|_{\tilde{\mathcal{F}}_t} = \tilde{\eta}_t(\phi),$$

which says that on  $\tilde{\mathcal{F}}_t$ ,

$$\begin{aligned} d\tilde{Q}^x &= \tilde{\eta}_t(\phi) d\tilde{P}^x \\ &= \frac{\phi(\tilde{Y}_{t \wedge \tau^B})}{\phi(x)} \exp\left(-\int_0^{t \wedge \tau^B} (\lambda_1 - (A-1)\beta)(\tilde{Y}_s) ds\right) d\Pi_x(\tilde{Y}) \\ &\quad \times \exp\left(-\int_0^t ((A-1)\beta)(\tilde{Y}_s) ds\right) dL^{\beta(\tilde{Y})} \prod_{v < \xi_{n_t}} p_{r_v}(\tilde{Y}_{\zeta_v}) \prod_{j: v_j \in O_v} dP_{t-\zeta_v}^{\tilde{Y}_{\zeta_v}}((\tau, M)_j^v) \\ &= d\Pi_x^\phi(\tilde{Y}) dL^{A\beta(\tilde{Y})}(\mathbf{n}) \prod_{v < \xi_{n_t}} \frac{p_{r_v}(\tilde{Y}_{\zeta_v})}{A(\tilde{Y}_{\zeta_v})} \prod_{j: v_j \in O_v} dP_{t-\zeta_v}^{\tilde{Y}_{\zeta_v}}((\tau, M)_j^v) \\ &= d\Pi_x^\phi(\tilde{Y}) dL^{A\beta(\tilde{Y})}(\mathbf{n}) \prod_{v < \xi_{n_t}} \hat{p}_{r_v}(\tilde{Y}_{\zeta_v}) \prod_{v < \xi_{n_t}} \frac{1}{r_v} \prod_{j: v_j \in O_v} dP_{t-\zeta_v}^{\tilde{Y}_{\zeta_v}}((\tau, M)_j^v). \end{aligned}$$

Thus the change of measure from  $\tilde{P}^x$  to  $\tilde{Q}^x$  has three effects: the spine will be changed to a Hunt process under  $\Pi_x^\phi$ , its fission times will be increased and the distribution of its family sizes will be sized-biased. More precisely, under  $\tilde{Q}^x$ ,

(i) the spine process  $\tilde{Y}_t$  moves according to the measure  $\Pi_x^\phi$ ;

(ii) the fission times along the spine occur at an accelerated intensity  $(A\beta)(\tilde{Y}_t)dt$ ;

(iii) at the fission time of node  $v$  on the spine, the single spine particle is replaced by a random number  $r_v$  of offspring with size-biased offspring distribution  $\hat{P}(\tilde{Y}_{\zeta_v}) := \left(\hat{p}_k(\tilde{Y}_{\zeta_v})\right)_{k \geq 1}$ ,

where  $\hat{p}_k(y)$  is defined by  $\hat{p}_k(y) := \frac{kp_k(y)}{A(y)}$ ,  $k = 1, 2, \dots$ ,  $y \in E$ ;

(iv) the spine is chosen uniformly from the  $r_v$  particles at the fission point  $v$ ;

(v) each of the remaining  $r_v - 1$  particles  $v_j \in O_v$  gives rise to independent subtrees  $(\tau, M)_j^v$  which evolve as independent subtrees determined by the probability measure  $P^{\tilde{Y}_{\zeta_v}}$  shifted to the time of creation.

We define a measure  $Q^x$  on  $(\tilde{\mathcal{T}}, \mathcal{F})$  by

$$Q^x := \tilde{Q}^x|_{\mathcal{F}}.$$

It follows from Theorem 6.4 in [8] and its proof that  $Q^x$  is a martingale change of measure by the martingale  $M_t(\phi)$ :

$$\frac{dQ^x}{dP^x} \Big|_{\mathcal{F}_t} = M_t(\phi).$$

**Theorem 2.9 (Spine decomposition)** *We have the following spine decomposition for the martingale  $M_t(\phi)$ :*

$$\tilde{Q}^x \left[ \phi(x)M_t(\phi) \Big| \tilde{\mathcal{G}} \right] = \phi(\tilde{Y}_t)e^{-\lambda_1 t} + \sum_{u < \xi_{n_t}} (r_u - 1)\phi(\tilde{Y}_{\zeta_u})e^{-\lambda_1 \zeta_u}. \quad (2.22)$$

**Proof.** We first decompose the martingale  $\phi(x)M_t(\phi)$  as

$$\phi(x)M_t(\phi) = e^{-\lambda_1 t}\phi(\tilde{Y}_t) + e^{-\lambda_1 t} \sum_{u \in L_t, u \neq \xi_{n_t}} \phi(Y_u(t)).$$

The individuals  $\{u \in L_t, u \neq \xi_{n_t}\}$  can be partitioned into subtrees created from fissions along the spines. That is, each node  $u < \xi_{n_t}$  in the spine  $\xi$  has given birth at time  $\zeta_u$  to  $r_u$  offspring among which one has been chosen as a node of the spine whilst the other  $r_u - 1$  individuals go off to make the subtree  $(\tau, M)_j^u$ . Put

$$X_t^j = \sum_{v \in L_t, v \in (\tau, M)_j^u} \delta_{Y_v(t)}(\cdot), \quad t \geq \zeta_u.$$

$(X_t^j, t \geq \zeta_u)$  is a  $(Y, \beta, \psi)$ -branching Hunt process with birth time  $\zeta_u$  and starting point  $\tilde{Y}_{\zeta_u}$ . Then

$$\phi(x)M_t(\phi) = e^{-\lambda_1 t}\phi(\tilde{Y}_t) + \sum_{u < \xi_{n_t}} \sum_{j: u_j \in O_u} M_t^{u,j}(\phi)\phi(\tilde{Y}_{\zeta_u})e^{-\lambda_1 \zeta_u}, \quad (2.23)$$

where

$$M_t^{u,j}(\phi) := e^{-\lambda_1(t-\zeta_u)} \frac{\langle \phi, X_{t-\zeta_u}^j \rangle}{\phi(\tilde{Y}_{\zeta_u})}$$

is, conditional on  $\tilde{\mathcal{G}}$ , a  $\tilde{P}^x$ -martingale on the subtree  $(\tau, M)_j^u$ , and therefore

$$\tilde{P}^x(M_t^{u,j}(\phi) | \tilde{\mathcal{G}}) = 1.$$

Thus taking  $\tilde{Q}^x$  conditional expectation of (2.23) gives

$$\tilde{Q}^x \left[ \phi(x) M_t^\phi | \tilde{\mathcal{G}} \right] = \phi(\tilde{Y}_t) e^{-\lambda_1 t} + \sum_{u < \xi_{n_t}} (r_u - 1) \phi(\tilde{Y}_{\zeta_u}) e^{-\lambda_1 \zeta_u},$$

which completes the proof.  $\square$

### 3 Proof of the main result

First, we give two lemmas. The first lemma is basically [6, Theorem 4.3.3].

**Lemma 3.1** *Suppose that  $\mathbf{P}$  and  $\mathbf{Q}$  are two probability measures on a measurable space  $(\Omega, \mathcal{F}_\infty)$  with filtration  $(\mathcal{F}_t)_{t \geq 0}$ , such that for some nonnegative martingale  $Z_t$ ,*

$$\left. \frac{d\mathbf{Q}}{d\mathbf{P}} \right|_{\mathcal{F}_t} = Z_t.$$

*The limit  $Z_\infty := \limsup_{t \rightarrow \infty} Z_t$  therefore exists and is finite almost surely under  $\mathbf{P}$ . Furthermore, for any  $F \in \mathcal{F}_\infty$*

$$\mathbf{Q}(F) = \int_F Z_\infty d\mathbf{P} + \mathbf{Q}(F \cap \{Z_\infty = \infty\}),$$

*and consequently,*

- (a)  $\mathbf{P}(Z_\infty = 0) = 1 \iff \mathbf{Q}(Z_\infty = \infty) = 1$
- (b)  $\int Z_\infty d\mathbf{P} = \int Z_0 d\mathbf{P} \iff \mathbf{Q}(Z_\infty < \infty) = 1.$

Now we are going to give a lemma which is the key to the proof of Theorem 1.6. To state this lemma, we need some more notation. Note that under  $\tilde{Q}^x$ , given  $\tilde{\mathcal{G}}$ ,  $N_t := \{ \{(\zeta_{\xi_i}, r_{\xi_i}) : i = 0, 1, 2, \dots, n_t - 1\} : t \geq 0 \}$  is a Poisson point process with instant intensity measure  $(A\beta)(\tilde{Y}_t) dt d\hat{P}(\tilde{Y}_t)$  at time  $t$ , where for each  $y \in E$ ,  $\hat{P}(y)$  is the size-biased probability measure on  $\mathbb{N}$  defined in Lemma 2.6. To simplify notation,  $\zeta_{\xi_i}$  and  $r_{\xi_i}$  will be denoted as  $\zeta_i$  and  $r_i$ , respectively.

Recall that  $l(x) = \sum_{i=2}^\infty (i\phi(x)) \log^+(i\phi(x)) p_i(x)$ .



**Lemma 3.2** (1) If  $\int_E \tilde{\phi}(y)\beta(y)l(y)m(dy) < \infty$ , then

$$\sum_{i=0}^{\infty} e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) < \infty, \quad \tilde{Q}^x - a.s.$$

(2) If  $\int_E \tilde{\phi}(y)\beta(y)l(y)m(dy) = \infty$ , then

$$\limsup_{i \rightarrow \infty} e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) = \infty, \quad \tilde{Q}^x - a.s.$$

**Proof.** (1) For any  $\epsilon > 0$ ,

$$\begin{aligned} & \sum_{i=0}^{\infty} e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) \\ &= \sum_i e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) \leq e^{\epsilon \zeta_i}\}} + \sum_i e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) > e^{\epsilon \zeta_i}\}} \\ &=: I + II. \end{aligned} \tag{3.24}$$

Note that (1.9) implies that there is a constant  $c > 0$  such that for any  $t > c$  and any nonnegative measurable function  $f$  with  $\|f\|_{\infty} \leq 1$ ,

$$\frac{1}{2} \int_E \phi(y) \tilde{\phi}(y) f(y) m(dy) \leq \int_E p^{\phi}(t, x, y) f(y) m(dy) \leq 2 \int_E \phi(y) \tilde{\phi}(y) f(y) m(dy), \quad x \in E. \tag{3.25}$$

Then,

$$\begin{aligned} & \tilde{Q}^x \left[ \sum_i I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) > e^{\epsilon \zeta_i}\}} \right] \\ &= \Pi_x^{\phi} \left[ \int_0^{\infty} \beta(\tilde{Y}_s) \left( \sum_{k=2}^{\infty} k p_k(\tilde{Y}_s) I_{\{k \phi(\tilde{Y}_s) > e^{\epsilon s}\}} \right) ds \right] \\ &= \int_0^{\infty} ds \int_E p^{\phi}(s, x, y) m(dy) \left[ \beta(y) \sum_{k=2}^{\infty} k p_k(y) I_{\{k \phi(y) > e^{\epsilon s}\}} \right] \\ &= \int_0^c ds \int_E p^{\phi}(s, x, y) m(dy) \left[ \beta(y) \sum_{k=2}^{\infty} k p_k(y) I_{\{k \phi(y) > e^{\epsilon s}\}} \right] \\ & \quad + \int_c^{\infty} ds \int_E p^{\phi}(s, x, y) m(dy) \left[ \beta(y) \sum_{k=2}^{\infty} k p_k(y) I_{\{k \phi(y) > e^{\epsilon s}\}} \right] \end{aligned} \tag{3.26}$$

Applying Fubini's theorem and using the assumption that  $A$  and  $\beta$  are bounded, we get

$$\int_0^c ds \int_E p^{\phi}(s, x, y) m(dy) \left[ \beta(y) \sum_{k=2}^{\infty} k p_k(y) I_{\{k \phi(y) > e^{\epsilon s}\}} \right] \leq \|\beta A\|_{\infty} \int_0^c ds \leq C_1, \tag{3.27}$$

where  $C_1$  is positive constant which only depends on  $c$ . Using (3.25), we get

$$\int_c^{\infty} ds \int_E p^{\phi}(s, x, y) m(dy) \left[ \beta(y) \sum_{k=2}^{\infty} k p_k(y) I_{\{k \phi(y) > e^{\epsilon s}\}} \right]$$

$$\begin{aligned}
&\leq 2 \int_E m(dy) \phi(y) \tilde{\phi}(y) \beta(y) \sum_{k=2}^{\infty} k p_k(y) \int_0^{\frac{1}{\varepsilon} \log^+ [k\phi(y)]} ds \\
&= \frac{2}{\varepsilon} \int_E \tilde{\phi}(y) \beta(y) l(y) m(dy).
\end{aligned} \tag{3.28}$$

Combining (3.26), (3.27) and (3.28), we have

$$\tilde{Q}^x \left[ \sum_i I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) > e^{\varepsilon \zeta_i}\}} \right] \leq C_1 + \frac{2}{\varepsilon} \int_E \tilde{\phi}(y) \beta(y) l(y) m(dy).$$

Therefore, the condition  $\int_E \tilde{\phi}(y) \beta(y) l(y) m(dy) < \infty$  implies that

$$\sum_i I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) > e^{\varepsilon \zeta_i}\}} < \infty, \quad \tilde{Q}^x - \text{a.s.}$$

for all  $\varepsilon > 0$ . Then we have

$$II < \infty, \quad \tilde{Q}^x - \text{a.s.} \tag{3.29}$$

Meanwhile for  $\varepsilon < \lambda_1$ ,

$$\begin{aligned}
\tilde{Q}^x(I) &= \tilde{Q}^x \left[ \sum_i e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) I_{\{r_i \phi(\tilde{Y}_{\zeta_i}) \leq e^{\varepsilon \zeta_i}\}} \right] \\
&= \Pi_x^\phi \int_0^\infty dt e^{-\lambda_1 t} \phi(\tilde{Y}_t) \beta(\tilde{Y}_t) A(\tilde{Y}_t) \sum_{k=2}^{\infty} k \hat{p}_k(\tilde{Y}_t) I_{\{k\phi(\tilde{Y}_t) \leq e^{\varepsilon t}\}} \\
&= \Pi_x^\phi \int_0^\infty dt e^{-\lambda_1 t} \phi(\tilde{Y}_t) \beta(\tilde{Y}_t) A(\tilde{Y}_t) \sum_{k=2}^{\infty} k \frac{k}{A(\tilde{Y}_t)} p_k(\tilde{Y}_t) I_{\{k\phi(\tilde{Y}_t) \leq e^{\varepsilon t}\}} \\
&\leq \Pi_x^\phi \int_0^\infty dt e^{-(\lambda_1 - \varepsilon)t} \beta(\tilde{Y}_t) \sum_{k=2}^{\infty} k p_k(\tilde{Y}_t) I_{\{k\phi(\tilde{Y}_t) \leq e^{\varepsilon t}\}} \\
&\leq C_2 \int_0^\infty e^{-(\lambda_1 - \varepsilon)t} dt < \infty,
\end{aligned}$$

where in the second to the last inequality we used the assumption that  $\beta$  and  $A$  are bounded and  $C_2$  is a positive constant. Then we have

$$I < \infty, \quad \tilde{Q}^x - \text{a.s.} \tag{3.30}$$

Combining (3.24), (3.29) and (3.30), we see that  $\sum_{i=0}^{\infty} e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) < \infty$ ,  $\tilde{Q}^x - \text{a.s.}$

(2) It is enough to prove that for any  $K > 1$ ,

$$\limsup_{i \rightarrow \infty} e^{-\lambda_1 \zeta_i} r_i \phi(\tilde{Y}_{\zeta_i}) > K, \quad \tilde{Q}^x - \text{a.s.} \tag{3.31}$$

For any fixed  $K > 1$ , define  $\gamma(t, y) := \beta(y) \sum_k k p_k(y) I_{\{k\phi(y) > K e^{\lambda_1 t}\}}$ . Since for any  $x \in E$ ,

$$\Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt = \int_0^T dt \int_E m(dy) p^\phi(t, x, y) \gamma(t, y)$$

$$\leq \int_0^T dt \int_E m(dy) p^\phi(t, x, y) \beta(y) A(y) < \infty,$$

we have  $\int_0^T \gamma(t, \tilde{Y}_t) dt < \infty$ ,  $\Pi_x^\phi$  - a.s.. Note that for any  $T \in (0, \infty)$ , conditional on  $\sigma(\tilde{Y})$ ,

$$\# \left\{ i : \zeta_i \in (0, T]; r_i > K \phi(\tilde{Y}_{\zeta_i})^{-1} e^{\lambda_1 \zeta_i} \right\}$$

is a Poisson random variable with intensity  $\int_0^T \gamma(t, \tilde{Y}_t) dt$  a.s. Hence, to prove (3.31), we just need to prove

$$Z_\infty =: \int_0^\infty \gamma(t, \tilde{Y}_t) dt = \infty, \quad \Pi_x^\phi - \text{a.s.} \quad (3.32)$$

Recall the choice of the constant  $c > 0$  in the statement above the inequalities (3.25). Applying Fubini's theorem and (3.25), we get

$$\begin{aligned} \Pi_x^\phi Z_\infty &= \int_0^\infty dt \int_E p^\phi(t, x, y) \gamma(t, y) m(dy) \\ &\geq \int_c^\infty dt \int_E p^\phi(t, x, y) \gamma(t, y) m(dy) \\ &\geq \frac{1}{2} \int_c^\infty dt \int_E \phi(y) \tilde{\phi}(y) \gamma(t, y) m(dy) =: \frac{1}{2} A_\infty. \end{aligned} \quad (3.33)$$

Exchanging the order of integration in  $A_\infty$ , we get that

$$\begin{aligned} A_\infty &\geq \int_E \phi(y) \tilde{\phi}(y) \beta(y) m(dy) \sum_k k p_k(y) I_{\{k > K \phi(y)^{-1}\}} \left[ \frac{1}{\lambda_1} \log(k \phi(y)) - \frac{\log K}{\lambda_1} - c \right]^+ \\ &\geq C_3 \int_E \phi(y) \tilde{\phi}(y) \beta(y) m(dy) \sum_k k \log(k \phi(y)) p_k(y) I_{\{k > K \phi(y)^{-1}\}} - C_4, \end{aligned} \quad (3.34)$$

where  $C_3 = 1/\lambda_1$  and  $C_4 = \|\beta A\|_\infty (\log K + c\lambda_1)/\lambda_1$ . The assumption that  $\int_E \tilde{\phi}(y) \beta(y) l(y) m(dy) = \infty$  says that

$$\int_E \phi(y) \tilde{\phi}(y) \beta(y) m(dy) \sum_k k \log(k \phi(y)) p_k(y) I_{\{k > \phi(y)^{-1}\}} = \infty.$$

Since

$$\begin{aligned} &\int_E \phi(y) \tilde{\phi}(y) \beta(y) m(dy) \sum_k k \log(k \phi(y)) p_k(y) I_{\{\phi(y)^{-1} < k \leq K \phi(y)^{-1}\}} \\ &\leq \log K \int_E \phi(y) \tilde{\phi}(y) \beta(y) A(y) m(dy) < \infty, \end{aligned}$$

it follows from (3.34) that

$$A_\infty = \infty. \quad (3.35)$$

Then by (3.33),

$$\Pi_x^\phi Z_\infty = \infty. \quad (3.36)$$

For any finite time  $T > 0$ , put  $Z_T = \int_0^T \gamma(t, \tilde{Y}_t) dt$  and

$$\Lambda_T = \int_c^T dt \int_E \phi(y) \tilde{\phi}(y) \gamma(t, y) m(dy).$$

Then  $\lim_{T \rightarrow \infty} Z_T = Z_\infty$ , and  $\lim_{T \rightarrow \infty} \Lambda_T = A_\infty = \Pi_x^\phi Z_\infty = \infty$ .

An argument similar to (3.33) yields that there exists a constant  $C_5$  which is independent of  $T$  and sufficiently large, such that

$$\frac{1}{C_5} \Lambda_T \leq \Pi_x^\phi Z_T \leq C_5 \Lambda_T. \quad (3.37)$$

By the Paley-Zygmund inequality (see, for instance, [6, Ex. 1.3.8]),

$$\Pi_x^\phi \left( Z_T \geq \frac{1}{2} \Pi_x^\phi Z_T \right) \geq \frac{(\Pi_x^\phi Z_T)^2}{4 \Pi_x^\phi [Z_T^2]}. \quad (3.38)$$

Now we estimate  $\Pi_x^\phi (Z_T^2)$ .

$$\begin{aligned} \Pi_x^\phi Z_T^2 &= \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \int_0^T \gamma(s, \tilde{Y}_s) ds \\ &= 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \int_t^T \gamma(s, \tilde{Y}_s) ds \\ &= 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \int_t^{(t+c) \wedge T} \gamma(s, \tilde{Y}_s) ds + 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \int_{(t+c) \wedge T}^T \gamma(s, \tilde{Y}_s) ds \\ &=: III + IV. \end{aligned}$$

By the Markov property of  $\tilde{Y}$ , we have

$$\begin{aligned} IV &= 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \Pi_{\tilde{Y}_t}^\phi \int_{(c+t) \wedge T}^T \gamma(s, \tilde{Y}_{s-t}) ds \\ &= 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \Pi_{\tilde{Y}_t}^\phi \int_{(c+t) \wedge T - t}^{T-t} \gamma(u+t, \tilde{Y}_u) du \\ &\leq 2 \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \Pi_{\tilde{Y}_t}^\phi \int_c^T \gamma(u, \tilde{Y}_u) du \\ &= 2 \Pi_x^\phi \int_0^T dt \gamma(t, \tilde{Y}_t) \int_c^T du \int_E m(dy) p^\phi(u, \tilde{Y}_t, y) \gamma(u, y), \end{aligned}$$

where in above inequality we used the fact that  $\gamma(s, y)$  is decreasing in  $s$ , which is obvious by the definition of  $\gamma$ . Using (3.25) and (3.37), we see that

$$IV \leq 4 \Lambda_T \Pi_x^\phi Z_T \leq 4 C_5 (\Pi_x^\phi Z_T)^2.$$

Meanwhile,

$$III \leq 2 \Pi_x^\phi \int_0^T dt \gamma(t, \tilde{Y}_t) \int_t^{(t+c) \wedge T} (A\beta)(\tilde{Y}_s) ds$$

$$\begin{aligned}
&\leq 2c\|\beta A\|_\infty \Pi_x^\phi \int_0^T \gamma(t, \tilde{Y}_t) dt \\
&= C_6 \Pi_x^\phi Z_T,
\end{aligned}$$

where  $C_6 = 2c\|\beta A\|_\infty$ . Therefore, there exists a constant  $C > 0$  which does not depend on  $T$ , such that

$$\Pi_x^\phi(Z_T^2) \leq C(\Pi_x^\phi(Z_T))^2.$$

Then by (3.38) we get

$$\Pi_x^\phi\left(Z_T \geq \frac{1}{2}\Pi_x^\phi Z_T\right) \geq (4C)^{-1},$$

and therefore,

$$\Pi_x^\phi\left(Z_\infty \geq \frac{1}{2}\Pi_x^\phi Z_T\right) \geq \Pi_x^\phi\left(Z_T \geq \frac{1}{2}\Pi_x^\phi Z_T\right) \geq (4C)^{-1} > 0.$$

Since  $\lim_{T \rightarrow \infty} \Lambda_T = \infty$ , (3.37) and the above inequality imply that  $\Pi_x^\phi(Z_\infty = \infty) > 0$ . Since for any  $t_0 > 0$ ,  $\int_0^{t_0} \gamma(t, \tilde{Y}_t) dt \leq \|\beta A\|_\infty t_0 < \infty$ , the event  $\Gamma := \{Z_\infty = \infty\} = \left\{\int_0^\infty \gamma(t, \tilde{Y}_t) dt = \infty\right\}$  is invariant, that is,  $\theta_t^{-1}(\Gamma) = \Gamma$  for all  $t \geq 0$ , where  $\{\theta_t : t \geq 0\}$  are the shift operators of the process  $\tilde{Y}$ . It follows from (1.9) and [22, Proposition X.3.9] that the process  $\tilde{Y}$  is Harris-recurrent. Thus it follows from [22, Propositions X.3.6 and X.3.10] that  $\Pi_x^\phi(Z_\infty = \infty) = 1$ , which is (3.32). And our second conclusion follows.  $\square$

**Proof of Theorem 1.6.** The proof heavily depends on the decomposition (2.22).

When  $\int_E \tilde{\phi}(x)\beta(x)l(x)m(dx) < \infty$ , the first conclusion of Lemma 3.2 says that

$$\sup_{t>0} \tilde{Q}^x \left[ \phi(x)M_t(\phi) \Big| \tilde{\mathcal{G}} \right] \leq \sum_{u \in \xi} r_u \phi(\tilde{Y}_{\zeta_u}) e^{-\lambda_1 \zeta_u} + \|\phi\|_\infty < \infty.$$

Fatou's lemma for conditional probability implies that  $\liminf_{t \rightarrow \infty} M_t(\phi) < \infty$ ,  $\tilde{Q}^x$ -a.s. The Radon-Nikodym derivative tells us that  $M_t(\phi)^{-1}$  is a nonnegative supermartingale under  $Q^x$  and therefore has a finite limit  $Q^x$ -a.s. So  $\lim_{t \rightarrow \infty} M_t(\phi) = M_\infty < \infty$ ,  $Q^x$ -a.s. Lemma 3.1 implies that in this case,

$$P^x[M_\infty(\phi)] = \lim_{t \rightarrow \infty} P^x[M_t(\phi)] = 1.$$

When  $\int_E \tilde{\phi}(x)\beta(x)l(x)m(dx) = \infty$ , using the second conclusion in Lemma 3.2, we can get under  $\tilde{Q}^x$ ,

$$\limsup_{t \rightarrow \infty} \phi(x)M_t(\phi) \geq \limsup_{t \rightarrow \infty} \phi(\tilde{Y}_{\zeta_{n_t}})(r_{n_t} - 1)e^{-\lambda_1 \zeta_{n_t}} = \infty.$$

This yields that  $M_\infty(\phi) = \infty$ ,  $Q^x$ -a.s. Using Lemma 3.1 again, we get  $M_\infty(\phi) = 0$ ,  $P^x$ -a.s. The proof is finished.  $\square$

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