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SOMEKAWA'S K-GROUPS AND VOEVODSKY'S HOM GROUPS (PRELIMINARY VERSION)

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ABSTRACT. We construct a surjective homomorphism from Somekawa's K-group associated to a finite collection of semi-abelian varieties over a perfect field to a corresponding Hom group in Voevodsky's triangulated category of effective motivic complexes.

CONTENTS

1. INTRODUCTION

In this note, we construct an epimorphism

$$K(k; G_1, \dots, G_n) \twoheadrightarrow \operatorname{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, G_1[0] \otimes \cdots \otimes G_n[0]) \tag{1.1}$$

where k is a perfect field, $G_1, \dots, G_n$ are semi-abelian k-varieties, the left-hand-side is the abelian group defined by K. Kato and studied by M. Somekawa in [6] and on the right hand side, the tensor product $G_1[0] \otimes \cdots \otimes G_n[0]$ is computed in Voevodsky's triangulated category of effective motivic complexes [11] or alternately in his category of homotopy invariant Nisnevich sheaves with transfers (ibid.). This has been announced in [8, Rk 10 (b)] and is used in [13, Th. 3.9].

I expect (1.1) to be bijective. This would provide an affirmative answer to a version of Somekawa's expectation in the introduction of his paper (probably the closest answer, as long as one does not have

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an abelian category of mixed motives at hand). The method I have in mind to prove injectivity involves defining a group

$$K_{-1}(k; G_1, \dots, G_n)$$

modelled on Voevodsky's construction $(-)_{-1}$ on sheaves. However the construction of this group appears more subtle than I initially thought, so I felt it might be useful to already release this much of the story.

Recall that $K(k; G_1, \dots, G_n)$ is defined as a quotient of a larger group

$$(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(k)$$

where $\overset{M}{\otimes}$ is computed in the category of cohomological Mackey functors [3]. Our strategy will be as follows:

(1) Construct a surjective homomorphism

$$(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(k) \to \operatorname{Hom}_{\mathbf{DM}_-^{\text{eff}}}(\mathbf{Z}, G_1[0] \otimes \dots \otimes G_n[0]). \tag{1.2}$$

This is achieved in §§2.12 and 3.6.

(2) Show that (1.2) factors through the Kato-Somekawa relations, yielding (1.1). This is achieved in Theorem 5.1.

I wish to thank Takao Yamazaki for his interest and encouragement to pursue this work.

2. Mackey functors and presheaves with transfers

2.1. A *Mackey functor* over k is a contravariant additive (i.e., commuting with coproducts) functor A from the category of étale k-schemes to the category of abelian groups, provided with a covariant structure verifying the following exchange condition: if

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{f} & X \end{array}$$

is a cartesian square of étale k-schemes, then the diagram

$$\begin{array}{ccc} A(Y') & \xrightarrow{f'^*} & A(Y) \\ {\scriptstyle g'_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ A(X') & \xrightarrow{f^*} & A(X) \end{array}$$

commutes. Here, * denotes the contravariant structure while $_*$ denotes the covariant structure. The Mackey functor A is *cohomological* if we further have

$$f_* f^* = \deg(f)$$

for any $f : X' \to X$, with X connected. We denote by **Mack** the abelian category of Mackey functors, and by $\mathbf{Mack}_c$ its full subcategory of cohomological Mackey functors.

2.2**.** Classically [9, (1.4)], a Mackey functor may be viewed as a contravariant additive functor on the category **Span** of "spans" on étale k-schemes, defined as follows: objects are étale k-schemes. A morphism from X to Y is an equivalence class of diagram (span)

$$X \xleftarrow{g} Z \xrightarrow{f} Y. \tag{2.1}$$

Composition of spans is defined via fibre product in an obvious manner. If A is a Mackey functor, the corresponding functor on **Span** has the same value on objects, while its value on a span (2.1) is given by $g_* f^*$.

Note that **Span** is a preadditive category: one may add (but not substract) two morphisms with same source and target. We may as well view a Mackey functor as an additive functor on the associated additive category $\mathbf{Z}\,\mathbf{Span}$.

2.3**.** Let **Cor** be Voevodsky's category of finite correspondences on smooth k-schemes, denoted by $SmCor(k)$ in [11, §2.1]. The category $\mathbf{Z}\,\mathbf{Span}$ is isomorphic to its full subcategory consisting of smooth k-schemes of dimension 0 (= étale k-schemes). In particular, any presheaf with transfers in the sense of Voevodsky [11, Def. 3.1.1] restricts to a Mackey functor over k. By [10, Cor. 3.15], the restriction of a *homotopy invariant* presheaf with transfers yields a cohomological Mackey functor. In other words, we have exact functors

$$\rho : \mathbf{PST} \to \mathbf{Mack} \tag{2.2}$$

$$\rho : \mathbf{HI} \to \mathbf{Mack}_c \tag{2.3}$$

where **PST** denotes the category of presheaves with transfers (contravariant additive functors from **Cor** to abelian groups) and **HI** is its full subcategory consisting of homotopy invariant presheaves with transfers.

2.4**.** There is a tensor product of Mackey functors $\overset{M}{\otimes}$, originally defined by L. G. Lewis (unpublished): it extends naturally the symmetric monoidal structure $(X, Y) \mapsto X \times_K Y$ on $\mathbf{Z}\,\mathbf{Span}$ via the additive Yoneda embedding (see §A.6). If either A or B is cohomological, $A \overset{M}{\otimes} B$ is cohomological.

This tensor product is the same as the one defined in [3]: this follows from (A.2) and the fact that $\mathbf{Z}\,\mathbf{Span}$ is rigid, all objects being self-dual (indeed, $\mathbf{Z}\,\mathbf{Span}$ is canonically isomorphic to the category of Artin Chow motives with integral coefficients).

2.5. There is a tensor product on presheaves with transfers defined exactly in the same way [11, p. 236].

2.6. By definition, the functor (2.2) equals i^*, where i is the inclusion $\mathbf{Z}\,\mathbf{Span} \to \mathbf{Cor}$. This inclusion has a left adjoint π_0 (scheme of constants). Both functors i and π_0 are symmetric monoidal: for π_0, reduce to the case where k is separably closed.

2.7. By §§A.2 and A.8, this implies that (2.2) is symmetric monoidal. In other words, if $\mathcal{F}$ and $\mathcal{G}$ are presheaves with transfers, then

$$\rho\mathcal{F} \overset{M}{\otimes} \rho\mathcal{G} \simeq \rho(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.4}$$

2.8. The inclusion functor $\mathbf{HI} \to \mathbf{PST}$ has a left adjoint h_0, and the symmetric monoidal structure of $\mathbf{PST}$ induces one on $\mathbf{HI}$ via h_0. In other words, if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}$, we define

$$\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G} = h_0(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.5}$$

Note that (2.3) is *not* symmetric monoidal (since it is the restriction of (2.2)).

2.9. For any $\mathcal{F} \in \mathbf{PST}$, the unit morphism $\mathcal{F} \to h_0(\mathcal{F})$ induces a surjection

$$\mathcal{F}(k) \to h_0(\mathcal{F})(k).$$

This is obvious from the formula $h_0(\mathcal{F}) = \operatorname{Coker}(C_1(\mathcal{F}) \to \mathcal{F})$.

2.10. We shall also need to work with Nisnevich sheaves with transfers. We denote by $\mathbf{NST}$ the category of Nisnevich sheaves with transfers (objects of $\mathbf{PST}$ which are sheaves in the Nisnevich topology). By [11, Th. 3.1.4], the inclusion functor $\mathbf{NST} \to \mathbf{PST}$ has an exact left adjoint $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}$ (sheafification). The category $\mathbf{NST}$ then inherits a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{NST}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{\mathrm{Nis}}.$$

Similarly, we define $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST}$. The sheafification functor restricts to an exact functor $\mathbf{HI} \to \mathbf{HI}_{\mathrm{Nis}}$ [11, Th. 3.1.11], and $\mathbf{HI}_{\mathrm{Nis}}$ gets a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G})_{\mathrm{Nis}}.$$

To summarise, all functors in the following commutative diagram are symmetric monoidal:

$$\begin{array}{ccc} \mathbf{PST} & \xrightarrow{\mathrm{Nis}} & \mathbf{NST} \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_0^{\mathrm{Nis}}} \\ \mathbf{HI} & \xrightarrow{\mathrm{Nis}} & \mathbf{HI}_{\mathrm{Nis}}\,. \end{array} \tag{2.6}$$

where each functor is left adjoint to the corresponding inclusion.

2.11. Let $\mathcal{F}$ be a presheaf on Sm/k, and let $\mathcal{F}_{\mathrm{Nis}}$ be the associated Nisnevich sheaf. Then we have an isomorphism

$$\mathcal{F}(k) \xrightarrow{\sim} \mathcal{F}_{\mathrm{Nis}}(k). \tag{2.7}$$

Indeed, any covering of the Nisnevich topology on $\operatorname{Spec} k$ refines to a trivial covering. In particular, the functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}(k)$ is exact.

This applies in particular to a presheaf with transfers and the associated Nisnevich sheaf with transfers.

2.12. If G is an abelian k-group scheme whose identity component is a quasi-projective variety, then G has a canonical structure of Nisnevich sheaf with transfers ([7, proof of Lemma 3.2] completed by [1, Lemma 1.3.2]). This applies in particular to semi-abelian varieties. In particular, if $G_1, \dots, G_n$ are semi-abelian varieties, (2.4) yields a canonical isomorphism

$$(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(k) \simeq (G_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} G_n)(k) \tag{2.8}$$

where the G_i are considered on the left as Mackey functors, and on the right as presheaves with transfers.

Since the G_i are semi-abelian varieties, they are homotopy invariant. Therefore, composing (2.8) with the unit morphism $Id \Rightarrow h_0^{\mathrm{Nis}}$ from (2.6) and taking (2.5) into account, we get a canonical morphism

$$(G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_n)(k) \to (G_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G_n)(k). \tag{2.9}$$

which is surjective by §2.9.

3. Preseheaves with transfers and motives

3.1. The left adjoint h_0^{Nis} in (2.6) "extends" to a left adjoint C_* of the inclusion

$$\mathbf{DM}_-^{\mathrm{eff}} \to D^-(\mathbf{NST})$$

where the left hand side is Voevodsky's triangulated category of effective motivic complexes [11, §3, esp. Prop. 3.2.3].

More precisely, $\mathbf{DM}^{\text{eff}}_-$ is defined as the full subcategory of objects of $D^-(\mathbf{NST})$ whose cohomology sheaves are homotopy invariant. The canonical t-structure of $D^-(\mathbf{NST})$ induces a t-structure on $\mathbf{DM}^{\text{eff}}_-$, with heart $\mathbf{HI}_{\text{Nis}}$. The functor C_* is right exact with respect to these t-structures, and if $\mathcal{F} \in \mathbf{NST}$, then $H_0(C_*(\mathcal{F})) = h_0^{\text{Nis}}(\mathcal{F})$.

3.2**.** The tensor structure of §2.10 on $\mathbf{NST}$ extends to one on $D^-(\mathbf{NST})$ [11, p. 206]. Via C_*, this tensor structure descends to a tensor structure on $\mathbf{DM}^{\text{eff}}_-$ [11, p. 210], which will simply be denoted by $\otimes$. The relationship between this tensor structure and the one of §2.10 is as follows: is $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\text{Nis}}$, then

$$\mathcal{F} \otimes_{\mathbf{HI}_{\text{Nis}}} \mathcal{G} = H^0(\mathcal{F}[0] \otimes \mathcal{G}[0]) \tag{3.1}$$

where $\mathcal{F}[0], \mathcal{G}[0]$ are viewed as complexes of Nisnevich sheaves with transfers concentrated in degree 0.

We shall need the following lemma, which is not explicit in [11]:

3.3. **Lemma.** *The tensor product* $\otimes$ *of* $\mathbf{DM}^{\text{eff}}_-$ *is right exact with respect to the homotopy* t*-structure.*

Proof. By definition,

$$C \otimes D = C_*(C \overset{L}{\otimes} D)$$

for $C, D \in \mathbf{DM}^{\text{eff}}_-$, where $\overset{L}{\otimes}$ is the tensor product of $D^-(\mathbf{NST})$ defined in [11, p. 206]. We want to show that, if C and D are concentrated in degrees ≤ 0, then so is $C \otimes D$. Using the canonical left resolutions of loc. cit., it is enough to do it for C and D of the form $C_*(L(X))$ and $C_*(L(Y))$ for two smooth schemes X, Y. Since C_* is symmetric monoidal, we have

$$C_*(L(X)) \otimes C_*(L(Y)) \xleftarrow{\sim} C_*(L(X) \overset{L}{\otimes} L(Y)) = C_*(L(X \times Y))$$

and the claim is obvious in view of the formula for C_* [11, p. 207]. □

3.4**.** Let $C \in \mathbf{DM}^{\text{eff}}_-$. For any $X \in Sm/k$ and any $i \in \mathbf{Z}$, we have

$$\mathbb{H}^i_{\text{Nis}}(X, C) \simeq \operatorname{Hom}_{\mathbf{DM}^{\text{eff}}_-}(M(X), C[i])$$

where $M(X) = C_*(L(X))$ is the motive of X computed in $\mathbf{DM}^{\text{eff}}_-$ (cf. [11, Prop. 3.2.7]).

Specialising to the case $X = \operatorname{Spec} k$ ($M(X) = \mathbf{Z}$) and taking §2.11 into account, we get

$$\operatorname{Hom}_{\mathbf{DM}^{\text{eff}}_-}(\mathbf{Z}, C[i]) \simeq H^i(C)(k). \tag{3.2}$$

Combining (3.1), (2.7) and (3.2), we get:

3.5. **Lemma.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then we have a canonical isomorphism*

$$(\mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n)(k) \simeq \operatorname{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]).$$

□

3.6. Combining Lemma 3.5 with (2.9), we get the announced homomorphism (1.2). However, we shall mainly work with presheaves with transfers in the sequel, hence use (2.9) rather than (1.2).

4. Presheaves with transfers and local symbols

4.1. Given a presheaf with transfers $\mathcal{G}$, recall from [10, p. 96] the presheaf with transfers $\mathcal{G}_{-1}$ defined by the formula

$$\mathcal{G}_{-1}(U) = \operatorname{Coker}\left(\mathcal{G}(U \times \mathbf{A}^1) \to \mathcal{G}(U \times (\mathbf{A}^1 - \{0\}))\right).$$

Suppose that $\mathcal{G}$ is homotopy invariant. Let $X \in Sm/k$ (connected), $K = k(X)$ and $x \in X$ be a point of codimension 1. By [10, Lemma 4.36], there is a canonical isomorphism

$$\mathcal{G}_{-1}(k(x)) \simeq H^1_x(X, \mathcal{G}_{\mathrm{Zar}}) \tag{4.1}$$

yielding a canonical map

$$\partial_x : \mathcal{G}(K) \to \mathcal{G}_{-1}(k(x)). \tag{4.2}$$

The following lemma follows from the construction of the isomorphisms (4.1). It is part of the general fact that $\mathcal{G}$ defines a cycle module in the sense of Rost (cf. [2, Prop. 5.4.64]).

4.2. **Lemma.** *a) Let $f : Y \to X$ be a dominant morphism, with Y smooth and connected. Let $L = k(Y)$, and let $y \in Y^{(1)}$ be such that $f(y) = x$. Then the diagram*

$$\begin{array}{ccc}
\mathcal{G}(L) & \xrightarrow{(\partial_y)} & \mathcal{G}_{-1}(k(y)) \\
{\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle ef^*} \\
\mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x))
\end{array}$$

commutes, where e is the ramification index of v_y relative to v_x.
b) If f is finite surjective, the diagram

$$\begin{array}{ccc}
\mathcal{G}(L) & \xrightarrow{(\partial_y)} & \bigoplus_{y \in f^{-1}(x)} \mathcal{G}_{-1}(k(y)) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathcal{G}(K) & \xrightarrow{\partial_x} & \mathcal{G}_{-1}(k(x))
\end{array}$$

commutes. □

4.3. **Proposition.** *Let* $\mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$. *There is a canonical isomorphism*

$$\mathcal{G}_{-1} = \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{G}).$$

Proof. This may not be the most economic proof, but it is quite short. The statement means that $\mathcal{G}_{-1}$ represents the functor

$$\mathcal{H} \mapsto \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathcal{G}).$$

By [10, Lemma 4.35], we have

$$\mathcal{G}_{-1} = \mathrm{Coker}(\mathcal{G} \to p_* p^* \mathcal{G})$$

where $p : \mathbf{A}^1 - \{0\} \to \operatorname{Spec} k$ is the structural morphism and p_*, p^* are computed with respect to the Zariski topology. By [10, Th. 5.7], we may replace the Zariski topology by the Nisnevich topology. Moreover, by [10, Prop. 5.4 and Prop. 4.20], we have $R^i p_* p^* \mathcal{G} = 0$ for $i > 0$, hence $p_* p^* \mathcal{G}[0] \xrightarrow{\sim} R p_* p^* \mathcal{G}[0]$.

By [11, Prop. 3.2.8], we have

$$R p_* p^* \mathcal{G}[0] = \underline{\mathrm{Hom}}(M(\mathbf{A}^1 - \{0\}), \mathcal{G}[0])$$

where $\underline{\mathrm{Hom}}$ is the (partially defined) internal Hom of $\mathbf{DM}_-^{\mathrm{eff}}$. By [11, Prop. 3.5.4] (Gysin triangle) and homotopy invariance, we have an exact triangle, split by any rational point of $\mathbf{A}^1 - \{0\}$:

$$\mathbf{Z}(1)[1] \to M(\mathbf{A}^1 - \{0\}) \to \mathbf{Z} \xrightarrow{+1}$$

To get a canonical splitting, we may choose the rational point $1 \in \mathbf{A}^1 - \{0\}$.

By [11, Cor. 3.4.3], we have an isomorphism $\mathbf{Z}(1)[1] \simeq \mathbf{G}_m[0]$. Hence, in $\mathbf{DM}_-^{\mathrm{eff}}$, we have an isomorphism

$$\mathcal{G}_{-1}[0] \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m[0], \mathcal{G}[0]).$$

Let $\mathcal{H} \in \mathbf{HI}_{\mathrm{Nis}}$. We get:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathcal{H}[0], \mathcal{G}_{-1}[0]) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathcal{H}[0] \otimes \mathbf{G}_m[0], \mathcal{G}[0])\\ &\quad \simeq \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(H^0(\mathcal{H}[0] \otimes \mathbf{G}_m[0]), \mathcal{G}) =: \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathcal{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathcal{G})\end{aligned}$$

as desired (see (3.1)). For the second isomorphism, we have used the right exactness of $\otimes$ (Lemma 3.3). □

4.4. *Remark.* The proof of Proposition 4.3 also shows that, in $\mathbf{DM}_-^{\mathrm{eff}}$, we have an isomorphism

$$\underline{\mathrm{Hom}}(\mathbf{G}_m[0], \mathcal{G}[0]) \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{G})[0]$$

where the left $\underline{\mathrm{Hom}}$ is computed in $\mathbf{DM}_-^{\mathrm{eff}}$ and the right $\underline{\mathrm{Hom}}$ is computed in $\mathbf{HI}_{\mathrm{Nis}}$. In particular, $\underline{\mathrm{Hom}}(\mathbf{G}_m[0], -) : \mathbf{DM}_-^{\mathrm{eff}} \to \mathbf{DM}_-^{\mathrm{eff}}$ is t-exact.

4.5. **Proposition.** *Let C be a smooth, proper, connected curve over k, with function field K. There exists a canonical homomorphism*

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(C, \mathcal{G}) \to \mathcal{G}_{-1}(k)$$

such that, for any $x \in C$, the composition

$$\mathcal{G}_{-1}(k(x)) \simeq H^1_x(C, \mathcal{G}) \to H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\mathrm{Tr}_C} \mathcal{G}_{-1}(k)$$

equals the transfer map $\mathrm{Tr}_{k(x)/k}$ associated to the finite surjective morphism $\operatorname{Spec} k(x) \to \operatorname{Spec} k$.

Proof. By [11, Prop. 3.2.7], we have

$$H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(C, \mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(M(C), \mathcal{G}[1]).$$

The structural morphism $C \to \operatorname{Spec} k$ yields a morphism of motives $M(C) \to \mathbf{Z}$ which, by Poincaré duality, yields a canonical morphism

$$\mathbf{G}_m[1] \simeq \mathbf{Z}(1)[2] \to M(C).$$

(One may view this morphism as the image of the canonical morphism $\mathbb{L} \to h(C)$ in the category of Chow motives.)

Therefore, by Proposition 4.3 and Remark 4.4, we get a map

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(X, \mathcal{G}) \to \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{G}_m[1], \mathcal{G}[1]) = \mathcal{G}_{-1}(k).$$

It remains to prove the claimed compatibility. Let $M^x(C)$ be the motive of C with supports in x, defined as $C_*(\mathrm{Coker}(L(C - \{x\}) \to L(C))$. Let $\mathbf{Z}_{k(x)} = M(\operatorname{Spec} k(x))$. By [11, proof of Prop. 3.5.4], we have an isomorphism $M^x(C) \simeq \mathbf{Z}_{k(x)}(1)[2]$, and we have to show that the composition

$$\mathbf{Z}(1)[2] \to M(C) \xrightarrow{g_x} \mathbf{Z}_{k(x)}(1)[2]$$

is $\mathrm{Tr}_{k(x)/k}$, up to twisting and shifting. To see this, we observe that g_x is the image of the morphism of Chow motives

$$h(C) \to h(\operatorname{Spec} k(x))(1)$$

dual to the morphism $h(\operatorname{Spec} k(x)) \to h(C)$ induced by the inclusion $\operatorname{Spec} k(x) \to C$: this is easy to check from the definition of g_x in [11] (observe that in this special case, $Bl_x(C) = C$ and that we may use a variant of the said construction replacing $C \times \mathbf{A}^1$ by $C \times \mathbf{P}^1$ to stay within smooth projective varieties). The conclusion now follows from the fact that the composition

$$\operatorname{Spec} k(x) \to C \to \operatorname{Spec} k$$

is the structural morphism of $\operatorname{Spec} k(x)$. □

4.6. **Proposition** (Reciprocity)**.** *Let C be a smooth, proper, connected curve over k, with function field K. Then the sequence*

$$\mathcal{G}(K) \xrightarrow{(\partial_x)} \bigoplus_{x\in C} \mathcal{G}_{-1}(k(x)) \xrightarrow{\sum_x \operatorname{Tr}_{k(x)/k}} \mathcal{G}_{-1}(k)$$

is a complex.

Proof. This follows from Proposition 4.5, since the composition

$$\mathcal{G}(K) \to \bigoplus_{x\in C} H^1_x(C,\mathcal{G}) \xrightarrow{g_x} H^1(C,\mathcal{G})$$

is 0. □

4.7**.** If $\mathcal{F},\mathcal{G}$ are presheaves with transfers, there is a bilinear morphism of presheaves with transfers (i.e. a natural transformation over $\mathbf{PST}\times\mathbf{PST}$):

$$\begin{aligned}&\mathcal{F}(U)\otimes\mathcal{G}_{-1}(V) = \\ &\quad \operatorname{Coker}\big(\mathcal{F}(U)\otimes\mathcal{G}(V\times\mathbf{A}^1)\to\mathcal{F}(U)\otimes\mathcal{G}(V\times(\mathbf{A}^1-\{0\}))\big)\to\\ &\operatorname{Coker}\big((\mathcal{F}\otimes_{\mathbf{PST}}\mathcal{G})(U\times V\times\mathbf{A}^1)\to(\mathcal{F}\otimes_{\mathbf{PST}}\mathcal{G})(U\times V\times(\mathbf{A}^1-\{0\}))\big)\\ &\qquad\qquad = (\mathcal{F}\otimes_{\mathbf{PST}}\mathcal{G})_{-1}(U\times V)\end{aligned}$$

which induces a morphism

$$\mathcal{F}\otimes_{\mathbf{PST}}\mathcal{G}_{-1}\to(\mathcal{F}\otimes_{\mathbf{PST}}\mathcal{G})_{-1}. \tag{4.3}$$

In particular, for $\mathcal{G}=\mathbf{G}_m$, we get a morphism $\mathcal{F}\to(\mathcal{F}\otimes_{\mathbf{PST}}\mathbf{G}_m)_{-1}$.

4.8. **Theorem.** *Suppose $\mathcal{F}\in\mathbf{HI}_{\mathrm{Nis}}$. Then*
a) The composition

$$\mathcal{F}\to(\mathcal{F}\otimes_{\mathbf{PST}}\mathbf{G}_m)_{-1}\to(\mathcal{F}\otimes_{\mathbf{HI}_{\mathrm{Nis}}}\mathbf{G}_m)_{-1}$$

is the unit map of the adjunction between $-\otimes_{\mathbf{HI}_{\mathrm{Nis}}}\mathbf{G}_m$ and $(-)_{-1}$ stemming from Proposition 4.3.
b) This composition is an isomorphism.

Proof. a) is an easy bookkeeping. For b), we compute again in $\mathbf{DM}_-^{\mathrm{eff}}$. By Proposition 4.3, we are considering the morphism in $\mathbf{HI}_{\mathrm{Nis}}$

$$\mathcal{F}\to\underline{\operatorname{Hom}}(\mathbf{G}_m,\mathcal{F}\otimes_{\mathbf{HI}_{\mathrm{Nis}}}\mathbf{G}_m). \tag{4.4}$$

Consider the corresponding morphism in $\mathbf{DM}_-^{\mathrm{eff}}$

$$\mathcal{F}[0]\to\underline{\operatorname{Hom}}(\mathbf{G}_m[0],\mathcal{F}[0]\otimes\mathbf{G}_m[0]).$$

As recalled in the proof of Proposition 4.3, we have $\mathbf{G}_m[0]=\mathbf{Z}(1)[1]$, hence the above morphism amounts to

$$\mathcal{F}[0]\to\underline{\operatorname{Hom}}(\mathbf{Z}(1),\mathcal{F}[0](1))$$

which is an isomorphism by the cancellation theorem [12]. A fortiori, (4.4), which is (by Remark 4.4) the H^0 of this isomorphism, is an isomorphism. □

4.9. **Notation.** Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$ and $\mathcal{H} = \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}$. Let X, K, x be as in §4.1. For $(a, b) \in \mathcal{F}(K) \times \mathcal{G}(K)$, we denote by $a \cdot b$ the image of $a \otimes b$ in $\mathcal{H}(K)$ by the map

$$\mathcal{F}(K) \otimes \mathcal{G}(K) \to \mathcal{H}(K).$$

4.10. **Proposition** (cf. [2, Prop. 5.5.27])**.** *Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, and consider the morphism induced by* (4.3)

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1} \xrightarrow{\Phi} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}.$$

Let X, K, x be as in §4.1. Then the diagram

$$\begin{array}{ccc}
\mathcal{F}(\mathcal{O}_{X,x}) \otimes \mathcal{G}(K) & \longrightarrow & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})(K) \\
\Big\downarrow{\scriptstyle i_x^* \otimes \partial_x} & & \\
\mathcal{F}(k(x)) \otimes \mathcal{G}_{-1}(k(x)) & & \Big\downarrow{\scriptstyle \partial_x} \\
\Big\downarrow & & \\
(\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1})(k(x)) & \xrightarrow{\Phi} & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}(k(x))
\end{array}$$

commutes, where i_x^* is induced by the reduction map $\mathcal{O}_{X,x} \to k(x)$. In other words, with Notation 4.9 we have the identity

$$\partial_x(a \cdot b) = \Phi(i_x^* a \cdot \partial_x b) \tag{4.5}$$

for $(a, b) \in \mathcal{F}(\mathcal{O}_{X,x}) \times \mathcal{G}(K)$.

4.11. **Corollary.** *Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$; let X, K, x be as in §4.1 and let $(a, f) \in \mathcal{F}(K) \times K^*$. Let $\mathcal{H} = \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m$, and consider the element $a \cdot f \in \mathcal{H}(K)$ as in Notation 4.9.*
a) Suppose that $a \in \mathrm{Im}(\mathcal{F}(\mathcal{O}_{X,x}) \to \mathcal{F}(K))$. Then we have

$$\partial_x(a \cdot f) = v_x(f)\bar{a}$$

where ∂_x is the map of (4.2) *and $\bar{a}$ is the image of a in $\mathcal{F}(k(x))$. Here we have used the isomorphism $\mathcal{H}_{-1} \simeq \mathcal{F}$ of Theorem 4.8.*
b) Suppose that $v_x(f - 1) > 0$. Then $\partial_x(a \cdot f) = 0$.

Proof. a) This follows from Proposition 4.10 (applied with $\mathcal{G} = \mathbf{G}_m$) and Theorem 4.8. b) This follows again from Proposition 4.10, after switching the rôles of $\mathcal{F}$ and $\mathcal{G}$. □

4.12. **Proposition.** *Let G be a semi-abelian variety and let $\mathcal{H} = G \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m$, so that $\mathcal{H}_{-1} = G$ by Theorem 4.8. Let X, K, x be as in §4.1, and let $(g, f) \in G(K) \times K^*$. As in Corollary 4.11, write $g \cdot f$ for the image of $g \otimes f$ in $\mathcal{H}(K)$. Then $\partial_x(g \cdot f) = \partial_x(g, f)$, where $\partial_x(-,-)$ is Somekawa's local symbol* [6, (1.1)] *(generalising the Rosenlicht-Serre local symbol).*

Proof. Up to base-changing from k to $\bar{k}$ (see Lemma 4.2 a)), we may assume k algebraically closed. We then have to show that $\partial_x(g \cdot f)$ is the Rosenlicht-Serre local symbol of [5, Ch. III, Def. 2]. In this definition, Condition i) is obvious, Condition ii) is Corollary 4.11 b), Condition iii) is Corollary 4.11 a) and Condition iv) is Proposition 4.6. The conclusion now follows from the uniqueness of the local symbol [5, Ch. III, Prop. 1]. □

5. The factorisation

5.1. **Theorem.** *The homomorphism* (2.9) *factors through Somekawa's relations. Consequently, we get a surjective homomorphism* (1.1).

Proof. Let C/k be a smooth proper an connected curve. Let $K = k(C)$, and, for all $i \in [1, n]$, let $g_i \in G_i(K)$. We also give ourselves a rational function $h \in K^*$. We assume that, for any $c \in C$, there exists $i(c)$ such that $g_i \in G_i(\mathcal{O}_{C,c})$ for all $i \neq i(c)$. Let $\mathcal{F} = G_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G_n$. We must show that the element

$$\sum_{c \in C} \mathrm{Tr}_{k(c)/k}(g_1(c) \otimes \cdots \otimes \partial_c(g_{i(c)}, h) \otimes \cdots \otimes g_n(c))$$

of $(G_1 \overset{M}{\otimes} \ldots \overset{M}{\otimes} G_n)(k)$ goes to 0 in $\mathcal{F}(k)$ via (2.9), where ∂_c is Somekawa's local symbol.[1]

Consider the element $g = g_1 \otimes \cdots \otimes g_n \in \mathcal{F}(K)$. It follows from Corollary 4.11 a) and Proposition 4.12 that, for any $c \in C$, we have

$$g_1(c) \otimes \cdots \otimes \partial_c(g_{i(c)}, h) \otimes \cdots \otimes g_n(c) = \\ g_1(c) \otimes \cdots \otimes \partial_c(g_{i(c)} \otimes \{h\}) \otimes \cdots \otimes g_n(c) = \partial_c(g \otimes \{h\}).$$

The claim now follows from Proposition 4.6. □

[1] As was observed by W. Raskind, the signs appearing in [6, (1.2.2)] should not be there.

Appendix A. Extending monoidal structures

A.1. Let $\mathcal{A}$ be an additive category. We write $\mathcal{A}$–Mod for the category of contravariant additive functors from $\mathcal{A}$ to abelian groups. This is a Grothendieck abelian category. We have the additive Yoneda embedding

$$y_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}\text{–}\mathrm{Mod}$$

sending an object to the corresponding representable functor.

A.2. Let $f : \mathcal{A} \to \mathcal{B}$ be an additive functor. We have an induced functor $f^* : \mathcal{B}\text{–}\mathrm{Mod} \to \mathcal{A}\text{–}\mathrm{Mod}$ ("composition with f"). As in [SGA 4, Exp. 1, Prop. 5.1 and 5.4], the functor f^* has a left adjoint $f_!$ and a right adjoint f_* and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \mathcal{A}\text{–}\mathrm{Mod} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_!}\big\downarrow \\ \mathcal{B} & \xrightarrow{y_{\mathcal{B}}} & \mathcal{B}\text{–}\mathrm{Mod} \end{array}$$

is naturally commutative.

A.3. If f is fully faithful, then $f_!$ and f_* are fully faithful and f^* is a localisation, as in [SGA 4, Exp. 1, Prop. 5.6].

A.4. Suppose that f has a left adjoint g. Then we have natural isomorphisms

$$g^* \simeq f_!, \quad g_* \simeq f^*$$

as in [SGA 4, Exp. 1, Prop. 5.5].

A.5. Suppose further that f is fully faithful. Then $g^* \simeq f_!$ is fully faithful. From the composition

$$g^* g_* \Rightarrow Id_{\mathcal{A}\text{–}\mathrm{Mod}} \Rightarrow g^* g_!$$

of the unit with the counit, one then deduces a canonical morphism of functors

$$g_* \Rightarrow g_!.$$

A.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. Their *tensor product* is the category $\mathcal{A} \boxtimes \mathcal{B}$ whose objects are finite collections (A_i, B_i) with $(A_i, B_i) \in \mathcal{A} \times \mathcal{B}$, and

$$(\mathcal{A} \boxtimes \mathcal{B})((A_i, B_i), (C_j, D_j)) = \bigoplus_{i,j} \mathcal{A}(A_i, C_j) \otimes \mathcal{B}(B_i, D_j).$$

We have a "cross-product" functor

$$\boxtimes : \mathcal{A}\text{–}\mathrm{Mod} \times \mathcal{B}\text{–}\mathrm{Mod} \to (\mathcal{A} \boxtimes \mathcal{B})\text{–}\mathrm{Mod}$$

given by

$$(M \boxtimes N)((A_i, B_i)) = \bigoplus_i M(A_i) \otimes N(B_i).$$

A.7**.** Let $\mathcal{A}$ be provided with a biadditive bifunctor $\bullet : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$. We may view $\bullet$ as an additive functor $\mathcal{A}\boxtimes\mathcal{A} \to \mathcal{A}$. We may then extend $\bullet$ to $\mathcal{A}$–Mod by the composition

$$\mathcal{A}\text{–}\operatorname{Mod}\times\mathcal{A}\text{–}\operatorname{Mod} \xrightarrow{\boxtimes} (\mathcal{A}\boxtimes\mathcal{A})\text{–}\operatorname{Mod} \xrightarrow{\bullet_!} \mathcal{A}\text{–}\operatorname{Mod}.$$

This is an extension in the sense that the diagram

$$\begin{array}{ccc} \mathcal{A}\times\mathcal{A} & \xrightarrow{y_{\mathcal{A}}\times y_{\mathcal{A}}} & \mathcal{A}\text{–}\operatorname{Mod}\times\mathcal{A}\text{–}\operatorname{Mod} \\ {\scriptstyle \bullet\times\bullet}\downarrow & & \bullet\downarrow \\ \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \mathcal{A}\text{–}\operatorname{Mod} \end{array}$$

is naturally commutative.

If $\bullet$ is monoidal (resp. monoidal symmetric), then its associativity and commutativity constraints canonically extend to $\mathcal{A}$–Mod.

A.8**.** Let $\mathcal{A}, \mathcal{B}$ be two additive symmetric monoidal categories, and let $f : \mathcal{A} \to \mathcal{B}$ be an additive symmetric monoidal functor. The above definition shows that the functor $f_! : \mathcal{A}\text{–}\operatorname{Mod} \to \mathcal{B}\text{–}\operatorname{Mod}$ is also symmetric monoidal.

A.9**.** In §A.7, let us write $\bullet_! = \int$ for clarity. Let $P \in (\mathcal{A}\boxtimes\mathcal{A})\text{–}\operatorname{Mod}$. Then $\int P$ is the *left Kan extension of* P *along* $\bullet$ in the sense of [4, X.3]. This gives a formula for $\int P$ as a *coend* (ibid., Th. X.4.1); for $A \in \mathcal{A}$:

$$\int P(A) = \int^{(B,B')} \mathcal{A}(A, B\bullet B') \otimes P(B, B'). \tag{A.1}$$

In particular:

A.10. **Proposition.** *Suppose* $\mathcal{A}$ *rigid. Then* (A.1) *simplifies as*

$$\int P(A) = \int^{B} P(B, A\bullet B^*)$$

where B^* *is the dual of* $B \in \mathcal{A}$*. In particular, if* $P = M\boxtimes N$ *for* $M, N \in \mathcal{A}\text{–}\operatorname{Mod}$*, we have for* $A \in \mathcal{A}$*:*

$$(M\bullet N)(A) = \int^{B} M(B) \otimes N(A\bullet B^*) \tag{A.2}$$

which describes $M\bullet N$ *as a "convolution".*

Proof. Applying (A.1) and rigidity, we have

$$\int P(A) = \int^{(B,B')} \mathcal{A}(A, B \bullet B') \otimes P(B,B')$$
$$= \int^{(B,B')} \mathcal{A}(A \bullet B^*, B') \otimes P(B,B')$$
$$= \int^{B} P(B, A \bullet B^*)$$

because in the third formula, the variable B' is dummy (this simplification is not in Mac Lane!). □

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