

# Cluster algebras and representation theory

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**Abstract.** We apply the new theory of cluster algebras of Fomin and Zelevinsky to study some combinatorial problems arising in Lie theory. This is joint work with Geiss and Schröer (§3, 4, 5, 6), and with Hernandez (§8, 9).

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## 1. Introduction: two problems in Lie theory

Let  $\mathfrak{g}$  be a simple complex Lie algebra of type  $A$ ,  $D$ , or  $E$ . We denote by  $G$  a simply-connected complex algebraic group with Lie algebra  $\mathfrak{g}$ , by  $N$  a maximal unipotent subgroup of  $G$ , by  $\mathfrak{n}$  its Lie algebra. In [47], Lusztig has introduced the semicanonical basis  $\mathcal{S}$  of the enveloping algebra  $U(\mathfrak{n})$  of  $\mathfrak{n}$ . Using the duality between  $U(\mathfrak{n})$  and the coordinate ring  $\mathbb{C}[N]$  of  $N$ , one obtains a new basis  $\mathcal{S}^*$  of  $\mathbb{C}[N]$  which we call the *dual semicanonical basis* [22]. This basis has remarkable properties. For example there is a natural way of realizing every irreducible finite-dimensional representation of  $\mathfrak{g}$  as a subspace  $L(\lambda)$  of  $\mathbb{C}[N]$ , and  $\mathcal{S}^*$  is compatible with this infinite system of subspaces, that is,  $\mathcal{S}^* \cap L(\lambda)$  is a basis of  $L(\lambda)$  for every  $\lambda$ .

The definition of the semicanonical basis is geometric (see below §5). A priori, to describe an element of  $\mathcal{S}^*$  one needs to compute the Euler characteristics of certain complex algebraic varieties. Here is a simple example in type  $A_3$ . Let  $V = V_1 \oplus V_2 \oplus V_3$  be a four-dimensional graded vector space with  $V_1 = \mathbb{C}e_1$ ,  $V_2 = \mathbb{C}e_2 \oplus \mathbb{C}e_3$ , and  $V_3 = \mathbb{C}e_4$ . There is an element  $\varphi_X$  of  $\mathcal{S}^*$  attached to the nilpotent endomorphism  $X$  of  $V$  given by

$$Xe_1 = e_2, \quad Xe_2 = Xe_3 = 0, \quad Xe_4 = e_3.$$

Let  $\mathcal{F}_X$  be the variety of complete flags  $F_1 \subset F_2 \subset F_3$  of subspaces of  $V$ , which are graded (*i.e.*  $F_i = \bigoplus_j (V_j \cap F_i)$  ( $1 \leq i \leq 3$ )) and  $X$ -stable (*i.e.*  $XF_i \subset F_i$ ). The calculation of  $\varphi_X$  amounts to computing the Euler characteristics of the connected components of  $\mathcal{F}_X$ . In this case there are four components, two points and two projective lines, so these Euler numbers are 1, 1, 2, 2. Unfortunately, such a direct geometric computation looks rather hopeless in general.

**Problem 1.1.** *Find a combinatorial algorithm for calculating  $\mathcal{S}^*$ .*

To formulate the second problem we need more notation. Let  $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$  be the loop algebra of  $\mathfrak{g}$ , and let  $U_q(L\mathfrak{g})$  denote the quantum analogue of its enveloping algebra, introduced by Drinfeld and Jimbo. Here we assume that  $q \in \mathbb{C}^*$  is not a root of unity. The finite-dimensional irreducible representations of  $U_q(L\mathfrak{g})$  are of special importance because their tensor products give rise to trigonometric  $R$ -matrices, that is, to trigonometric solutions of the quantum Yang-Baxter equation with spectral parameters [38]. The question arises whether the tensor product of two given irreducible representations is again irreducible. Equivalently, one can ask whether a given irreducible can be factored into a tensor product of representations of strictly smaller dimensions.

For instance, if  $\mathfrak{g} = \mathfrak{sl}_2$  and  $V_n$  is its  $(n+1)$ -dimensional irreducible representation, the loop algebra  $L\mathfrak{sl}_2$  acts on  $V_n$  by

$$(x \otimes t^k)(v) = z^k xv, \quad (x \in \mathfrak{sl}_2, k \in \mathbb{Z}, v \in V_n).$$

Here  $z \in \mathbb{C}^*$  is a fixed number called the evaluation parameter. Jimbo [37] has introduced a simple  $U_q(L\mathfrak{sl}_2)$ -module  $W_{n,z}$ , which can be seen as a  $q$ -analogue of this evaluation representation. Chari and Pressley [7] have proved that  $W_{n,z} \otimes W_{m,y}$  is an irreducible  $U_q(L\mathfrak{sl}_2)$ -module if and only if

$$q^{n-m} \frac{z}{y} \notin \left\{ q^{\pm(n+m+2-2k)} \mid 0 < k \leq \min(n, m) \right\}.$$

In the other direction, they showed that every simple object in the category  $\text{mod } U_q(L\mathfrak{sl}_2)$  of (type 1) finite-dimensional  $U_q(L\mathfrak{sl}_2)$ -modules can be written as a tensor product of modules of the form  $W_{n_i, z_i}$  for some  $n_i$  and  $z_i$ . Thus the modules  $W_{n,z}$  can be regarded as the *prime* simple objects in the tensor category  $\text{mod } U_q(L\mathfrak{sl}_2)$ .

Similarly, for general  $\mathfrak{g}$  one would like to ask

**Problem 1.2.** *Find the prime simple objects of  $\text{mod } U_q(L\mathfrak{g})$ , and describe the prime tensor factorization of the simple objects.*

Both problems are quite hard, and we can only offer partial solutions. An interesting feature is that, in both situations, cluster algebras provide the natural combinatorial framework to work with.

## 2. Cluster algebras

Cluster algebras were invented by Fomin and Zelevinsky [16] as an abstraction of certain combinatorial structures which they had previously discovered while studying total positivity in semisimple algebraic groups. A nice introduction [14] to these ideas is given in these proceedings, with many references to the growing literature on the subject.

A cluster algebra is a commutative ring with a distinguished set of generators and a particular type of relations. Although there can be infinitely many generators and relations, they are all obtained from a finite number of them by means of an inductive procedure called *mutation*.

Let us recall the definition.<sup>1</sup> We start with the field of rational functions  $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$ . A *seed* in  $\mathcal{F}$  is a pair  $\Sigma = (\mathbf{y}, Q)$ , where  $\mathbf{y} = (y_1, \dots, y_n)$  is a free generating set of  $\mathcal{F}$ , and  $Q$  is a quiver (*i.e.* an oriented graph) with vertices labelled by  $\{1, \dots, n\}$ . We assume that  $Q$  has neither loops nor 2-cycles. For  $k = 1, \dots, n$ , one defines a new seed  $\mu_k(\Sigma)$  as follows. First  $\mu_k(y_i) = y_i$  for  $i \neq k$ , and

$$\mu_k(y_k) = \frac{\prod_{i \rightarrow k} y_i + \prod_{k \rightarrow j} y_j}{y_k}, \quad (1)$$

where the first (*resp.* second) product is over all arrows of  $Q$  with target (*resp.* source)  $k$ . Next  $\mu_k(Q)$  is obtained from  $Q$  by

- (a) adding a new arrow  $i \rightarrow j$  for every existing pair of arrows  $i \rightarrow k$  and  $k \rightarrow j$ ;
- (b) reversing the orientation of every arrow with target or source equal to  $k$ ;
- (c) erasing every pair of opposite arrows possibly created by (a).

It is easy to check that  $\mu_k(\Sigma)$  is a seed, and  $\mu_k(\mu_k(\Sigma)) = \Sigma$ . The *mutation class*  $\mathcal{C}(\Sigma)$  is the set of all seeds obtained from  $\Sigma$  by a finite sequence of mutations  $\mu_k$ . One can think of the elements of  $\mathcal{C}(\Sigma)$  as the vertices of an  $n$ -regular tree in which every edge stands for a mutation. If  $\Sigma' = ((y'_1, \dots, y'_n), Q')$  is a seed in  $\mathcal{C}(\Sigma)$ , then the subset  $\{y'_1, \dots, y'_n\}$  is called a *cluster*, and its elements are called *cluster variables*. Now, Fomin and Zelevinsky define the *cluster algebra*  $\mathcal{A}_\Sigma$  as the subring of  $\mathcal{F}$  generated by all cluster variables. Some important elements of  $\mathcal{A}_\Sigma$  are the *cluster monomials*, *i.e.* monomials in the cluster variables supported on a single cluster.

For instance, if  $n = 2$  and  $\Sigma = ((x_1, x_2), Q)$ , where  $Q$  is the quiver with  $a$  arrows from 1 to 2, then  $\mathcal{A}_\Sigma$  is the subring of  $\mathbb{Q}(x_1, x_2)$  generated by the rational functions  $x_k$  defined recursively by

$$x_{k+1}x_{k-1} = 1 + x_k^a, \quad (k \in \mathbb{Z}). \quad (2)$$

The clusters of  $\mathcal{A}_\Sigma$  are the subsets  $\{x_k, x_{k+1}\}$ , and the cluster monomials are the special elements of the form

$$x_k^l x_{k+1}^m, \quad (k \in \mathbb{Z}, l, m \in \mathbb{N}).$$

It turns out that when  $a = 1$ , there are only five different clusters and cluster variables, namely

$$x_{5k+1} = x_1, \quad x_{5k+2} = x_2, \quad x_{5k+3} = \frac{1+x_2}{x_1}, \quad x_{5k+4} = \frac{1+x_1+x_2}{x_1x_2}, \quad x_{5k} = \frac{1+x_1}{x_2}.$$

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<sup>1</sup>For simplicity we only consider a particular subclass of cluster algebras: the antisymmetric cluster algebras of geometric type. This is sufficient for our purpose.

For  $a \geq 2$  though, the sequence  $(x_k)$  is no longer periodic and  $\mathcal{A}_\Sigma$  has infinitely many cluster variables.

The first deep results of this theory shown by Fomin and Zelevinsky are:

**Theorem 2.1** ([16],[17]). (i) *Every cluster variable of  $\mathcal{A}_\Sigma$  is a Laurent polynomial with coefficients in  $\mathbb{Z}$  in the cluster variables of any single fixed cluster.*

(ii)  *$\mathcal{A}_\Sigma$  has finitely many clusters if and only if the mutation class  $\mathcal{C}(\Sigma)$  contains a seed whose quiver is an orientation of a Dynkin diagram of type  $A, D, E$ .*

One important open problem [16] is to prove that the coefficients of the Laurent polynomials in (i) are always positive. In §9 below, we give a (conjectural) representation-theoretical explanation of this positivity for a certain class of cluster algebras. More positivity results, based on combinatorial or geometric descriptions of these coefficients, have been obtained by Musiker, Schiffler and Williams [48], and by Nakajima [52].

### 3. The cluster structure of $\mathbb{C}[N]$

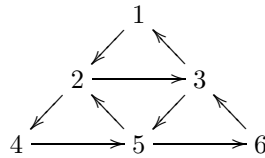
To attack Problem 1.1 we adopt the following strategy. We endow  $\mathbb{C}[N]$  with the structure of a cluster algebra<sup>2</sup>. Then we show that all cluster monomials belong to  $\mathcal{S}^*$ , and therefore we obtain a large family of elements of  $\mathcal{S}^*$  which can be calculated by the combinatorial algorithm of mutation.

In [2, §2.6] explicit initial seeds for a cluster algebra structure in the coordinate ring of the big cell of the base affine space  $G/N$  were described. A simple modification yields initial seeds for  $\mathbb{C}[N]$  (see [24]).

For instance, if  $G = SL_4$  and  $N$  is the subgroup of upper unitriangular matrices, one of these seeds is

$$((D_{1,2}, D_{1,3}, D_{12,23}, D_{1,4}, D_{12,34}, D_{123,234}), Q),$$

where  $Q$  is the triangular quiver:



Here, by  $D_{I,J}$  we mean the regular function on  $N$  which associates to a matrix its minor with row-set  $I$  and column-set  $J$ . Moreover, the variables

$$x_4 = D_{1,4}, \quad x_5 = D_{12,34}, \quad x_6 = D_{123,234}$$

are *frozen*, *i.e.* they cannot be mutated, and therefore they belong to every cluster. Using Theorem 2.1, it is easy to prove that this cluster algebra has finitely many clusters, namely 14 clusters and 12 cluster variables if we count the 3 frozen ones.

<sup>2</sup>Here we mean that  $\mathbb{C}[N] = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}$  for some cluster algebra  $\mathcal{A}$  contained in  $\mathbb{C}[N]$ .

In general however, that is, for groups  $G$  other than  $SL_n$  with  $n \leq 5$ , the cluster structure of  $\mathbb{C}[N]$  has infinitely many cluster variables. To relate the cluster monomials to  $\mathcal{S}^*$  we have to bring the preprojective algebra into the picture.

## 4. The preprojective algebra

Let  $\overline{Q}$  denote the quiver obtained from the Dynkin diagram of  $\mathfrak{g}$  by replacing every edge by a pair  $(\alpha, \alpha^*)$  of opposite arrows. Consider the element

$$\rho = \sum (\alpha\alpha^* - \alpha^*\alpha)$$

of the path algebra  $\mathbb{C}\overline{Q}$  of  $\overline{Q}$ , where the sum is over all pairs of opposite arrows. Following [29, 53], we define the *preprojective algebra*  $\Lambda$  as the quotient of  $\mathbb{C}\overline{Q}$  by the two-sided ideal generated by  $\rho$ . This is a finite-dimensional selfinjective algebra, with infinitely many isomorphism classes of indecomposable modules, except if  $\mathfrak{g}$  has type  $A_n$  with  $n \leq 4$ . It is remarkable that these few exceptional cases coincide precisely with the cases when  $\mathbb{C}[N]$  has finitely many cluster variables. Moreover, it is a nice exercise to verify that the number of indecomposable  $\Lambda$ -modules is then equal to the number of cluster variables.

This suggests a close relationship in general between  $\Lambda$  and  $\mathbb{C}[N]$ . To describe it we start with Lusztig's Lagrangian construction of the enveloping algebra  $U(\mathfrak{n})$  [46, 47]. This is a realization of  $U(\mathfrak{n})$  as an algebra of  $\mathbb{C}$ -valued constructible functions over the varieties of representations of  $\Lambda$ .

To be more precise, we need to introduce more notation. Let  $S_i$  ( $1 \leq i \leq n$ ) be the one-dimensional  $\Lambda$ -modules attached to the vertices  $i$  of  $\overline{Q}$ . Given a sequence  $\mathbf{i} = (i_1, \dots, i_d)$  and a  $\Lambda$ -module  $X$  of dimension  $d$ , we introduce the variety  $\mathcal{F}_{X, \mathbf{i}}$  of flags of submodules

$$\mathfrak{f} = (0 = F_0 \subset F_1 \subset \dots \subset F_d = X)$$

such that  $F_k/F_{k-1} \cong S_{i_k}$  for  $k = 1, \dots, d$ . This is a projective variety. Denote by  $\Lambda_{\mathbf{d}}$  the variety of  $\Lambda$ -modules  $X$  with a given dimension vector  $\mathbf{d} = (d_i)$ , where  $\sum_i d_i = d$ . Consider the constructible function  $\chi_{\mathbf{i}}$  on  $\Lambda_{\mathbf{d}}$  given by

$$\chi_{\mathbf{i}}(X) = \chi(\mathcal{F}_{X, \mathbf{i}})$$

where  $\chi$  denotes the Euler-Poincaré characteristic. Let  $\mathcal{M}_{\mathbf{d}}$  be the  $\mathbb{C}$ -vector space spanned by the functions  $\chi_{\mathbf{i}}$  for all possible sequences  $\mathbf{i}$  of length  $d$ , and let

$$\mathcal{M} = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} \mathcal{M}_{\mathbf{d}}.$$

Lusztig has endowed  $\mathcal{M}$  with an associative multiplication which formally resembles a convolution product, and he has shown that, if we denote by  $e_i$  the Chevalley generators of  $\mathfrak{n}$ , there is an algebra isomorphism  $U(\mathfrak{n}) \xrightarrow{\sim} \mathcal{M}$  mapping the product  $e_{i_1} \cdots e_{i_d}$  to  $\chi_{\mathbf{i}}$  for every  $\mathbf{i} = (i_1, \dots, i_d)$ .

Now, following [22, 23], we dualize the picture. Every  $X \in \text{mod } \Lambda$  determines a linear form  $\delta_X$  on  $\mathcal{M}$  given by

$$\delta_X(f) = f(X), \quad (f \in \mathcal{M}).$$

Using the isomorphisms  $\mathcal{M}^* \simeq U(\mathfrak{n})^* \simeq \mathbb{C}[N]$ , the form  $\delta_X$  corresponds to an element  $\varphi_X$  of  $\mathbb{C}[N]$ , and we have thus attached to every object  $X$  in  $\text{mod } \Lambda$  a polynomial function  $\varphi_X$  on  $N$ .

For example, if  $\mathfrak{g}$  is of type  $A_3$ , and if we denote by  $P_i$  the projective cover of  $S_i$  in  $\text{mod } \Lambda$ , one has

$$\varphi_{P_1} = D_{123,234}, \quad \varphi_{P_2} = D_{12,34}, \quad \varphi_{P_3} = D_{1,4}.$$

More generally, the functions  $\varphi_X$  corresponding to the 12 indecomposable  $\Lambda$ -modules are the 12 cluster variables of  $\mathbb{C}[N]$ .

Via the correspondence  $X \mapsto \varphi_X$  the ring  $\mathbb{C}[N]$  can be regarded as a kind of Hall algebra of the category  $\text{mod } \Lambda$ . Indeed the multiplication of  $\mathbb{C}[N]$  encodes extensions in  $\text{mod } \Lambda$ , as shown by the following crucial result. Before stating it, we recall that  $\text{mod } \Lambda$  possesses a remarkable symmetry with respect to extensions, namely,  $\text{Ext}_\Lambda^1(X, Y)$  is isomorphic to the dual of  $\text{Ext}_\Lambda^1(Y, X)$  functorially in  $X$  and  $Y$  (see [10, 25]). In particular  $\dim \text{Ext}_\Lambda^1(X, Y) = \dim \text{Ext}_\Lambda^1(Y, X)$  for every  $X, Y$ .

**Theorem 4.1** ([22, 25]). *Let  $X, Y \in \text{mod } \Lambda$ .*

- (i) *We have  $\varphi_X \varphi_Y = \varphi_{X \oplus Y}$ .*
- (ii) *Assume that  $\dim \text{Ext}_\Lambda^1(X, Y) = 1$ , and let*

$$0 \rightarrow X \rightarrow L \rightarrow Y \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Y \rightarrow M \rightarrow X \rightarrow 0$$

*be non-split short exact sequences. Then  $\varphi_X \varphi_Y = \varphi_L + \varphi_M$ .*

In fact [25] contains a formula for  $\varphi_X \varphi_Y$  valid for any dimension of  $\text{Ext}_\Lambda^1(X, Y)$ , but we will not need it here. As a simple example of (ii) in type  $A_2$ , one can take  $X = S_1$  and  $Y = S_2$ . Then we have the non-split short exact sequences

$$0 \rightarrow S_1 \rightarrow P_2 \rightarrow S_2 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow S_2 \rightarrow P_1 \rightarrow S_1 \rightarrow 0,$$

which imply the relation  $\varphi_{S_1} \varphi_{S_2} = \varphi_{P_2} + \varphi_{P_1}$ , that is, the elementary determinantal relation  $D_{1,2} D_{2,3} = D_{1,3} + D_{12,23}$  on the unitriangular subgroup of  $SL_3$ . More generally, the short Plücker relations in  $SL_{n+1}$  can be obtained as instances of (ii).

We note that Theorem 4.1 is the analogue for  $\text{mod } \Lambda$  of a formula of Caldero and Keller [6] for the cluster categories introduced by Buan, Marsh, Reineke, Reiten and Todorov [4] to model cluster algebras with an acyclic seed. Cluster categories are not abelian, but Keller [40] has shown that they are triangulated, so in this setting exact sequences are replaced by distinguished triangles.

## 5. The dual semicanonical basis $\mathcal{S}^*$

We can now introduce the basis  $\mathcal{S}^*$  of the vector space  $\mathbb{C}[N]$ . Let  $\mathbf{d} = (d_i)$  be a dimension vector. The variety  $\overline{E}_{\mathbf{d}}$  of representations of  $\overline{\mathbb{C}Q}$  with dimension vector  $\mathbf{d}$  is a vector space of dimension  $2 \sum d_i d_j$ , where the sum is over all pairs  $\{i, j\}$  of vertices of the Dynkin diagram which are joined by an edge. This vector space has a natural symplectic structure. Lusztig [46] has shown that  $\Lambda_{\mathbf{d}}$  is a Lagrangian subvariety of  $\overline{E}_{\mathbf{d}}$ , and that the number of its irreducible components is equal to the dimension of the degree  $\mathbf{d}$  homogeneous component of  $U(\mathfrak{n})$  (for the standard  $\mathbb{N}^n$ -grading given by the Chevalley generators). Let  $\mathcal{Z}$  be an irreducible component of  $\Lambda_{\mathbf{d}}$ . Since the map  $\varphi : X \mapsto \varphi_X$  is a constructible map on  $\Lambda_{\mathbf{d}}$ , it is constant on a Zariski open subset of  $\mathcal{Z}$ . Let  $\varphi_{\mathcal{Z}}$  denote this generic value of  $\varphi$  on  $\mathcal{Z}$ . Then, if we denote by  $\mathcal{I} = \sqcup_{\mathbf{d}} \mathcal{I}_{\mathbf{d}}$  the collection of all irreducible components of all varieties  $\Lambda_{\mathbf{d}}$ , one can easily check that

$$\mathcal{S}^* = \{\varphi_{\mathcal{Z}} \mid \mathcal{Z} \in \mathcal{I}\}$$

is dual to the basis  $\mathcal{S} = \{f_{\mathcal{Z}} \mid \mathcal{Z} \in \mathcal{I}\}$  of  $\mathcal{M} \cong U(\mathfrak{n})$  constructed by Lusztig in [47], and called by him the semicanonical basis.

For example, if  $\mathfrak{g}$  is of type  $A_n$  and  $N$  is the unitriangular subgroup in  $SL_{n+1}$ , then all the matrix minors  $D_{I,J}$  which do not vanish identically on  $N$  belong to  $\mathcal{S}^*$  [22]. They are of the form  $\varphi_X$ , where  $X$  is a subquotient of an indecomposable projective  $\Lambda$ -module.

More generally, suppose that  $X$  is a *rigid*  $\Lambda$ -module, *i.e.* that  $\text{Ext}_{\Lambda}^1(X, X) = 0$ . Then  $X$  is a generic point of the unique irreducible component  $\mathcal{Z}$  on which it sits, that is,  $\varphi_X = \varphi_{\mathcal{Z}}$  belongs to  $\mathcal{S}^*$ , so the calculation of  $\varphi_{\mathcal{Z}}$  amounts to evaluating the Euler characteristics  $\chi(\mathcal{F}_{X,\mathbf{i}})$  for every  $\mathbf{i}$  (of course only finitely many varieties  $\mathcal{F}_{X,\mathbf{i}}$  are non-empty). Thus in type  $A_3$ , the nilpotent endomorphism  $X$  of §1 can be regarded as a rigid  $\Lambda$ -module with dimension vector  $\mathbf{d} = (1, 2, 1)$ , and the connected components of  $\mathcal{F}_X$  are just the non-trivial varieties  $\mathcal{F}_{X,\mathbf{i}}$ , namely

$$\mathcal{F}_{X,(2,1,2,3)}, \quad \mathcal{F}_{X,(2,3,2,1)}, \quad \mathcal{F}_{X,(2,2,1,3)}, \quad \mathcal{F}_{X,(2,2,3,1)}.$$

Note however that if  $\mathfrak{g}$  is not of type  $A_n$  ( $n \leq 4$ ), there exist irreducible components  $\mathcal{Z} \in \mathcal{I}$  whose generic points are not rigid  $\Lambda$ -modules.

## 6. Rigid $\Lambda$ -modules

Let  $r$  be the number of positive roots of  $\mathfrak{g}$ . Equivalently  $r$  is the dimension of the affine space  $N$ . This is also the number of elements of every cluster of  $\mathbb{C}[N]$  (if we include the frozen variables). Geiss and Schröer have shown [28] that the number of pairwise non-isomorphic indecomposable direct summands of a rigid  $\Lambda$ -module is bounded above by  $r$ . A rigid module with  $r$  non-isomorphic indecomposable summands is called *maximal*. We will now see that the seeds of the cluster structure of  $\mathbb{C}[N]$  come from maximal rigid  $\Lambda$ -modules.

Let  $T = T_1 \oplus \cdots \oplus T_r$  be a maximal rigid module, where every  $T_i$  is indecomposable. Define  $B = \text{End}_\Lambda T$ , a basic finite-dimensional algebra with simple modules  $s_i$  ( $1 \leq i \leq r$ ). Denote by  $\Gamma_T$  the quiver of  $B$ , that is, the quiver with vertex set  $\{1, \dots, r\}$  and  $d_{ij}$  arrows from  $i$  to  $j$ , where  $d_{ij} = \dim \text{Ext}_B^1(s_i, s_j)$ .

**Theorem 6.1** ([23]). *The quiver  $\Gamma_T$  has no loops nor 2-cycles.*

Define  $\Sigma(T) = ((\varphi_{T_1}, \dots, \varphi_{T_r}), \Gamma_T)$ .

**Theorem 6.2** ([24]). *There exists an explicit maximal rigid  $\Lambda$ -module  $U$  such that  $\Sigma(U)$  is one of the seeds of the cluster structure of  $\mathbb{C}[N]$ .*

Let us now lift the notion of seed mutation to the category  $\text{mod } \Lambda$ .

**Theorem 6.3** ([23]). *Let  $T_k$  be a non-projective indecomposable summand of  $T$ . There exists a unique indecomposable module  $T_k^* \not\cong T_k$  such that  $(T/T_k) \oplus T_k^*$  is maximal rigid.*

We call  $(T/T_k) \oplus T_k^*$  the *mutation of  $T$  in direction  $k$* , and denote it by  $\mu_k(T)$ . The proof of the next theorem relies among other things on Theorem 4.1.

**Theorem 6.4** ([23]). (i) *We have  $\Sigma(\mu_k(T)) = \mu_k(\Sigma(T))$ , where in the right-hand side  $\mu_k$  stands for the Fomin-Zelevinsky seed mutation.*

(ii) *The map  $T \mapsto \Sigma(T)$  gives a one-to-one correspondence between the maximal rigid modules in the mutation class of  $U$  and the clusters of  $\mathbb{C}[N]$ .*

It follows immediately that the cluster monomials of  $\mathbb{C}[N]$  belong to  $\mathcal{S}^*$ . Indeed, by (ii) every cluster monomial is of the form

$$\varphi_{T_1}^{a_1} \cdots \varphi_{T_r}^{a_r} = \varphi_{T_1^{a_1} \oplus \cdots \oplus T_r^{a_r}}, \quad (a_1, \dots, a_r \in \mathbb{N}),$$

for some maximal rigid module  $T = T_1 \oplus \cdots \oplus T_r$ , and therefore belongs to  $\mathcal{S}^*$  because  $T_1^{a_1} \oplus \cdots \oplus T_r^{a_r}$  is rigid.

Thus the cluster monomials form a large subset of  $\mathcal{S}^*$  which can (in principle) be calculated algorithmically by iterating the seed mutation algorithm from an explicit initial seed. This is our partial answer to Problem 1.1.

Of course, these results also give a better understanding of the cluster structure of  $\mathbb{C}[N]$ . For instance they show immediately that the cluster monomials are linearly independent (a general conjecture of Fomin and Zelevinsky). Furthermore, they suggest the definition of new cluster algebra structures on the coordinate rings of unipotent radicals of parabolic subgroups of  $G$ , obtained in a similar manner from some appropriate Frobenius subcategories of  $\text{mod } \Lambda$  (see [26]). One can also develop an analogous theory for finite-dimensional unipotent subgroups  $N(w)$  of a Kac-Moody group attached to elements  $w$  of its Weyl group (see [3, 27]).

## 7. Finite-dimensional representations of $U_q(\mathbf{Lg})$

We now turn to Problem 1.2. We need to recall some known facts about the category  $\text{mod } U_q(\mathbf{Lg})^3$  of finite-dimensional modules over  $U_q(\mathbf{Lg})$ .

<sup>3</sup>We only consider modules of type 1, a mild technical condition, see e.g. [8, §12.2 B].



By construction,  $U_q(L\mathfrak{g})$  contains a copy of  $U_q(\mathfrak{g})$ , so in a sense the representation theory of  $U_q(L\mathfrak{g})$  is a refinement of that of  $U_q(\mathfrak{g})$ . Let  $\varpi_i$  ( $1 \leq i \leq n$ ) be the fundamental weights of  $\mathfrak{g}$ , and denote by

$$P = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i, \quad P_+ = \bigoplus_{i=1}^n \mathbb{N}\varpi_i,$$

the weight lattice and the monoid of dominant integral weights. It is well known that  $\text{mod } U_q(\mathfrak{g})$  is a semisimple tensor category, with simple objects  $L(\lambda)$  parametrized by  $\lambda \in P_+$ . In fact, every  $M \in \text{mod } U_q(\mathfrak{g})$  has a decomposition

$$M = \bigoplus_{\mu \in P} M_\mu \tag{3}$$

into eigenspaces for a commutative subalgebra  $A$  of  $U_q(\mathfrak{g})$  coming from a Cartan subalgebra of  $\mathfrak{g}$ . One shows that if  $M$  is irreducible, the highest weight occurring in (3) is a dominant weight  $\lambda$ ,  $\dim M_\lambda = 1$ , and there is a unique simple  $U_q(\mathfrak{g})$ -module with these properties, hence the notation  $M = L(\lambda)$ . For an arbitrary  $M \in \text{mod } U_q(\mathfrak{g})$ , the formal sum

$$\chi(M) = \sum_{\mu \in P} \dim M_\mu e^\mu$$

is called the *character* of  $M$ , since it characterizes  $M$  up to isomorphism.

When dealing with representations of  $U_q(L\mathfrak{g})$  one needs to introduce spectral parameters  $z \in \mathbb{C}^*$ , and therefore  $P$  and  $P_+$  have to be replaced by

$$\widehat{P} = \bigoplus_{1 \leq i \leq n, z \in \mathbb{C}^*} \mathbb{Z}(\varpi_i, z), \quad \widehat{P}_+ = \bigoplus_{1 \leq i \leq n, z \in \mathbb{C}^*} \mathbb{N}(\varpi_i, z).$$

It was shown by Chari and Pressley [7, 9] that finite-dimensional irreducible representations of  $U_q(L\mathfrak{g})$  were similarly determined by their highest  $l$ -weight  $\widehat{\lambda} \in \widehat{P}_+$  (where  $l$  stands for “loop”). This comes from the existence of a large commutative subalgebra  $\widehat{A}$  of  $U_q(L\mathfrak{g})$  containing  $A$ . If  $M \in \text{mod } U_q(L\mathfrak{g})$  is regarded as a  $U_q(\mathfrak{g})$ -module by restriction and decomposed as in (3), then every  $U_q(\mathfrak{g})$ -weight-space  $M_\mu$  has a finer decomposition into generalized eigenspaces for  $\widehat{A}$

$$M_\mu = \bigoplus_{\widehat{\mu} \in \widehat{P}} M_{\widehat{\mu}}$$

where the  $\widehat{\mu} = \sum_k m_{i_k}(\varpi_{i_k}, z_k)$  in the right-hand side all satisfy  $\sum_k m_{i_k} \varpi_{i_k} = \mu$ . The corresponding formal sum

$$\chi_q(M) = \sum_{\widehat{\mu} \in \widehat{P}} \dim M_{\widehat{\mu}} e^{\widehat{\mu}}$$

has been introduced by Frenkel and Reshetikhin [20] and called by them the  $q$ -character of  $M$ . It characterizes the class of  $M$  in the Grothendieck ring of

$\text{mod } U_q(L\mathfrak{g})$ , but one should be warned that this is not a semisimple category, so this is much coarser than an isomorphism class.

For instance, the 4-dimensional irreducible representation  $V_3$  of  $U_q(\mathfrak{sl}_2)$  with highest weight  $\lambda = 3\varpi_1$  has character

$$\chi(V_3) = Y^3 + Y^1 + Y^{-1} + Y^{-3}$$

if we set  $Y = e^{\varpi_1}$ . There is a family  $W_{3,z} \in \text{mod } U_q(L\mathfrak{sl}_2)$  of affine analogues of  $V_3$ , parametrized by  $z \in \mathbb{C}^*$ , whose  $q$ -character is given by

$$\chi_q(W_{3,z}) = Y_z Y_{zq^2} Y_{zq^4} + Y_z Y_{zq^2} Y_{zq^6}^{-1} + Y_z Y_{zq^4} Y_{zq^6}^{-1} + Y_{zq^2}^{-1} Y_{zq^4}^{-1} Y_{zq^6}^{-1},$$

where we write  $Y_a = e^{(\varpi_1, a)}$  for  $a \in \mathbb{C}^*$ . Thus  $W_{3,z}$  has highest  $l$ -weight

$$\widehat{\lambda} = (\varpi_1, z) + (\varpi_1, zq^2) + (\varpi_1, zq^4).$$

The reader can easily imagine what is the general expression of  $\chi_q(W_{n,z})$  for any  $(n, z) \in \mathbb{N} \times \mathbb{C}^*$ . It follows that there is a closed formula for the  $q$ -character of every finite-dimensional irreducible  $U_q(L\mathfrak{sl}_2)$ -module since, as already mentioned, every such module factorizes as a tensor product of  $W_{n_i, z_i}$  and the factors are given by a simple combinatorial rule [7].

The situation is far more complicated in general. In particular it is not always possible to endow an irreducible  $U_q(\mathfrak{g})$ -module with the structure of a  $U_q(L\mathfrak{g})$ -module. The only general description of  $q$ -characters of simple  $U_q(L\mathfrak{g})$ -modules, due to Ginzburg and Vasserot for type  $A$  [30] and to Nakajima in general [49], uses intersection cohomology of certain moduli spaces of representations of graded pre-projective algebras, called graded quiver varieties. This yields a Kazhdan-Lusztig type algorithm for calculating the irreducible  $q$ -characters [50], but this type of combinatorics does not easily reveal the possible factorizations of the  $q$ -characters.

## 8. The subcategories $\mathcal{C}_\ell$

It can be shown that Problem 1.2 for  $\text{mod } U_q(L\mathfrak{g})$  can be reduced to the same problem for some much smaller tensor subcategories  $\mathcal{C}_\ell$  ( $\ell \in \mathbb{N}$ ) which we shall now introduce.

Denote by  $L(\widehat{\lambda})$  the simple object of  $\text{mod } U_q(L\mathfrak{g})$  with highest  $l$ -weight  $\widehat{\lambda} \in \widehat{P}_+$ . Since the Dynkin diagram of  $\mathfrak{g}$  is a tree, it is a bipartite graph. We denote by  $I = I_0 \sqcup I_1$  the corresponding partition of the set of vertices, and we write  $\xi_i = 0$  (*resp.*  $\xi_i = 1$ ) if  $i \in I_0$  (*resp.*  $i \in I_1$ ). For  $\ell \in \mathbb{N}$ , let

$$\widehat{P}_{+, \ell} = \bigoplus_{1 \leq i \leq n, 0 \leq k \leq \ell} \mathbb{N}(\varpi_i, q^{\xi_i + 2k}).$$

We then define  $\mathcal{C}_\ell$  as the full subcategory of  $\text{mod } U_q(L\mathfrak{g})$  whose objects  $M$  have all their composition factors of the form  $L(\widehat{\lambda})$  with  $\widehat{\lambda} \in \widehat{P}_{+, \ell}$ . It is not difficult to

prove [33] that  $\mathcal{C}_\ell$  is a tensor subcategory, and that its Grothendieck ring  $K_0(\mathcal{C}_\ell)$  is the polynomial ring in the  $n(\ell + 1)$  classes of fundamental modules

$$[L(\varpi_i, q^{\xi_i + 2k})], \quad (1 \leq i \leq n, 0 \leq k \leq \ell).$$

For example, let  $W_{j,a}^{(i)}$  denote the simple object of  $\text{mod } U_q(L\mathfrak{g})$  with highest  $l$ -weight

$$(\varpi_i, a) + (\varpi_i, aq^2) + \cdots + (\varpi_i, aq^{2j-2}), \quad (i \in I, j \in \mathbb{N}^*, a \in \mathbb{C}^*),$$

a so-called *Kirillov-Reshetikhin module*. The  $q$ -characters of the Kirillov-Reshetikhin modules satisfy a nice system of recurrence relations, called  $T$ -system in the physics literature, which allows to calculate them inductively in terms of the  $q$ -characters of the fundamental modules  $L(\varpi_i, a)$ . This was conjectured by Kuniba, Nakanishi and Suzuki [44], and proved by Nakajima [51] (see also [31] for the non simply-laced cases). The  $q$ -characters of the fundamental modules can in turn be calculated by means of the Frenkel-Mukhin algorithm [19]. One should therefore regard the Kirillov-Reshetikhin modules as the most “accessible” simple  $U_q(L\mathfrak{g})$ -modules. There are  $n(\ell + 1)(\ell + 2)/2$  such modules in  $\mathcal{C}_\ell$ , namely:

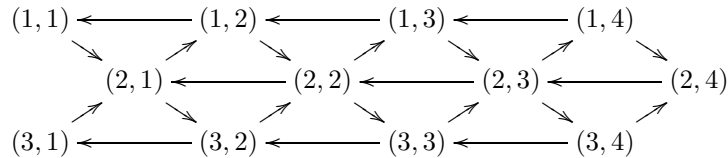
$$W_{j,q^{\xi_i + 2k}}^{(i)}, \quad (i \in I, 0 < j \leq \ell + 1, 0 \leq k \leq \ell + 1 - j).$$

## 9. The cluster algebras $\mathcal{A}_\ell$

Let  $Q$  denote the quiver obtained by orienting the Dynkin diagram of  $\mathfrak{g}$  so that every  $i \in I_0$  (*resp.*  $i \in I_1$ ) is a source (*resp.* a sink). We define a new quiver  $\Gamma_\ell$  with vertex set  $\{(i, k) \mid i \in I, 1 \leq k \leq \ell + 1\}$ . There are three types of arrows

- (a) arrows  $(i, k) \rightarrow (j, k)$  for every arrow  $i \rightarrow j$  in  $Q$  and every  $1 \leq k \leq \ell + 1$ ;
- (b) arrows  $(j, k) \rightarrow (i, k + 1)$  for every arrow  $i \rightarrow j$  in  $Q$  and every  $1 \leq k \leq \ell$ ;
- (c) arrows  $(i, k) \leftarrow (i, k + 1)$  for every  $i \in I$  and every  $1 \leq k \leq \ell$ .

For example, if  $\mathfrak{g}$  has type  $A_3$  and  $I_0 = \{1, 3\}$ , the quiver  $\Gamma_3$  is:



Let  $\mathbf{x} = \{x_{(i,k)} \mid i \in I, 1 \leq k \leq \ell + 1\}$  be a set of indeterminates corresponding to the vertices of  $\Gamma_\ell$ , and consider the seed  $(\mathbf{x}, \Gamma_\ell)$  in which the  $n$  variables  $x_{(i,\ell+1)}$  ( $i \in I$ ) are frozen. This is the initial seed of a cluster algebra  $\mathcal{A}_\ell \subset \mathbb{Q}(\mathbf{x})$ .

By Theorem 2.1, if  $\mathfrak{g}$  has type  $A_1$  then  $\mathcal{A}_\ell$  has finite cluster type  $A_\ell$ . Also, if  $\ell = 1$ ,  $\mathcal{A}_\ell$  has finite cluster type equal to the Dynkin type of  $\mathfrak{g}$ . Otherwise, except for a few small rank cases,  $\mathcal{A}_\ell$  has infinitely many cluster variables.

Our partial conjectural solution of Problem 1.2 can be summarized as follows (see [33] for more details):

**Conjecture 9.1.** *There is a ring isomorphism  $\iota_\ell : \mathcal{A}_\ell \xrightarrow{\sim} K_0(\mathcal{C}_\ell)$  such that*

$$\iota_\ell(x_{(i,k)}) = \left[ W_{k, q^{\varepsilon_i + 2(\ell + 1 - k)}}^{(i)} \right], \quad (i \in I, 1 \leq k \leq \ell + 1).$$

*The images by  $\iota_\ell$  of the cluster variables are classes of prime simple modules, and the images of the cluster monomials are the classes of all real simple modules in  $\mathcal{C}_\ell$ , i.e. those simple modules whose tensor square is simple.*

Thus, if true, Conjecture 9.1 gives a combinatorial description in terms of cluster algebras of the prime tensor factorization of every real simple module. Note that, by definition, the square of a cluster monomial is again a cluster monomial. This explains why cluster monomials can only correspond to real simple modules. For  $\mathfrak{g} = \mathfrak{sl}_2$ , all simple  $U_q(L\mathfrak{g})$ -modules are real. However for  $\mathfrak{g} \neq \mathfrak{sl}_2$  there exist *imaginary* simple  $U_q(L\mathfrak{g})$ -modules (i.e. simple modules whose tensor square is not simple), as shown in [45]. This is consistent with the expectation that a cluster algebra with infinitely many cluster variables is not spanned by its set of cluster monomials.

We arrived at Conjecture 9.1 by noting that the  $T$ -system equations satisfied by Kirillov-Reshetikhin modules are of the same form as the cluster exchange relations. This was inspired by the seminal work of Fomin and Zelevinsky [15], in which cluster algebra combinatorics is used to prove Zamolodchikov's periodicity conjecture for  $Y$ -systems attached to Dynkin diagrams. Kedem [39] and Di Francesco [13], Keller [41, 42], Inoue, Iyama, Kuniba, Nakanishi and Suzuki [34], have also exploited the similarity between cluster exchange relations and other types of functional equations arising in mathematical physics ( $Q$ -systems, generalized  $T$ -systems,  $Y$ -systems attached to pairs of simply-laced Dynkin diagrams). Recently, Inoue, Iyama, Keller, Kuniba and Nakanishi [35, 36] have obtained a proof of the periodicity conjecture for all  $T$ -systems and  $Y$ -systems attached to a non simply-laced quantum affine algebra.

As evidence for Conjecture 9.1, we can easily check that for  $\mathfrak{g} = \mathfrak{sl}_2$  and any  $\ell \in \mathbb{N}$ , it follows from the results of Chari and Pressley [7]. On the other hand, for arbitrary  $\mathfrak{g}$  we have:

**Theorem 9.2** ([33, 52]). *Conjecture 9.1 holds for  $\mathfrak{g}$  of type  $A, D, E$  and  $\ell = 1$ .*

This was first proved in [33] for type  $A$  and  $D_4$  by combinatorial and representation-theoretic methods, and soon after, by Nakajima [52] in the general case, by using the geometric description of the simple  $U_q(L\mathfrak{g})$ -modules. In both approaches, a crucial part of the proof can be summarized in the following chart:

$$\begin{array}{ccc} F\text{-polynomials} & \leftrightarrow & \text{quiver Grassmannians} \\ \updownarrow & & \updownarrow \\ q\text{-characters} & \leftrightarrow & \text{Nakajima quiver varieties} \end{array}$$

Here, the  $F$ -polynomials are certain polynomials introduced by Fomin and Zelevinsky [18] which allow to calculate the cluster variables in terms of a fixed initial seed. By work of Caldero-Chapoton [5], Fu-Keller [21] and Derksen-Weyman-Zelevinsky [11, 12],  $F$ -polynomials have a geometric description via Grassmannians of subrepresentations of some quiver representations attached to cluster variables: this is the upper horizontal arrow of our diagram. The lower horizontal arrow refers to the already mentioned relation between irreducible  $q$ -characters and perverse sheaves on quiver varieties established by Nakajima [49, 50]. In [33] we have shown that the  $F$ -polynomials for  $\mathcal{A}_1$  are equal to certain natural truncations of the corresponding irreducible  $q$ -characters of  $\mathcal{C}_1$  (the left vertical arrow), and we observed that this yielded an alternative geometric description of these  $q$ -characters in terms of ordinary homology of quiver Grassmannians. In [52] Nakajima used a Deligne-Fourier transform to obtain a direct relation between perverse sheaves on quiver varieties for  $\mathcal{C}_1$  and homology of quiver Grassmannians (the right vertical arrow), and deduced from it the desired connection with the cluster algebra  $\mathcal{A}_1$ .

The other main step in the approach of [33] is a certain tensor product theorem for the category  $\mathcal{C}_1$ . It states that a tensor product  $S_1 \otimes \cdots \otimes S_k$  of simples objects of  $\mathcal{C}_1$  is simple if and only if  $S_i \otimes S_j$  is simple for every pair  $1 \leq i < j \leq k$ . A generalization of this theorem to the whole category  $\text{mod } U_q(L\mathfrak{g})$  has been recently proved by Hernandez [32]. Note that the theorem of Hernandez is also valid for non simply-laced Lie algebras  $\mathfrak{g}$ , and thus opens the way to a similar treatment of Problem 1.2 in this case.

Conjecture 9.1 has also been checked for  $\mathfrak{g}$  of type  $A_2$  and  $\ell = 2$  [33, §13]. In that small rank case,  $\mathcal{A}_2$  still has finite cluster type  $D_4$ , and this implies that  $\mathcal{C}_2$  has only real objects. There are 18 explicit prime simple objects with respective dimensions

$$3, 3, 3, 3, 3, 3, 6, 6, 6, 6, 8, 8, 8, 10, 10, 15, 15, 35,$$

and 50 factorization patterns (corresponding to the 50 vertices of a generalized associahedron of type  $D_4$  [17]). Our proof in this case is quite indirect and uses a lot of ingredients: the quantum affine Schur-Weyl duality, Ariki's theorem for type  $A$  affine Hecke algebras [1], the coincidence of Lusztig's dual canonical and dual semicanonical bases of  $\mathbb{C}[N]$  in type  $A_4$  [22], and Theorem 6.4.

One remarkable consequence of Theorem 9.2 from the point of view of cluster algebras is that it immediately implies the positivity conjecture of Fomin and Zelevinsky for the cluster algebras  $\mathcal{A}_1$  with respect to any reference cluster (see [33, §2]). Conjecture 9.1 would similarly yield positivity for the whole class of cluster algebras  $\mathcal{A}_\ell$ .

## 10. An intriguing relation

Problem 1.1 and Problem 1.2 may not be as unrelated as it would first seem. For a suggestive example, let us take  $\mathfrak{g}$  of type  $A_3$ . In that case, the abelian category  $\text{mod } \Lambda$  has 12 indecomposable objects (which are all rigid), 3 of them

being projective-injective. On the other hand the tensor category  $\mathcal{C}_1$  has 12 prime simple objects (which are all real), 3 of them having the property that their tensor product with every simple of  $\mathcal{C}_1$  is simple. It is easy to check that  $\mathbb{C}[N]$  and  $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_1)$  are isomorphic as (complexified) cluster algebras with frozen variables. Therefore we have a unique one-to-one correspondence

$$X \leftrightarrow S$$

between rigid objects  $X$  of  $\text{mod } \Lambda$  and simple objects  $S$  of  $\mathcal{C}_1$  such that

$$\varphi_X \equiv [S],$$

that is, such that  $X$  and  $S$  project to the same cluster monomial. In this correspondence, direct sums  $X \oplus X'$  map to tensor products  $S \otimes S'$ . It would be interesting to find a general framework for relating in a similar way, via cluster algebras, certain additive categories such as  $\text{mod } \Lambda$  to certain tensor categories such as  $\mathcal{C}_1$ . We refer to [43] for a very accessible survey of these ideas.

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