Singular solutions to the heat equations with nonlinear absorption and Hardy potentials

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Abstract

We study the existence and nonexistence of singular solutions to the equation $u_t - \Delta u - \frac{\kappa}{|x|^2}u + |x|^\alpha u|u|^{p-1} = 0$, $p > 1$, in $\mathbb{R}^N \times [0, \infty)$, $N \geq 3$, with a singularity at the point $(0, 0)$, that is, nonnegative solutions satisfying $u(x, 0) = 0$ for $x \neq 0$, assuming that $\alpha > -2$ and $\kappa < \left(\frac{N-2}{2}\right)^2$. The problem is transferred to the one for a weighted Laplace-Beltrami operator with a non-linear absorbtion, absorbing the Hardy potential in the weight. A classification of a singular solution to the weighted problem either as a *source solution* with a multiple of the Dirac mass as initial datum, or as a unique *very singular solution*, leads to a complete classification of singular solutions to the original problem, which exist if and only if $p < 1 + \frac{2(2+\alpha)}{N+2+\sqrt{(N-2)^2-4\kappa}}$.

1 Introduction and main results

In this paper we study nontrivial nonnegative solutions in $\mathbb{R}^N \times [0, \infty)$ to the equation

$$u_t - \Delta u - \frac{\kappa}{r^2}u + r^\alpha u|u|^{p-1} = 0, \tag{1.1}$$

vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ that is, (non-trivial) nonnegative solutions to (1.1) satisfying

$$u(x, 0) = 0, \text{ that is, } \lim_{t\to 0} u(x, t) = 0 \text{ for } x \neq 0. \tag{1.2}$$

Here and below $r = |x|$, and we always assume that with $N \geq 3$, $\kappa < \left(\frac{N-2}{2}\right)^2$ and $\alpha > -2$. The behaviour of $u(x, t)$ as $(x, t) \to (0, 0)$ is not prescribed, so we study solutions with possible singularity at $(0, 0)$.

We remark here for further reference that there is no ambiguity in the last definition since for a solution u of (1.1), $u(x, t)dx \to 0$ as $t \to 0$ in the sense of weak-$*$ convergence of measures on $\mathbb{R}^N \setminus \{0\}$ if and only if $u(x, t) \to 0$ as $t \to 0$ locally uniformly in $x \in \mathbb{R}^N \setminus \{0\}$, by the same argument as in [5, Proof of Theorem 2, steps 2,3].

The nonlinear heat equation with absorption

$$u_t - \Delta u + u|u|^{p-1} = 0, \tag{1.3}$$

i.e. (1.1) with $\kappa = \alpha = 0$, with bounded measures as initial data was first studied by Brezis and Friedman in the seminal work [5], where it was proved that the solution to (1.3) with $u(x,0) = \varkappa\delta_0(x)$, a multiple of the Dirac mass at 0, exists and is unique if and only if $0 < p < 1 + \frac{2}{N}$. The solution obtained has roughly speaking the same behaviour as $t \to 0$ as the fundamental solution to the linear heat equation. Such solutions are referred to as *source type solutions* (**SS**).

In [6] for $1 < p < 1 + \frac{2}{N}$ a new nonlinear phenomenon was discovered, namely, a new solution to (1.3) satisfying (1.2) was found. This solution is more singular at $t \to 0$ than the fundamental solution, it is self-similar of the form $t^{-\frac{1}{p-1}} f(|x|/\sqrt{t})$, where f is a unique solution to a certain ordinary differential equation. This solution in [6], called *very singular solution*, was constructed by the shooting method. Later in [12] the existence of very singular solutions was proved by the variational approach. In [16] the very singular solution was shown to be a monotone limit of source type solutions. A classification of all positive singular solutions to (1.3) was given in [26]. The cited papers state that for $p \in \left(1, 1 + \frac{2}{N}\right)$ every singular solution to (1.1) satisfying (1.2) is either source type solution, satisfying $u(x,t)dx \to \varkappa\delta_0$ as $t \to 0$ in the sense of weak convergence of measures, with $\varkappa = \lim\limits_{t\to 0} \int\limits_{\{|x|<1\}} u(x,t)dx$, or u is the unique very singular solution (**VSS**), the only one satisfying $\lim\limits_{t\to 0} \int\limits_{\{|x|<1\}} u(x,t)dx = \infty$. For $p \geq 1 + \frac{2}{N}$ there are no non-trivial positive solutions to (1.3) satisfying (1.2). Recently the problem of singular solutions to (1.1) in the case $\kappa = 0$, $\alpha > -2$ was treated by Shishkov and Veron [27]. They showed that the qualitative picture is the same as for (1.3), but the critical exponent changes from $1 + \frac{2}{N}$ for equation (1.1) to $1 + \frac{2+\alpha}{N}$ for the equation $u_t - \Delta u + r^\alpha u|u|^{p-1} = 0$. In all the above result a crucial role is played by the following *a priori* estimates of Keller-Osserman type for a singular solution to (1.1) (with $\kappa = 0$), which is a generalization of the classical one due to Brezis and Friedman [5] in the case $\alpha = 0$:

$$u(x,t) \leq c\left(|x|^2 + t\right)^{-\frac{2+\alpha}{2(p-1)}}. \tag{1.4}$$

The critical exponent $1 + \frac{2+\alpha}{N}$, which reflects the nonexistence of singular solutions, can be seen as a result of comparing the behaviour at $(0,0)$ of the fundamental solution to the linear problem with estimate (1.4). This plausible argument can no longer be applied to (1.1) since the fundamental solution to the equation $u_t - \Delta u - \frac{\kappa}{r^2}u = 0$ does not exist at $x = 0$ (except for the case $\kappa = 0$) for the reasons explained below.

During the last decades there is growing interest to the elliptic and parabolic problems involving the inverse-square potential (Hardy potential), stemming from its criticality. Many qualitative properties of solutions are affected by the presence of the Hardy potential, which leads to occurrence of a number of interesting unusual phenomena [1, 3, 24, 29]. This is mainly due to the properties of the corresponding linear equation $u_t - \Delta u - \frac{\kappa}{r^2}u = 0$, which are significantly different from the properties of the heat equation. In particular, the linear equation does not have the fundamental solution at zero, i.e. the Cauchy problem with $\delta_0(x)$ as the initial datum has no solution, which can be seen from by now well known two-sided estimates for the corresponding heat kernel $p(t,x,y)$[21, 23, 25]:

$$\frac{c_1}{t^{\frac{N}{2}}}\left(\frac{|x|}{|x|+\sqrt{t}}\right)^{\lambda}\left(\frac{|x|}{|x|+\sqrt{t}}\right)^{\lambda} e^{-\frac{|x-y|^2}{c_2 t}} \leq p(t,x,y) \leq \frac{c_3}{t^{\frac{N}{2}}}\left(\frac{|x|}{|x|+\sqrt{t}}\right)^{\lambda}\left(\frac{|y|}{|y|+\sqrt{t}}\right)^{\lambda} e^{-\frac{|x-y|^2}{c_4 t}},$$

where here and below $\lambda = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \kappa}$ is the bigger root of the quadratic equation $\lambda^2 + \lambda(N-2) + \kappa = 0$ and c_1, c_2, c_3, c_4 are some positive constants. Moreover, for $\kappa > 0$ and $N \geq 3$ the Cauchy problem is not well posed in $L^p(\mathbb{R}^N)$, $p \in \left[1, \frac{N}{N+\lambda}\right)$ [22]. The semilinear equations with Hardy potentials and nonlinear excitation were recently studied in [24] and some interesting nonuniqueness phenomena were discovered, but to our knowledge the corresponding equation with nonlinear absorption, equation (1.1), has not been yet studied. This is exactly the aim the present paper. In the course of this study we will reveal several interesting phenomena peculiar to equation (1.1). In order to overcome the difficulties described above and to classify solutions to (1.1) satisfying (1.2) we will use the technique of transference to the weighted space, which is by now standard in the linear theory and is called the *ground state* transform (cf. [21, 24, 23, 25]). We will outline it here.

Below and further on we use the following notation for the weighted Lebesgue and Sobolev spaces. For a weight φ we denote

$$L^p_\varphi(\mathbb{R}^N) := \{f : \mathbb{R}^N \to \mathbb{R};\ \int_{\mathbb{R}^N} |f|^p \varphi dx < \infty\},$$
$$H^1_\varphi(\mathbb{R}^N) := \{f : \mathbb{R}^N \to \mathbb{R};\ \int_{\mathbb{R}^N} (|\nabla f|^2 + |f|^2)\varphi dx < \infty\}.$$

Let $h \in H^1_{loc}$ satisfy $\Delta h + \frac{\kappa}{r^2} h = 0$, that is, $h = r^\lambda$, with $\lambda > -\frac{N-2}{2}$ as above. The change of variables $\tilde{u} := u/h$ is a unitary operator $L^2 := L^2(\mathbb{R}^N, dx) \to L^2_{h^2} := L^2(\mathbb{R}^N, h^2 dx)$. Moreover, the quadratic form $\mathcal{E}(u) = \int |\nabla u|^2 dx - \int \frac{\kappa}{r^2} u^2 dx$ on L^2 is isomorphic to a quadratic form $\mathcal{E}_h(\tilde{u}) = \int |\nabla \tilde{u}|^2 h^2 dx$ on $L^2_{h^2}$, which is precisely stated in the next proposition

Proposition 1.1. *Let $\kappa < \frac{(N-2)^2}{4}$. Let $\mathcal{E}$ be the closed quadratic form in L^2 defined by*

$$\mathcal{E}(u) = \int |\nabla u|^2 dx - \int \frac{\kappa}{r^2} |u|^2 dx, \quad u \in H^1(\mathbb{R}^N)$$

and H be the associated self-adjoint operator, $H = -\Delta - \frac{\kappa}{r^2}$ (form-sum). Let $h \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be a positive weak solution to the equation $Hh = 0$.

Then the unitary map $U : L^2 \to L^2_{h^2}$, $Uu = \frac{u}{h}$, maps H to the operator $-\Delta_{h^2}$ associated with the form

$$\mathcal{E}_h(u) = \|\nabla u\|^2_{L^2_{h^2}}, \ u \in H^1_{h^2}(\mathbb{R}^N).$$

Proof. First observe that $h \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and that $h > 0$. Hence $h^{\pm 1} C^\infty_c(\mathbb{R}^N \setminus \{0\}) = C^\infty_c(\mathbb{R}^N \setminus \{0\})$. Note that $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is a core of the form $\mathcal{E}$. The image of $\mathcal{E}$ on $L^2_{h^2}$ is given by $\mathcal{E}_h(\varphi) = \mathcal{E}(h\varphi)$. For $\phi \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ one has

$$|\nabla(h\phi)|^2 = h^2|\nabla\phi|^2 + 2h\phi\nabla\phi\nabla h + \phi^2|\nabla h|^2 = h^2|\nabla\phi|^2 + \nabla(h\phi^2)\nabla h.$$

Taking into account that h is a weak solution to the equation $\Delta u + \frac{\kappa}{r^2} u = 0$, we obtain

$$\mathcal{E}(h\phi) = \int \left(|\nabla(h\phi)|^2 - \frac{\kappa}{r^2} h^2\phi^2\right) dx = \int h^2|\nabla\phi|^2 dx + \left(\nabla(h\phi^2)\cdot\nabla h - \frac{\kappa}{r^2}(h\phi^2)h\right) dx = \|\phi\|^2_{L^2_{h^2}}.$$

Since $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is invariant under multiplication by $h^{\pm 1}$, it is also a core for $\mathcal{E}_h$. The assertion follows. □

As a result, the operator $-\Delta - \frac{\kappa}{r^2}$ on L^2 is isomorphic to the weighted Laplacian $-\Delta_{h^2} := -\frac{1}{h^2}\nabla \cdot h^2 \nabla$ on $L^2_{h^2}$. Recall that $h = r^\lambda$. So equation (1.1) takes the form

$$\tilde{u}_t - r^{-2\lambda}\nabla(r^{2\lambda}\nabla\tilde{u}) + r^\beta \tilde{u}|\tilde{u}|^{p-1} = 0, \tag{1.5}$$

with $\beta := \alpha + \lambda(p-1)$.

This motivates the following problem about singular solutions with the weighted Laplacian which is of independent interest.

Assume that u is a weak solution to the equation

$$\partial_t u - r^{-2\lambda}\nabla \cdot (r^{2\lambda}\nabla u) + r^\beta |u|^{p-1}u = 0, \tag{1.6}$$

satisfying

$$\int_{\mathbb{R}^N} u(t)\theta\, h^2 dx \to 0 \quad \text{as } t \to 0, \text{ for all } \theta \in C_c(\mathbb{R}^N \setminus \{0\}). \tag{1.7}$$

We say that $u \in L^2_{loc}\Big((0,\infty);\ H^1_{loc}(\mathbb{R}^N, h^2 dx)\Big) \cap L^{p+1}_{loc}\Big(\mathbb{R}^N \times (0,\infty), r^\beta h^2 dx dt\Big)$ is a weak solution to (1.6) if it satisfies the integral identity

$$\int u(t_1)\zeta(t_1)h^2 dx + \int_{t_0}^{t_1}\int (\nabla u \cdot \nabla\zeta)h^2 dx\, dt + \int_{t_0}^{t_1}\int r^\beta |u|^{p-1}u\zeta h^2 dx\, dt$$

$$= \int u(t_0)\zeta(t_0)h^2 dx + \int_{t_0}^{t_1}\int u\partial_t\zeta\, h^2 dx\, dt \tag{1.8}$$

for all $\zeta \in C^1\big((0,\infty);\ C_c(\mathbb{R}^N)\big) \cap L^2_{loc}\big((0,\infty);\ H^1_{h^2}\big)$ and $0 < t_0 < t_1 < \infty$.

The main results concerning the solutions to (1.6) satisfying (1.7) are collected in the following theorem.

Theorem 1.2. *Let $p > 1$. Let λ, β be any real numbers such that $\lambda > -\frac{N-2}{2}$, $\beta > -2$. Denote $p^* = 1 + \frac{2+\beta}{N+2\lambda}$. Then*

(a) for any weak solution u to (1.6) satisfying (1.7) the following Keller-Osserman type estimate holds: there exists $c > 0$ such that for all $x \in \mathbb{R}^N$ and $t > 0$

$$|u(x,t)| \le c\left(|x|^2 + t\right)^{-\frac{2+\beta}{2(p-1)}};$$

(b) for every singular solution to (1.6) there exists $\varkappa \in [0,\infty]$ such that $u(t)h^2 dx \to \varkappa\delta_0$ as $t \to 0$ in the weak-$*$ topology of Radon measures;

(c) for $p \ge p^*$, the only solution to (1.6) satisfying (1.7) is zero;

(d) for $p < p^*$ and $\varkappa \in (0,\infty]$ there exists a unique singular solution $u_\varkappa$ to (1.6) satisfying $u_\varkappa(t)h^2 dx \to \varkappa\delta_0$ as $t \to 0$ in the weak-$*$ topology of Radon measures;

(e) for $p < p^*$ and $\varkappa \in (0,\infty)$, $u_\varkappa$ satisfies $u_\varkappa(\cdot,t) - \varkappa p^{h^2}(t,\cdot,0) \to 0$ in $L^1\left(\mathbb{R}^N, r^{2\lambda}dx\right)$ as $t \to 0$, where p^{h^2} is the heat kernel for the weighted Laplacian $\Delta_{h^2} := r^{-2\lambda}\nabla \cdot r^{2\lambda}\nabla$;

(f) for $p < p^*$, u_∞ is self-similar, that is $u_\infty(x,t) = t^{-\frac{2+\beta}{2(p-1)}} v(x/\sqrt{t})$, with $v(x) \le Ce^{-|x|^2/8}$ for some $C > 0$.

The proof of Theorem 1.2 is given in Sections 2–5. In fact, the results for the case of the initial data $\varkappa\delta$ are obtained as a special case of Radon measures as initial data, as it is done in [5]. We extend the results of Brezis and Friedman [5] and Véron [30, Chapter 6, Theorem 6.12] to the case of equations with the generator of a symmetric ultracontractive Feller semigroup in place of the Laplacian. This allows for a much wider range of applications such as the fractional Laplacian, symmetric subelliptic operators and many more (for further examples see, e.g. [13]).

The solution to the original problem (1.1), (1.2) is contained in the next corollary, which is a pull back of Theorem 1.2.

Corollary 1.3. *Let $p > 1$, $\kappa < \left(\frac{N-2}{2}\right)^2$, $\lambda = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 - \kappa}$, $\alpha > -2$. Denote $p^{**} = 1 + \frac{2+\alpha}{N+\lambda}$. Then*

a) for $p \ge p^{**}$, there are no singular solutions to (1.1). More precisely, the only solution to (1.1) satisfying (1.2) is zero;

(b) for $p < p^{**}$ and $\varkappa \in (0,\infty]$, there exists a unique singular solution $u_\varkappa$ to (1.1) satisfying $\lim\limits_{t\to 0} \int_{\{|x|<\rho\}} u_\varkappa(x,t)|x|^\lambda dx = \varkappa$ for all $\rho > 0$. The map $\varkappa \to u_\varkappa$ is a bijection between $(0,\infty]$ and the set of nontrivial singular solutions to (1.1);

(c) for $p < p^{**}$, the very singular solution u_∞ is self-similar, $u_\infty(x,t) = t^{-\frac{2+\alpha}{2(p-1)}} v(x/\sqrt{t})$ with $v(x) \le C|x|^\lambda e^{-|x|^2/8}$ for some $C > 0$.

Remark 1.4. *The above corollary shows that the Lebesgue measure does not allow for a classification of singular solutions to (1.1). To demonstrate this let α, κ, λ and p^{**} be as in the preceding corollary.*

1. *For $\kappa < 0$ (hence $\lambda > 0$) and $p \in (1, p^{**})$ every non-trivial positive singular solution u to (1.1) satisfies $u(x,t) = O(|x|^\lambda)$ as $x \to 0$ for all $t > 0$, and $\int_{\{|x|<\rho\}} u(x,t)dx \to \infty$ as $t \to 0$ for all $\rho > 0$.*
2. *for $\kappa > 0$ (hence $\lambda < 0$) and $p \in (1, p^{**})$, given $\varkappa \in (0,\infty)$, one has $\int_{\{|x|<\rho\}} u_\varkappa(x,t)dx \to 0$ as $t \to 0$ for all $\rho > 0$. Moreover, $\int_{\{|x|<\rho\}} u_\infty(x,t)dx \to 0$ as $t \to 0$ for all $\rho > 0$ if $p \in (1 + \frac{2+\alpha}{N}, p^{**})$. So in this case we have the initial datum zero with nonzero solution, and we encounter the* **non-uniqueness** *phenomenon.*
3. *For $\kappa > 0$ (hence $\lambda < 0$) one has $\int_{\{|x|<\rho\}} u_\infty(x,t)dx \to \infty$ for all $\rho > 0$ if $p \in (1, 1 + \frac{2+\alpha}{N})$.*
4. *For $\kappa > 0$ (hence $\lambda < 0$) and $\rho > 0$ one has $\int_{\{|x|<\rho\}} u_\infty(x,t)dx \to c < \infty$ for $p = 1 + \frac{2+\alpha}{N}$. The limit c is independent of ρ. So this is the only case with the initial datum $c\delta_0$.*

Further on we use the following notation. For $p \in (1,\infty)$, p' is the conjugate exponent, that is $p' = \frac{p}{p-1}$. $\mathbf{1}_X$ stands for the characteristic function of the set X, $B_R := \{x \in \mathbb{R}^N : |x| \le R\}$, $D_R := \{(x,t) \in \mathbb{R}^N \times (0,\infty) : R^2 \le |x|^2 + t \le 4R_0^2\}$.

For $\delta > 0$, let T_δ denote the Steklov average, $T_\delta u(t) = \frac{1}{2\delta}\int\limits_{t-\delta}^{t+\delta} u(s)ds = \frac{1}{2\delta}\int\limits_{-\delta}^{\delta} u(s+t)ds$.

We finish this section with the proposition classifying singular solutions to (1.6) i.e, the solutions satisfying (1.7). This is an analogue of [17, Lemma 1.1].

Proposition 1.5. *Let $u \in L^2_{loc}\Big((0,\infty);\ H^1_{loc}(\mathbb{R}^N, h^2dx)\Big) \cap L^{p+1}_{loc}\Big(\mathbb{R}^N \times (0,\infty), r^\beta h^2 dxdt\Big)$ be a non-trivial positive solution to (1.6) satisfying (1.7). Then, for any $\rho > 0$, there exists the limit*

$$\lim_{t\to 0}\int\limits_{B_\rho} u(x,t)|x|^{2\lambda}dx =: \varkappa \le +\infty.$$

The limit is independent of $\rho > 0$.

For the proof we need the following lemma which will also be used further on.

Lemma 1.6. *Let $u \in L^2_{loc}\Big((0,\infty);\ H^1_{loc}(\mathbb{R}^N, h^2dx)\Big) \cap L^{p+1}_{loc}\Big(\mathbb{R}^N \times (0,\infty), r^\beta h^2 dxdt\Big)$ be a solution (sub-solution) to (1.6) satisfying (1.7). Let $\tilde u$ denote the continuation of u into the semi-space $\mathbb{R}^N \times (-\infty, 0)$ by zero. Then, for every domain Ω such that $\overline{\Omega} \Subset \mathbb{R}^N \setminus \{0\}$, the function $\tilde u$ is a solution (sub-solution) to (1.6) in $\Omega \times \mathbb{R}$ and, moreover, $u \in L^\infty_{loc}\Big(\mathbb{R}^N \times (0,\infty)\Big)$.*

In particular, if u is a solution to (1.6) then $\tilde u \in C^{2,1}(\Omega\times\mathbb{R})$ and $u(x,t) \to 0$ as $t \to 0$ uniformly in $x \in \Omega$.

Proof. Given Ω such that $\overline{\Omega} \Subset \mathbb{R}^N \setminus \{0\}$, observe that $h, r^\beta \in C^\infty(\Omega)$ and there exists a constant $c > 1$ such that $\frac{1}{c} < h^2, r^\beta < c$ on Ω. Hence the first assertion follows from [5, Proof of Theorem 2, steps 2,3].

To prove the second assertion, consider a cylinder $B_R \times (t_0, t_1)$, $R > 0$, $0 < t_0 < t_1$. Then there exists $\tau \in (0, \frac{1}{2}t_0)$ such that $u(\tau) \in H^1_{h^2}(B_{2R})$. Moreover, u is bounded on $\partial B_{2R} \times (\tau, 2t_1)$, by the first assertion. Let the function w be the solution to the problem

$$\begin{cases} \partial_t w - h^{-2}\nabla\cdot(h^2\nabla w) = 0 & \text{in } B_{2R}\times(\tau, 2t_1),\\ w(x,t) = u(x,t), & (x,t) \in B_{2R}\times\{\tau\} \cup \partial B_{2R}\times(\tau,2t_1). \end{cases}$$

Then w is bounded on $B_R \times (t_0, t_1)$ [15] and, by the maximum principle, $|u| \le |w|$. □

Proof of Proposition 1.5. First we show that if the limit exists, then it is independent of ρ. Indeed, for $R > \rho$,

$$\lim_{t\to 0}\int\limits_{B_R\setminus B_\rho} u\, h^2 dx = 0$$

since $u(x,t) \to 0$ as $t \to 0$ uniformly in $x \in B_R \setminus B_\rho$, by Lemma 1.6.

Now we show the existence of the limit. Note that $u = (u-1)^+ + u \wedge 1$. Given $\rho > 0$, Lemma 1.6 implies that there exists $T_\rho > 0$ such that $u(x,t) < 1$ for all $x \in B_\rho \setminus B_{\rho/2}$, $t \in [0, T_\rho]$. Hence $u_1 := (u-1)^+\mathbf{1}_{B_\rho} \in L^2_{loc}\Big((0,T_\rho);\ H^1_{h^2}(\mathbb{R}^N)\Big)$.

Next we integrate (1.6) over the set $\{u_1 > 0\}$. To do this, consider the sequence $(\xi_n)_n$, $\xi_n : \mathbb{R}^+ \to \mathbb{R}^+$, $\xi_n(s) := (ns)^+ \wedge 1$. Then $(\xi_n)_n$ is a sequence of bounded non–decreasing Lipschitz functions approximating $\mathbf{1}_{(0,\infty)}$, so that $\xi_n(u_1)$ can be used as a test function for (1.6). For $0 < s < t < \infty$ we have

$$\int_s^t\int_{B_\rho} \xi_n(u_1)\partial_t u\, h^2 dx\, dt = -\int_s^t\int_{B_\rho} \nabla\xi_n(u_1)\cdot\nabla u\, h^2 dx\, dt - \int_s^t\int_{B_\rho} \xi(u_1)u^p r^\beta h^2 dx\, dt.$$

Since $\nabla\xi_n(u_1)\cdot\nabla u \ge 0$, it follows that $\int_s^t\int_{B_\rho}\xi_n(u_1)\partial_t u r^{2\lambda}dx\,dt \le 0$. Note that $\xi_n(u_1)\partial_t u = \partial_t\Xi_n(u_1)$, where $\Xi_n(s)=\int_0^s\xi_n(\tau)d\tau \to s^+$ as $n\to\infty$. Then

$$\int_{B_\rho}\Xi_n(u_1)(t)r^{2\lambda}dx \le \int_{B_\rho}\Xi_n(u_1)(s)r^{2\lambda}dx.$$

Passing to the limit as $n\to\infty$ we obtain that

$$\int_{B_\rho}(u-1)^+(t)h^2dx \le \int_{B_\rho}(u-1)^+(s)h^2dx.$$

Due to this monotonicity,

$$\int_{B_\rho}(u-1)^+(t)h^2dx \to \varkappa \le +\infty \text{ as } t\to 0.$$

Finally, $\int_{B_\rho}u\wedge 1\,h^2dx\to 0$ by the Lebesgue dominated convergence theorem. □

Remark 1.7. *If $\int_{B_\rho}u(t)r^{2\lambda}dx\to\varkappa<\infty$ as $t\to 0$ then $u(t)h^2dx\to\varkappa\delta_0$ as $t\to 0$ in the weak-∗ topology of Radon measures. Indeed, for any $\theta\in C_c(\mathbb{R}^N)$ there exists $R>0$ such that* $\operatorname{supp}\theta\subset B_R$*, and, for any $\epsilon>0$ there exists $\rho>0$ such that $|\theta(x)-\theta(0)|<\epsilon$ for all $x\in B_\rho$. Then*

$$\int_{\mathbb{R}^N}\theta u(t)h^2dx=\theta(0)\int_{B_R}u(t)h^2dx+\int_{B_R\setminus B_\rho}(\theta-\theta(0))u(t)h^2dx+\int_{B_\rho}(\theta-\theta(0))u(t)h^2dx. \tag{1.9}$$

Now $\int_{B_R\setminus B_\rho}(\theta-\theta(0))u(t)h^2dx\to 0$ as $t\to 0$ since $u(x,t)\to 0$ as $t\to 0$ uniformly in $x\in B_R\setminus B_\rho$ and

$$\limsup_{t\to 0}\left|\int_{B_\rho}(\theta-\theta(0))u(t)h^2dx\right|\le\epsilon\varkappa.$$

Therefore it follows from (1.9) *that, for any $\epsilon>0$,*

$$\limsup_{t\to 0}\left|\int_{\mathbb{R}^N}\theta u(t)h^2dx-\theta(0)\varkappa\right|<\epsilon\varkappa.$$

Proposition 1.5 gives rise to the following definition giving classification to singular solutions to (1.6).

Definition 1.8. A non-trivial positive solution u to (1.6) satisfying (1.7) is called a *source-type solution* (SS) if $\int_{B_\rho}u(t)r^{2\lambda}dx\to\varkappa$ with some finite $\varkappa>0$. The solution u is called a *very singular solution* (VSS) if $\int_{B_\rho}u(t)r^{2\lambda}dx\to\infty$.

The rest of the paper is organized as follows. In Section 2 we prove a-priory estimates of Keller-Osserman type and show that in the critical and supercritical range of values of p the only solution to equation (1.6) satisfying (1.7) is zero. In Section 3 we study general linear inhomogeneous evolution equations with a generator of a Feller semigroup and with Radon measures in the right hand side and as initial data. These results are applied in Section 4, where the general semilinear equations with Radon measures as initial data are studied. The results are then applied to equation (1.6). Very singular solutions to equation (1.6) are discussed in Section 5. Finally, in Appendix we give a version of the Hardy inequality and provide an auxiliary compactness result.

2 A-priori estimates and nonexistence result

We start with a-priori estimates for sub-solutions to (1.6) similar to that obtained in [5]. We use the notation

$$D_{\rho,R} := \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+ \ : \ \rho^2 < |x|^2 + t < R^2\}.$$

Proposition 2.1. *Let $0 < R_0 < \frac{1}{4}R_1$ and u be a sub-solution to (1.6) in the paraboloid layer D_{R_0,R_1} such that $u(x,t) \to u_0$ as $t \to 0$ in $L^2_{h^2}(B_{R_1} \setminus B_{R_0})$. Then, for all $R_0 < \rho < \frac{1}{2}R < \frac{1}{2}R_1$,*

$$\iint\limits_{D_{2\rho,R}} \left[|\nabla u|^2 + r^\beta u^{p+1}\right] h^2 dx\, dt \le c\left(\rho^{N+2\lambda-2\frac{2+\beta}{p-1}} + R^{N+2\lambda-2\frac{2+\beta}{p-1}} + \|u_0\|^2_{L^2_{h^2}}\right). \tag{2.1}$$

Moreover, for $u_0 = 0$ one has, $u(x,t) \le c(|x|^2+t)^{-\frac{2+\beta}{2(p-1)}}$ for $4R_0^2 < |x|^2 + t < R_1^2$.

Proof. In the proof we use some of ideas from [8]. Let $\phi \in C^1(\mathbb{R})$, $\mathbf{1}_{(4,\infty)} \le \phi \le \mathbf{1}_{(1,\infty)}$, $|\phi'| \le c\phi^\alpha$, with $\alpha < 1$ to be chosen later. Let

$$\xi = \phi\left(\frac{|x|^2+t}{\rho^2}\right), \eta = \phi\left(5 - \frac{|x|^2+t}{R^2}\right) \text{ and } \zeta = \xi\eta.$$

Then

$$\begin{aligned} &\mathbf{1}_{D_{2\rho,R}} \le \zeta \le \mathbf{1}_{D_{\rho,2R}}, \\ &|\partial_t\zeta| \le \frac{c}{\rho^2}\mathbf{1}_{D_{\rho,2\rho}}\xi^\alpha\eta + \frac{c}{R^2}\mathbf{1}_{D_{R,2R}}\xi\eta^\alpha, \\ &|\nabla\zeta| \le \frac{c}{\rho}\mathbf{1}_{D_{\rho,2\rho}}\xi^\alpha\eta + \frac{c}{R}\mathbf{1}_{D_{R,2R}}\xi\eta^\alpha. \end{aligned} \tag{2.2}$$

We set $u(x,t) := u_0(x)$ for $t \le 0$, $x \in B_{R_1}$ and choose $T_\delta(\zeta^2(T_\delta u))$ as a test function in (1.6) on $D_{\rho,2R}$. Note that it is a legitimate test function, by Lemma 1.6. Further on we denote $w = T_\delta u$. Then we obtain

$$\begin{aligned} &\iint |\nabla w\zeta|^2 h^2 dxdt + \iint r^\beta T_\delta(|u|^{p-1}u)\zeta w\zeta h^2 dxdt \\ &\qquad \le \iint w^2\zeta\partial_t\zeta\, h^2 dxdt + \iint w^2|\nabla\zeta|^2\, h^2 dxdt + \int\limits_{B_{2R}\setminus B_\rho} w^2(0)\zeta^2(0)h^2 dx. \end{aligned}$$

Passing to the limit as $\delta \to 0$ in the last inequality, we may replace w with u. Now we estimate the first two integrals in the right hand side using (2.2). By the Young inequality, for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\begin{aligned} u^2\zeta|\partial_t\zeta| \le& u^2\zeta\left[\frac{c}{\rho^2}\mathbf{1}_{D_{\rho,2\rho}}\xi^\alpha\eta + \frac{c}{R^2}\mathbf{1}_{D_{R,2R}}\xi\eta^\alpha\right] \\ \le& \varepsilon\left(r^{-2}u^2\zeta^2 + r^\beta|u|^{p+1}\zeta^2\right) + c_\varepsilon r^{\frac{2}{1-\alpha}-2-2\frac{2+\beta}{p-1}}\left(\rho^{-\frac{2}{1-\alpha}}\mathbf{1}_{D_{\rho,2\rho}} + R^{-\frac{2}{1-\alpha}}\mathbf{1}_{D_{R,2R}}\right), \\ u^2|\nabla\zeta|^2 \le& u^2\left[\frac{c}{\rho^2}\mathbf{1}_{D_{\rho,2\rho}}\xi^{2\alpha}\eta^2 + \frac{c}{R^2}\mathbf{1}_{D_{R,2R}}\xi^2\eta^{2\alpha}\right] \\ \le& \varepsilon\left(r^{-2}u^2\zeta^2 + r^\beta|u|^{p+1}\zeta^2\right) + c_\varepsilon r^{\frac{2}{1-\alpha}-2-2\frac{2+\beta}{p-1}}\left(\rho^{-\frac{2}{1-\alpha}}\mathbf{1}_{D_{\rho,2\rho}} + R^{-\frac{2}{1-\alpha}}\mathbf{1}_{D_{R,2R}}\right). \end{aligned}$$

Note that, by Hardy inequality (A.1), for all $t > 0$,

$$\int r^{-2}u^2(t)\zeta^2(t)h^2 dx \le \frac{4}{(N+2\lambda-2)^2}\int |\nabla u\zeta|^2(t)h^2 dx.$$

Choose now $\alpha \in (\frac{2}{p+1}, 1)$ such that

$$\frac{2}{1-\alpha} - 2 - 2\frac{2+\beta}{p-1} + 2\lambda + N > 0.$$

Then by a direct calculation

$$\iint\limits_{D_{R,2R}} R^{-\frac{2}{1-\alpha}} r^{\frac{2}{1-\alpha}-2-2\frac{2+\beta}{p-1}} h^2 dxdt \leq R^{N+2\lambda-2\frac{2+\beta}{p-1}}$$

which implies that

$$\iint |\nabla u\zeta|^2 h^2 dxdt + \iint r^\beta |u|^{p+1} \zeta^2 h^2 dxdt \leq c\rho^{N+2\lambda-2\frac{2+\beta}{p-1}} + cR^{N+2\lambda-2\frac{2+\beta}{p-1}} + c \int\limits_{B_{2R}\setminus B_\rho} u^2(0)\zeta^2(0) h^2 dx,$$

which completes the proof of the first assertion.

To prove the second assertion, note that by the mean value inequality for sub-solutions (see Theorem 3.8 below) we have that

$$\sup_{D_{5/2\rho,7/2\rho}} |u| \leq C \left(⨌_{D_{2\rho,4\rho}} u^2 h^2 dxdt \right)^{\frac{1}{2}}.$$

Using the Hardy inequality and (2.1) with $R = 4\rho$, we have

$$\iint_{D_{2\rho,4\rho}} u^2 h^2 dxdt \leq \iint |u\zeta|^2 h^2 dxdt \leq \rho^2 \iint |u\zeta|^2 r^{-2} h^2 dxdt \tag{2.3}$$

$$\leq c\rho^2 \iint |\nabla(u\zeta)|^2 h^2 dxdt \leq c\rho^{N+2\lambda+2-2\frac{2+\beta}{p-1}}. \tag{2.4}$$

Hence

$$\left(⨌_{D_{2\rho,4\rho}} u^2 h^2 dxdt \right)^{\frac{1}{2}} \leq c\rho^{-\frac{2+\beta}{p-1}}.$$

□

Corollary 2.2. *Let $p < 1 + \frac{2+\beta}{N+2\lambda}$ and let u be a solution to (1.6) such that $u(t) \to \varkappa\delta_0$ as $t \to 0$ in weak-$*$ topology of Radon measures. Then $u \in L^2_{loc}\big((0,T); H^1_{h^2}\big)$ for all $T > 0$ and $u(x,t) \leq \varkappa p_t^{h^2}(x,0)$ for a.a. $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where p^{h^2} is the fundamental solution of the linear equation $(\partial_t - \Delta_{h^2})w = 0$. In particular, $u \in L^{p+1}\big(\mathbb{R}^N \times (0,T), r^\beta h^2 dx\, dt\big)$ for all $T > 0$.*

Proof. The first assertion follows from (2.1), setting $R \to \infty$ and choosing ρ arbitrary small. The proof of the second assertion literally follows the argument [18].Namely, let $u(t) \to \varkappa\delta_0$ as $t \to 0$ in weak-$*$ topology of Radon measures. Let $u^{(\tau)}$ be the solution to the initial value problem

$$\begin{cases} (\partial_t - \Delta_{h^2})u^{(\tau)} = 0, & t > \tau, \\ u^{(\tau)}(\tau) = u(\tau). \end{cases}$$

Then, by the maximum principle, $u^{(\tau)}(x,t) \geq u(x,t)$ for a.a. $(x,t) \in \mathbb{R}^N \times (\tau,\infty)$. So, for $0 < \tau' < \tau$ one has $u^{(\tau')}(\tau) \geq u(\tau) = u^{(\tau)}(\tau)$. Hence, by the maximum principle, $u^{(\tau')}(x,t) \geq u^{(\tau)}(x,t)$ a.e. on $\mathbb{R}^N \times (\tau,\infty)$. So $u^{(\tau)} \uparrow u^{(0)}$ as $\tau \downarrow 0$ and $u^{(0)}(x,t) = \varkappa p_t^{h^2}(x,0)$ since $u^{(\tau)}(\tau) = u(\tau) \to \varkappa\delta_0$ as $t \to 0$. □

The next lemma reduces the proof of the second assertion of Theorem 1.2 to the critical case $p = p^*$.

Lemma 2.3. *Let u be a sub-solution to (1.6) satisfying (1.7). Then, for $1 < q \le p$, the function*

$$\left(\frac{p-1}{q-1}\right)^{\frac{1}{p-1}} |u|^{\frac{p-1}{q-1}}$$

is a sub-solution to the equation $\partial_t w - h^{-2}\nabla\cdot(h^2\nabla w) + r^\beta w|w|^{q-1} = 0$.

Proof. Denote $\varkappa = \frac{p-1}{q-1} \ge 1$, T_δ the Steklov average and $u_\delta := T_\delta u$. For $\varepsilon > 0$ and $\zeta \in C_c^{2,1}\Big(\mathbb{R}^N \times (0,\infty)\Big)$, $\zeta \ge 0$, choose the following test function for (1.6):

$$T_\delta\Big(\zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}}\Big).$$

Note that this is a legitimate test function since u is locally bounded. Then the following inequality holds:

$$\begin{aligned} &\iint \partial_t u_\delta \zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}} h^2 dxdt + \iint \nabla u_\delta \nabla(\zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}}) h^2 dxdt \\ &\qquad \le - \iint \zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}} r^\beta T_\delta(|u|^{p-1}u) h^2 dxdt. \end{aligned} \tag{2.5}$$

Denote $V_\varepsilon(u) := \frac{1}{\varkappa}\Big((u^2+\varepsilon)^{\frac{\varkappa}{2}} - \varepsilon^{\frac{\varkappa}{2}}\Big)$. Then

$$\partial_t u_\delta \zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}} = \zeta \partial_t V_\varepsilon(u_\delta)$$

and

$$\nabla u_\delta \nabla(\zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}}) = \nabla V_\varepsilon(u_\delta)\nabla\zeta + \zeta\Big((\varkappa-1)u_\delta^2+\varepsilon\Big)(u_\delta^2+\varepsilon)^{\frac{\varkappa-4}{2}}|\nabla u_\delta|^2.$$

Hence it follows from (2.5) that

$$-\iint V_\varepsilon(u_\delta)\partial_t\zeta h^2 dxdt + \iint \nabla V_\varepsilon(u_\delta)\nabla\zeta h^2 dxdt \le -\iint \zeta u_\delta (u_\delta^2+\varepsilon)^{\frac{\varkappa-2}{2}} r^\beta T_\delta(|u|^{p-1}u) h^2 dxdt.$$

Passing to the limit as $\varepsilon \to 0$ and then as $\delta \to 0$ we obtain

$$-\iint |u|^\varkappa \partial_t\zeta h^2 dxdt + \iint \nabla|u|^\varkappa \nabla\zeta h^2 dxdt + \varkappa\iint \zeta|u|^{p+\varkappa-1} h^2 dxdt \le 0.$$

Hence the assertion follows. □

Remark 2.4. *Lemma 2.3 is a parabolic version of [19, Proposition 1.1].*

The following theorem establishes the removability of singularity at $(0,0)$ for the critical case.

Theorem 2.5. *Let $p = 1 + \frac{2+\beta}{N+2\lambda}$. Let $0 \le u \in L^\infty_{loc}(\mathbb{R}^N \times \mathbb{R}^1_+ \setminus (0,0))$ be such that*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^1_+} \Big(r^\beta u^p \zeta - u\zeta_t - u\Delta_{h^2}\zeta\Big) h^2 dx\, dt \le 0, \quad \zeta \in C_c^{2,1}. \tag{2.6}$$

Then $u = 0$.

Proof. Let $\xi \in \mathcal{C}^1(\mathbb{R}^1_+)$ be such that

$$\mathbf{1}_{[2,\infty)} \le \xi \le \mathbf{1}_{[1,\infty)}, \quad |\xi'|, |\xi''| \le c\xi^{\frac{1}{p}}. \tag{2.7}$$

Let $0 < \rho \ll R < \infty$ and define

$$\xi_\rho(x,t) = \xi\left(\frac{t+|x|^2}{\rho^2}\right), \quad \eta_R(x,t) = 1 - \xi\left(\frac{t+|x|^2}{R^2}\right). \tag{2.8}$$

We take $\zeta = \xi_\rho \eta_R$ as a test function in (2.6). It is easy to see that

$$\operatorname{supp}\zeta = \{(x,t) \,:\, \rho^2 \le t + |x|^2 \le 2R^2\}.$$

Using (2.7) one verifies directly that

$$|\partial_t \zeta| + |\Delta_{h^2}\zeta| \le c\frac{1}{\rho^2}\xi_\rho^{\frac{1}{p}}\eta_R \mathbf{1}_{\{\rho^2 \le t+|x|^2 \le 2r^2\}} + c\frac{1}{R^2}\xi_\rho \eta_R^{\frac{1}{p}}\mathbf{1}_{\{R^2 \le t+|x|^2 \le 2R^2\}}.$$

Thus we have

$$\begin{aligned}
I :=& \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} r^\beta u^p \zeta h^2 dx\, dt \le \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u(|\partial_t \zeta| + |\Delta_{h^2}\zeta|) h^2 dx\, dt \\
\le& c\rho^{-2} \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u \xi_\rho^{\frac{1}{p}} \eta_R \mathbf{1}_{\{\rho^2 \le t+|x|^2 \le 2\rho^2\}} h^2 dx\, dt \\
&+ cR^{-2} \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u \xi_\rho \eta_R^{\frac{1}{p}} \mathbf{1}_{\{R^2 \le t+|x|^2 \le 2R^2\}} h^2 dx\, dt \\
:=& I_1 + I_2.
\end{aligned} \tag{2.9}$$

By the Young inequality

$$I_1 \le c\rho^{-2} \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u(\xi_\rho \eta_R)^{\frac{1}{p}} \mathbf{1}_{\{\rho^2 \le t+|x|^2 \le 2\rho^2\}} h^2 dx\, dt \le \frac{1}{4} I + c\rho^{N+2-\frac{2p}{p-1}+2\lambda-\frac{\beta}{p-1}}.$$

Similarly,

$$I_2 \le cR^{-2} \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u(\xi_\rho \eta_R)^{\frac{1}{p}} \mathbf{1}_{\{R^2 \le t+|x|^2 \le 2R^2\}} dx\, dt \le \frac{1}{4} I + cR^{N+2-\frac{2p}{p-1}+2\lambda-\frac{\beta}{p-1}}.$$

Hence for every $\rho > 0$ and $R > 2\rho$ we obtain

$$I \le c(\rho^{N+2-\frac{2p}{p-1}+2\lambda-\frac{\beta}{p-1}} + R^{N+2-\frac{2p}{p-1}+2\lambda-\frac{\beta}{p-1}}) = c \quad \text{as} \quad N + 2 - \frac{2p}{p-1} + 2\lambda - \frac{\beta}{p-1} = 0.$$

Passing to the limits $\rho \to 0$ and $R \to \infty$ we conclude that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^1_+} r^\beta u^p \zeta h^2 dx\, dt < \infty. \tag{2.10}$$

Now we return to (2.9). Estimating I_1 and I_2 by means of the Young inequality and using (2.10) we have

$$\begin{aligned}
I_1 \;&\le\; c\rho^{-2} \iint_{\mathbb{R}^N \times \mathbb{R}^1_+} u \xi_\rho^{\frac{1}{p}} \mathbf{1}_{\{\rho^2 \le t+|x|^2 \le 2\rho^2\}} h^2 dx\, dt \\
&\le\; c\left(\iint_{\mathbb{R}^N \times \mathbb{R}^1_+} \mathbf{1}_{\{\rho^2 \le t+|x|^2 \le 2\rho^2\}} r^\beta u^p \zeta h h^2 dx\, dt\right)^{\frac{1}{p}} \to 0 \quad \text{as } \rho \to 0.
\end{aligned} \tag{2.11}$$

Similarly we see that $I_2 \to 0$ as $R \to \infty$. Hence we conclude from (2.9) that $I = 0$ which implies that $u = 0$. □

Now assertion (c) of Theorem 1.2 follows from Theorem 2.5, Lemma 2.3 and the corresponding parabolic version of the Kato inequality (see, e.g. [30, Chap. 6]).

3 Linear equation with a generator of an ultra-contractive Feller semigroup

In this section we study an abstract inhomogeneous evolution equation with measures as initial data.

In this section $\Omega \subset \mathbb{R}^N$ is a domain and γ is a positive Radon measure on Ω and we denote $L^p := L^p(\Omega, d\gamma)$. Let $T > 0$ and $Q = \Omega \times [0,T]$. We also denote $L^p(Q) = L^p(Q, d\gamma\, dt)$ and naturally identify $L^p(Q) = L^p([0,T]; L^p)$.

In the sequel we also use the notation $C_0(\Omega), C_b(\Omega)$ for the spaces of continuous function vanishing at infinity and at the boundary of Ω and bounded continuous function, respectively. $\mathcal{M}(\Omega)$ stands for finite signed Radon measures on Ω.

Let $(\mathcal{E}, \mathcal{F})$ be a closed symmetric Dirichlet form on L^2, $-\mathcal{L}$ the associated self-adjoint operator in L^2, and $S = (S(t))_{t\geq 0}$ the associated symmetric Markov semigroup on L^2, i.e. $\|S(t)f\|_\infty \leq \|f\|_\infty$ for any $t \geq 0$), $S(t) = e^{\mathcal{L}t}$. The domain $\mathcal{F}$ of the form $\mathcal{E}$ is a real Hilbert space with the norm $\|f\|_{\mathcal{F}} = (\mathcal{E}(f))^{\frac{1}{2}}$. We refer the reader to [10, 13] for the definition and properties.

The action of the semigroup S on the measure μ is defined in a standard way by the following identity

$$\int_\Omega (S(t)\mu)\phi d\gamma = \int_\Omega (S(t)\phi) d\mu, \quad \phi \in C_0(\Omega). \tag{3.1}$$

We start with the following simple statement.

Proposition 3.1. *Let $\psi : (0,\infty) \to (0,\infty)$ be a non-increasing function. Assume that*

$$\|S(t)\|_{L^1 \to L^\infty} \leq \psi(t), \ t > 0 \tag{3.2}$$

and

$$S(t)C_0(\Omega) \subset C_b(\Omega) \text{ and } S(t)\mathbf{1} \in C_b(\Omega). \tag{3.3}$$

Then $S(t)$, $t > 0$, is a bounded operator $S(t) : \mathcal{M}(\Omega) \to C_b(\Omega) \cap L^1 \cap \mathcal{F}$ and

$$\|S(t)\|_{\mathcal{M}\to L^q} \leq \psi^{\frac{1}{q'}}(t), \quad 1 \leq q \leq \infty. \tag{3.4}$$

Moreover, for every $t > 0$, $S(t)$ is an integral operator with a positive bounded symmetric kernel $p_t(x,y)$ which is continuous in each of the variables x, y, t and

$$p_t(x,y) \leq \psi(t) \text{ and } \int_\Omega p_t(x,y)\gamma(dy) \leq 1. \tag{3.5}$$

For every $t > 0$, the operator $S(t)$ maps weak-$*$-convergent sequences in $\mathcal{M}(\Omega)$ into strongly convergent sequences in $C_b(\Omega) \cap L^1 \cap \mathcal{F}$.

Proof. By the Riesz-Thorin interpolation theorem it follows from (3.2) that $\|S(t)\|_{L^p\to L^q} \leq \psi^{\frac{1}{p}-\frac{1}{q}}(t)$, $1 \leq p \leq q \leq \infty$. Since $C_0(\Omega) \cap L^p$ is dense in L^p, by (3.3), $S(t) : L^p \to C_b(\Omega)$. Hence, $\|S(t)\|_{L^p\to C_b} \leq \psi^{\frac{1}{p}}(t)$, $t > 0$. By duality $S(t) : C_b(\Omega)^* \to L^q$, $1 \leq q \leq \infty$. In particular, $S(t) : \mathcal{M}(\Omega) \to L^q$ and $\|S(t)\|_{\mathcal{M}\to L^q} \leq \psi^{\frac{1}{q'}}(t)$, $1 \leq q \leq \infty$. By the simple factorization $S(t) = S(t/2)S(t/2) : \mathcal{M}(\Omega) \to L^q \to C_b(\Omega)$, and $\|S(t)\|_{\mathcal{M}\to C_b} \leq \psi(t)$, $t > 0$.

Similarly, $S(t) = S(t/2)S(t/2) : \mathcal{M}(\Omega) \to L^2 \to \mathcal{F}$.

The second assertion follows from the first one taking $p_t(x, y) = (S(t)\delta_y)(x)$.

To prove the last assertion we first show that S is a strong Feller semigroup, that is, for $t > 0$, $S(t)$ maps bounded Borel measurable functions into continuous ones. To this end it suffices to verify that $x \mapsto p_t(x, \cdot)$ is a continuous function from Ω to L^1 for all $t > 0$. If $\gamma(\Omega) < \infty$ this immediately follows from the fact that $p_t(x, y)$ is continuous in x for all $t > 0$ and $y \in \Omega$ and the bound $0 \le p_t(x, y) \le \psi(t)$. In case $\gamma(\Omega) = \infty$ to verify the assumptions of the Vitali theorem it suffices to show that, for every $x_n \to x$ in Ω as $n \to \infty$ and every $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset \Omega$ and $N_\varepsilon \in \mathbb{N}$ such that

$$\int_{\Omega\setminus K_\varepsilon} p_t(x_n, y)\gamma(dy) < \varepsilon \text{ for all } n > N_\varepsilon.$$

Given $x_n \to x$ in Ω as $n \to \infty$ and $\varepsilon > 0$, let $K_\varepsilon \subset \Omega$ be such that

$$\int_{\Omega\setminus K_\varepsilon} p_t(x, y)\gamma(dy) < \frac{\varepsilon}{2}.$$

Note that $\mathbf{1}_{K_\varepsilon} \in L^1$ so $S_t\mathbf{1}_{K_\varepsilon} \in C_b(\Omega)$. Since $S_t\mathbf{1} \in C_b(\Omega)$, we conclude that $S_t\mathbf{1}_{\Omega\setminus K_\varepsilon} = S_t\mathbf{1} - S_t\mathbf{1}_{K_\varepsilon} \in C_b(\Omega)$. In particular,

$$\left| \int_{\Omega\setminus K_\varepsilon} p_t(x_n, y)\gamma(dy) - \int_{\Omega\setminus K_\varepsilon} p_t(x, y)\gamma(dy) \right| \to 0 \text{ as } n \to \infty.$$

Now choose N_ε such that the above variable is less then $\frac{\varepsilon}{2}$ for $n > N_\varepsilon$. Thus S is strongly Feller.

Now let $\mu_n \to \mu$ as $n \to \infty$ in the sense of weak-$*$ convergence in $\mathcal{M}(\Omega)$. Then, for every Borel measurable E,

$$\int_E \big(S(t)\mu_n\big)d\gamma = \int_\Omega (S(t)\mathbf{1}_E)d\mu_n \to \int_\Omega (S(t)\mathbf{1}_E)d\mu = \int_E \big(S(t)\mu\big)d\gamma \text{ as } n \to \infty,$$

since $S(t)\mathbf{1}_E \in C_b(\Omega)$. Hence $S(t)\mu_n \to S(t)\mu$ as $n \to \infty$ weakly in L^1. Since $\big(S(t)\mu_n\big)(x) = \int p_t(x, y)\mu_n(dy)$ and p_t is bounded and continuous in y, we conclude that $S(t)\mu_n \to S(t)\mu$ as $n \to \infty$, pointwise in Ω. Hence $S(t)\mu_n \to S(t)\mu$ as $n \to \infty$ strongly in L^1. The strong convergence in $\mathcal{F}$ and in $C_b(\Omega)$ follows from the factorization argument. □

Let us introduce the convolution operator $\mathcal{T}$ on $L^p(Q)$ by $(\mathcal{T}f)(x, t) = \int_0^t (S(t-s)f)(x, s)ds$. In the next two propositions we collect the required properties of $\mathcal{T}$.

Proposition 3.2. *Let condition* (3.2) *hold. The following assertions hold*

1) $\mathcal{T}$ is a completely continuous operator on $L^1(Q)$;

2) $\mathcal{T}$ is a bounded operator $L^2(Q) \to L^2((0, T); \mathcal{F})$ and $L^2(Q) \to L^2((0, T); D(\mathcal{L}))$.

Proof. Note that $\mathcal{T}$ is an integral operator on $L^1(Q)$ and

$$(\mathcal{T}f)(x, t) = \iint_Q k(x, t; y, s)f(y, s)\gamma(dy)\, ds$$

with $k(x,t;y,s) = \mathbf{1}_{(0,\infty)}(t-s)p_{t-s}(x,y)$. Since

$$\iint\limits_{Q} k(x,t;y,s)\gamma(dx)\,dt = \int\limits_0^T \mathbf{1}_{(0,\infty)}(t-s)\int\limits_{\Omega} p_{t-s}(x,y)\gamma(dx)\,dt \le \int\limits_s^T dt \le T$$

for a.a. $(y,s)\in Q$, it follows by the Dunford–Pettis lemma (see e.g. [11, Lemma III.11]) that $\mathcal{T}$ is completely continuous on $L^1(Q)$.

To prove the next assertion, observe that

$$\|(\mathcal{T}f)(t)\|_{\mathcal{F}} \le c\int_0^t \frac{1}{\sqrt{t-s}}\|f\|_{L^2}(s)ds \quad \text{and} \quad \|(\mathcal{L}\mathcal{T}f)(t)\|_{L^2} \le c\int_0^t \frac{1}{t-s}\|f\|_{L^2}(s)ds.$$

Since the integral operators with the kernels $K_0(t,s) = \frac{1}{\sqrt{t-s}}$ and $K_1(t,s) = \frac{1}{t-s}$ are bounded on $L^2(0,T)$, the second assertion follows. □

Proposition 3.3. *Let the conditions of Proposition 3.1 be fulfilled. In addition assume that*

$$S(t)\phi(x)\to\phi(x) \text{ as } t\to 0 \text{ for all } x\in\Omega, \phi\in C_0(\Omega). \tag{3.6}$$

Then $\mathcal{T}$ can be uniquely extended to a bounded operator from $\mathcal{M}(Q)$ to $L^\infty\left((0,T);\, L^1\right)$. Moreover, $\mathcal{T}L^1(Q)\subset C\left([0,T];\, L^1\right)$.

Proof. First, observe that, $S(t)\mu$ is strongly continuous in $L^1\subset\mathcal{M}(\Omega)$ for all $t>0$, for every $\mu\in\mathcal{M}(\Omega)$. Moreover, it follows from (3.6) that $S(t)\mu$ is w-$*$ continuous continuous at $t=0$.

Now let $m\in\mathcal{M}(Q)$ and $m=\mu_t\otimes\nu$ be its disintegration into $\nu\in\mathcal{M}([0,T])$ and a function $t\mapsto\mu_t\in\mathcal{M}(\Omega)$ such that $t\mapsto\mu_t(F)$ is ν-measurable for all Borel sets F (see [2, Theorem 2.28]). So $t\mapsto\mu_t$ is a weakly ν-measurable function from $[0,T]$ to $\mathcal{M}(\Omega)$. Hence the function $s\mapsto S(t-s)\mu_s$ is also a weakly ν-measurable function from $[0,t]$ to $\mathcal{M}(\Omega)$. Since $S(s)\mathcal{M}(\Omega)\subset L^1$ for $s>0$, we conclude that $s\mapsto S(t-s)\mu_s$ is separably valued, hence it is (strongly) ν-measurable by the Pettis measurability theorem (see [11, Theorem 2.2]). So we define the extension of $\mathcal{T}$ on $\mathcal{M}(Q)$ by

$$(\mathcal{T}m)(t) := \int\limits_{[0,t]} S(t-s)\mu_s\nu(ds), \tag{3.7}$$

where the right hand side is a Bochner integral.

Moreover, $\mathcal{T}:\mathcal{M}(Q)\to L^\infty((0,T);\,\mathcal{M}(\Omega))$ is bounded. Indeed,

$$\|(\mathcal{T}m)(t)\|_{\mathcal{M}(\Omega)} \le \int_0^t \|S(t-s)\mu_s\|_{\mathcal{M}(\Omega)}\,|\nu|(ds) \le \int_0^t \|\mu_s\|_{\mathcal{M}(\Omega)}\,|\nu|(ds) = \|m\|_{\mathcal{M}(Q)}.$$

Hence the extension is unique.

Now, let $\nu=\nu_c+\sum c_k\delta_{t_k}$ be the decomposition of ν into the continuous and the atomic parts. Then $\int_0^t S(t-s)\mu_s\nu_c(ds)\in L^1$ since $S(t-s)\mu_s\in L^1$ for all $s\in[0,t)$, and

$$\int_0^t S(t-s)\mu_s\sum c_k\delta_{t_k}(ds) = \sum_{t_k\le t} c_k S(t-t_k)\mu_{t_k}.$$

The latter belongs to L^1 for all $t\ne t_k$, $k=1,2,\ldots$. So $\mathcal{T}m(t)\in L^1$ for a.a $t\in[0,T]$.

Finally, we show that if $\nu = \nu_c$ then $\mathcal{T}m \in C\big([0,T];\, L^1\big)$, which will prove the last assertion. Indeed,

$$\mathcal{T}m(t+h) - \mathcal{T}m(t) = \int\limits_t^{t+\delta} S(t+\delta-s)\mu_s\nu(ds) + \int\limits_0^t \Big[S(t+\delta-s) - S(t-s)\Big]\mu_s\nu(ds).$$

Then

$$\Big\|\int\limits_t^{t+\delta} S(t+\delta-s)\mu_s\nu(ds)\Big\|_{L^1} \le \int\limits_t^{t+\delta} \|\mu_s\|_{\mathcal{M}(\Omega)}|\nu|(ds) = |m|\big(\Omega\times(t,t+\delta)\big) \to 0 \text{ as } \delta\to 0.$$

Further,

$$\Big\|\big[S(t+\delta-s) - S(t-s)\big]\mu_s\Big\|_{L^1} \to 0 \text{ as } \delta\to 0 \text{ for all } s\in[0,t).$$

Moreover, $\Big\|\big[S(t+\delta-s) - S(t-s)\big]\mu_s\Big\|_{L^1} \le 2\|\mu_s\|_{\mathcal{M}(\Omega)}$. Thus

$$\Big\|\int\limits_0^t \Big[S(t+\delta-s) - S(t-s)\Big]\mu_s\nu(ds)\Big\|_{L^1} \to 0 \text{ as } \delta\to 0.$$

□

Remark 3.4. *For further use we observe that, for $\eta \in C\big([0,T];\, L^1\cap C_b\big)$,*

$$(\mathcal{T}^*\eta)(t) = \int\limits_t^T S(s-t)\eta(s)ds,$$

where the right hand side is a Bochner integral. Note that $\mathcal{T}^*$ is a bounded operator on $C\big([0,T];\, L^1\cap C_b\big)$, by the argument similar to the one in the proof of the preceding proposition.

Definition 3.5. Let $m \in \mathcal{M}(Q)$ and $\mu\in\mathcal{M}(\Omega)$. We say that $u \in L^2_{loc}\Big((0,\infty);\,\mathcal{F}\Big)\cap L^1(Q)$ is a solution (sub-solution) to the problem

$$(\partial_t - \mathcal{L})u = m, \quad u(0) = \mu \tag{3.8}$$

if the following integral identity (inequality) holds

$$\int\limits_\Omega u(t_1)\zeta(t_1)\gamma(dx) - \int\limits_{t_0}^{t_1}\int\limits_\Omega u\partial_t\zeta\,\gamma(dx)\,dt + \int\limits_{t_0}^{t_1}\mathcal{E}\big(u(t),\zeta(t)\big)dt$$

$$= (\le) \int\limits_{t_0}^{t_1}\int\limits_\Omega \zeta\, m(dxdt) + \int\limits_\Omega u(t_0)\zeta(t_0)\gamma(dx) \tag{3.9}$$

and

$$\lim_{t\to 0}\,(\limsup_{t\to 0}) \int\limits_\Omega u(t)\zeta(t)\gamma(dx) = (\le) \int\limits_\Omega \zeta(0)d\mu \tag{3.10}$$

for all $t_1 > t_0 > 0$ and $\zeta\in W$, $\zeta\ge 0$, where

$$W := \Big\{\zeta\in C_b(Q)\cap L^2_{loc}\Big((0,\infty);\,\mathcal{F}\Big)\cap W^{1,\infty}_{loc}\Big((0,\infty);\,L^\infty\Big)\Big\}.$$

The next lemma provides the representation of the solution to (3.8).

Lemma 3.6. *Let u be a solution (sub-solution) to* (3.8)*. Then $u = S\mu + \mathcal{T}m$ $(u \le S\mu + \mathcal{T}m)$.*

Proof. We prove the assertion for solutions, the proof for sub-solutions being completely the same. Let $\eta \in C\big([0,T];\, L^1 \cap C_b\big)$, $\lambda > 0$. Denote

$$\eta_\lambda(t) := (\mathrm{I} - \lambda\mathcal{L})^{-1}\eta(t), t \in [0,T] \text{ and } \zeta_\lambda := \mathcal{T}^*\eta_\lambda.$$

Then $\eta_\lambda, \zeta_\lambda \in C\big([0,T];\, L^1 \cap C_b\big)$. Since $\mathcal{L}\eta_\lambda(t) = \frac{1}{\lambda}\eta(t) - \frac{1}{\lambda}\eta_\lambda(t)$ for all $t \in [0,T]$, we conclude that $\mathcal{L}\eta_\lambda(\cdot)$, $\mathcal{L}\zeta_\lambda(\cdot) \in C\big([0,T];\, L^1 \cap C_b\big)$. Hence $\partial_t\zeta_\lambda = -\mathcal{L}\zeta_\lambda - \eta_\lambda$. In particular, $\zeta_\lambda \in W$.

Testing (3.9) by ζ_λ and noticing that $\xi_\lambda(T) = 0$ we obtain

$$-\int_{t_0}^T \int_\Omega u\partial_t\zeta_\lambda d\gamma\, dt + \int_{t_0}^T \mathcal{E}(u,\zeta_\lambda)dt = \int_{t_0}^T \int_\Omega \zeta_\lambda dm + \int_\Omega u(t_0)\zeta_\lambda(t_0)d\gamma.$$

Note that

$$\mathcal{E}(u,\zeta_\lambda) = -\int_\Omega u\mathcal{L}\zeta_\lambda d\gamma = \int_\Omega u(\partial_t\zeta_\lambda + \eta_\lambda)d\gamma.$$

Hence, passing to the limit as $t_0 \to 0$ we obtain that

$$\int_0^T\int_\Omega u\eta_\lambda d\gamma\, dt = \int_0^T \int_\Omega \zeta_\lambda dm + \int_\Omega \zeta_\lambda(0)d\mu = \int_0^T\int_\Omega (\mathcal{T}m)\eta_\lambda d\gamma\, dt + \int_0^T \int_\Omega (S\mu)\eta_\lambda d\gamma\, dt,$$

where the last equality follows from (3.1) and the definition of $\mathcal{T}$. Finally, observe that $\eta_\lambda \to \eta$ as $\lambda \to 0$ pointwise, so passing to the limit in λ, we have

$$\int_0^T\int_\Omega u\eta\, d\gamma\, dt = \int_0^T\int_\Omega (\mathcal{T}m + S\mu)\eta\, d\gamma dt.$$

Hence the assertion follows. □

The next proposition gives a version of a maximum principle. It is an extension of [5, Lemma 3].

Proposition 3.7. *Let $f \in L^1(Q)$, $\mu \in \mathcal{M}(\Omega)$, and u be a solution to* (3.8)*. Then, for $t \in (0,T)$,*

$$\int_\Omega u^+(t)d\gamma \le \int_0^t\int_\Omega f\mathbf{1}_{\{u>0\}}d\gamma ds + \int_\Omega d\mu^+,$$

$$\int_\Omega |u(t)|d\gamma \le \int_0^t\int_\Omega f\,\mathrm{sgn}(u)d\gamma ds + \int_\Omega d|\mu|.$$

Proof. Note that $u = S\mu + \mathcal{T}f$, by Lemma 3.6. It suffices to prove the inequalities for $f \in L^1(Q)\cap L^2(Q)$ since $\mathcal{T}$ is a bounded operator on $L^1(Q)$. By Proposition 3.2 $u \in L^2_{loc}\Big((0,T);\, \mathcal{F}\Big)$, $\partial_t u$, $\mathcal{L}u \in L^2_{loc}\Big((0,T);\, L^2\Big)$ and

$$(\partial_t - \mathcal{L})u = f. \tag{3.11}$$

Now we prove the first estimate. Denote $v_k(s) := (ks)^+ \wedge 1$, $k = 1, 2, \ldots$ Then v_k is Lipschiz, non-decreasing, $v_k(0) = 0$ and $v_k \to \mathbf{1}_{\{s>0\}}$ as $k \to \infty$. Hence $v_k(u) \in L^2_{loc}\Big((0,T);\, \mathcal{F}\Big)$ (cf. [13, Theorem 1.4.1]).

We claim that $\mathcal{E}\left(v_k(u), u\right) \geq 0$. Indeed, recall that, for all $u, v \in \mathcal{F}$ one has $\mathcal{E}(u,v) = \lim\limits_{\lambda\to\infty} \mathcal{E}^\lambda(u,v)$, where

$$\mathcal{E}^\lambda(u,v) = \mathcal{E}\left(u, \lambda(\lambda - \mathcal{L})^{-1} v\right)$$

is the approximation of $\mathcal{E}$. By [13, (1.4.8)], there exist positive measures $\mu_\lambda \in \mathcal{M}(\Omega)$ and $\sigma_\lambda \in \mathcal{M}(\Omega \times \Omega)$ such that

$$\mathcal{E}^\lambda(u) = \int_\Omega u^2 \mu_\lambda(dx) + \iint_{\Omega\times\Omega} \left(u(x) - u(y)\right)^2 \sigma_\lambda(dx, dy).$$

Then it is straightforward that $\mathcal{E}^\lambda(\rho(u), u) \geq 0$ for all $\lambda > 0$ and all Lipschiz monotone ρ such that $\rho(0) = 0$. Hence passing to the limit as $\lambda \to \infty$, we conclude that $\mathcal{E}(v_k(u), u) \geq 0$.

Now multiply (3.11) by $v_k(u)$ in L^2 to obtain that

$$\int\limits_\Omega v_k(u(s))\partial_t u(s)\, d\gamma \leq \int\limits_\Omega v_k(u) f\, d\gamma.$$

Integrating the latter in s over the interval (τ, t) we obtain

$$\int\limits_\Omega V_k\big(u(t)\big) d\gamma \leq \int_\tau^t \int\limits_\Omega v_k(u) f\, d\gamma ds + \int\limits_\Omega V_k\big(u(\tau)\big) d\gamma,$$

where $V_k(s)$ is the primitive of $v_k(s)$, $V_k(s) \uparrow s^+$ as $k \uparrow \infty$. So, for $0 < \tau < t$, it follows that

$$\int\limits_\Omega u^+(t) d\gamma \leq \int\limits_\tau^t \int\limits_\Omega f \mathbf{1}_{\{u>0\}} d\gamma ds + \int\limits_\Omega u^+(\tau) d\gamma.$$

It remains to pass to the limit $\tau \to 0$. By Lemma 3.6 using positivity of S and $\mathcal{T}$, we have that $u^+(\tau) = (S(\tau)\mu + (\mathcal{T}f)(\tau))^+ \leq S(\tau)\mu^+ + (\mathcal{T}f^+)(\tau)$ and

$$\int\limits_\Omega (\mathcal{T}f^+)(\tau) d\gamma = \int\limits_0^\tau \int\limits_\Omega S(\tau - s) f^+(s) d\gamma ds \leq \int\limits_0^\tau \|f(s)\|_{L^1} ds \to 0 \text{ as } \tau \to 0.$$

So, as $\tau \to 0$ we arrive at the first assertion.

To prove the second assertion, note that $v = (-u)$ is the solution to the problem $(\partial_t - \mathcal{L})v = -f$, $v(0) = -\mu$. Hence

$$\int\limits_\Omega u^-(t) d\gamma \leq -\int\limits_0^t \int\limits_\Omega f \mathbf{1}_{\{u<0\}} d\gamma ds + \int\limits_\Omega d\mu^-.$$

□

We conclude this section by recalling two results on the parabolic equation with a weighted Laplacian.

Linear equation for a weighted Laplacian. Here we consider a special of the measure $d\gamma = h^2 dx$ and the operator $\mathcal{L}u = -h^{-2}\mathrm{div}(h^2\nabla u)$, where as before $h(x) = |x|^\lambda$ with $\lambda > \frac{2-N}{2}$. Namely, we state the Mean-value inequality and the heat kernel estimates for the linear equation

$$\partial_t u - h^{-2}\mathrm{div}(h^2\nabla u) = 0. \tag{3.12}$$

Theorem 3.8 (Mean-value inequality)**.** *There exists a constant $C > 0$ such that, for all $(x,t) \in \mathbb{R}^{N+1}$, $r > 0$, $q > 0$ and a weak positive (sub-)solution u to* (3.12) *in the cylinder $Q_{2r}^{(x,t)} := B_{2r}(x) \times (t - 4r^2, t + 4r^2)$, the following inequality holds: for $Q^- := B_{r/2}(x) \times (t - 2r^2, t)$ and $Q^+ := B_r(x) \times (t + 3r^2, t + 4r^2)$,*

$$\sup_{Q^-} u \le C \left(⨌_{Q^+} u^q \right)^{\frac{1}{q}},$$

where the average integral in the right hand side is by measure $h^2 dx dt$.

Theorem 3.9. *Let k be the fundamental solution k to the equation* (3.12)*. Then for all $\delta > 0$ there exists $c_\delta > 0$ such that for all $x, y \in \mathbb{R}^N$ and $t > 0$ the following estimate holds:*

$$k(t,x,y) \le c_\delta t^{-\frac{N+2\lambda}{2}} e^{-\frac{|x-y|^2}{4(1+\delta)t}} \left(\frac{|x|}{\sqrt{t}} + 1 \right)^{-\lambda} \left(\frac{|y|}{\sqrt{t}} + 1 \right)^{-\lambda}. \tag{3.13}$$

The detailed exposition of these and related results can be found in [15, 25].

4 Source solutions

Here we use the same notation as in the previous section. In this section we construct solutions to an abstract semilinear equation with measures as initial data. We closely follow ideas from [30, Chapter 6].

Consider the solution of the non-linear equation

$$(\partial_t - \mathcal{L})\, u(x,t) + g\,(x, u(x,t)) = 0, \quad u(0) = \mu \in \mathcal{M}(\Omega), \tag{4.1}$$

where $\mathcal{L}$ is as in the previous section and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable in x for all $r \in \mathbb{R}$, continuous in r for a.a. $x \in \Omega$ (the Caratheodory conditions), non-decreasing in r and vanishing at $r = 0$ for a.a $x \in \Omega$. We denote $G : u \mapsto g\,(x, u(x))$ the correspondent monotone homogeneous Nemytskii operator. So a weak solution to the problem (4.1), e.g. $(\partial_t - \mathcal{L} + G)u = 0$, $u(0) = \mu$, is $u \in L^1(Q) \cap L^2_{loc}((0,T); \mathcal{F})$ such that $Gu \in L^1(Q)$ and $(\partial_t - \mathcal{L})u = -Gu$, $u(0) = \mu$ in the sense of Definition 3.5. In particular,

$$u \; = S\mu - \mathcal{T}Gu. \tag{4.2}$$

Proposition 4.1. *Let $\mu_1, \mu_2 \in \mathcal{M}(\Omega)$, $\mu_1 \le \mu_2$, $g_1(x,r) \ge g_2(x,r)$ for all $r \in \mathbb{R}$ and a.a. x, G_1, G_2 be the corresponding Nemytskii operators and $u_j \in L^1(Q)$ be solutions to the problems $(\partial_t - \mathcal{L} + G_j)u_j = 0$, $u_j(0) = \mu_j$, $j = 1, 2$. Then $u_1 \le u_2$ pointwise for a.a. $(x,t) \in Q$.*

Proof. Let $w = u_1 - u_2$. Then w satisfies $(\partial_t - \mathcal{L})w = -(G_1u_1 - G_2u_2)$, $w(0) = -(\mu_2 - \mu_1) \le 0$. By Proposition 3.7, for $t > 0$,

$$\int_\Omega w^+(t) h^2 dx \le - \int_0^t \int_\Omega (G_1u_1 - G_2u_2)\mathbf{1}_{\{w>0\}} d\gamma\, ds.$$

However, $w > 0$ implies $u_1 > u_2$ and hence $G_1u_1 \ge G_1u_2 \ge G_2u_2$. So the above yields $w^+ = 0$ and $u_1 \le u_2$. ☐

The next corollary is a straightforward consequence of Proposition 4.1.

Corollary 4.2. *Let $\mu \in \mathcal{M}(\Omega)$, G be a monotone homogeneous Nemytskii operator. There exists at most one solution to the problem $(\partial_t - \mathcal{L} + G)u = 0$, $u(0) = \mu$. The solution satisfies the estimates*

$$-S\mu^- \le u \le S\mu^+$$

and

$$\int_\tau^T \|u(s)\|_{\mathcal{F}}^2\, ds \le \frac{1}{2}\psi(\tau)\|\mu\|_{\mathcal{M}(\Omega)}^2. \tag{4.3}$$

Proof. The first assertion is clear from Proposition 4.1. The second assertion follows from the comparison of the solution to the problem $(\partial_t - \mathcal{L} + G)u = 0$, $u(0) = \mu$, with the solutions to the problems $(\partial_t - \mathcal{L} + G^\mp)v^\pm = 0$, $v^\pm(0) = \pm\mu^\pm$, where $G^\mp$ is the Nemytskii operator corresponding to the function $\mathbb{1}_{[\mp r \ge 0]} g(x, r)$. Note that $v^\pm = \pm S\mu^\pm$ and that $\mathbb{1}_{[-r\ge 0]} g(x, r) \le g(x, r) \le \mathbb{1}_{[r\ge 0]} g(x, r)$. Hence the pointwise estimate follows.

Now we prove (4.3). For $\lambda > 0$ let $\zeta_\lambda(t) := S(\lambda)u(t)$, $t \in [0, T]$. Since $u \in C\big([0, T];\, L^1\big) \cap L^2_{loc}\big((0, T);\, \mathcal{F}\big)$, one has $\zeta_\lambda \in C\big([0, T];\, L^1 \cap C_b\big) \cap L^2_{loc}\big((0, T);\, \mathcal{F}\big)$. Moreover, differentiating the equation $\zeta_\lambda(t) = S(\lambda + t)\mu - S(\lambda)(\mathcal{T}Gu)(t)$, we obtain

$$\big(\partial_t\zeta_\lambda\big)(t) = \mathcal{L}S(\lambda)u(t) - S(\lambda)\big(Gu\big)(t),\ t \in (0, T).$$

Hence $\zeta_\lambda \in W^{1,1}_{loc}\big((0, T);\, L^\infty\big)$. Since $|u(t)| \le S(t)|\mu|$, $t \in (0, T]$, we conclude that $u \in L^\infty_{loc}\big((0, T);\, L^\infty\big)$. Hence (3.9) holds with $\zeta = \zeta_\lambda$. For $\tau \in (0, T)$ we have

$$\frac{1}{2}\big\|\zeta_{\lambda/2}(T)\big\|_{L^2}^2 + \int_\tau^T \mathcal{E}\big(\zeta_{\lambda/2}(t)\big)\, dt + \int_\tau^T\int_\Omega \zeta_\lambda G u\, d\gamma dt = \frac{1}{2}\big\|\zeta_{\lambda/2}(\tau)\big\|_{L^2}^2.$$

Passing to the limit as $\lambda \to 0$, we arrive at

$$\frac{1}{2}\big\|u(T)\big\|_{L^2}^2 + \int_\tau^T \mathcal{E}\big(u(t)\big)\, dt + \int_\tau^T\int_\Omega u\, Gu\, d\gamma dt = \frac{1}{2}\big\|u(\tau)\big\|_{L^2}^2.$$

Finally, observe that $u\, Gu \ge 0$ a.e. and that $\big\|u(\tau)\big\|_{L^2}^2 \le \big\|S(\tau)|\mu|\big\|_{L^2}^2 \le \psi(\tau)\|\mu\|_{\mathcal{M}(\Omega)}^2$. ◻

Proposition 4.3. *Let $g_n(x, r) \to g(x, r)$ for a.a. x and locally uniformly in $r \in \mathbb{R}$, as $n \to \infty$. Let G_n, G be the corresponding monotone homogeneous Nemytskii operators. Let $\mu_n, \mu \in \mathcal{M}(\Omega)$ be such that $\mu_n \to \mu$ in the weak-$*$ topology of $\mathcal{M}(\Omega)$. In addition assume that*

$$w := \sup_n \big[G_n S\mu_n^+ - G_n(-S\mu_n^-)\big] \in L^1(Q). \tag{4.4}$$

Let u_n be the solutions to the problems

$$(\partial_t - \mathcal{L} + G)u_n = 0, \quad u_n(0) = \mu_n, \quad n \in \mathbb{N}. \tag{4.5}$$

Then $u_n \to u$ in $L^1(Q)$ as $n \to \infty$, and u is the solution to the problem $(\partial_t - \mathcal{L} + G)u = 0$, $u(0) = \mu$.

Proof. First note that the sequence $(\mu_n)_n$ is bounded in $\mathcal{M}(\Omega)$ since it is weak-$*$ convergent. Let $M = \sup_n \|\mu_n\|_{\mathcal{M}(\Omega)} < \infty$. Now we have to pass to the limit in (4.5).

Since $|G_n u_n| \le w \in L^1(Q)$ and by Proposition 3.7,

$$\|G_n u_n\|_{L^1(Q)} \le M,$$

the sequence $(G_n u_n)_n$ is a pre-compact set in the weak topology in $L^1(Q)$. By Proposition 3.2, $\mathcal{T}$ is a completely continuous operator on $L^1(Q)$. Moreover, $S\mu_n \to S\mu$ by Proposition 3.1. Therefore the sequence $(\mathcal{T} G u_n)_n$, and hence the sequence $(u_n)_n$ are compact in $L^1(Q)$. Moreover, due to (4.3), $(u_n)_n$ is weakly compact in $L^2_{\text{loc}}\big((0,T);\,\mathcal{F}\big)$. Let (u_{n_l}) be a sub-sequence of $(u_n)_n$ convergent, in $L^1(Q)$ strongly, in $L^2_{\text{loc}}\big((0,T);\,\mathcal{F}\big)$ weakly and a.e. on Q to a limit u. Note that $|u_{n_l}(t)| \le S(t)|\mu_{n_l}| \le M\psi(t)$ a.e. by Corollary 4.2 and (3.4). Since for all $t > 0$ and a.a. $x \in \Omega$ one has $g_n(x,r) \to g(x,r)$ as $n \to \infty$ uniformly in $r \in [-M\psi(t), M\psi(t)]$, we conclude that $G_{n_l} u_{n_l} \to Gu$ as $l \to \infty$ a.e. on Q. So $G_{n_l} u_{n_l} \to Gu$ in $L^1(Q)$, by the Lebesgue dominated convergence theorem. Hence we can pass to the limit in the equality $u_{n_l} = S\mu_{n_l} - \mathcal{T} G_{n_l} u_{n_l}$ as $l \to \infty$ and obtain that $u = S\mu - \mathcal{T} G u$. Moreover, since $u_{n_l} \to u$ as $l \to \infty$ weakly in $L^2_{\text{loc}}\big((0,T);\,\mathcal{F}\big)$, it follows that u satisfies (3.9) with $f = -Gu$ for all $\zeta \in W$. Hence $(\partial_t - \mathcal{L} + G)u = 0$ and $u(0) = \mu$. By Corollary 4.2, the solution to the latter equation is unique so $(u_n)_n$ has a unique limit point u. Hence $u_n \to u$ in $L^1(Q)$ strongly and in $L^2_{\text{loc}}\Big((0,T);\,\mathcal{F}\Big)$ weakly. □

The following is a straightforward consequence of Proposition 4.3.

Corollary 4.4. *Let G be a monotone homogeneous Nemytskii operator, $\mu_n \to \mu$ in weak-$*$ topology of $\mathcal{M}(\Omega)$, $\mu_n \ge 0$, $\operatorname{supp}(\mu_n) \subset B_r$ and $\|\mu_n\|_{\mathcal{M}(\Omega)} \le c$. Let u_n be the solution to (4.5). Set $s_c(x,t) := c \sup_{y \in B_r} p_t(x,y)$. Assume that*

$$Gs_c \in L^1(Q). \tag{4.6}$$

Then $u_n \to u$ in $L^1(Q)$ as $n \to \infty$, and u is the solution to the problem $(\partial_t - \mathcal{L} + G)u = 0$, $u(0) = \mu$.

The next theorem is the main result of this section.

Theorem 4.5. *Let (3.2) and (3.3) hold. Let $\mu \in \mathcal{M}(\Omega)$ satisfy the condition*

$$\iint_Q \left[GS\mu^+ - G(-S\mu^-)\right] d\gamma dt < \infty. \tag{4.7}$$

Then there exists a unique solution $u = u_\mu$ to the Cauchy problem

$$\begin{cases} (\partial_t - \mathcal{L} + G)\, u = 0, \\ u(0) = \mu. \end{cases} \tag{4.8}$$

Moreover, $[u_\mu(t) - S(t)\mu] \to 0$ in L^1 as $t \to 0$.

Proof. First we consider g such that $\bar{g} \in L^1$ with $\bar{g}(x) := \sup_r |g(x,r)|$, $x \in \Omega$. Denote $H(x,r) := \int_0^r g(x,s)ds$. Then H is a convex positive sub-linear function in r for a.a. $x \in \Omega$. Consider the functional

$$J(u) := \frac{1}{2}\mathcal{E}(u) + \int_\Omega H\left(x, u(x)\right) \gamma(dx), \;\; u \in \mathcal{F}.$$

Then $\delta J = -\mathcal{L} + G$. By [28, Theorem III.4.1, Proposition III.4.2], for $j = 1, 2, 3, \ldots$ there exists a unique solution $u_j \in L^2\Big((0,T);\, \mathcal{F}\Big)$ to the Cauchy problem

$$\begin{cases} (\partial_t - \mathcal{L} + G)u_j = 0, \\ u_j(0) = \mu_j \in L^1 \cap L^\infty. \end{cases} \tag{4.9}$$

Moreover, $u_j \in L^\infty\left((0,T), \mathcal{F}\right) \cap W^{1,2}\Big((0,T);\, L^2\Big)$.

If $\mu_j \to \mu$ as $j \to \infty$ in the sense of weak-$*$ convergence of measures, then, by Proposition 4.3, $u_j \to u$ as $j \to \infty$ in $L^1(Q)$, and u is the unique solution to (4.8). Indeed, we have to verify condition (4.4). However,

$$\sup_n \left[G_n S\mu_n^+ - G_n(-S\mu_n^-)\right] \le \bar{g} \in L^1.$$

Hence the assertion follows.

For a general g, let $E_k \subset \Omega$ be an increasing sequence of subsets of finite measure such that $\Omega = \cup E_k$. For $k = 1, 2, 3, \ldots$, let $g_k(x,r) := \mathbf{1}_{E_k} \mathrm{sgn}\left(g(x,r)\right)\left(|g(x,r)| \wedge k\right)$, let G_k be the corresponding Nemytskii operator and let u_k be the solution to the equation $(\partial_t - \mathcal{L} + G_k)u_k = 0$, $u_k(0) = \mu$ constructed above. Then, by Corollary 4.2, $-S\mu^- \le u_k \le S\mu^+$, and hence

$$|G_k u_k| \le |Gu_k| \le \begin{cases} GS\mu^+, & u_k \ge 0, \\ -G(-S\mu^-), & u_k < 0, \end{cases} \le GS\mu^+ - G(-S\mu^-). \tag{4.10}$$

Since $GS\mu^+ - G(-S\mu^-) \in L^1(Q)$, Proposition 4.3 implies that $u_k \to u$ as $k \to \infty$ in $L^1(Q)$, and u is the solution to (4.8).

To prove the last assertion, note that $S\mu - u = \mathcal{T}Gu$. So, by (4.10),

$$\int_\Omega |u(t) - S\mu(t)| d\gamma \le \int_0^t \int_\Omega \left[GS\mu^+ - G(-S\mu^-)\right] d\gamma d\tau \to 0 \text{ as } t \to 0.$$

□

The next corollary together with the last assertion of the previous theorem provide the proof of assertions (d) and (e) of Theorem 1.2 for $\varkappa < \infty$.

Corollary 4.6. *Let $0 < p < 1 + \frac{2+\beta}{N+2\lambda}$. Then the problem*

$$\begin{cases} \partial_t u - h^{-2} div\big(h^2 \nabla u\big) + r^\beta |u|^{p-1} u & \text{in } \mathbb{R}^N, \\ u(0) = \varkappa \delta_0 \end{cases}$$

has a unique solution $u_\varkappa$ for every $\varkappa > 0$. Conversely, for $1 < p < 1 + \frac{2+\beta}{N+2\lambda}$ and $\varkappa \in (0, \infty)$, if u is a solution to (1.6) satisfying $u(t) \to \varkappa\delta_0$ as $t \to 0$ in the sense of weak-$*$ convergence of measures, then $u = u_\varkappa$.

Proof. In this case $\mathcal{E}(u) = \|\nabla u\|^2_{L^2_{h^2}}$ is the bilinear quadratic form in $L^2_{h^2}$ with $C^1_c(\mathbb{R}^N)$ as its core. (It follows from Lemma A.1 that $\Big(\mathcal{E}, C^1_c(\mathbb{R}^N)\Big)$ is closable in $L^2_{h^2}$.) Let S denote the corresponding semigroup and k its integral kernel. By Theorem 3.9, k obeys the estimate (3.13). Now we verify the assumption of Theorem 4.5:

$$\int_0^T\int_{\mathbb{R}^N} r^\beta |S(t)\varkappa\delta_0|^p h^2 dxdt = \varkappa^p \int_0^T\int_{\mathbb{R}^N} |x|^{\beta+2\lambda}|k(t,x,0)|^p dxdt$$
$$\leq \int_0^T\int_{\mathbb{R}^N} |x|^{\beta+2\lambda} c_\delta t^{-\frac{p(N+2\lambda)}{2}} e^{-\frac{p|x|^2}{4(1+\delta)t}} \left(\frac{|x|}{\sqrt{t}}+1\right)^{-p\lambda} dxdt$$
$$\leq c_{\delta,p}\int_0^T t^{\frac{\beta-(p-1)(N+2\lambda)}{2}} dt \int_{\mathbb{R}^N} |\xi|^{\beta+2\lambda} e^{-\frac{p|\xi|^2}{4(1+\delta)}} (|\xi|+1)^{-p\lambda} d\xi.$$

The integral in t converges since $\frac{\beta+2\lambda-(p-1)N-2p\lambda}{2} > -1$, that is, $p < 1+\frac{2+\beta}{N+2\lambda}$. The integral in ξ converges since $\beta+2\lambda+N = (\beta+2)+(2\lambda+N-2) > 0$.

The second assertion follows from Corollaries 2.2 and 4.2 $\square$

5 Very singular solutions

In this section we construct a very singular solution to (1.6) and prove its uniqueness. Throughout the section we assume that

$$1 < p < p^* = 1 + \frac{2+\beta}{N+2\lambda}.$$

We start this section by showing that every very singular solution (VSS) if it exists, dominates pointwise every source type solution (SS). The next proposition is an analogue of [17, Lemma 1.3].

Proposition 5.1. *Let v be a VSS and u be a SS to* (1.6)*, respectively. Then $u \leq v$ pointwise for a.a. $(x,t) \in \mathbb{R}^N \times (0,T)$.*

Proof. Let $\int_{B_1} u(t)h^2dx \to \varkappa < \infty$ as $t \to 0$. Let $\tau_0 > 0$ be such that $\int_{B_1} v(t)h^2dx > \varkappa$ for all $0 < t \leq \tau_0$. Then, for $\tau \in (0,\tau_0)$ there exists $\varphi_\tau \in L^1_{h^2}$ such that $0 \leq \varphi_\tau \leq v(\tau)\mathbf{1}_{B_1}$ and $\|\varphi_\tau\|_{L^1_{h^2}} = \varkappa$.

Let $u^{(\tau)}$ be the solution to the problem

$$(\partial_t - \Delta_{h^2})u + r^\beta u^p = 0, \quad u(0) = \varphi_\tau.$$

Thanks to Proposition 2.1 it is easy to check that

$$v \in L^1(\mathbb{R}^N \times (\tau,t), h^2dxdt) \cap L^p(\mathbb{R}^N \times (\tau,t), r^\beta h^2dxdt).$$

Then by Proposition 4.1

$$u^{(\tau)}(t) \leq v(t+\tau), \quad t > 0. \tag{5.1}$$

Since $\|u^{(\tau)}(0)\|_{L^1_{h^2}} = \varkappa$ and $\operatorname{supp} u^{(\tau)}(0) \subset \operatorname{supp} v(\tau)$, it follows that $u^{(\tau)}(0)h^2dx \to \varkappa\delta_0$ in weak-$*$ topology of $\mathcal{M}(\Omega)$. Hence $u^{(\tau)} \to u_\varkappa$ in $L^1_{h^2}(Q)$ by Corollary 4.4, where (3.13) is used to verify (4.6). Then (5.1) implies that $u_\varkappa \leq v$. $\square$

The above leads to an immediate construction of the minimal VSS.

Corollary 5.2. $u_\infty := \lim_{\varkappa\to\infty} u_\varkappa$ *is the minimal VSS, where* $u_\varkappa$ *is the solution from Corollary 4.6.*

Proof. Using Proposition 2.1 one can easily verify that the above limit exists and is a solution to (1.6), (1.7). □

In the next proposition we follow the construction from [18, Theorem 4.1].

Proposition 5.3. $U_\infty(x,t) := \sup\{u(x,t)\ :\ u\ \ \text{is a positive singular solution}\}$ *is the maximal VSS.*

Proof. Let u be a solution for (1.6), (1.7). It follows from Lemma 1.6 and Proposition 2.1 that, for all $R>0$ one has $u\in C^{2,1}(\mathbb{R}^N\setminus B_R\times[0,T))$ and $u(x,t)\to 0$ as $|x|\to\infty$ uniformly in t. Moreover, by Proposition 2.1

$$u(x,t)\le c(|x|^2+t)^{-\frac{2+\beta}{2(p-1)}},\quad (x,t)\in\mathbb{R}^N\times(0,\infty), \tag{5.2}$$

with a constant $c>0$ independent of u. Let v be the solution of the linear inhomogeneous problem

$$\begin{cases}(\partial_t-\Delta_{h^2})v=0 & \text{in } \mathbb{R}^N\setminus B_R\times(0,\infty),\\ v(x,0)=0, & x\in\mathbb{R}^N\setminus B_R,\\ v(x,t)=cR^{-\frac{2+\beta}{p-1}}, & x\in\partial B_R, t>0.\end{cases}$$

Then, by the maximum principle, $u\le v$. Note that v enjoys the estimate

$$0\le v(x,t)\le C_R\int_0^t\int_{[R<|x|<2R]} p_s^{h^2}(x,y)dy\,ds, \tag{5.3}$$

where $C_R>0$ is a constant and $p_t^{h^2}(\cdot,\cdot)$ is the fundamental solution to the linear equation $(\partial_t-\Delta_{h^2})u=0$. Hence all u and U_∞ satisfy (5.3) with v replaced by u and U_∞, respectively. Note also that U_∞ satisfies the estimate (5.2) with u replaced by U_∞. In particular, $U_\infty(\tau)\in L^1_{h^2}$ and $U_\infty\in L^1(\mathbb{R}^N\times(\tau,T),\,h^2dxdt)\cap L^p(\mathbb{R}^N\times(\tau,T),\,r^\beta h^2dxdt)$ for all $\tau>0$.

By Theorem 4.5 for $t>\tau$ the problem

$$\begin{cases}(\partial_t-\Delta_{h^2})u+r^\beta u^p=0, & t>\tau,\\ u(\tau)=U_\infty(\tau)\end{cases}$$

has a unique solution $u^{(\tau)}$.

For every singular solution u we have that $u^{(\tau)}(\tau)\ge u(\tau)$. Therefore by Proposition 4.1 $u^{(\tau)}(t)\ge u(t)$, and hence $u^{(\tau)}(t)\ge U_\infty(t)$ for all $t\ge\tau$. Moreover, for $\tau'\le\tau$ one has

$$u^{(\tau')}(\tau)\ge U_\infty(\tau)=u^{(\tau)}(\tau).$$

Using Proposition 4.1 again we obtain that

$$u^{(\tau')}(t)\ge u^{(\tau)}(t),\quad \tau'\le\tau\le t.$$

By Proposition 2.1 it follows that, for all $t_0 > 0$ and $\tau < \frac{t_0}{2}$, with $\rho := \sqrt{t_0 - \tau} \geq \sqrt{\frac{t_0}{2}}$

$$\iint\limits_{t>t_0} |\nabla u^{(\tau)}|^2 h^2 dx\, dt \leq c \left(\int\limits_{|x|>\rho} U_\infty(x,\tau)^2 h^2 dx + \rho^{N+2\lambda-2\frac{2+\beta}{p-1}} \right) \leq c t_0^{\frac{N}{2}+\lambda-\frac{2+\beta}{p-1}}.$$

So $(\nabla u^{(\tau)})_\tau$ is bounded in $L^2_{loc}(\mathbb{R}^N \times (0,\infty), h^2 dx\, dt)$ uniformly in τ. Hence $u_\tau \uparrow u$ as $\tau \downarrow 0$. Now passing to the limit in τ it is easy to see that

$$(\partial_t - \Delta_{h^2})u + r^\beta u^p = 0.$$

Furthermore, by (5.3) for $x \neq 0$ we have that $u(x,t) \to 0$ as $t \to 0$. Thus u is a singular solution, and hence $u \leq U_\infty$. Since $u \geq u_\tau \geq U_\infty$, we conclude that $u = U_\infty$. □

Lemma 5.4. *The minimal VSS u_∞ and and the maximal VSS U_∞ to* (1.6) *are self-similar. More precisely,*

$$u_\infty(x,t) = t^{-\sigma} v_\infty(\frac{x}{\sqrt{t}}), \quad U_\infty(x,t) = t^{-\sigma} V_\infty(\frac{x}{\sqrt{t}})$$

where v_∞ and V_∞ are positive solutions to the problem

$$\begin{cases} -K^{-1}\nabla(K\nabla v) - \sigma v + r^\beta v|v|^{p-1} = 0, \\ r^{2\sigma} v \to 0 \quad \text{as } r \to \infty, \end{cases} \tag{5.4}$$

with $\sigma = \frac{\beta+2}{2(p-1)}$ and $K = r^{2\lambda} e^{\frac{r^2}{4}}$.

Proof. Let u be a singular solution to (1.6). Then $T_\rho u$ defined by $T_\rho u(x,t) := \rho^{\frac{\beta+2}{p-1}} u(\rho x, \rho^2 t)$, is another singular solution to (1.6). Moreover, $T_\rho u_c = u_{c_\rho}$ with $c_\rho = c\rho^{\frac{\beta+2}{p-1}-N-2\lambda}$. Hence by definition $T_\rho U_\infty = U_\infty$ and $T_\rho u_\infty = u_\infty$. Now the assertion follows with $\rho = t^{-\frac{1}{2}}$. □

Proposition 5.5. *A VSS to* (1.6) *is unique, i.e. $U_\infty = u_\infty$.*

Proof. It suffices to show that $V_\infty \leq v_\infty$. Let $w := V_\infty - v_\infty$. Note that w is a sub-solution to the equation

$$-K^{-1}\nabla(K\nabla w) - \sigma w + r^\beta v_\infty^{p-1} w = 0.$$

Since $-K^{-1}\nabla(K\nabla v_\infty) - \sigma v_\infty + r^\beta v_\infty^p = 0$, it follows from [10, Theorem 4.1] that

$$\int |\nabla\theta|^2 K\, dx + \int \left(r^\beta v_\infty^{p-1} - \sigma \right) \theta^2 K\, dx \geq 0 \text{ for all } \theta \in C_c^{0,1}(\mathbb{R}^N).$$

Since $r^{2\sigma} V_\infty \to 0$ as $r \to \infty$ it follows that, for a sufficiently large R, one has $w(x) \leq R^{-2\sigma}$ for all x, $|x| > R$. By the weak maximum principle, we infer that $w(x) \leq R^{-2\sigma}$ for all x, $|x| < R$. Hence $w \leq 0$. □

As the positive solution to (5.4) produces a VSS, it is clear that (5.4) has a unique positive solution. To find it one can use a variational approach almost identical to that in [12]. Namely, one considers the nonlinear functional J on the Banach space $X := H^1_K(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N, r^\beta K dx)$,

$$J(\theta) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\theta|^2 K\, d\xi + \frac{1}{p+1}\int_{\mathbb{R}^N} |\theta|^{p+1} r^\beta K\, d\xi - \frac{\mu}{2}\int_{\mathbb{R}^N} |\theta|^2 K\, d\xi. \tag{5.5}$$

To show that J is bounded below we need the following auxiliary result.

Lemma 5.6. *For any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|\theta\|^2_{L^2_K} \le \varepsilon \left(\|\nabla\theta\|^2_{L^2_K} + \|\theta\|^{p+1}_{L^2_{r^\beta K}} \right) + C_\varepsilon,\ \theta \in X.$$

Proof. For $\varepsilon > 0$, choose $R_\varepsilon > r_\varepsilon > 0$ such that

$$\frac{16}{R_\varepsilon^2} + \frac{4r_\varepsilon^2}{(N-2+2\lambda)^2} < \frac{\varepsilon}{2}.$$

Then, by Lemma A.1, we have

$$\int\limits_{B_{r_\varepsilon} \cup \mathbb{R}^N \setminus B_{R_\varepsilon}} |\theta|^2 K\,dx \le \frac{\varepsilon}{2}\|\nabla\theta\|^2_{L^2_K}.$$

Now, by the Young inequality,

$$\int\limits_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\theta|^2 K\,dx \le \frac{\varepsilon}{2} \int\limits_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |\theta|^{p+1} K\,dx + C_{p,\varepsilon} \int\limits_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} K\,dx.$$

□

Now we are ready to show the existence of a non-trivial minimizer of the functional J. This follows immediately from the next proposition.

Proposition 5.7. *The functional J defined in (5.5) is bounded below and lower semi-continuous with respect to the weak topology. Moreover, $J(\theta) \to \infty$ as $\|\theta\|_X \to \infty$.*

Proof. Let $\theta_n \to \theta$ weakly in X. Then $\theta_n \to \theta$ strongly in L^2_K since $H^1_K \overset{compact}{\hookrightarrow} L^2_K$, by Corollary A.2. So $\liminf \|\theta_n\|_{H^1_K} \ge \|\theta\|_{H^1_K}$, and due to the lower semi-continuity of the L^p-norm w.r.t. the weak convergence (see, e.g. [20]) $\liminf \|\theta_n\|_{L^{p+1}_{r^\beta K}} \ge \|\theta\|_{L^{p+1}_{r^\beta K}}$ and $\lim \|\phi_n\|_{L^2_K} = \|\phi\|_{L^2_K}$. The last two assertions follow directly from Lemma 5.6. □

Next we show that the minimizer is nontrivial and can be chosen non-negative.

Proposition 5.8. *Let $\mu > \frac{N+2\lambda}{2}$. Then there exists a non-trivial minimizer of J which can be chosen nonnegative.*

Proof. Note that $J(0) = 0$. Let $\tau > 0$. Set $\phi = \tau e^{-\frac{r^2}{4}}$. Then

$$E(\tau\phi) = (\frac{N+2\lambda}{2} - \mu)\tau^2 \int \phi^2 K\,dx + \tau^{p+1} \int \phi^{p+1} r^\beta K\,dx.$$

Now it clear that there exists $\tau > 0$ such that $J(\tau\phi) < 0$, hence zero is not a minimizer. The last assertion follows from the fact that $J(\theta) = J(|\theta|)$. □

The minimizer is exponentially decaying at infinity, which is shown in the next proposition.

Proposition 5.9. *Let $v \in H^1_K(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N, r^\beta K dx)$ be a solution to*

$$-K^{-1}\nabla(K\nabla v) - \mu v + r^\beta v|v|^{p-1} = 0. \tag{5.6}$$

There exists $C > 0$ such that

$$|v| \le Ce^{-\frac{r^2}{8}},\ \text{on } \mathbb{R}^N \setminus B_1.$$

Proof. We follow [12] simplifying the arguments. Let $w := ve^{|x|^2/8}$. Then w satisfies the equation

$$-\frac{1}{h^2}\nabla(h^2\nabla w) + Vw = 0 \tag{5.7}$$

with $V := \frac{N+2\lambda}{4} - \mu + \frac{r^2}{16} + e^{-\frac{(p-1)r^2}{8}} r^\beta |w|^{p-1}$. One can choose $R > 0$ such that $V(x) \geq 0$ on $|x| > R$. It is easily seen that solutions to (5.7) are locally bounded outside the point $x = 0$. Let $M := \sup_{1<|x|<R} w(x)$. Looking at (5.7) on $|x| > 1$ and taking $\varphi := (w - M)^+$ as a test function we obtain that $\varphi = 0$. Changing w by $-w$ in (5.7) we see that $|w(x)| \leq M$, which proves the assertion. □

A Appendix: Hardy-type inequality and compact embedding

Lemma A.1. *For $\lambda > \frac{2-N}{2}$, $\alpha \geq 0$, $K = r^{2\lambda}e^{\alpha r^2}$, $\theta \in C_c^\infty(\mathbb{R}^N)$, there holds*

$$\int_{\mathbb{R}^N} |\nabla\theta|^2 K dx \geq \int_{\mathbb{R}^N} \left(\alpha^2 r^2 + \alpha(N + 2\lambda) + \left(\frac{N-2+2\lambda}{2}\right)^2 \frac{1}{r^2}\right) |\theta|^2 K dx. \tag{A.1}$$

Proof. First, notice that $\operatorname{div}(x|x|^q) = (N + q)|x|^q$ for all $q > -N$. Now let $v = r^\lambda e^{\frac{\alpha}{2}r^2}\theta$. Then $v \in H^1(\mathbb{R}^N)$ and

$$\nabla v = r^\lambda e^{\frac{\alpha}{2}|x|^2}\nabla\theta + x\left(\alpha + \frac{\lambda}{r^2}\right) v.$$

Hence we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\theta|^2 K dx &= \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} \left(\alpha^2 r^2 + 2\alpha\lambda + \frac{\lambda^2}{r^2}\right) v^2 dx - \int_{\mathbb{R}^N} (\nabla v^2) \cdot x \left(\alpha + \frac{\lambda}{r^2}\right) dx \\ &= \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} \left(\alpha^2 r^2 + \alpha(N + 2\lambda) + \frac{\lambda^2 + (N-2)\lambda}{r^2}\right) v^2 dx.\end{aligned}$$

Now the assertion follows from the standard Hardy inequality. □

Corollary A.2. *Let $\lambda > -\frac{N-2}{2}$, $\alpha > 0$, $K = e^{\alpha r^2} r^{2\lambda}$. Then $H^1(\mathbb{R}^N, Kdx)$ is compactly embedded into $L^2(\mathbb{R}^N, Kdx)$.*

Proof. It suffices to prove that, given $v_n \to 0$ weakly in $H^1(\mathbb{R}^N, Kdx)$, the sequence $(v_n)_n$ converges to 0 strongly in $L^2(\mathbb{R}^N, Kdx)$.

Let $m := \sup_n \|v_n\|_{H^1}$. We use the following decomposition: for $0 < r_0 < R_0$,

$$\int_{\mathbb{R}^N} v^2 K dx = \int_{r<r_0} v^2 K dx + \int_{r_0<r<R_0} v^2 K dx + \int_{r>R_0} v^2 K dx. \tag{A.2}$$

Fix $\varepsilon > 0$ and choose r_0 and R_0 such that

$$2mr_0^2 \left(\frac{N-2+2\lambda}{2}\right)^{-2} < \varepsilon \text{ and } \frac{2m}{\alpha^2 R_0^2} < \varepsilon.$$

Then by Lemma A.1

$$\begin{aligned}\int_{r<r_0} v^2 K dx + \int_{r>R_0} v^2 K dx &\leq r_0^2 \int_{\mathbb{R}^N} \frac{v^2}{r^2} K dx + \frac{1}{R_0^2} \int_{\mathbb{R}^N} r^2 v^2 K dx \\ &\leq \left(r_0^2 \left(\frac{N-2+2\lambda}{2}\right)^{-2} + \frac{16}{R_0^2}\right) m < \varepsilon.\end{aligned}$$

Finally, the sequence (v_n) is bounded in $H^1(B_{R_0} \setminus B_{r_0}, Kdx) = H^1(B_{R_0} \setminus B_{r_0}, dx)$. So $v_n \to 0$ weakly in $H^1(B_{R_0} \setminus B_{r_0}, dx)$, hence $v_n \to 0$ strongly in $L^2(B_{R_0} \setminus B_{r_0}, dx) = L^2(B_{R_0} \setminus B_{r_0}, Kdx)$. Thus, for all $\varepsilon > 0$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} v_n^2 K dx < \varepsilon.$$

The same argument implies the second assertion. $\square$

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