Pfaffian formulae for one dimensional coalescing and annihilating systems

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Abstract

A variety of probabilities concerning coalescing and annihilating systems on R are shown to be given by pfaffians of two particle probabilities. As an application, an exact asymptotic for the n-point density function for coalescing particles is derived.

1 Introduction

The study of single species reaction diffusion systems $A + A \to A$ (coalescence) and $A + A \to 0$ (annihilation) originated in non-equilibrium statistical mechanics (see [10]), but has now a large mathematical literature (see, for example, [2], [4], [5], [7]). In one dimension the systems exhibit strongly non-mean field behaviour due to correlation effects. In this paper we give several examples showing that this correlation structure can be encoded algebraically in a pfaffian structure. Note that the embedding of annihilating random walks as domain boundaries for a Glauber model makes pfaffian formulae quite reasonable due to the free fermion structure of the Glauber model. We concentrate on Brownian particles, but we expect our results hold for a large variety of spatial motion processes, and the pfaffian structure seems to arise from two basic underlying mechanisms: linearly ordered particle motion and instantaneous reactions.

Outline of paper. In section 2 we recall certain facts about coalescing and annihilating Brownian motions: the thinning relation; some duality formulae; the maximal entrance laws. In section 3 we show that the product moment for annihilating Brownian motions

$$(x_1, \ldots, x_{2n}) \to E^{A}_{(x_1,\ldots,x_{2n})} \left[\prod_i g(X^i_t) \right],$$

for bounded measurable g, is given by a pfaffian. We then show a variety of other probabilities are given by pfaffians (see (15,18,21,23)), either by exploiting duality formulae or by using algebraic results for pfaffians. We remark in sections 2 and 3 on extensions of our formulae to more general motion processes, including for example Markovian motions that are suitably non-degenerate.

In section 4 we improve the asymptotics for the n-particle density function $\rho_t^{(n)}$ for coalescing Brownian motions defined by

$$P\,[\text{there exist particles in } dx_1, \ldots, dx_n \text{ at time } t] = \rho_t^{(n)}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

In [9] we showed, under a variety of initial conditions, the large time estimate

$$\rho_t^{(n)}(x_1, \ldots, x_n) \approx t^{-\frac{n}{2} - \frac{n(n-1)}{4}} \prod_{1 \le i < j \le n} |x_i - x_j|$$

in that $\rho_t^{(n)}$ is bounded between upper and lower constant multiples of the right hand side for large t. The non-linear factor in the power of t illustrates the non mean-field behaviour due to correlations. In this paper we show that, in some cases, the true asymptotic holds as $t \to \infty$, with a constant given by the pfaffian of a certain matrix.

2 Brief review of some facts for one-dimensional coalescing and annihilating Brownian motions

We consider, at first, initial conditions that have only finitely many particles. This paper describes only the one dimensional distributions, that is at a fixed $t > 0$, of any remaining particles. We list the positions of the particles at time t as $(X_t^i : i \in I_t)$. The exact details of the labeling system I_t will not be important for us, and indeed our results all relate only to the empirical measure N_t defined by

$$N_t(A) = \sum_{i \in I_t} \chi(X_t^i \in A) \quad \text{for measurable } A \subseteq \mathrm{R}.$$

For the case of annihilating particles, if the initial number of particles is even then it remains so for all time. To a list (x_i) of an even number $2n$ of disjoint positions we may associate the open set

$$S((x_i)) = (\hat{x}_1, \hat{x}_2) \cup \ldots \cup (\hat{x}_{2n-1}, \hat{x}_{2n})$$

where $\hat{x}_1 < \ldots < \hat{x}_{2n}$ are the ordered positions. Some of the formulae for annihilating particles are then most easily stated in terms of the set valued process

$$S_t = S\left((X_t^i : i \in I_t)\right).$$

Notation. We write $P^C_{(x_1,\ldots,x_n)}$ to indicate that we are considering (instantaneously) coalescing Brownian motions started from initial positions $x_1, \ldots, x_n$. When the particles are annihilating we change the superscript from C to A. When the initial positions are random we change the subscript to Ξ, where Ξ is the law of $(X_0^i : i \in I_0)$.

2.1 The thinning relation

The formulae about coalescing systems in the paper will always come with an analogue for annihilating systems. The close link between the two systems has often been observed. For this paper the formulae can usually be derived from the following thinning relation. For a list of positions $(x_1, \ldots, x_n)$ we let $\Theta(x_1, \ldots, x_n)$ be the random subset formed by thinning at rate 1/2, that is by removing each position independently with probability 1/2. We may also thin a random set of positions, for example $\Theta(X_t^i : i \in I_t)$, with the understanding that the randomness in the thinning is independent of the randomness in the set of positions. We write $\Theta(\Xi)$ for the law of the thinned set of positions that initially have law Ξ. Then the thinning relation between coalescing and annihilating Brownian motions is the following equality in distribution:

$$(X_t^i : i \in I_t) \text{ under } P^A_{\Theta(\Xi)} \quad \stackrel{\mathcal{D}}{=} \quad \Theta(X_t^i : i \in I_t) \text{ under } P^C_{\Xi}. \tag{1}$$

Such a thinning relation is discussed in Arratia [1] for the scaled limit of reacting random walks, and is related to results in many later papers. It can be deduced from the duality relations (2) and (3) in subsection 2.2. However there is a simple colouring proof in ben Avraham and Brunet [3] that bears repetition here. After the paths of a coalescing system have been realized, independently add random colours as follows. Initially colour each particle red or blue independently with probability 1/2. At coalescences the colours evolve according to the rules $R+R \to R$, $B+B \to R$ and $R+B \to B$. Then the resulting system of blue particles evolves as an annihilating system. Moreover the colour of a particle at time t depends on whether there were a odd or even number of ancestors at time zero that were coloured red. Since distinct particles have disjoint sets of ancestors, the colour of all particles at any time $t > 0$ remains independently red or blue with equal probability. The thinning relation follows. This argument makes it clear that the result holds much more widely, since the exact nature of the motion process is not relevant, nor is the mechanism of reaction (for example it holds for delayed reactions, when the reactions are controlled by the intersection local time).

2.2 Duality formulae

We use two duality formulae. For $a_1 < a_2 < \ldots < a_{2m}$ let $I_k = (a_k, a_{k+1})$ for $k = 1, \ldots, 2m-1$. Then for disjoint (x_i)

$$P^C_{(x_i)}\left[N_t(I_1) = N_t(I_3) = \ldots = N_t(I_{2m-1}) = 0\right] = P^A_{(a_i)}\left[S_t \cap (x_i) = \emptyset\right]. \tag{2}$$

The annihilating analogue of this is, writing $|A|$ for the cardinality of a set A,

$$E^A_{(x_i)}\left[(-1)^{N_t(I_1 \cup I_3 \cup \ldots \cup I_{2m-1})}\right] = E^A_{(a_i)}\left[(-1)^{|S_t \cap (x_i)|}\right]. \tag{3}$$

There are various ways to see these formulae, but for coalescing systems a key construction is the Brownian web and its coupling with the dual Brownian web, first considered by Arratia and explored in Toth and Werner [13] (and subsequent papers). We need only part of the Brownian web as follows. For a fixed $t > 0$, there is a system of coalescing

Brownian motions starting from every $x \in \mathrm{Q}$ and running over the time interval $[0,t]$, and a coupled system of backwards coalescing Brownian paths starting at time t at all $x \in \mathrm{Q}$ and running back to time zero. In fact, the Brownian web has particles starting at all space-time points (s,x) but we will not need this, and it is enough to establish (2) first for rational (x_i) and (a_i). The key property is that, almost surely, none of the forward paths cross any of the backwards paths. (A discrete version of this coupling, that is using simple coalescing simple random walks, is easy to construct - see the appendix in [11] - and illustrates this non-crossing property). From this non-crossing property one sees that the event that $N_t((a,b)) = 0$ for the forward coalescing system is almost surely equal to the event that the open interval formed by pair of backwards particles starting at a and b does not contain any of the initial forwards particles. The coalescing duality (2) follows immediately, once one notes that S_t may be replaced by its closure and that annihilating the backwards particles when they meet will not affect this closure.

The annihilating duality (3) follows from (2) and the thinning relation. Note the fact that if $B(n,1/2)$ is a Bernouilli variable (with n trials and parameter 1/2) then $E[(-1)^{B(n,1/2)}] = 0$. Then thinning and (2) show that

$$\begin{aligned} E^A_{\Theta(x_i)}\left[(-1)^{N_t(I_1 \cup I_3 \cup \ldots \cup I_{2m-1})}\right] &= P^C_{(x_i)}\left[N_t(I_1) = N_t(I_3) = \ldots = N_t(I_{2m-1}) = 0\right] \\ &= P^A_{(a_i)}\left[S_t \cap (x_i) = \emptyset\right] \\ &= E^A_{(a_i)}\left[(-1)^{|S_t \cap \Theta(x_i)|}\right]. \end{aligned}$$

One may then argue by induction on the number n of the initial particles $(x_1, \ldots, x_n)$. When $n = 1$ the above identity reduces to (3) for a single particle. For general n the identity is a mixture of copies of (3) for initial conditions that are subsets of (x_i). But all but one of the copies will involve $n-1$ or less particles allowing an inductive proof. Note also that (2) follows from (3) - a weighted sum of (3) according to the distribution of $\Theta(x_i)$ yields (2).

Remark. Other coalescing duality formulae, such as those in Xiong and Zhou [14], also follow from the Brownian web and its dual, but their proof shows that one may also establish them using the Markov generator duality, as explained in section 4.4 of Ethier and Kurtz, and thus bypass the Brownian web. In particular this generator technique may be extended to show analogous dualities for more general spatial motions, where the web construction does not (as yet) exist. Formally the generator proof shows that the dualities (2) and (3) will hold for instantly reacting continuous Markovian motions, where the motion on the right hand side must be the image of the motion on the left hand side under reflection $x \to -x$. Furthermore the maximal entrance laws constructed in the next section should follow once some moment control is established, which will require some non-degeneracy of the spatial motion to ensure enough reactions take place.

2.3 Maximal entrance laws

One may start coalescing systems from infinitely many particles at time zero. A natural state space for the empirical measure is the set $\mathcal{M}_{LFP}(\mathrm{R})$ of locally finite point measures on R, which is a closed subset of the space locally finite measures under the topology of vague convergence of measures. The reactions ensure that the point masses only have mass one, and so we consider the (measurable) subset $\mathcal{M}_0$ of those measures of the form

$$\mu = \sum_i \delta_{x_i} \quad \text{where } (x_i) \text{ is locally finite in R and has disjoint elements.} \tag{4}$$

(To obtain a process with continuous paths, which does not concern us in this paper, one can quotient $\mathcal{M}_{LFP}$ by the minimal relation that ensures $\mu + 2\delta_x \sim \mu + \delta_x$.)

There is a Feller Markov transition kernel $p_t(\mu, d\nu)$ on $\mathcal{M}_0$. Moreover, there is a maximal entrance law, intuitively starting with one particle at every site (as in the Brownian web). This can be characterized by passing to the limit in (2) as (x_i) increase to become dense in the real line. This entrance law, which we denote by P^C_∞, has one dimensional distributions satisfying

$$P^C_\infty \left[N_t(I_1) = N_t(I_3) = \ldots = N_t(I_{2m-1}) = 0\right] = P^A_{(a_i)}\left[\tau < t\right] \tag{5}$$

where τ is the time for complete extinction of the annihilating system. This characterizes the one dimensional laws on $\mathcal{M}_0$, and these laws are an entrance law for the Markov transition kernel described above. By the scaling property of Brownian motions we have $P^A_{(Ta_i)}[\tau < T^2 t]$ is independent of $T > 0$. Using (5) this translates into a scaling for the entrance law

$$\text{The law of } (T^{-1} X^i_{tT^2} : i \in I_{tT^2}) \text{ is independent of } T > 0 \text{ under } P^C_\infty.$$

Many suitably spread out and non-degenerate initial conditions are attracted to the maximal entrance law as $t \to \infty$. For a large class of initial conditions (x_i), the law of $(T^{-1} X^i_{T^2 t} : i \in I_{T^2 t})$ under $P^C_{(x_i)}$ converges in distribution, as $T \to \infty$, to the law of $(X^i_t : i \in I_t)$ under P^C_∞. Indeed, using the extension of (2) to countable (x_i), this follows from

$$\begin{aligned}
& P^C_{(x_i)} \left[(T^{-1} X^i_{T^2 t} : i \in I_{T^2 t}) \cap I_k = \emptyset \text{ for } k = 1, 3, \ldots, 2m-1\right] \\
&\quad = P^C_{(x_i)} \left[N_{tT^2}(TI_1) = N_{tT^2}(TI_3) = \ldots = N_{tT^2}(TI_{2m-1}) = 0\right] \\
&\quad = P^A_{(Ta_i)} \left[S_{tT^2} \cap (x_i) = \emptyset\right] \\
&\quad = P^A_{(a_i)} \left[S_t \cap (T^{-1} x_i) = \emptyset\right] \\
&\quad \to P^A_{(a_i)} \left[\tau < t\right] \\
&\quad = P^C_\infty \left[N_t(I_1) = N_t(I_3) = \ldots = N_t(I_{2m-1}) = 0\right].
\end{aligned} \tag{6}$$

The third equality comes from Brownian scaling and the final equality is (5). The convergence holds for deterministic (x_i) for which $(T^{-1} x_i)$ become dense in any finite interval $[a, b]$ as $T \to \infty$. A large class of random initial conditions will clearly also work, for example non-zero stationary and spatially ergodic.

For annihilating systems a Markov transition kernel can also be constructed, using (3) and it's extension to countable (x_i) as a means of characterization. We can define an entrance law P_∞^A for annihilating system by taking the thinned copy of the entrance law for the coalescing system. This satisfies the formula

$$E_\infty^A\left[(-1)^{N_t(I_1\cup I_3\cup\ldots\cup I_{2m-1})}\right] = P_{(a_i)}^A\left[\tau < t\right] \tag{7}$$

which again determines one dimensional laws on $\mathcal{M}_0$ that form an entrance law for the annihilating system. The domain of attraction of this entrance law is more delicate. The example in section 3 of Bramson and Griffeath [4] suggests that different approximations to a maximal entrance law may yield different laws at times $t > 0$ (their example uses varying intensities of nearby pairs at time zero). For initial conditions that fill the lattice $\lambda^{-1}\mathrm{Z}$, or that are Poisson with intensity λ, the one-dimensional distributions converge as $\lambda \to \infty$ to those of the entrance measure, or for a fixed λ the large time distribution rescales to those of the entrance law as above.

Since we found it difficult to find a full account in the literature, we give, in the appendix, a brief sketch of the proofs of the results in this subsection.

3 Pfaffians abound

We write $\mathrm{Pf}\,(a_{ij} : 1 \le i < j \le 2n)$ for the pfaffian of the real anti-symmetric matrix whose elements are a_{ij} for $i < j$. We give a short review of the facts we shall need about pfaffians (mostly proved in [12] section 2).

The determinant of an anti-symmetric matrix of odd order is zero. Suppose A is an anti-symmetric $2n \times 2n$ matrix. Then $det(A)$ is the square of a polynomial of degree n in the matrix elements, called the pfaffian of A and written as $\mathrm{Pf}\,(A)$. One can define the pfaffian as a suitable sum over permutations of products of matrix elements. Indeed,

$$\mathrm{Pf}\,(A) = \sum_{\sigma\in\Sigma_{2n}} \mathrm{sgn}(\sigma) a_{i_1,j_1} a_{i_2,j_2} \ldots a_{i_n,j_n} \tag{8}$$

where Σ_{2n} is the set of permutations σ of $\{1,2,\ldots,2n\}$ given by $\sigma(2k-1) = i_k,\ \sigma(2k) = j_k$ for $k = 1,\ldots,n$ for which the choices $(i_k),(j_k)$ satisfy $i_k < j_k$ for all k and $i_1 < i_2 < \ldots < i_n$. A convenient way to calculate the sign of such a permutation is via *crossings*. The quadruple i_k, j_k, i_l, j_l is called crossed if $i_k < i_l < j_k < j_l$. Then the sign of $\sigma \in \Sigma_{2n}$ equals $(-1)^M$ where M is the number of crossings. To visualize these crossings easily one can embed the integers $1,\ldots,2n$ into the x-axis of the plane and join i_k to j_k for each k with a loop in the upper half plane.

It is worth recording the smallest cases:

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a \qquad \mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

The explicit 4×4 case was used to guess many of the pfaffian formulae in this paper.

Pfaffians have many similar properties to determinants. It follows from the definition that $\mathrm{Pf}(\lambda_i \lambda_j a_{ij}) = \mathrm{Pf}(a_{ij}) \prod_k \lambda_k$. For any $2n \times 2n$ matrix B the product $B^T A B$ is anti-symmetric and $\mathrm{Pf}(B^T A B) = \det(B) \mathrm{Pf}(A)$. The pfaffian can be decomposed along a row, or column, of the matrix. For example if A is a $2n \times 2n$ anti-symmetric matrix it satisfies the recursion, for any $i \in \{1, 2, \ldots, 2n\}$,

$$\mathrm{Pf}(A) = \sum_{j=2}^{2n} a_{ij} (-1)^{i+j+1} \mathrm{Pf}(A^{(i,j)}) \tag{9}$$

where $A^{(i,j)}$ is the $(2n-2) \times (2n-2)$ submatrix formed by removing the ith and jth row and column. We will also use (see (17)) a decomposition formula for the pfaffian of a sum $\mathrm{Pf}(A + B)$.

A very convenient tool for manipulating pfaffians is the Berezin integral. We provide arguments that avoid this tool in this paper, and so do not describe the rules for manipulating these integrals. However they were used repeatedly while exploring these results, and in the next section we show how the Berezin integral can considerably shorten the argument. A very readable account of Berezin integrals can be found in Itzykson and Drouffe [8]. The key property linking the Berezin integral to pfaffians is (compare with the normalizing determinant for multi-dimensional Gaussian integrals)

$$\mathrm{Pf}(a_{ij} : i \leq i < j \leq 2n) = \int d\psi_{2n} \ldots d\psi_1 e^{-\frac{1}{2} \sum_{i,j=1}^{2n} \psi_i a_{ij} \psi_j}. \tag{10}$$

3.1 Product moment pfaffians

We start with the product moment, defined for bounded measurable $g : \mathrm{R} \to \mathrm{R}$ and disjoint (x_i) by

$$m_t^{(n)}(x_1, \ldots, x_n) = E^A_{(x_1, \ldots, x_n)} \left[\prod_{i \in I_t} g(X_t^i) \right]$$

where the product over an empty set, occurring when all the particles have been annihilated, is defined to have value 1. Note that $m_t^{(1)}(x)$ is given by the Brownian semigroup applied to g. We also set $m^{(0)} \equiv 1$. To show that $m^{(2n)}$ is given by a pfaffian, we shall give a p.d.e. derivation similar in spirit to that showing the Karlin McGregor formula for the transition density for non-intersecting Brownian motions is given by a determinant. However, one can in fact prove that the various pfaffians in this section hold true in different ways and starting from different places.

Let $V_n \subseteq \mathrm{R}^n$ be the open cell $\{x : x_1 < x_2 < \ldots < x_n\}$. On $(0, \infty) \times V_n$ the function $m_t^{(n)}(x)$ solves the heat equation, and we must examine the boundary conditions. For $n \geq 2$ and when g is bounded and continuous, the functions $m^{(n)}$ are continuous on $[0, \infty) \times V_n$ and extend to a continuous function in $C((0, \infty) \times \overline{V}_n)$. There are lots of pieces to the boundary of V_n, but the most important are the faces $F_{i,n}$ defined by $x_i = x_{i+1}$ and where the remaining x_k are disjoint. On $F_{i,n}$ the continuous extension agrees with the lower order moment $m^{(n-2)}(x^{(i,i+1)})$, where $x^{(i,j)} \in \mathrm{R}^{n-2}$ is the $(n-2)$-tuple formed by removing x_i and x_j from $(x_1, \ldots, x_n)$. This can be seen by showing that near the boundary the hitting time between particles starting at x_i and x_{i+1} is likely to occur before any other collision and before time t. On other parts of the boundary the extension agrees with other lower moments.

The system of heat equations for $(m^{(n)} : n = 1, 2, \ldots)$

$$\begin{cases} \frac{\partial}{\partial t} m_t^{(n)}(x) & = & \Delta m_t^{(n)}(x) & \text{on } (0,\infty) \times V_n, \\ m_t^{(n)}(x) & = & m_t^{(n-2)}(x^{(i,i+1)}) & \text{for } x \in F_{i,n} \text{ and } i = 1, \ldots, n-1, \\ m_0^{(n)}(x) & = & \prod_{i=1}^n g(x_i) & \text{for } x \in V_n, \end{cases}$$

forms a closed system, in that each equation has boundary conditions formed by equations of lower order. Note that, typically, the initial condition does not match the boundary conditions. However, taking g bounded and smooth, the system has unique solutions in $C^{1,2}([0,\infty) \times V_n) \cap C((0,\infty) \times \overline{V}_n)$. It is enough to specify boundary conditions only on each face $F_{i,n}$ - the Feynman-Kac formula makes it clear the other parts of the boundary of V_n do not affect the value of $m^{(n)}$.

We claim, when $x_1 < x_2 < \ldots < x_{2n}$, that the even moments $m_t^{(2n)}(x)$ are given by the pfaffian

$$m_t^{(2n)}(x_1, \ldots, x_{2n}) = \tag{11}$$

This pfaffian determines $m^{(2n)}$ completely since it is symmetric in the variables $x_1, \ldots, x_{2n}$. We shall check this by showing the pfaffian solves the system of heat equations above. It is enough, by approximation, to check this for bounded smooth g. Note that (11) holds when $t = 0$ since

$$m_0^{(2n)}(x_1, \ldots, x_{2n}) = \prod_{i=1}^{2n} g(x_i) = \mathrm{Pf}\left(g(x_i)g(x_j)\right).$$

The pfaffian in (11) is a finite sum of product terms, see (8), of the form

$$\mathrm{sgn}(\sigma) m_t^{(2)}(x_{i_1}, x_{j_1}) m_t^{(2)}(x_{i_2}, x_{j_2}) \ldots m_t^{(2)}(x_{i_n}, x_{j_n})$$

where σ is a permutation given by $\sigma(2k-1) = i_k$, $\sigma(2k) = j_k$ for $k = 1, \ldots, n$. Since $m_t^{(2)}(x,y)$ satisfies the heat equation on $[0,\infty) \times \{x < y\}$, each product term lies in $C^{1,2}([0,\infty) \times V_{2n})$ and satisfies the heat equation on $[0,\infty) \times V_{2n}$. Since $m_t^{(2)}(x,y)$ extends continuously to $(0,\infty) \times \{(x,y) : x \le y\}$, the pfaffian extends continuously to $(0,\infty) \times \overline{V}_{2n}$. By the uniqueness for the system of heat equations, it remains to check that the pfaffian satisfies the required boundary conditions on each face $F_{i,2n}$. We show the argument for the face $F_{1,2n}$ where $x_1 = x_2$ (other faces are similar).

We may argue inductively, and suppose that $m^{(k)}$ is given by the pfaffian for $k = 0, 2, \ldots, 2n-2$. Our quickest proof is using the representation (10) in terms of Berezin integrals. This gives

$$\mathrm{Pf}\left(m_t^{(2)}(x_i, x_j) \,:\, 1 \le i < j \le 2n\right)\Big|_{x_1 = x_2}$$

$$
\begin{aligned}
&= \int d\psi_{2n} \dots d\psi_1 e^{-\frac{1}{2}\sum_{i,j=1}^{2n} \psi_i m_t^{(2)}(x_i,x_j)\psi_j}\Big|_{x_1=x_2} \\
&= \int d\psi_{2n} \dots d\psi_1 e^{-\frac{1}{2}\sum_{i,j=3}^{2n} \psi_i m_t^{(2)}(x_i,x_j)\psi_j} e^{-(\psi_1+\psi_2)\sum_{k=3}^{2n} m_t^{(2)}(x_1,x_k)\psi_k}.
\end{aligned}
$$

The sum $M = \sum_{k=3}^{2n} m_t^{(2)}(x_1,x_k)\psi_k$ is independent of ψ_1 and ψ_2 and the $d\psi_2 d\psi_1$ integral becomes (using the rules for Berezin integrals)

$$
\int d\psi_2 d\psi_1 e^{-(\psi_1+\psi_2)M} = \int d\psi_2 d\psi_1 (1-\psi_2\psi_1)(1-(\psi_1+\psi_2)M) = 1.
$$

This simplification leaves only $\int d\psi_{2n} \dots d\psi_3 e^{-\frac{1}{2}\sum_{i,j=3}^{2n} \psi_i m_t^{(2)}(x_i,x_j)\psi_j}$ which is the Berezin integral for $m^{(2n-2)}(x_3,\dots,x_{2n})$.

An argument that avoids Berezin integrals is as follows. Using the recursive relation for pfaffians (9) we see that the pfaffian in (11) equals

$$
\sum_{k=2}^{2n} (-1)^k m_t^{(2)}(x_1,x_k) m_t^{(2n-2)}(x^{(1,k)}).
$$

Since $m_t^{(2)}(x_1,x_2)$ extends to the function 1 on $x_1 = x_2$, it remains only to check that

$$
\sum_{k=3}^{2n} (-1)^k m_t^{(2)}(x_1,x_k) m_t^{(2n-2)}(x^{(1,k)}) \tag{12}
$$

vanishes when $x_1 = x_2$ and $t > 0$. But this follows from expressing $m^{(2n-2)}$ using (8). Indeed, fix $j,k \geq 3$. Then for an expression of the form

$$
m_t^{(2)}(x_1,x_k) m_t^{(2)}(x_2,x_j) m_t^{(2)}(x_{i_2},x_{j_2}) \dots m_t^{(2)}(x_{i_{n-1}},x_{j_{n-1}})
$$

arising from the kth term in (12), where $\{i_2,j_2,\dots,i_{n-1},j_{n-1}\} = \{3,4,\dots,2n\} \setminus \{j,k\}$, there is a corresponding term

$$
m_t^{(2)}(x_1,x_j) m_t^{(2)}(x_2,x_k) m_t^{(2)}(x_{i_2},x_{j_2}) \dots m_t^{(2)}(x_{i_{n-1}},x_{j_{n-1}})
$$

arising from the jth term in (12). These terms agree on $x_1 = x_2$ and a careful check of the signs of the permutations, and the factors $(-1)^j$ and $(-1)^k$ in (12), shows they will cancel. One way to do this check is to compare the sign of

$$
\sigma = \begin{pmatrix} 2 & 3 & 4 & 5 & \dots & k-1 & k+1 & \dots & 2n-1 & 2n \\ 2 & j & i_2 & i_3 & \dots & \dots & \dots & \dots & i_{n-1} & j_{n-1} \end{pmatrix}
$$

with that of

$$
\sigma' = \begin{pmatrix} 2 & 3 & 4 & 5 & \dots & j-1 & j+1 & \dots & 2n-1 & 2n \\ 2 & k & i_2 & i_3 & \dots & \dots & \dots & \dots & i_{n-1} & j_{n-1} \end{pmatrix}
$$

by counting crossings. The loop joining 2 to j in σ must be replaced by a loop joining 2 to k in σ'. This may affect crossings with any of the loops emanating from sites between

j and k, and will do so unless a pair of them are joined to each other. There are $|k-j|-1$ sites between j and k so it will change the parity of the number of crossings exactly when $|k-j|$ is even.

For odd moments there is also a pfaffian representation, namely, when $x_1 < x_2 < \ldots < x_{2n-1}$,

$$m_t^{(2n-1)}(x_1, \ldots, x_{2n-1}) = \mathrm{Pf}\left(m_t^{(2)}(x_i, x_j) : 0 \le i < j \le 2n-1\right) \tag{13}$$

where we adopt the convention that $m_t^{(2)}(x_0, x_k) = m_t^{(1)}(x_k)$. This pfaffian involves a linear combination of terms of the form

$$\mathrm{sgn}(\sigma) m_t^{(1)}(x_{j_1}) m^{(2)}(x_{i_2}, x_{j_2}) \ldots m^{(2)}(x_{i_n}, x_{j_n})$$

which again shows that it solves the heat equation when $[0, \infty) \times V_{2n-1}$. The recursive pfaffian relation gives

$$m_t^{(2n-1)}(x) = \sum_{k=1}^{2n-1} (-1)^k m_t^{(1)}(x_k) m_t^{(2n-2)}(x^{(k)}).$$

Expanding the pfaffian along its first row using (9) we obtain for $x = (x_1, \ldots, x_{2n-1}) \in V_{2n-1}$

$$\mathrm{Pf}\left(m_t^{(2)}(x_i, x_j) : 0 \le i < j \le 2n-1\right) = \sum_{k=1}^{2n-1} (-1)^{k+1} m_t^{(1)}(x_k) m_t^{(2n-2)}(x^{(k)}) \tag{14}$$

where we again write superscripts $x^{(i,j,\ldots)}$ to mean that we remove the indicated co-ordinates. The terms with $k = 1$ and $k = 2$ cancel on the face $F_{2n-1,1}$ where $x_1 = x_2$. Moreover on this face, for $k \ge 3$, $m_t^{(2n-2)}(x^{(k)}) = m_t^{(2n-4)}(x^{(1,2,k)})$ so that the pfaffian in (14) becomes

$$\sum_{k=3}^{2n-1} (-1)^{k+1} m_t^{(1)}(x_k) m_t^{(2n-4)}(x^{(1,2,k)}).$$

But this is the decomposition of $m_t^{(2n-3)}(x^{(1,2)})$ along the first row, and this shows the boundary conditions are correct on $F_{2n-1,1}$. Other faces are similar.

Remark. Since our proof relies only on uniqueness for the underlying system of heat equations, the extension of these product moment pfaffians to more general spatial motions looks quite straightforward, for example to Markovian spatial motions that are continuous and suitably non-degenerate. The pfaffians in the next section would then also follow for these more general motions, just by algebraic manipulation, once maximal entrance laws characterized by (5) and (7) are established. We anticipate that a detailed treatment will be given in the Warwick thesis [15].

3.2 Other pfaffians

Fixing $a_1 < \ldots < a_{2m}$ and choosing $g(x) = (-1)^{\sum_i \chi(x \le a_i)}$ in (11) we see that both sides of the duality (3) are pfaffians in the variables (x_i). Choosing $g = 0$, recalling that an empty product takes the value 1, we see that $P^A_{(x_i)}[\tau < t]$ is a pfaffian. The entrance law dualities (5) and (7) show that

$$P^C_\infty\left[N_t(I_1) = N_t(I_3) = \ldots = N_t(I_{2m-1}) = 0\right] = E^A_\infty\left[(-1)^{N_t(I_1 \cup I_3 \cup \ldots \cup I_{2m-1})}\right] \tag{15}$$

are pfaffians in the variables (a_i). The entries in this last pfaffian are explicit since

$$P^C_\infty\left[N_t\left((a_j, a_k)\right) = 0\right] = E^A_\infty\left[(-1)^{N_t((a_j,a_k))}\right] = P^A_{(a_i,a_j)}\left[\tau < t\right]$$

are all equal to (by Brownian hitting probabilities)

$$2F((2t)^{-1/2}(a_j - a_i)) \text{ where } F(x) = \textstyle\int_x^\infty (2\pi)^{-1/2} \exp(-y^2/2)\, dy. \tag{16}$$

We now show, starting from the pfaffian (15), that certain other probabilities, which are perhaps more informative, can also be expressed as reasonably simple pfaffians. We exploit a formula (see [12]) for the pfaffian of a sum of two $2n \times 2n$ anti-symmetric matrices A and B:

$$\mathrm{Pf}\,(A + B) = \sum_J (-1)^{|J|/2} (-1)^{s(J)} \mathrm{Pf}\,(A|_J) \mathrm{Pf}\,(B|_{J^c}) \tag{17}$$

where: the sum is over all subsets $J \subseteq \{1, 2, \ldots, 2n\}$ with an even number of terms; $J^c = \{1, 2, \ldots, 2n\} \setminus J$; $s(J) = \sum_{j \in J} j$ (and $s(\emptyset) = 0$); and where $A|_J$ means the submatrix of A formed by the rows and columns indexed by elements of J (and the pfaffian of the empty matrix is taken to have value 1). We give three examples, and in each case E is the $2n \times 2n$ anti-symmetric matrix with elements $e_{ij} = P^C_\infty[N_t\left((a_j, a_k)\right) = 0]$ as in (16).

I. Let I be the $2n \times 2n$ anti-symmetric matrix with entries 1 above the diagonal. Then $\mathrm{Pf}\,(I) = 1$. Note also, for a $2n \times 2n$ anti-symmetric matrix A, that $\mathrm{Pf}\,(-A) = (-1)^n \mathrm{Pf}\,(A)$. The formula (17) specializes to

$$\mathrm{Pf}\,(I - A) = \sum_J (-1)^{s(J)} \mathrm{Pf}\,(A|_J).$$

We combine this with a simple combinatorial identity (which can be checked by induction on n): suppose that $(m_{j,k} : 1 \leq j < k \leq n)$ satisfy the collapsing product $m_{j,k} m_{k,l} = m_{j,l}$ for all j, k, l; then

$$\begin{aligned}
\prod_{k=1}^{n-1} (1 + m_{k,k+1}) &= 1 + \sum_{1 \leq k_1 < k_2 \leq n} m_{k_1,k_2} \\
&\quad + \sum_{1 \leq k_1 < k_2 < k_3 < k_4 \leq n} m_{k_1,k_2} m_{k_3,k_4} \\
&\quad + \sum_{1 \leq k_1 < k_2 < k_3 < k_4 < k_5 < k_6 \leq n} m_{k_1,k_2} m_{k_3,k_4} m_{k_5,k_6} + \ldots \\
&= \sum_J m_J
\end{aligned}$$

where the final sum is over all subsets J of $\{1, 2, \ldots, n\}$ of even size, and if $J = \{k_1, \ldots, k_{2m}\}$ where $k_1 < \ldots < k_{2m}$ then $m_J = m_{k_1,k_2} m_{k_3,k_4} \ldots m_{k_{2m-1},k_{2m}}$ (and with

$m_\emptyset = 1$). If n is even then the last term of this series is $m_{1,2}m_{3,4}\dots m_{n-1,n}$. Note that $\bar{m}_{j,k} = \alpha^{k-j} m_{j,k}$ also satisfy $\bar{m}_{j,k}\bar{m}_{k,l} = \bar{m}_{j,l}$ and applying the above for $\bar{m}$ one obtains a decomposition for $\prod_{k=1}^{n-1}(1+\alpha m_{k,k+1})$. In particular for $\alpha = -1$ we get

$$\prod_{k=1}^{N-1}(1 - m_{k,k+1}) = \sum_J (-1)^{s(J)} m_J.$$

Now apply this with $m_{j,k} = \chi(N_t((a_j,a_k)) = 0)$. These satisfy the collapsing products almost surely under the probability P^C_∞. The pfaffian (15) shows that $E^C_\infty[m_J] = \mathrm{Pf}(E|_J)$ and so

$$\begin{aligned} & P^C_\infty\left[N_t(I_k) > 0 \text{ for } k = 1,\dots,2n-1\right] \\ = \;& E^C_\infty\left[\prod_{k=1}^{2n-1}(1 - m_{k,k+1})\right] \\ = \;& \sum_J (-1)^{s(J)} E^C_\infty\left[m_J\right] \\ = \;& \sum_J (-1)^{s(J)} \mathrm{Pf}(E|_J) \\ = \;& \mathrm{Pf}(I - E). \end{aligned} \tag{18}$$

We may apply the same argument for the annihilating case taking $m_{j,k} = (-1)^{N_t((a_j,a_k))}$, where $1 - m_{j,k} = 2\chi(N_t((a_j,a_k)) \text{ is odd})$, to find

$$P^A_\infty\left[N_t(I_k) \text{ is odd for } k = 1,2,\dots,2n-1\right] = 2^{1-2n}\mathrm{Pf}(I - A). \tag{19}$$

II. Let $O = O_{2n}$ be the $2n \times 2n$ anti-symmetric matrix formed by n copies of the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ down the diagonal and zeros elsewhere. Then $\mathrm{Pf}(O_{2n}) = 1$ and $\mathrm{Pf}(O_{2n}|_J) = 0$ unless $O_{2n}|_J$ is a copy of O_{2m} for some $m \in \{0,1,\dots,n\}$. This occurs either if J is empty or if J is of the form

$$\begin{aligned} J_1 \;=\; & \{2k_1 - 1, 2k_1, 2k_2 - 1, 2k_2, \dots, 2k_m - 1, 2k_m\} \\ & \text{for some } 1 \le k_1 < \dots < k_m \le n. \end{aligned} \tag{20}$$

Then formula (17) specializes to

$$\mathrm{Pf}(O - A) = \sum_{J_1} (-1)^{|J_1|/2} \mathrm{Pf}(A|_{J_1})$$

where the sum is over all J_1 of the form in (20) (including the empty set). We use another combinatorial identity, also straightforward by induction on n:

$$\prod_{k=1}^{n}(1 - m_{2k-1,2k}) = \sum_{J_1} (-1)^{|J_1|/2} m_{J_1}$$

where the sum is over all J_1 of the form in (20) (including the empty set). Arguing as in the previous example we find

$$P^C_\infty\left[N_t((a_j,a_k)) > 0 \text{ for } k = 1,3,5,\dots,2n-1\right] = \mathrm{Pf}(O - E). \tag{21}$$

As in (19), there is an analogous result for annihilating systems.

III. Let $\hat{O} = \hat{O}_{2n}$ be the $2n \times 2n$ anti-symmetric matrix with entries

$$\hat{O}_{ij} = \begin{cases} +1 & \text{if } i = 2,4,\dots,2n-2 \text{ and } j = i+1, \\ -1 & \text{if } j = 2,4,\dots,2n-2 \text{ and } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\hat{O}$ also has copies of the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in some places down the diagonal and zeros elsewhere. Then $\mathrm{Pf}\,(\hat{O}_{2n}) = 0$ and $\mathrm{Pf}\,(\hat{O}_{2n}|_J) = 0$ unless $\hat{O}_{2n}|_J$ is a copy of O_{2m} for some $m \in \{0,1,\dots,n-1\}$. This occurs either if J is empty or if J is of the form

$$\begin{aligned} J_2 \;=\; & \{2k_1, 2k_1+1, 2k_2, 2k_2+1, \dots, 2k_m, 2k_m+1\} \\ & \text{for some } 1 \le k_1 < \dots < k_m \le n-1. \end{aligned} \tag{22}$$

Then formula (17) specializes to

$$\mathrm{Pf}\,(A - \hat{O}) = \sum_{J_2} (-1)^{|J_2|/2} \mathrm{Pf}\,(A|^c_{J_2})$$

where the sum is over all J_2 of the form in (22) (including the empty set). The required combinatorial identity is

$$\prod_{k=1}^{n-1} (1 - m_{2k,2k+1}) \prod_{k=1}^{n} m_{2k-1,2k} = \sum_{J_2} (-1)^{|J_2|/2} m_{J_2}$$

where the sum is over all J_2 of the form in (20) (including the empty set). As in the earlier examples we find

$$\begin{aligned} & P^C_\infty\,[N_t(I_k) = 0 \text{ for } k = 1,3,\dots,2k-1 \text{ and } N_t(I_k) > 0 \text{ for } k = 2,4,\dots,2k-2] \\ & = \;\; \mathrm{Pf}\,(E - \hat{O}). \end{aligned} \tag{23}$$

Remarks. **1.** An alternative starting point, used by ben-Avraham et al. (see [2], [3]), is to show the empty interval probabilities $P^C_\infty[N_t(I_1) = N_t(I_3) = \dots = N_t(I_{2m-1}) = 0]$ satisfy heat equations in the variables (a_i), though the connection with pfaffians does not seem to have been noted.

2. Suppose $(M_{xy} : x \le y)$ is a bounded continuous field that satisfies $E[M_{x_1x_2}M_{x_3x_4}] = \mathrm{Pf}\,(E[M_{x_ix_j}] : 1 \le i < j \le 4)$ for $x_1 < \dots < x_4$. Then by continuity $E[M_{xx}M_{xx}] = \mathrm{Pf}\,(E[M_{xx}]) = E[M_{xx}]^2$ and so M_{xx} must be deterministic. This restriction prevented us exploring for some pfaffians which were not true.

3. The linear ordering of particles is crucial. For Brownian particles on a one-dimensional torus the product moment pfaffians hold true. The maximal entrance laws also hold as in (5) but where the stopping time τ is the first time that the open set S_t vanishes. This is not the same time as the first total annihilation time, since particles may annihilate by meeting in either direction on the torus. Thus the pfaffians in this section do not follow for motions on the torus.

4 Asymptotics for the n-point density function

We work in this section under the entrance measure P^C_∞ which, as explained in section 2.3, reflects the large t behaviour for many other initial conditions. The n-point density function $\rho^{(n)}_t(x)$ is a Lebesgue density for the measure $E^C_\infty[N_t(dx_1)\dots N_t(dx_n)]$ on V_n. The existence of this density, defined almost everywhere, and the simple bound

$$\rho^{(n)}_t(x) \le C_n t^{-n/2} \quad \text{for all } t>0 \text{ and } x \in V_n \tag{24}$$

is discussed in [9]. Furthermore there we established the following upper bound: for all $L>0$ there exists $C_L<\infty$ so that

$$\rho^{(n)}_t(x_1,\dots,x_n) \le C_L t^{-\frac{n}{2}-\frac{n(n-1)}{4}} \prod_{1\le i<j\le n} |x_i-x_j| \quad \text{for all } t>0 \text{ and } |x_i| \le Lt^{1/2}. \tag{25}$$

By thinning the corresponding density for annihilating systems differs only by a multiplicative factor 2^{-n}, so we need consider only the coalescing case.

We shall analyze first a modified density function $\tilde{\rho}^{(2n)}_t(x)$ for $x \in V_{2n}$, which is a density for the measure

$$E^C_\infty\left[N_t(dx_1)\dots N_t(dx_{2n})\chi(N_t(I_k)=0 \text{ for } k=1,3,\dots,2n-1)\right]$$

where we recall that $I_k=(x_k,x_{k+1})$. We claim that

$$\tilde{\rho}^{(2n)}_t(x_1,\dots,x_{2n}) = (2\pi t^2)^{-n/2}\mathrm{Pf}\left(\phi\left((x_j-x_i)/2t^{1/2}\right) : 1\le i<j\le 2n\right) \tag{26}$$

where $\phi(z)=z\exp(-z^2/2)$. To see this we use, for $\mu=\sum_i \delta_{x_i}$ a locally finite point measure with disjoint atoms, the distributional derivative on V_{2n}

$$\begin{aligned}
&\partial_{x_1}\dots\partial_{x_{2n}}\chi(\mu(x_k,x_{k+1})>0 \text{ for } k=1,2,\dots,2n-1) \\
&= \chi(\mu(x_k,x_{k+1})>0 \text{ for } k=1,3,5\dots,2n-1)\mu(dx_1)\dots\mu(dx_{2n}).
\end{aligned} \tag{27}$$

We illustrate how to check this by showing that, in the distributional sense on V_2,

$$\partial_x\chi(\mu(x,y)>0) = -\chi(\mu(x,y)=0)\mu(dx)\,dy.$$

Indeed, if f is smooth and compactly supported in V_2 then

$$\begin{aligned}
&\int_{\mathrm{R}^2} f(x,y)\chi(\mu(x,y)=0)\mu(dx)\,dy \\
&= \sum_i \int_{\mathrm{R}} f(x_i,y)\chi(\mu(x_i,y)=0,\, x_i<y)\,dy \\
&= \int_{\mathrm{R}^2} \partial_x f(x,y)\left(\sum_i \chi(\mu(x_i,y)=0,\, x<x_i<y)\right) dx\,dy \\
&= \int_{\mathrm{R}^2} \partial_x f(x,y)\chi(\mu(x,y)>0)\,dx\,dy
\end{aligned}$$

since at most one term in the sum over i is non-zero. Iterating such calculations leads to (27). From (27) we have, for smooth f compactly supported in V_{2n},

$$\begin{aligned}
&\int_{V_{2n}} f(x_1,\dots,x_{2n})\tilde{\rho}_t^{(2n)}(x_1,\dots,x_{2n})dx_1\dots dx_{2n} \\
&= E^C_\infty\left[\int_{V_{2n}} f(x_1,\dots,x_{2n})\chi(N_t(I_k)=0 \text{ for } k=1,3,\dots,2n-1)N_t(dx_1)\dots N_t(dx_{2n})\right] \\
&= E^C_\infty\left[\int_{V_{2n}} \partial_{x_1}\dots\partial_{x_{2n}} f(x_1,\dots,x_{2n})\chi(N_t(I_k)>0 \text{ for } k=1,2,\dots,2n-1)dx_1\dots dx_{2n}\right] \\
&= \int_{V_{2n}} \partial_{x_1}\dots\partial_{x_{2n}} f(x_1,\dots,x_{2n})\mathrm{Pf}\,(1-2F((2t)^{-1/2}(x_j-x_i)))\,dx_1\dots dx_{2n} \\
&= (2\pi t^2)^{-n/2}\int_{V_{2n}} f(x_1,\dots,x_{2n})\mathrm{Pf}\,(\phi((2t)^{-1/2}(x_j-x_i)))\,dx_1\dots dx_{2n}.
\end{aligned}$$

In the last step we have passed the derivatives onto the pfaffian, which is smooth since F is smooth. We have used, for twice differentiable $\Psi : \mathrm{R}\to\mathrm{R}$, that

$$\partial_{x_1}\dots\partial_{x_{2n}}\mathrm{Pf}\,(\Psi(x_j-x_i)) = \mathrm{Pf}\,(-\Psi''(x_j-x_i))$$

which follows from the permutation expansion (8) of the pfaffian. This yields the claim (26).

The key tool for the asymptotics will be the following lemma, proved at the end of this section, that gives an expansion for a pfaffian whose entries are close to the zero of an odd function.

Lemma 1 *Let $\phi : R\to R$ be an odd function that is analytic at zero. Then for any $n\geq 1$ there exist $\epsilon(n,\phi)>0$ and $C(n,\phi)<\infty$ so that for $y\in V_n$ with $|y|\leq\epsilon(n,\phi)$*

$$Pf(\phi(y_j-y_i) : 1\leq i<j\leq 2n) = \prod_{1\leq i<j\leq 2n}(y_j-y_i)\,Pf(J^{(2n)}(\phi)+R^{(2n)}(y))$$

where $J^{(2n)}(\phi)$ is the constant anti-symmetric matrix with entries

$$J^{(2n)}_{ij}(\phi) = (-1)^{j-1}\frac{1}{(i-1)!(j-1)!}\frac{d^{i+j-2}\phi}{dx^{i+j-2}}(0) \quad \text{for } 1\leq i<j\leq 2n, \tag{28}$$

and the remainder $R^{(2n)}(y)$ is a anti-symmetric matrix satisfying

$$\left|R^{(2n)}_{ij}(y)\right|\leq C(n,\phi)|y| \quad \text{for all } i,j \text{ and } |y|\leq\epsilon(n,\phi).$$

We apply this lemma to the pfaffian in (26) with $\phi(z)=ze^{-z^2/2}$ and with $y_i=(2t)^{-1/2}x_i$ for t large enough. Expanding the pfaffian $\mathrm{Pf}\,(J^{(2n)}(\phi)+R^{(2n)}((2t)^{-1/2}x))$ using (17) we find only one term, namely $\mathrm{Pf}\,(J^{(2n)}(\phi))$, that does not decay as $t\to\infty$. This shows that

$$\lim_{t\to\infty} t^{n^2+\frac{n}{2}}\tilde{\rho}_t^{(2n)}(x_1,\dots,x_{2n}) = c_{2n}\prod_{1\leq i<j\leq 2n}(x_j-x_i)\,\mathrm{Pf}\,(J^{(2n)}(\phi))$$

where $c_{2n} = \pi^{-n/2} 2^{-n^2}$.

To obtain the same estimate for $\rho^{(2n)}$ we estimate the difference as follows.

$$\begin{aligned}
0 &\leq \rho_t^{(2n)}(x_1,\ldots,x_{2n}) - \tilde{\rho}_t^{(2n)}(x_1,\ldots,x_{2n}) \\
&= E_\infty^C \left[N_t(dx_1)\ldots N_t(dx_{2n})\chi(N_t(I_k) > 0 \text{ for some } k = 1,3,\ldots,2n-1) \right] \\
&\leq \sum_{k=1}^{2n-1} E_\infty^C \left[N_t(dx_1)\ldots N_t(dx_{2n}) N_t(I_k) \right] \\
&= \sum_{k=1}^{2n-1} \int_{I_k} \rho_t^{(2n+1)}(x_1,\ldots,x_k,z,x_{k+1},\ldots,x_{2n}) dz.
\end{aligned}$$

Each term in this sum is of a smaller order in t by (25).

Examination of the proof shows that we need not let the values of $x_1,\ldots,x_{2n}$ be fixed, and that in fact

$$\sup_{|x_i| << t^{1/2}} \left| \frac{\rho_t^{(2n)}(x_1,\ldots,x_{2n})}{t^{-n-\frac{n(2n-1)}{2}} \prod_{1\leq i<j\leq 2n} |x_i - x_j|} - c_{2n} \mathrm{Pf}\left(J^{(2n)}(\phi)\right) \right| \to 0 \quad \text{as } t \to \infty,$$

where $|x_i| << t^{1/2}$ means that we may take the supremum over any positions $(x_i(t))$ provided that $\sup_i |x_i(t)| t^{-1/2} \to 0$ as $t \to \infty$.

Proof of Lemma 1. Let Φ be the $2n \times 2n$ anti-symmetric matrix with entries given by $\Phi_{ij} = \phi(y_j - y_i)$. The aim is to show, for small y, that

$$\Phi = V^T(J+R)V$$

where J and R are as in the lemma (with n fixed and suppressed) and V is the $2n \times 2n$ Vandermond matrix given by $V_{ij} = y_j^{i-1}$. Since $\det(V) = \prod_{1\leq i<j\leq 2n}(y_j - y_i)$, the conclusion then holds from $\mathrm{Pf}(V^T(J+R)V) = \det(V)\mathrm{Pf}(J+R)$

For small $|y|$ we expand by analyticity (writing $\phi^k(0)$ for the kth derivative of ϕ at zero)

$$\begin{aligned}
\Phi_{ij} &= \sum_{n=0}^{\infty} \frac{1}{n!} \phi^n(0)(y_j - y_i)^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \phi^n(0) y_j^k (-y_i)^{n-k} \\
&= \sum_{k,l=0}^{\infty} \frac{1}{k!\,l!} \phi^{k+l}(0) y_j^k (-y_i)^l \\
&= \sum_{k,l=1}^{\infty} y_i^{l-1} y_j^{k-1} J_{lk} \qquad (29)
\end{aligned}$$

where we have rearranged using $l = n - k$ in the penultimate equality. Note that

$$(V^T J V)_{ij} = \sum_{k,l=1}^{2n} V_{li} V_{kj} J_{lk} = \sum_{k,l=1}^{2n} y_i^{l-1} y_j^{k-1} J_{lk}.$$

It remains to re-express the remaining terms in (29) as the desired remainder.

Recall the symmetric polynomials $\sigma_k^{2n}(y)$ defined for $y \in \mathrm{R}^{2n}$ by

$$\prod_{k=1}^{2n}(y_k - \lambda) = \sum_{k=0}^{2n}(-1)^k \sigma_k^{2n}(y)\lambda^{2n-k}. \tag{30}$$

Note that σ_k^{2n} is a polynomial of order k. Since $\sigma_0^{2n} \equiv 1$ we may choose $\lambda = y_i$ to see that

$$0 = y_i^{2n} + \sum_{k=1}^{2n}(-1)^k \sigma_k^{2n}(y) y_i^{2n-k} \qquad \text{for } i = 1, \ldots, 2n.$$

Multiplying by y_i^p we see that

$$y_i^{p+2n} = \sum_{k=1}^{2n}(-1)^{k+1}\sigma_k^{2n}(y) y_i^{p+2n-k} \qquad \text{for } i = 1, \ldots, 2n \text{ and } p = 0, 1, \ldots. \tag{31}$$

By iterating this we may express y_i^{p+2n} for $p \geq 0$ as a mixture of $1, y_i, y_i^2, \ldots, y_i^{2n-1}$, as follows:

$$y_i^{p+2n} = \sum_{k=1}^{2n} \tau_k^{2n,p+2n}(y) y_i^{k-1} \qquad \text{for } i = 1, \ldots, 2n \text{ and } p = 0, 1, \ldots \tag{32}$$

where $\tau_k^{2n,p+2n}(y)$ is a polynomial of order $p + 2n - k + 1$. Using this substitution in the remaining terms of (29), that is where k or l is at least $2n + 1$, we find (formally) that

$$\left(\sum_{k,l=2n+1}^{\infty} + \sum_{k=1}^{2n} \sum_{l=2n+1}^{\infty} + \sum_{l=1}^{2n} \sum_{k=2n+1}^{\infty} \right) y_i^{l-1} y_j^{k-1} J_{lk} = \sum_{p,q=1}^{2n} y_i^{p-1} y_j^{q-1} R_{pq}(y)$$

where

$$\begin{aligned} R_{pq}(y) \quad = \quad & \sum_{k,l=2n+1}^{\infty} \tau_p^{2n,l-1}(y) J_{kl} \tau_q^{2n,k-1}(y) \\ & + \sum_{k=1}^{2n} \sum_{l=2n+1}^{\infty} \tau_p^{2n,l-1}(y) J_{ql} + \sum_{l=1}^{2n} \sum_{k=1}^{\infty} \tau_q^{2n,k-1}(y) J_{kp}. \end{aligned} \tag{33}$$

Note the lowest order of the polynomial entries in the terms for R_{pq} is of order 1. In the appendix we check that this rearrangement of (29) is valid when $|y|$ is suitably small and that the required error bound $|R_{pq}(y)| \leq C(n, \phi)|y|$ holds.

5 Appendix

5.1 Details for section 2.3

We give a few details on (one approach to) the results surveyed in section 2.3. For coalescing systems one can use monotonicity, adding initial particles one by one, to construct the infinite system. This is not available for annihilating systems, so we sketch a weak convergence argument that applies to both systems.

One can control moments by bounds on the n-point density function. Indeed $\rho_t^{(n)}(x)$, the density for the measure $E^C_{(x_i)}[N_t(dx_1)\dots N_t(dx_n)]$ on V_n, depends on the initial condition, but satisfies the bound $\rho_t^{(n)}(x) \le C_n t^{-n/2}$ uniformly over all possible finite initial conditions (x_i). This follows by duality for $n=1$ and by anti-correlation for $n>1$ (see [9]). It follows that $E^C_{(x_i)}[N^p_t(a,b)]$ is bounded, for each $t,p>0$, $a,b \in \mathrm{R}$, uniformly over finite initial conditions (x_i).

Fix $\mu \in \mathcal{M}_0$ and take finite measures μ_n so that $\mu_n \to \mu$ (recall we are using vague convergence). The moment bounds above imply that the laws of N_t on $\mathcal{M}_{LFP}$ under $P^C_{\mu_n}$ are tight. Take a subsequence n' along which they converge to a limit, which we denote Q. The functions

$$\nu \to F_{(a_i)}(\nu) := \chi(\nu(I_1) = \nu(I_3) = \ldots = \nu(I_{2n-1}) = 0)$$

are discontinuous on $\mathcal{M}_{LFP}$. However the first moment bounds $E^C_{(x_i)}[N_t(a,b)] \le C(t)(b-a)$ hold also for the limit law Q and imply that $\nu(\{a_i\}) = 0$, $Q(d\nu)$ almost surely. This shows that Q does not charge the discontinuity set of $F_{(a_i)}$. Then we may pass to the limit in (2) to deduce that

$$\int F_{(a_i)}(\nu) Q(d\nu) = P^A_{(a_i)}\left[S_t \cap \mathrm{supp}(\mu) = \emptyset\right]. \tag{34}$$

These functionals do not characterize a law on $\mathcal{M}_{LFP}$, but they do characterize a law that is supported on $\mathcal{M}_0$. To see this note that for $\nu \in \mathcal{M}_0$

$$\nu([x,y]) = \lim_{N\to\infty} \sum_k \chi\left(\nu([x,y] \cap (\frac{k}{N}, \frac{k+1}{N}]) > 0\right).$$

From this one may use (34) to find $\int \nu([x_1,y_1])\dots\nu([x_n,y_n])Q(d\nu)$ which, by the moment bounds, determine Q. To see that Q is supported on $\mathcal{M}_0$ note that

$$P^C_{\mu_n}\left[N_t(a,b) \ge 2\right] \le \int_a^b \int_a^b \rho_t^{(2)}(x_1,x_2)\, dx_1 dx_2 \le C(t)(b-a)^2.$$

This bound holds uniformly over n and hence also for the limit law Q. Then the conclusion follows from the usual covering argument, for instance

$$\begin{aligned} & Q\left[\ \mu(\{x\} > 1 \text{ for some } x \in [-L,L]\right] \\ \le & \sum_{k=-LN}^{LN} Q\left[\mu([k/N,(k+1)/N]) \ge 2\right] \le C(L,t)N^{-1}. \end{aligned}$$

Thus the law Q is determined and we may define $p_t(\mu, d\nu)$ to equal $Q(d\nu)$.

The remainder of the results in section 2.3 follow using similar tools. For example, for the continuity of $\mu \to p_t(\mu, d\nu)$, that is the Feller property, suppose that $\mu_n \to \mu$ in $\mathcal{M}_0$.

The moment bounds, which still hold for infinite initial conditions, imply the tightness of $p_t(\mu_n, d\nu)$. Passing to the limit in

$$\int F_{(a_i)}(\nu) p_t(\mu_n, d\nu) = P^A_{(a_i)} \left[S_t \cap \text{supp}(\mu_n) = \emptyset \right].$$

shows that any limit point of $p_t(\mu_n, d\nu)$ must be $p_t(\mu, d\nu)$. The semigroup property, for bounded continuous $F : \mathcal{M}_{LFP} \to \mathrm{R}$,

$$\int F(\nu) p_{t+s}(\mu, d\nu) = \int \int F(\nu') p_s(\nu, d\nu') p_t(\mu, d\nu), \tag{35}$$

which is valid for finite measures μ extends by approximation, using the Feller property, to hold for $\mu \in \mathcal{M}_0$. The same tightness and characterization methods establish the existence of a law characterized by (5), and that many initial laws are attracted to it as in (6). That (5) determines an entrance law can be established by passing to the limit in (35) along $\mu = \sum_k \delta_{\lambda^{-1} k}$ as $\lambda \to \infty$.

The annihilating case follows the same lines, with moments controlled since the n-point density and moments for annihilating systems are bounded by the corresponding coalescing system. The coalescing duality formula (2) is replaced by the annihilating duality formula (3), and to see that this will characterize the law note that for $\nu \in \mathcal{M}_0$

$$\nu([x,y]) = \lim_{N \to \infty} \sum_k \left(1 - (-1)^{\nu\left([x,y] \cap (\frac{k}{N}, \frac{k+1}{N}]\right)} \right).$$

5.2 Details for section 4

Here we give the error estimates for the pfaffian expansion Lemma 1.

The product (30) that defines the symmetric polynomials σ^{2n}_k yields a total of 2^{2n} monomials so we have the simple bound $|\sigma^{2n}_k(y)| \leq 2^{2n} |y|^k$. The expansion (31) must be iterated at most p times to derive (32) and this leads to to the bound

$$|\tau^{2n,p+2n}_k(y)| \leq (2n 2^{2n})^p |y|^{p+2n-k+1}. \tag{36}$$

Using this we may bound the size of the remainder terms given in (33). For example

$$\begin{aligned}
& \sum_{k,l=2n+1}^{\infty} |\tau^{2n,l-1}_p(y)| \, |J_{kl}| \, |\tau^{2n,k-1}_q(y)| \\
\leq \; & \sum_{k,l=2n+1}^{\infty} \frac{1}{(k-1)!(l-1)!} \, |\phi^{k+l-2}(0)| \, (2n 2^{2n})^{l+k-4n-2} |y|^{l+k-p-q} \\
\leq \; & |y|^2 \sum_{k,l=2n+1}^{\infty} \frac{1}{(k-1)!(l-1)!} \, |\phi^{k+l-2}(0)| \, (2n 2^{2n} \epsilon)^{l+k-4n-2} \quad \text{when } |y| \leq \epsilon \\
\leq \; & |y|^2 \sum_{r=4n}^{\infty} \sum_{|s| \leq r-4n} \frac{2^r}{r!} \, |\phi^r(0)| \, (2n 2^{2n} \epsilon)^{r-4n}
\end{aligned}$$

$$\text{using } r = k+l-2,\ s = k-l \text{ and } \tfrac{k!\,l!}{(k+l)!} \geq 2^{-k-l}$$
$$\leq\ 2^{4n}|y|^2 \sum_{r=4n}^{\infty} \frac{1}{r!}\,|\phi^r(0)|\,2r(4n2^{2n}\epsilon)^{r-4n}.$$

Choosing $\epsilon = \epsilon(n,\phi)$ so that $4n2^{2n}\epsilon$ lies in the radius of convergence of ϕ we obtain a convergent series. Similarly

$$\begin{aligned}
\sum_{l=2n+1}^{\infty} |\tau_p^{2n,l-1}(y)|\,|J_{ql}| \ &\leq\ \sum_{l=2n+1}^{\infty} \frac{1}{(q-1)!\,(l-1)!}\,|\phi^{q+l-2}(0)|\,(2n2^{2n})^{l-2n-1}|y|^{l-p} \\
&\leq\ |y| \sum_{l=2n+1}^{\infty} \frac{1}{(l-1)!}\,|\phi^{q+l-2}(0)|\,(2n2^{2n}\epsilon)^{l-2n-1} \\
&\leq\ |y| \sum_{l=2n+1}^{\infty} \frac{1}{(q+l-2)!}\,|\phi^{q+l-2}(0)|\,(2n2^{2n}\epsilon)^{l-2n-1}(l+2n)^{2n} \\
&\leq\ C(n,\phi)|y|.
\end{aligned}$$

A similar bound holds for the final term in (33). Combining the estimates yields the desired error bound on $R^{(2n)}_{pq}$. Moreover these bounds show the absolute convergence that justifies the rearrangement of the series (29) used in Lemma 1 provided that $|y| \leq \epsilon(n,\phi)$.

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