

## ON SUMS OF RAMANUJAN SUMS

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## 1. INTRODUCTION

In this note, we study the moments of the average of the Ramanujan sum. To be precise, we want to evaluate the sums

$$C_k(x, y) = \sum_{n \leq y} \left( \sum_{q \leq x} c_q(n) \right)^k,$$

where  $k$  is a positive integer,  $x$  and  $y$  are large real numbers, and

$$(1.1) \quad c_q(n) = \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} e(-an/q) = \sum_{d|(q, n)} d\mu(q/d)$$

is the Ramanujan sum. Our interest in this question stems from an old attempt [1] to apply Fourier techniques to Diophantine approximations of reals by sums of rational numbers. At the time, we were surprised that information on the asymptotic behavior of  $C_k(x, y)$  appeared to be missing from the literature. Of course, it is easy to evaluate  $C_k(x, y)$  in some cases. For example, elementary arguments based on the second formula in (1.1) yield the asymptotic formulas

$$C_1(x, y) = y + O(x^{2+\epsilon})$$

and

$$C_2(x, y) = \frac{yx^2}{2\zeta(2)} + O(x^{4+\epsilon}),$$

for any fixed  $\epsilon > 0$ . However, these bounds (as well as other simple things we have tried) are only of interest when  $y \geq x^{2+\epsilon}$ . In our application to Diophantine approximations, on the other hand, we were interested in  $C_2(x, y)$  when  $x^{1+\epsilon} \leq y \leq x^{2+\epsilon}$ . Some numerical experimentation suggested that in that range the order of  $C_2(x, y)$  is still  $yx^2$ , though the coefficient  $\frac{1}{2\zeta(2)}$  seemed somewhat off when  $y$  is close to  $x^2$ . In the present note, we prove, among other things, that those empirical observations are true. Our first result is an asymptotic formula for  $C_1(x, y)$ .

**Theorem 1.1.** *Let  $x$  be a large real number and  $y \geq x$ . Then*

$$C_1(x, y) = y - \frac{x^2}{4\zeta(2)} + O(xy^{1/3} \log x + x^3 y^{-1}).$$

The proof of Theorem 1.1 uses exponential sum estimates and can be easily modified to yield different estimates for the remainder. The remainder claimed in the statement of the theorem is the result of applying one of the simplest exponential sum bounds—the exponential pair  $(\frac{1}{2}, \frac{1}{2})$  from van der Corput’s method [2]. More sophisticated exponential pairs will result in replacing the error term  $xy^{1/3}$  by different ones, which will be sharper for

certain choices of the relative sizes of  $x$  and  $y$ . The interested reader will have no trouble obtaining such improvements by appropriate modifications of (2.5) below. Since our main focus is on maximizing the “support” of the result, we prefer the simple version above.

Theorem 1.1 demonstrates that when  $x = o(y)$ , the Ramanujan sum  $c_q(n)$  is  $o(1)$  on average over  $q \leq x$  and  $n \leq y$ . Our next theorem shows that this is no longer the case when the average  $C_1(x, y)$  is replaced by the mean square sum  $C_2(x, y)$ .

**Theorem 1.2.** *Let  $x$  be a large real number,  $y \geq x$ , and  $B > 10$  be fixed.*

(i) *If  $y \geq x^2(\log x)^B$ , then*

$$C_2(x, y) = \frac{yx^2}{2\zeta(2)} + O(yx^2(\log x)^{10}(x^{-1/2} + x^2y^{-1})).$$

(ii) *If  $x(\log x)^{2B} \leq y \leq x^2(\log x)^B$ , then*

$$C_2(x, y) = \frac{yx^2}{2\zeta(2)}(1 + 2\kappa(u)) + O(yx^2(\log x)^{10}(x^{-1/2} + (y/x)^{-1/2})),$$

where  $u = \log(yx^{-2})$ , and  $\kappa(u)$  is defined by (3.20) below and satisfies the inequalities

$$\kappa(u) > -0.4, \quad \kappa(u) \ll \exp(-|u|^{3/5-\epsilon}),$$

for any fixed  $\epsilon > 0$ . In particular,  $\kappa(u) = o(1)$  as  $|u| \rightarrow \infty$ .

The hypothesis  $B > 10$  and the powers of the logarithms in the ranges for  $y$  in the statement of the theorem are chosen to ensure that the error terms are of a smaller order of magnitude than the main term. Also, the lower bound for  $\kappa(u)$  is far from sharp and has been chosen so to simplify the arguments in §4. In fact, by increasing the amount of numeric calculations in §4 by an order of magnitude or two, it can be shown that  $\kappa(u) > -0.3$ . We chose not to pursue such a sharper bound here for the sake of clarity. However, for the benefit of the reader, we should mention that the more elaborate numerical computations that we carried suggest that  $\kappa(u)$  is decreasing when  $u < u_0$  and increasing when  $u > u_0$ , where  $u_0 \approx 1.63$ ; its minimum value is approximately  $\kappa(1.63) \approx -0.2943$ . It also seems that  $\kappa(u)$  is always negative.

## 2. THE FIRST MOMENT

Using (1.1), we have

$$C_1(x, y) = \sum_{n \leq y} \sum_{q \leq x} \sum_{\substack{d|q \\ d|n}} d\mu(q/d) = \sum_{n \leq y} \sum_{\substack{dk \leq x \\ d|n}} d\mu(k).$$

Interchanging order of summation, we obtain

$$\begin{aligned} (2.1) \quad C_1(x, y) &= \sum_{dk \leq x} d\mu(k) \sum_{\substack{n \leq y \\ d|n}} 1 = \sum_{dk \leq x} d\mu(k) \left[ \frac{y}{d} \right] \\ &= y \sum_{dk \leq x} \mu(k) - \frac{1}{2} \sum_{dk \leq x} d\mu(k) - \sum_{dk \leq x} d\mu(k)\psi(y/d) \\ &= C_{1,1}(x, y) - C_{1,2}(x, y) - C_{1,3}(x, y), \quad \text{say.} \end{aligned}$$

Here, for a real  $t$ ,  $[t]$  is the integral part of  $t$  and  $\psi(t) = t - [t] - \frac{1}{2}$  is the saw-tooth function. It is easy to see that

$$(2.2) \quad C_{1,1}(x, y) = y \sum_{m \leq x} \mu(k) = y \sum_{m \leq x} \sum_{k|m} \mu(k) = y.$$

Further,

$$(2.3) \quad \begin{aligned} C_{1,2}(x, y) &= \frac{1}{2} \sum_{dk \leq x} d\mu(k) = \frac{1}{2} \sum_{k \leq x} \mu(k) \sum_{d \leq x/k} d \\ &= \frac{1}{2} \sum_{k \leq x} \mu(k) \left( \frac{x^2}{2k^2} + O\left(\frac{x}{k}\right) \right) \\ &= \frac{x^2}{4} \sum_{k \leq x} \frac{\mu(k)}{k^2} + O(x \log x) = \frac{x^2}{4\zeta(2)} + O(x \log x). \end{aligned}$$

It remains to estimate  $C_{1,3}(x, y)$ . By partial summation,

$$(2.4) \quad C_{1,3}(x, y) \ll \sum_{k \leq x} \sum_{j=0}^{\infty} N_j \sup_{\mathbf{I}} \left| \sum_{d \in \mathbf{I}} \psi(y/d) \right|,$$

where  $N_j = N_{j,k} = (x/k)2^{-j}$  and the supremum is over all subintervals  $\mathbf{I}$  of  $(N_j, 2N_j]$ . We remark that the sum over  $j$  is, in fact, finite and has  $O(\log x)$  terms. We now use the following lemma, which is a special case of [2, Lemma 4.3] (see [2] or [4] for the definition of exponential pairs).

**Lemma 2.1.** *Suppose that  $(\varkappa, \lambda)$  is an exponential pair and that  $\mathbf{I}$  is a subinterval of  $(N, 2N]$ . Then*

$$\sum_{n \in \mathbf{I}} \psi(y/n) \ll y^{\varkappa/(\varkappa+1)} N^{(\lambda-\varkappa)/(\varkappa+1)} + N^2 y^{-1}.$$

We apply Lemma 2.1 with the exponential pair  $(\varkappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$  to the sum over  $d$  on the right side of (2.4). This yields

$$(2.5) \quad \begin{aligned} C_{1,3}(x, y) &\ll \sum_{k \leq x} \sum_{j=0}^{\infty} (N_j y^{1/3} + N_j^3 y^{-1}) \\ &\ll \sum_{k \leq x} ((x/k)y^{1/3} + (x/k)^3 y^{-1}) \\ &\ll (xy^{1/3} \log x + x^3 y^{-1}). \end{aligned}$$

Theorem 1.1 now follows easily from (2.1)–(2.5).

### 3. THE SECOND MOMENT

Recall the identity

$$(3.1) \quad \sum_{q=1}^{\infty} \frac{c_q(n)}{q^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \quad (\operatorname{Re}(s) > 1),$$

where  $\zeta(s)$  denotes the Riemann zeta-function and  $\sigma_z(n) = \sum_{d|n} d^z$ . We note that without loss of generality, we may assume that  $x, y \in \mathbb{Z} + \frac{1}{2}$ . Then, using (3.1) and the truncated Perron formula [3, Corollary 5.3], we get

$$(3.2) \quad \sum_{q \leq x} c_q(n) = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} \frac{\sigma_{1-s}(n) x^s}{\zeta(s) s} ds + E_1(x, n).$$

Here,  $T$  is a real parameter at our disposal,  $\alpha > 1 + (\log x)^{-1}$  is a real number, and the remainder  $E_1(x, n)$  satisfies

$$E_1(x, n) \ll \frac{x^\alpha}{T} \sum_{q=1}^{\infty} \frac{|c_q(n)|}{q^\alpha} + \sum_{x/2 < q < 2x} |c_q(n)| \min \left( 1, \frac{x}{T|x-q|} \right).$$

If we assume further that  $\alpha < 1 + 3(\log x)^{-1}$ , we can show that

$$(3.3) \quad E_1(x, n) \ll (x/T) \sigma_0(n) \log x.$$

Applying (3.2) and (3.3) with  $\alpha = \alpha_j = 1 + j(\log x)^{-1}$ ,  $j = 1, 2$ , we obtain

$$(3.4) \quad \left( \sum_{q \leq x} c_q(n) \right)^2 = \frac{1}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} F(s_1, s_2, n) ds_2 ds_1 + E_2(x, n),$$

where

$$F(s_1, s_2, n) = \frac{\sigma_{1-s_1}(n) \sigma_{1-s_2}(n) x^{s_1+s_2}}{\zeta(s_1) \zeta(s_2) s_1 s_2}$$

and

$$E_2(x, n) \ll (x^2/T) \sigma_0(n)^2 \log x.$$

Summing (3.4) over  $n$ , we deduce

$$(3.5) \quad C_2(x, y) = \frac{1}{(2\pi i)^2} \int_{\alpha_1-iT}^{\alpha_1+iT} \int_{\alpha_2-iT}^{\alpha_2+iT} \frac{G(s_1, s_2; y) x^{s_1+s_2}}{\zeta(s_1) \zeta(s_2) s_1 s_2} ds_2 ds_1 + O\left(\frac{yx^2 L^4}{T}\right),$$

where  $L = \log(xy)$  and

$$G(s_1, s_2; y) = \sum_{n \leq y} \sigma_{1-s_1}(n) \sigma_{1-s_2}(n).$$

We now recall Ramanujan's identity

$$(3.6) \quad \sum_{n=1}^{\infty} \frac{\sigma_a(n) \sigma_b(n)}{n^s} = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)},$$

valid for  $\operatorname{Re}(s) > \max\{1, 1+\operatorname{Re}(a), 1+\operatorname{Re}(b), 1+\operatorname{Re}(a+b)\}$ . From (3.6), by another application of Perron's formula, we get

$$(3.7) \quad G(s_1, s_2; y) = \frac{1}{2\pi i} \int_{\alpha-3iT}^{\alpha+3iT} H(s_1, s_2, w) \frac{y^w}{w} dw + O\left(\frac{yL^4}{T}\right),$$

where  $\alpha = 1 + 3(\log y)^{-1}$ ,  $y \in \mathbb{Z} + \frac{1}{2}$ , and

$$H(s_1, s_2, w) = \frac{\zeta(w) \zeta(w+s_1-1) \zeta(w+s_2-1) \zeta(w+s_1+s_2-2)}{\zeta(2w+s_1+s_2-2)}.$$

Let  $\Gamma(\alpha, \beta, T)$  denote the contour consisting of the line segments  $[\alpha-iT, \beta-iT]$ ,  $[\beta-iT, \beta+iT]$  and  $[\beta+iT, \alpha+iT]$ . We now move the integration in (3.7) to  $\Gamma(\alpha, \frac{1}{2}, 3T)$ ; let us denote the

respective integrals  $I_1, I_2$  and  $I_3$ . When  $\text{Re}(s_j) = \alpha_j$ , the integrals  $I_1$  and  $I_3$  over the two horizontal line segments are bounded above by

$$(3.8) \quad I_1, I_3 \ll \zeta(\alpha_1 + \alpha_2 - 1)L^4 \int_{1/2}^{\alpha} T^{1-2\sigma} y^{\sigma} d\sigma \ll L^5 (y^{1/2} + yT^{-1}).$$

Here, we have used the standard convexity bound (see [4, eqn. (5.1.4)] for a slightly weaker version)

$$(3.9) \quad |\zeta(\sigma + it)| \ll (|t| + 2)^{(1-\sigma)/2} \log(|t| + 2) \quad (\tfrac{1}{2} \leq \sigma \leq 1).$$

Furthermore, by Hölder's inequality,

$$(3.10) \quad I_2 \ll y^{1/2} \zeta(\alpha_1 + \alpha_2 - 1) \prod_{0 \leq j, k \leq 1} M_4(\tfrac{1}{2} + j(\alpha_1 - 1) + k(\alpha_2 - 1), 5T),$$

where for  $\tfrac{1}{2} \leq \sigma \leq 1$ ,

$$M_4^4(\sigma, T) = \int_{-T}^T |\zeta(\sigma + it)|^4 \frac{dt}{1 + |t|}.$$

Appealing to the fourth-moment estimates for  $\zeta(s)$  [4, §7.5 & §7.6], we deduce from (3.10) that

$$(3.11) \quad I_2 \ll y^{1/2} L^6.$$

Combining (3.7), (3.8) and (3.11), we obtain

$$(3.12) \quad G(s_1, s_2; y) = \sum_{j=1}^4 R_j(s_1, s_2; y) + O(L^6 (y^{1/2} + yT^{-1})),$$

where  $R_j(s_1, s_2; y)$  are the four residues of the integrand in (3.7):

$$\begin{aligned} R_1(s_1, s_2; y) &= \frac{y\zeta(s_1)\zeta(s_2)\zeta(s_1 + s_2 - 1)}{\zeta(s_1 + s_2)}, \\ R_2(s_1, s_2; y) &= \frac{y^{2-s_1}\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_2)}{(2-s_1)\zeta(2-s_1+s_2)}, \\ R_3(s_1, s_2; y) &= \frac{y^{2-s_2}\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta(s_1)}{(2-s_2)\zeta(2+s_1-s_2)}, \\ R_4(s_1, s_2; y) &= \frac{y^{3-s_1-s_2}\zeta(3-s_1-s_2)\zeta(2-s_2)\zeta(2-s_1)}{(3-s_1-s_2)\zeta(4-s_1-s_2)}. \end{aligned}$$

Substituting (3.12) into the right side of (3.5), we get

$$(3.13) \quad C_2(x, y) = \sum_{j=1}^4 C_{2,j}(x, y) + O(yx^2 L^{10} (y^{-1/2} + T^{-1})),$$

where

$$C_{2,j}(x, y) = \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} \frac{R_j(s_1, s_2; y) x^{s_1+s_2}}{\zeta(s_1)\zeta(s_2) s_1 s_2} ds_2 ds_1.$$

We now proceed to evaluate the integrals  $C_{2,j}(x, y)$ , starting with  $C_{2,1}(x, y)$ . We first move the integration over  $s_2$  to the contour  $\Gamma(\alpha_2, \frac{1}{2}, T)$ ; let us denote the integrals over the three

line segments  $J_{1,1}$ ,  $J_{1,2}$  and  $J_{1,3}$ . The contributions from the integrals over the two horizontal line segments are bounded using (3.9):

$$\begin{aligned} J_{1,1}, J_{1,3} &\ll \frac{xyL}{T} \int_{-T}^T \frac{dt}{1+|t|} \int_{1/2}^{\alpha_2} T^{(1-\sigma)/2} x^\sigma d\sigma \\ &\ll xyL^2(xT^{-1} + 1). \end{aligned}$$

Furthermore, using estimates for the second moment of  $\zeta(s)$  in the critical strip, we find that the contribution from the integral over the line  $\text{Re}(s_2) = \frac{1}{2}$  is

$$\begin{aligned} J_{1,2} &\ll yx^{3/2} \int_{-T}^T \int_{-T}^T \frac{|\zeta(\alpha_1 - \frac{1}{2} + i(t_1 + t_2))|}{(|t_1| + 1)(|t_2| + 1)} dt_1 dt_2 \\ &\ll yx^{3/2} \int_{-2T}^{2T} |\zeta(\alpha_1 - \frac{1}{2} + iu)| \int_{-T}^T \frac{dt du}{(|t| + 1)(|t - u| + 1)} \\ &\ll yx^{3/2} L \int_{-2T}^{2T} |\zeta(\alpha_1 - \frac{1}{2} + iu)| \frac{du}{|u| + 1} \ll yx^{3/2} L^3. \end{aligned}$$

Hence, accounting for the residue at  $s_2 = 2 - s_1$ , we have

$$\begin{aligned} (3.14) \quad C_{2,1}(x, y) &= \frac{yx^2}{\zeta(2)} \frac{1}{2\pi i} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \frac{ds_1}{s_1(2 - s_1)} + O(yx^2 L^3(x^{-1/2} + T^{-1})), \\ &= \frac{yx^2}{2\zeta(2)} + O(yx^2 L^3(x^{-1/2} + T^{-1})). \end{aligned}$$

To estimate  $C_{2,4}(x, y)$ , we move the integration over  $s_2$  to the contour  $\Gamma(\alpha_2, \beta, T)$ , where  $\beta = \frac{5}{2} - \alpha_1$ ; let us denote the respective integrals  $J_{4,1}$ ,  $J_{4,2}$  and  $J_{4,3}$ . Similarly to the estimation of  $J_{1,j}$ , we have

$$J_{4,1}, J_{4,3} \ll yx^2 L^5(T^{-1} + (x/yT)^{1/2}), \quad J_{4,2} \ll y^{1/2} x^{5/2} L^5.$$

Since the integrand is holomorphic in the region between the two contours, we deduce that

$$(3.15) \quad C_{2,4}(x, y) \ll yx^2 L^5((x/y)^{1/2} + T^{-1}).$$

Similarly, by moving the integration to  $\Gamma(\alpha_2, \frac{3}{2}, T)$ , we find that

$$(3.16) \quad C_{2,3}(x, y) \ll yx^2 L^5((x/y)^{1/2} + T^{-1}),$$

and

$$(3.17) \quad C_{2,2}(x, y) = \frac{yx^2}{\zeta(2)} K(x, y) + O(yx^2 L^5((x/y)^{1/2} + T^{-1})),$$

where

$$K(x, y) = \frac{1}{2\pi i} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \frac{\zeta(2 - s) (yx^{-2})^{1-s}}{\zeta(s) s^2(2 - s)} ds.$$

The first term on the right side of (3.17) arises from the pole of  $R_2(s_1, s_2; y)$  at  $s_2 = s_1$ . Thus, by (3.13)–(3.17),

$$(3.18) \quad C_2(x, y) = \frac{yx^2}{\zeta(2)} \left( \frac{1}{2} + K(x, y) \right) + O(yx^2 L^{10}(x^{-1/2} + (x/y)^{1/2} + T^{-1})).$$

It remains to evaluate the integral  $K(x, y)$ . To that end, we argue differently depending on the size of  $yx^{-2}$ .

*Case 1:*  $yx^{-2} \geq L^B$  for some fixed  $B > 10$ . Then we move the integration to the contour  $\Gamma(\alpha_1, \beta, T)$ , where  $\beta = 2 - L^{-1}$ ; let us denote the respective integrals  $K_1(x, y)$ ,  $K_2(x, y)$  and  $K_3(x, y)$ . Then

$$K_1(x, y), K_3(x, y) \ll T^{-2},$$

and

$$K_2(x, y) \ll y^{-1}x^2L \int_{-T}^T |\zeta(2 - \beta - it)| \frac{dt}{(1 + |t|)^3} \ll y^{-1}x^2L.$$

Therefore, in this case, on choosing  $T = x$ , we obtain

$$(3.19) \quad C_2(x, y) = \frac{yx^2}{2\zeta(2)} + O(yx^2L^{10}(x^{-1/2} + x^2y^{-1})).$$

This completes the proof of part (i) of the theorem.

*Case 2:*  $yx^{-2} \leq L^B$ . Then we move the integration to the contour  $\Gamma(\alpha_1, 1, T)$ . Let us denote the respective integrals  $K_4(x, y)$ ,  $K_5(x, y)$  and  $K_6(x, y)$ . Then, by the bounds for  $\zeta(s)$  in [4, §6.19],

$$K_4(x, y), K_6(x, y) \ll T^{-2},$$

and

$$K_5(x, y) = \kappa(u) + O(T^{-2}),$$

where  $u = \log(yx^{-2})$  and  $\kappa(u)$  is the Fourier integral

$$(3.20) \quad \kappa(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(it)e^{-itu} dt, \quad f(s) = \frac{\zeta(1-s)}{\zeta(1+s)} \frac{1}{(1+s)^2(1-s)}.$$

We choose  $T = x$  and get the asymptotic formula

$$(3.21) \quad C_2(x, y) = \frac{yx^2}{\zeta(2)} \left( \frac{1}{2} + \kappa(u) \right) + O(yx^2L^{10}((x/y)^{1/2} + x^{-1/2})).$$

Thus, to complete the proof of part (ii) of the theorem, we need only establish the desired properties of  $\kappa(u)$ . We examine  $\kappa(u)$  in the next section.

#### 4. ESTIMATION OF $\kappa(u)$

We now study the function  $\kappa(u)$ . Suppose first that  $|u| \rightarrow \infty$ . For a fixed  $\epsilon > 0$  and a real  $t$ , we define

$$\delta(t) = C_\epsilon (\log(|t| + 2))^{-2/3-\epsilon}.$$

By the standard bounds for  $\zeta(s)$  near the edge of the critical strip [4, p. 135],

$$|f(\sigma + it)| \ll \frac{\delta(t)^{-2}}{(1 + |t|)^3} \quad \text{when } |\sigma| \leq \delta(t).$$

Applying Cauchy's integral formula for the circle  $\gamma_t$ ,  $|s - it| = \delta(t)$ , we deduce that

$$|f^{(k)}(it)| = \left| \frac{k!}{2\pi i} \int_{\gamma_t} \frac{f(s)}{(s - it)^{k+1}} ds \right| \ll \frac{k! \delta(t)^{-k-2}}{(1 + |t|)^3} \quad (k \geq 1).$$

Hence, a  $k$ -fold integration by parts gives

$$\kappa(u) \ll \frac{1}{|u|^k} \left| \int_{-\infty}^{\infty} f^{(k)}(it) e^{-itu} dt \right| \ll \frac{k! C_\epsilon^{-k-2}}{|u|^k} \int_2^\infty (\log t)^{(2/3+\epsilon)(k+2)} t^{-3} dt.$$

Using the inequality

$$\int_2^\infty (\log t)^\alpha t^{-3} dt \leq \int_0^\infty z^\alpha e^{-2z} dz = \frac{\Gamma(\alpha+1)}{2^{\alpha+1}},$$

we deduce that

$$\kappa(u) \ll \frac{k! m_k!}{2^{m_k} (C_\epsilon |u|)^k} \ll \frac{k^{(5/3+\epsilon)k}}{(C_\epsilon |u|)^k},$$

where  $m_k = \lceil (2/3 + \epsilon)(k + 2) \rceil$ . Finally, we set  $k = \lfloor C'_\epsilon |u|^{1/(5/3+\epsilon)} \rfloor$ , and we get

$$\kappa(u) \ll \exp(-|u|^{3/5-\epsilon}).$$

Next, we focus on the case when  $|u|$  is bounded. The integral  $\kappa(u)$  is differentiable, and

$$|\kappa'(u)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|t| dt}{(1+t^2)^{3/2}} \leq \frac{1}{2}.$$

Hence,

$$(4.1) \quad |\kappa(u) - \kappa(v)| \leq \frac{|u - v|}{2}.$$

Since the integrand is bounded above by  $|t|^{-3}$ , we have

$$(4.2) \quad \kappa(u) = \frac{1}{2\pi} \int_{-60}^{60} f(it) e^{-itu} dt + \vartheta_u 10^{-4},$$

where  $|\vartheta_u| \leq \frac{1}{2}$ . Let  $\kappa_0(u)$  denote the integral on the right side of (4.2). By partial integration,

$$|\kappa_0(u)| \leq \frac{1}{2\pi|u|} \left( |f(60i)| + |f(-60i)| + \int_{-60}^{60} |f'(it)| dt \right) \leq \frac{4.2}{2\pi|u|} \leq 0.67|u|^{-1}.$$

Combining the latter inequality and (4.2), we deduce that  $|\kappa(u)|$  when  $|u| \geq 1.7$ . It remains to estimate  $\kappa(u)$  when  $|u| \leq 1.7$ . In view of (4.1) and (4.2), it suffices to tabulate  $\kappa_0(u)$  for a set of suitably chosen values  $u_1, u_2, \dots$ , with  $|u_j| \leq 1.7$ . To this end, a quick calculation using *Mathematica* yields the following table.

$u_j$	$\kappa_0(u_j)$	covered range
-1	-0.0277	$-1.7 \leq u \leq -0.3$
0.3	-0.0984	$-0.3 \leq u \leq 0.9$
1.1	-0.2586	$0.9 \leq u \leq 1.3$
1.5	-0.2925	$1.3 \leq u \leq 1.7$

Each row of this table lists a value of  $\kappa_0(u_j)$  and a range of the variable  $u$ , where we can infer the bound  $\kappa(u) \geq -0.4$  from (4.1), (4.2) and the listed value of  $\kappa_0(u_j)$ . For example, when  $0.9 \leq u \leq 1.3$ , we have

$$\kappa(u) \geq \kappa(1.1) - 0.5|u - 1.1| > \kappa_0(1.1) - 0.1001 > -0.36.$$

Clearly, this completes the proof of part (ii) of Theorem 1.2.



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