On fractional parts of powers of real numbers close to 1

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Abstract. *We prove that there exist arbitrarily small positive real numbers ε such that every integral power $(1+\varepsilon)^n$ is at a distance greater than $2^{-17}\varepsilon|\log\varepsilon|^{-1}$ to the set of rational integers. This is sharp up to the factor $2^{-17}|\log\varepsilon|^{-1}$. We also establish that the set of real numbers $\alpha > 1$ such that the sequence of fractional parts $(\{\alpha^n\})_{n\geq 1}$ is not dense modulo 1 has full Hausdorff dimension.*

1. Introduction

Throughout this note, $\{\cdot\}$ denotes the fractional part and $||\cdot||$ the distance to the nearest integer. In 1935, Koksma [4] established that the sequence $(\{\alpha^n\})_{n\geq 1}$ is uniformly distributed modulo 1, for almost all (with respect to the Lebesgue measure) real numbers α greater than 1. However, very little is known on the distribution of $(\{\alpha^n\})_{n\geq 1}$ for a specific real number α greater than 1. If α is a Pisot number, that is, an algebraic integer greater than 1 all of whose Galois conjugates except α lie in the open unit disc, then $||\alpha^n||$ tends to 0 as n tends to infinity, and the limit points of $(\{\alpha^n\})_{n\geq 1}$ are contained in $\{0,1\}$. Pisot and Salem [8] established that if α is a Salem number, that is, an algebraic integer greater than 1 all of whose Galois conjugates except α and $1/\alpha$ lie on the unit circle, then $(\{\alpha^n\})_{n\geq 1}$ is dense but not uniformly distributed modulo 1. We do not know any explicit transcendental real number α larger than 1 for which the sequence $(\{\alpha^n\})_{n\geq 1}$ is not uniformly distributed modulo 1.

In the present note, we are concerned with the set E composed of the real numbers $\alpha > 1$ for which $(\{\alpha^n\})_{n\geq 1}$ is not dense modulo 1. In 1948, Vijayaraghavan [9] established that, for every real numbers a and b with $1 < a < b$, the intersection $E \cap (a,b)$ is uncountable. Noticing that, in the proof of his Theorem 2, the parameter η should be taken equal to $\delta/(1+b+\ldots+b^{h-1})$ and not to $1/(1+b+\ldots+b^{h-1})$, the following quantitative statement follows from his proof.

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Theorem V1. *There exist arbitrarily small positive real numbers ε such that*

$$\inf_{n\geq 1} \|(1+\varepsilon)^n\| > \varepsilon^{2/\varepsilon}.$$

In the same paper, Vijayaraghavan [9] also showed that, for any interval I of positive length contained in $[0,1]$, there are uncountably many α all of whose integral powers are lying in I modulo 1. This result was recently reproved by Dubickas [2]. Theorem 1 of [9] includes the following statement.

Theorem V2. *Let $H \geq 3$ be an integer. For every $\delta > 2/H$ and every interval I of length δ, there exists α in $(H, H+1)$ such that $\{\alpha^n\}$ lies in I for every $n \geq 1$.*

The first purpose of the present note is to significantly improve Theorem V1, by means of a suitable modification of a method introduced by Peres and Schlag [7] (see also [5, 6]), based on the Lovász local lemma.

Surprisingly, it seems that no metric result is known on the size of the set E. The second aim of this note is to give a suitable adaptation of Vijayaraghavan's proof of Theorem V2 for showing that E has full Hausdorff dimension.

2. Main results

Our first result is a considerable improvement of Theorem V1.

Theorem 1. *There exist arbitrarily small positive real numbers ε such that*

$$\inf_{n\geq 1} \|(1+\varepsilon)^n\| > 2^{-17}\, \varepsilon\, |\log \varepsilon|^{-1}.$$

Theorem 1 is sharp up to the factor $2^{-17}|\log \varepsilon|^{-1}$, since the above infimum is clearly at most equal to ε, when $\varepsilon < 1/2$. The numerical constant 2^{-17} occurring in Theorem 1 can certainly be reduced, but we have made no effort in this direction.

The Peres–Schlag method is an inductive construction. Roughly speaking, at each step k, we remove finitely many intervals, which have (in all known applications until now) essentially the same length. The novelty in the present application of the method is that these intervals are far from having the same length: here, at step k, the quotient of the longest length by the smallest one grows exponentially in k. Consequently, the original approach of Peres and Schlag does not allow us to prove Theorem 1, and we have to perform a more complicated induction.

In [1] we have combined the Peres–Schlag method with the mass distribution principle to show that, in many situations, the exceptional set constructed by means of the Peres–Schlag method has full Hausdorff dimension. A similar approach allows us to establish that, for every small positive ε, the Hausdorff dimension of

$$\left\{\varepsilon' \in (\varepsilon, 2\varepsilon) : \inf_{n\geq 1} \|(1+\varepsilon')^n\| > c\, \varepsilon\, |\log \varepsilon|^{-1}\right\}$$

tends to 1 when c tends to 0. Brief explanations are given at the end of the proof of Theorem 1.

The proof of Theorem 1 can be readily adapted to give the more general following statement.

Theorem 2. *Let M be a positive real number. For any non-zero real number ξ in $[-M, M]$ and for any sequence $(\eta_n)_{n\ge1}$ of real numbers, there exist a positive number γ, depending only on M, and arbitrarily small positive real numbers ε such that*

$$\inf_{n\ge1} \|\xi(1+\varepsilon)^n + \eta_n\| > \gamma\,\varepsilon\,|\log\varepsilon|^{-1}.$$

Our last result implies that the set of real numbers greater than 1 all of their integral powers stay, modulo one, in a given interval of positive length is rather big. It strengthens Corollary 5 of [2].

Theorem 3. *Let ξ be a positive real number. Let $\varepsilon < 1$ be a positive real number. Let $(a_n)_{n\ge1}$ be a sequence of real numbers satisfying $0 \le a_n < 1-\varepsilon$ for $n \ge 1$. The set of real numbers α such that $a_n \le \{\xi\alpha^n\} \le a_n + \varepsilon$ for every $n \ge 1$ has full Hausdorff dimension.*

Theorem V2 suggests the next question, which seems to be quite difficult.

Question. *Let ε be a positive real number. Are there arbitrarily large real numbers α such that α is not a Pisot number and all the fractional parts $\{\alpha^n\}$, $n \ge 1$, are lying in an interval of length ε/α ?*

Dubickas [2] gave an alternative proof of a version of Theorem V2 in which the lower bound $2/H$ is replaced by $8/H$.

Throughout the present paper, λ denotes the Lebesgue measure. Furthermore, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote respectively the largest integer smaller than or equal to x and the smallest integer greater than or equal to x.

3. Proof of Theorem 1

First, note that if the real numbers ε, δ and the positive integers k, m satisfy $0 < \varepsilon, \delta < 1/5$ and

$$|(1+\varepsilon)^k - m| \le \delta,$$

then we get

$$\left|\frac{(1+\varepsilon)^k}{m} - 1\right| \le \frac{\delta}{m}$$

and

$$\left|\log(1+\varepsilon) - \frac{\log m}{k}\right| \le \frac{2\delta}{km}.$$

1. Dangerous sets.

Let t be a large positive integer and set

$$\eta = 2^{-t}, \qquad \psi = \frac{\eta}{2^{14}\log(1/\eta)} = \frac{1}{2^{t+14}t\log 2}. \tag{3.1}$$

Our aim is to find a real number ξ such that

$$2^{-t} \le \xi \le 2^{-t+1}$$

and, for every $k \ge 1$,

$$\xi \notin A_k = \bigcup_{m=\lfloor e^{\eta k} \rfloor}^{\lceil e^{2\eta k} \rceil} A_{k,m}, \quad \text{where} \quad A_{k,m} = \left(\frac{\log m}{k} - \frac{\psi}{km}, \frac{\log m}{k} + \frac{\psi}{km} \right). \tag{3.2}$$

Setting $\varepsilon := e^{\xi} - 1$, this proves our theorem in view of the preliminary observation. Indeed, ε then satisfies

$$|(1+\varepsilon)^k - m| \ge \frac{\psi}{2},$$

for every positive integers k, m.

In the union occurring in (3.2), the integer m varies between $\lfloor e^{\eta k} \rfloor$ and $\lceil e^{2\eta k} \rceil$. Since the quotient of these two numbers depends on k, we cannot use the Peres–Schlag method as it was applied in [7, 5, 6]. Fortunately, it is possible to adapt it to prove our theorem.

In the sequel, we use an inductive process to establish that

$$[2^{-t}, 2^{-t+1}] \setminus \bigcup_{k \ge 1} \bigcup_{m=\lfloor e^{\eta k} \rfloor}^{\lceil e^{2\eta k} \rceil} A_{k,m}$$

is non-empty. Put

$$h = (t+6)2^{t+6} = \frac{2^6}{\eta \log 2} \log \frac{2^6}{\eta} \tag{3.3}$$

and

$$k_n = hn, \quad \text{for } n \ge 1.$$

2. Initial steps.

We construct real numbers

$$W_1, W_2, \ldots, W_{n-1}, \ldots \in [\eta, 2\eta], \qquad W_n \le W_{n+1}, \qquad n \ge 1,$$

and positive integers

$$l_1, l_2, \ldots, l_n, \ldots, \quad w_1, w_2, \ldots, w_n, \ldots$$

in such a way that

$$W_n = \frac{w_n}{2^{l_n}}, \qquad n \ge 1,$$

and

$$J_n = \left[W_n, W_n + \frac{1}{2^{l_n}} \right] \subset J_{n-1} = \left[W_{n-1}, W_{n-1} + \frac{1}{2^{l_{n-1}}} \right] \subset [\eta, 2\eta], \qquad n \ge 2.$$

Let l_1 be such that

$$2^{-l_1} < \frac{2\psi}{h\lceil e^{2\eta h}\rceil} \leq 2^{-l_1+1},$$

and observe that

$$l_1 \geq 5t, \tag{3.4}$$

if t is large enough. Then, for each set $A_{k,m}$ with $k \leq k_1$, we consider the shortest dyadic interval $\hat{A}^1_{k,m}$ of the form

$$\left(\frac{a_1}{2^{l_1}}, \frac{a_2}{2^{l_1}}\right), \qquad a_1, a_2 \in \mathbf{Z},$$

which covers the interval $A_{k,m}$, and we define

$$\hat{A}^1_k := \bigcup_{m=\lfloor e^{\eta k}\rfloor}^{\lceil e^{2\eta k}\rceil} \hat{A}^1_{k,m} \supset A_k.$$

The choice of l_1 implies that

$$\lambda(\hat{A}^1_{k,m}) \leq 4\lambda(A_{k,m}),$$

for $\lfloor e^{\eta k}\rfloor \leq m \leq \lceil e^{2\eta k}\rceil$.

Furthermore,

$$\sum_{k=1}^{h} \sum_{m=\lfloor e^{\eta k}\rfloor}^{\lceil e^{2\eta k}\rceil} \frac{2\psi}{mk} \leq \psi h 2^{-t+3} \leq 2^{-5}2^{-t}.$$

By (3.4), this shows that there exists $J_1 := [W_1, W_1 + 2^{-l_1}]$ such that $J_1 \cap \hat{A}^1_k = \emptyset$ for every $k \leq k_1$ and

$$W_1 \geq (1 + 2^{-6})\eta. \tag{3.5}$$

We set

$$l_2 = \left\lceil \log_2 \frac{4k_2 \exp\left(\left(W_1 + 2^{-l_1}\right) k_2\right)}{\psi} \right\rceil,$$

and, for $n \geq 3$, we define l_n by

$$l_n = \left\lceil \log_2 \frac{4k_n \exp\left(\left(W_{n-2} + 2^{-l_{n-2}}\right) k_n\right)}{\psi} \right\rceil. \tag{3.6}$$

Let $n \geq 2$ be an integer. Instead of the interval $A_{k,m}$, where $k \leq k_n$, we consider the shortest dyadic interval $\hat{A}^n_{k,m}$ of the form

$$\left(\frac{a_1}{2^{l_n}}, \frac{a_2}{2^{l_n}}\right), \quad a_1, a_2 \in \mathbf{Z},$$

which covers the interval $A_{k,m}$. Define

$$\hat{A}^n_k := \bigcup_{m=\lfloor e^{\eta k}\rfloor}^{\lceil e^{2\eta k}\rceil} \hat{A}^n_{k,m} \supset A_k.$$

Let $n \geq 1$ be an integer. We check that

$$2t \leq l_{n+1} - l_n \leq 2h. \tag{3.7}$$

In particular, we have $l_{n+1} \geq l_n$, thus

$$\hat{A}^n_{k,m} \supset \hat{A}^{n+1}_{k,m} \supset A_{k,m}.$$

Here, we should note that W_{n+1} is not defined yet, but it has to satisfy

$$\left[W_{n+1}, W_{n+1} + \frac{1}{2^{l_{n+1}}}\right] \subset J_n. \tag{3.8}$$

We claim that, for any such choice of W_{n+1}, we have

$$\lambda(\hat{A}^{n+1}_{k,m}) \leq 4\lambda(A_{k,m}) \tag{3.9}$$

for every $k \leq k_{n+1}$ and for every integer m such that

$$A_{k,m} \cap J_{n-1} \neq \emptyset. \tag{3.10}$$

The reason for (3.9) to be valid is as follows. Given an integer k, we define $m_1 = m_1(k)$ to be the maximal m for which (3.10) holds. Then

$$m_1(k) \leq \exp\left(\left(W_{n-1} + 2^{-l_{n-1}}\right) k + 1\right).$$

From (3.6) it follows that

$$\lambda(A_{k,m}) = \frac{2\psi}{mk} \geq \frac{2\psi}{m_1(k_{n+1})k_{n+1}} \geq \frac{2}{2^{l_{n+1}}},$$

for $k \leq k_{n+1}$, and (3.9) holds for any possible value of W_{n+1} satisfying (3.8).

3. Inductive assumption.

We describe the inductive assumption of our version of the Peres–Schlag method. It consists, for $n \geq 2$, of the following two points ($\mathbf{i_n}$) and ($\mathbf{ii}_n$), that have to be satisfied by an interval J:

($\mathbf{i_n}$) $\quad J \cap \hat{A}^{n-1}_k = \emptyset, \quad$ for every $k \leq k_{n-1}$.

($\mathbf{ii_n}$) $\quad \lambda\big(J \setminus \big(\bigcup_{k\leq k_n} \hat{A}^n_k\big)\big) \geq \lambda(J)/2.$

Estimating

$$\sum_{k=h+1}^{2h} \sum_{m=\lfloor e^{kW_1}\rfloor}^{\lceil e^{k(W_1+2^{-l_1})}\rceil} \frac{2\psi}{mk} \leq \psi h 2^{-l_1} \leq 2^{-l_1-6},$$

our choice of l_2 implies that

$$\lambda\left(J_1 \setminus \left(\bigcup_{k\leq k_2} \hat{A}^2_k\right)\right) \geq \frac{\lambda(J_1)}{2}.$$

We have thus checked that ($\mathbf{i_2}$) and ($\mathbf{ii_2}$) hold for the interval J_1.

4. Independence shift.

This subsection is devoted to the proof of a key lemma for the inductive step.

Lemma 1. *For $n \geq 2$ and k satisfying $k_n \leq k \leq k_{n+1}$, we have*

$$\lambda(J_{n-1} \cap \hat{A}_k^{n+1}) < 16\psi\lambda(J_{n-1}).$$

Proof. It follows from (3.9) that it is enough to establish that

$$\lambda(J_{n-1} \cap A_k) < 4\psi\lambda(J_{n-1}), \tag{3.11}$$

for $k \geq k_n$. Recall that

$$A_k = \bigcup_{m=\lfloor e^{\eta k}\rfloor}^{\lceil e^{2\eta k}\rceil} \left(\frac{\log m}{k} - \frac{\psi}{km}, \frac{\log m}{k} + \frac{\psi}{km}\right).$$

Let $m_0, m_0+1, \ldots, m_0+t = m_1$ be the integers m for which the interval $A_{k,m}$ has non-empty intersection with the segment J_{n-1}. Then

$$\lambda(J_{n-1} \cap A_k) \leq \sum_{m=m_0}^{m_1} \frac{2\psi}{km}$$

and

$$m_0 \geq e^{W_{n-1}k} - 1 \geq \frac{e^{W_{n-1}k_n}}{2}.$$

We check that

$$\max_{j=0,\ldots,t-1} \left(\frac{\log(m_0+j+1)}{k} - \frac{\log(m_0+j)}{k}\right) \leq \frac{1}{m_0 k} \leq \frac{2}{k_n e^{W_{n-1}k_n}}.$$

Furthermore, since $2^{-l_{n-1}}k_n \leq 1$, we get

$$\begin{aligned} k_n e^{W_{n-1}k_n} &\geq k_{n-1} e^{(W_{n-3}+2^{-l_{n-3}})k_{n-1}} e^{-2^{-l_{n-1}}k_n} e^{W_{n-1}h} \\ &\geq (\psi 2^{l_{n-1}}) \cdot \frac{1}{4} \cdot \frac{64}{\psi} \\ &\geq 2^{l_{n-1}+4} \geq \frac{16}{\lambda(J_{n-1})}, \end{aligned}$$

for $n \geq 4$. We check below that the inequality

$$k_n e^{W_{n-1}k_n} \geq \frac{16}{\lambda(J_{n-1})} \tag{3.12}$$

also holds for $n = 2$ and $n = 3$.

For $n = 2$, inequality (3.12) is satisfied as soon as

$$2he^{2hW_1}\psi \geq 2^5 he^{2\eta h},$$

that is, using (3.5), as soon as

$$e^{2^{-5}\eta h}\psi \geq 2^4.$$

The latter inequality is a direct consequence of (3.1) and (3.3), provided that t is sufficiently large.

For $n = 3$, inequality (3.12) holds as soon as

$$e^{3hW_2} \geq 2^5 e^{2(W_1+2^{-l_1})h},$$

which, by (3.4), holds for t sufficiently large.

Consequently, for $n \geq 2$, at least two centers of the intervals $A_{k,m}$ are lying inside J_{n-1} and

$$\lambda(J_{n-1}) \geq \frac{\log(m_1/m_0)}{k} - \frac{2\psi}{k_n m_0}.$$

Thus, we get

$$\lambda(J_{n-1} \cap A_k) \leq \sum_{m=m_0}^{m_1} \frac{\psi}{km} \leq \psi\lambda(J_{n-1}) + \frac{2\psi}{k_n m_0}. \tag{3.13}$$

Since

$$\frac{1}{k_n m_0} \leq \frac{1}{k_n(e^{W_{n-1}k_n} - 1)} \leq \frac{1}{2^{l_{n-1}}} = \lambda(J_{n-1}),$$

the lemma follows from (3.11) and (3.13). □

5. Inductive step.

Let $n \geq 2$ be an integer and J_{n-1} be an interval such that $(\mathbf{i_n})$ and $(\mathbf{ii_n})$ hold with $J = J_{n-1}$. We consider the set

$$J_{n-1} \setminus \left(\bigcup_{k=k_{n-1}+1}^{k_n} \hat{A}_k^n \right) = \bigcup_{\nu=1}^{T} I^\nu,$$

where $T \geq 1$ and the I^ν are distinct intervals of the form

$$I^\nu = \left[\frac{a^\nu}{2^{l_n}}, \frac{a^\nu + 1}{2^{l_n}} \right].$$

We see that

$$\lambda(I^\nu) = \frac{1}{2^{l_n}}, \qquad I^\nu \cap \hat{A}_k = \emptyset,$$

for $\nu = 1, \ldots, T$ and for $k \leq k_n$. For a given index ν consider the set

$$I^\nu \setminus \bigcup_{k=k_n+1}^{k_{n+1}} \hat{A}_k^{n+1}.$$

We see that

$$\lambda\left(I^\nu \setminus \bigcup_{k=k_n+1}^{k_{n+1}} \hat{A}_k^{n+1}\right) \geq \frac{1}{2^{l_n}} - \sum_{k=k_n+1}^{k_{n+1}} \lambda\left(I^\nu \cap \hat{A}_k^{n+1}\right),$$

thus

$$\sum_{\nu=1}^{T} \lambda\left(I^\nu \setminus \bigcup_{k=k_n+1}^{k_{n+1}} \hat{A}_k^{n+1}\right) \geq \frac{T}{2^{l_n}} - \sum_{\nu=1}^{T} \sum_{k=k_n+1}^{k_{n+1}} \lambda\left(I^\nu \cap \hat{A}_k^{n+1}\right).$$

But

$$\begin{aligned} \sum_{\nu=1}^{T} \sum_{k=k_n+1}^{k_{n+1}} \lambda\left(I^\nu \cap \hat{A}_k^{n+1}\right) &\leq \sum_{k=k_n+1}^{k_{n+1}} \lambda\left(J_{n-1} \cap \hat{A}_k^{n+1}\right) \\ &\leq 16\psi h \lambda\left(J_{n-1}\right), \end{aligned}$$

by Lemma 1. We deduce from the inductive assumption (**ii**) that

$$\lambda(J_{n-1}) \leq 2\lambda\left(J_{n-1} \setminus \left(\bigcup_{k=k_{n-1}+1}^{k_n} A_k\right)\right).$$

Furthermore, we have

$$\frac{T}{2^{l_n}} = \lambda\left(J_{n-1} \setminus \left(\bigcup_{k=k_{n-1}+1}^{k_n} A_k\right)\right)$$

and, by (3.1) and for t large enough,

$$32\psi h \leq \frac{1}{2}.$$

Consequently,

$$\begin{aligned} \sum_{\nu=1}^{T} \lambda\left(I^\nu \setminus \bigcup_{k=k_n+1}^{k_{n+1}} \hat{A}_k^{n+1}\right) &\geq \frac{1}{2} \cdot \lambda\left(J_{n-1} \setminus \left(\bigcup_{k=k_{n-1}+1}^{k_n} A_k\right)\right) \\ &= \frac{1}{2} \cdot \sum_{\nu=1}^{T} \lambda(I^\nu). \end{aligned}$$

Thus, there exists $\nu_0 = 1, \ldots, T$ such that

$$\lambda\left(I^{\nu_0} \setminus \bigcup_{k=k_n+1}^{k_{n+1}} \hat{A}_k^{n+1}\right) \geq \frac{1}{2}\lambda(I^{\nu_0}).$$

We put $J_n = I^{\nu_0}$. We have shown that ($\mathbf{i_n}$) and ($\mathbf{ii_n}$) are satisfied with $J = J_n$.

6. Conclusion.

The sequence $(J_n)_{n\geq 1}$ is a decreasing (with respect to inclusion) sequence of non-empty compact intervals. Consequently, the intersection $\bigcap_{n\geq 1} J_n$ is non-empty. By construction, if ξ is in $\bigcap_{n\geq 1} J_n$, then ξ avoids every interval $\hat{A}_k^n$. This completes the proof of Theorem 1.

We can slightly modify our construction to end up with an uncountable intersection $\bigcap_{n\geq 1} J'_n$. Indeed, in view of (3.7), the integer T occurring in the inductive step is not too small. Thus, at each step n, we have at least two choices for the interval J_n, and we let J'_n be the union of two such suitable intervals.

Furthermore, a (small) positive δ being given, we see that there are indeed at least $\lfloor (1-\delta)T \rfloor$ suitable choices for J_n at each step n, provided that the value 2^{14} in (3.1) is replaced by a larger number, say $\kappa(\delta)$, depending only on δ and tending to infinity as δ tends to 0. Thus, we have a Cantor type construction and the Hausdorff dimension of the resulting set $\mathcal{C}_\delta$ can be bounded from below by means of the mass distribution principle, as was done in [1]. Replacing the value 2^6 in (3.3) by $\sqrt{\kappa(\delta)}$, it follows from a rapid calculation using (3.7) that the Hausdorff dimension of $\mathcal{C}_\delta$ tends to 1 as δ approaches 0. This establishes the metrical statement enounced after Theorem 1. $\square$

4. Proof of Theorem 3

We adapt the proof of Theorem 1 of [9]. For simplicity, we only treat the case where $\xi = 1$. Set $b_n = a_n + \varepsilon$ for $n \geq 1$. Without any loss of generality, we assume that $\varepsilon \leq 1/2$. Let η be a positive real number with $\eta < 1$. Let H be an integer such that $\varepsilon H > H^\eta > 2$ and put $c = \lfloor H^\eta \rfloor - 2$.

Set $I_1 = [H + a_1, H + b_1]$. This is our Step 1. Since

$$(H + b_1)^2 - (H + a_1)^2 \geq 2H^\eta \geq c + 2,$$

there is an integer j_1 such that $j_1, \ldots, j_1 + c$ are in the interval $[(H + a_1)^2, (H + b_1)^2]$. For $h = j_1, \ldots, j_1 + c - 1$, let $I_{2,h}$ be the interval $[\sqrt{h + a_2}, \sqrt{h + b_2}]$. Since

$$(H + a_1)^2 \leq h + a_2 < h + b_2 \leq (H + b_1)^2,$$

the interval $I_{2,h}$ is included in I_1. By construction, every real number ξ in $I_{2,h}$ is such that $\{\xi\}$ and $\{\xi^2\}$ are in $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Let E_2 be the union of the c intervals $I_{2,h}$. This completes Step 2.

We continue this process. Let $h = j_1, \ldots, j_1 + c - 1$. Since

$$(\sqrt{h + b_2})^3 - (\sqrt{h + a_2})^3 \geq \big((\sqrt{h + b_2})^2 - (\sqrt{h + a_2})^2\big)\sqrt{h + a_2} \geq H\varepsilon \geq c + 2,$$

there is an integer j_2 such that $j_2, \ldots, j_2 + c$ are in the interval $[(h + a_2)^{3/2}, (h + b_2)^{3/2}]$. For $i = j_2, \ldots, j_2 + c - 1$, let $I_{2,h,i}$ be the interval $[(i + a_3)^{1/3}, (i + b_3)^{1/3}]$. By construction, $I_{2,h,i}$ is included in $I_{2,h}$. Proceeding in this way, we construct at Step 3 a union E_3 of

c^2 sub-intervals of I_1, whose elements ξ have the property that $\{\xi\}, \{\xi^2\}$ and $\{\xi^3\}$ are in $[a_1, b_1]$, $[a_2, b_2]$ and $[a_3, b_3]$, respectively.

Continuing further in the same way, for $j \geq 4$, we construct at Step j a set E_j which is the union of c^{j-1} closed intervals of length approximately equal to

$$\asymp (H^j + b_j)^{1/j} - (H^j + a_j)^{1/j} \asymp \varepsilon H^{-j+1}/j.$$

Each of these intervals gives birth to c intervals at the next step. Furthermore, two different intervals at Step j are separated by at least H^{-j+1}/j times an absolute positive constant. The set

$$\mathcal{C}_\eta = \bigcap_{j \geq 1} E_j$$

is a Cantor type set, whose elements have the property that, for $n \geq 1$, the fractional part of their n-th power lies in $[a_n, b_n]$. The Hausdorff dimension of $\mathcal{C}_\eta$ can be bounded from below by using the mass distribution principle, as given, e.g., in Chapter 4 of [3]. We get that the dimension of $\mathcal{C}_\eta$ is at least equal to $(\log c)/(\log H)$ and, since η can be taken arbitrarily close to 1, our theorem is proved. $\square$

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