A LOCALIZATION IN MV-ALGEBRAS

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Abstract. In this document we consider a way of localizing an MV-algebra. Given any prime filter F we find a local MV-algebra which has the same poset of prime filters as the poset of prime filters comparable to F.

1. Introduction

A local MV-algebra is one with a single maximal implication filter. Such algebras are of interest in the representation theory of MV-algebras (see [7] for example).

The set of prime implication filters of an MV-algebra forms a spectral root system, ordered by set-inclusion. The existence of a unique maximal filter is equivalent to the stem of this root system being nonempty. (The stem is the set $\text{Stem} = \{P \mid P \text{ is a prime filter comparable to every other prime filter}\}$.) Whenever the stem is non-empty it has a least element, the Conrad filter (defined below). This filter can be characterized in several ways, as we show in section 2 below. This work is heavily based on work of Conrad on lattice-ordered groups (see [5]), recasting his material in terms of implication filters in MV-algebras.

In the last section we consider how to invert this characterization to get a prime filter into the stem of an MV-algebra. This localization takes a prime implication filter P and finds a quotient in which the maximal filter over P is the unique maximal filter, and the prime filter structure of the quotient is isomorphic to the set of prime filters comparable to P.

In most of what follows the filters are taken to be implication filters rather than lattice filters. We recall that an implication filter is a lattice filter closed under powers.

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Given an MV-algebra $\mathcal{L}$, there are several sets of filters that we are interested in:

$$\begin{aligned}
\mathrm{PSpec} &= \{P \mid P \text{ is a prime implication filter of } \mathcal{L}\}\\
&= \text{ the prime spectrum;}\\
\mathrm{PSpec}(F) &= \{P \mid P \text{ is a prime implication filter of } \mathcal{L} \text{ comparable to } F\}\,;\\
\mu\mathrm{S} &= \{P \mid P \text{ is a minimal prime filter of } \mathcal{L}\}\\
&= \text{the minimal spectrum;}\\
\mu\mathrm{S}(F) &= \{P \mid P \text{ is a minimal prime filter of } \mathcal{L} \text{ comparable to } F\}\,.
\end{aligned}$$

Our notation usually follows that of [3] with the exception that we use $\otimes$ instead of $\odot$.

2. Counits

Definition 2.1. *$u \in \mathcal{L}$ is a* counit *iff $u < 1$ and there exists some $v < 1$ with $u \vee v = 1$.*

Definition 2.2. *The* Conrad filter *of an MV-algebra is the implication filter generated by the counits.*

We usually denote it by $\mathcal{N}(\mathcal{L})$ or N.

If $N = \mathcal{N}(\mathcal{L})$ then N is prime as $a \vee b = 1$, $a, b < 1$ implies a and b are counits and so in N.

All implication filters that contain N form a chain. The following lemma provides an alternative characterization of the prime filters in this chain.

Lemma 2.3. *Let P be a prime implication filter. Then P contains all counits iff for all $x \notin P$ and all $p \in P$ $p \geq x$.*

Proof. Suppose that $x \notin P$ and $y \in P$ with $x \nleq y$. We know that $(x \to y) \vee (y \to x) = 1$.

As $x \nleq y$ we have $x \to y < 1$, and $y \nleq x$ implies $y \to x < 1$ and so $y \to x$ is a co-unit.

If it is in P then so is $x \wedge y = (y \to x) \otimes y$, contradicting $x \notin P$. Thus P cannot contain all co-units.

Conversely if a is a co-unit and $a \vee b = 1$ for some $b > 0$. One of a or b is in P (as P is prime). If $a \notin P$ then $a \leq b$ which is impossible, so $a \in P$. □

A slight variation of this proof lets us see that filters are incomparable because of counits.

Lemma 2.4. *Let P and Q be incomparable implication filters. Then there is a counit in $Q \setminus P$.*

Proof. Suppose not, ie every counit in Q is also in P. As P and Q are incomparable we can find $x \in Q \setminus P$ and $y \in P \setminus Q$. Thus $x \nleq y$ and $y \nleq x$ and so $x \to y < 1$ and $y \to x < 1$ and $(x \to y) \vee (y \to x) = 1$. So $y \to x$ is a counit in Q and (by assumption) must be in P. As $y \in P$ we now have $x \wedge y = y \otimes (y \to x) \in P$ contradicting $x \notin P$. □

The next two results show that N is actually the minimum prime filter comparable to all prime filters.

Proposition 2.5. *Let P be any prime implication filter that does not contain all counits. Then there is a prime implication filter incomparable to P.*

Proof. As P does not contain all counits we know that there is some $g \notin P$ that is not below P, ie there is some $p \in P$ with $p \ngeq g$. Of course $g \ngeq p$. Thus $g \to p < 1$ and $p \to g < 1$ and $(g \to p) \vee (p \to g) = 1$.

As $(p \to g) \otimes (p \vee g) = g$ we must have $p \to g \notin P$.

Let Q be maximal avoiding $g \to p$. Then Q is prime and as $(g \to p) \vee (p \to g) = 1 \in Q$ we have $p \to g \in Q \setminus P$. By construction $g \to p \in P \setminus Q$ and so these two ideals are incomparable. □

Proposition 2.6. *If P is a prime implication filter then either $N \subseteq P$ or $P \subseteq N$.*

Proof. If P is not a subset of N then we can find $p \in P \setminus N$. $p \notin N$ implies p is below N and so $N \subseteq [p, 1] \subseteq P$. □

Thus N is the minimum prime implication filter comparable to all others. The existence of such a filter implies that N is a proper filter, as if we have a minimal prime implication filter F comparable to all others then it must contain all counits – by proposition 2.5 and so N exists and so F equals N.

Since any desired root system is the root system of an MV-algebra ([4]), we see that it is possible to have non-trivial N.

Proposition 2.7. *N is a minimal prime implication filter iff $N = \{1\}$.*

Proof. The right to left implication is immediate.

If N is minimal then it is the unique minimal implication filter and so must equal $\{1\}$ – as we know the intersection of all minimal implication filters is $\{1\}$.

Or just notice that $\mathcal{L}$ embeds into $\prod_{m \in \mu S} \mathcal{L}/m = \mathcal{L}/N$ is linearly ordered, and so $\mathcal{L}$ is linearly ordered which implies $N = \{1\}$. □

We also note that if N is proper then there is a unique maximal implication filter – the one that contains N. We also have the converse.

Proposition 2.8. *If there is exactly one maximal proper implication filter then it contains all counits.*

Proof. Let M be the maximum implication filter. Let $a, b < 1$ with $a \vee b = 1$. Let $F_b = \{x \mid x \vee b = 1\}$. Then $0 \notin F_b$, $a \in F_b$ and it is easy to see that F_b is a lattice filter. Also, $x \in F_b$ implies $x^n \vee b \geq x^n \vee b^n = (x \vee b)^n = 1$ and so F_b is an implication filter. Hence $a \in F_b \subseteq M$. □

Thus, if there is a maximum implication filter M then $N \subseteq M$ and N is proper.

3. Localization

Let P be a prime implication filter. We seek a quotient of $\mathcal{L}$ in which P contains the Conrad filter. The construction we give below also preserves the structure of PSpec(P).

Definition 3.1. *Let P be a prime implication filter. Then*

$$\ell(P) = [\![\{x \to p \mid p \in P \text{ and } x \notin P\}]\!].$$

Because of lemma 2.4 we need to quotient out by at least $\ell(P)$ in order to make P contain all counits in a quotient.

It is clear that $\ell(P) \subseteq P$ as $x \to p \geq p$ for any $p \in P$. In general this inclusion is strict, with the only exception being minimal prime filters.

Lemma 3.2. *P is minimal prime iff $\ell(P) = P$.*

Proof. If P is minimal prime and $p \in P$ then there is some $t \notin P$ with $t \vee p = 1$. Therefore $t \to p = 1 \to p = p \in \ell(P)$.

If $\ell(P) = P$ and $p \in P$ then $p \geq x \to p'$ for some $x \notin P$ and $p' \in P$. Now $p' \to x \notin P$ else we would have $p' \otimes (p' \to x) = p' \wedge x \in P$ and so $x \in P$. Also $p \vee (p' \to x) \geq (x \to p') \vee (p' \to x) = 1$. Thus P must be minimal. □

The next few lemmas show the relationship of $\ell(P)$ to the minimal filters below P.

Lemma 3.3. *If $m \subseteq P$ is minimal prime then $\ell(P) \subseteq m$.*

Proof. Let $x \notin P$ and $p \in P$. Then $p \otimes (p \to x) = p \wedge x$ implies $p \to x \notin P$ and so is not in m. But $(x \to p) \vee (p \to x) = 1 \in m$ and m is prime, so $x \to p \in m$. □

Lemma 3.4. *Let $p \in P \setminus \ell(P)$. Then there is some minimal prime filter $m \subseteq P$ with $p \notin m$.*

Proof. Look in $\mathcal{L}/\ell(P)$. Then $[p] \neq 1$ and is in $P/\ell(P)$. We also know that the Conrad filter of $\mathcal{L}/\ell(P)$ is contained in $P/\ell(P)$ – since $x \notin P$ and $p \in P$ implies $x \to p \in \ell(P)$ and so $x \leq p \mod \ell(P)$. All minimal filters must be subsets of the Conrad filter and so take M to be a minimal prime filter of $\mathcal{L}/\ell(P)$ that avoids $[p] < [1]$. Then $M \subseteq P/\ell(P)$ and so the preimage M' gives a prime subfilter of P that avoids p.

Any minimal filter of $\mathcal{L}$ contained in M' works. □

Theorem 3.5.

$$\ell(P) = \bigcap \{m \mid m \in \mu S \text{ and } m \subseteq P\}.$$

Proof. By lemma 3.3 we know that LHS⊆RHS.

From lemma 3.4 we know that $p \notin$LHS implies $p \notin$RHS, i.e. RHS⊆LHS. □

We can now define the localization of an MV-algebra at a prime implication filter.

Definition 3.6. *Let P be a prime implication filter of an MV-algebra $\mathcal{L}$. Then the* localization of $\mathcal{L}$ at P *is the MV-algebra $\mathcal{L}/\ell(P)$.*

If $Q \subseteq P$ are two prime implication filters then we have $\{m \mid m \in \mu S \text{ and } m \subseteq Q\} \subseteq \{m \mid m \in \mu S \text{ and } m \subseteq P\}$ and so $\ell(P) \subseteq \ell(Q)$ (from the theorem). Hence there is a natural MV-morphism $\mathcal{L}/\ell(P) \to \mathcal{L}/\ell(Q)$.

And finally a universal property of this localization.

We recall that if $f\colon \mathcal{L} \to \mathcal{M}$ is an MV-morphism then the *shell* of f is

$$\mathrm{sh}(f) = f^{-1}[1] = \{x \mid f(x) = 1\}$$

is an implication filter in $\mathcal{L}$.

Theorem 3.7. *Let P be any filter and $f\colon \mathcal{L} \to \mathcal{M}$ such that $sh(f) \subseteq P$ and $\mathscr{N}(\mathcal{M}) \subseteq f[P]\uparrow$.*

Then $\ell(P) \subseteq sh(f)$.

Proof. Let $x \notin P$ and $p \in P$. If $f(x) \notin f[P]$ then $f(x) \leq f(p)$ and so $f(x \to p) = 1$, i.e. $x \to p \in \mathrm{sh}(f)$.

If $f(x) \in f[P]$ then for some $p \in P$ we have $x \to p$ and $p \to x$ both in the shell of f and hence in P. But then $x \wedge p = p \otimes (p \to x) \in P$ – contradiction. □

From the theorem we see that if f takes P to a filter containing all counits then f factorizes through $\mathcal{L}/\ell(P)$, and so, in some sense, $\mathcal{L}/\ell(P)$ is the largest quotient in which P contains all counits (or dominates its complement).

The assumption that $\mathrm{sh}(f) \subseteq P$ is essential, else the theorem yields only that the smaller set $\ell(P \vee \mathrm{sh}(f)) \subseteq \mathrm{sh}(f)$. Indeed if P, Q are incomparable prime filters then $\mathscr{N}(\mathcal{L}/Q) = \{1\} \subseteq P/Q$ but if $q \in Q \setminus P$ and $p \in P \setminus Q$ then $q \to p \in \ell(P) \setminus Q$ – else $p \wedge q = q \otimes (q \to p) \in Q$, contradicting $p \notin Q$.

Lemma 3.8. *Let F be a prime filter. Then $\ell(P) \subseteq F$ iff F is comparable to P.*

Proof. If $P \subseteq F$ then $\ell(P) \subseteq P \subseteq F$. If $F \subseteq P$ then $\ell(P) \subseteq \ell(F) \subseteq F$.

Conversely, if $\ell(P) \subseteq F$ then $F/\ell(P)$ is prime in $\mathcal{L}/\ell(P)$ and so comparable to $P/\ell(P)$. Hence $F = \eta^{-1}[F/\ell(P)$ is comparable to $\eta^{-1}[P/\ell(P)] = P$. ◻

Theorem 3.9. *PSpec(P) is order-isomorphic to PSpec$(\mathcal{L}/\ell(P))$.*

Proof. We know that $\mathrm{PSpec}(\mathcal{L}/\ell(P))$ is order-isomorphic to $\{F \mid F \text{ is a prime filter with } \ell(P) \subseteq F\}$ and from the lemma the latter set is $\mathrm{PSpec}(P)$. ◻

References

[1] C. G. Bailey, *The Prime Filters of an MV-Algebra*, in preparation, available at `arXiv:0907.3328v1[math.RA]`.

[2] Dumitru Buşneag and Dana Piciu, *Localization of MV -algebras and lu-groups*, Algebra Universalis **50** (2003), 359–380.

[3] R. Cignoli, I. M. D'Ottaviano, and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer, 2000.

[4] R. Cignoli and A. Torrens, *The Poset of Prime ℓ-ideals of an Abelian ℓ-group with a Strong Unit*, Journal of Algebra **184** (1996), 604–612.

[5] Paul Conrad, *The lattice of all convex ℓ-subgroups of a lattice-ordered group*, Czechoslovak Mathematical Journal **15** (1965), 101–123.

[6] ______, *Some Structure Theorems for Lattice-Ordered Groups*, Transactions of the American Mathematical Society **99** (1961), 212–240.

[7] A. Di Nola, I. Esposito, and B. Gerla, *Local algebras in the representation of MV-algebras*, Algebra Universalis **56** (2007), 133–164.

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