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A mean curvature estimate for cylindrically bounded submanifolds

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Abstract

We extend the estimate obtained in [1] for the mean curvature of a cylindrically bounded proper submanifold in a product manifold with an Euclidean space as one factor to a general product ambient space endowed with a warped product structure.

Let (L^ℓ, g_L) and (P^n, g_P) be complete Riemannian manifolds of dimension ℓ and n, respectively, where L^ℓ is non compact. Then, let $N^{n+\ell} = L^\ell \times_\rho P^n$ be the product manifold $L^\ell \times P^n$ endowed with the warped product metric $ds^2 = dg_L + \rho^2 dg_P$ for some positive warping function $\rho \in C^\infty(L)$.

Let $B_P(r_0)$ denote the geodesic ball with radius r_0 centered at a reference point $o \in P^n$. We assume that the radial sectional curvatures in $B_P(r_0)$ along the geodesics issuing from o are bounded as $K_P^{\mathrm{rad}} \leq b$ for some constant $b \in \mathbb{R}$, and that $0 < r_0 < \min\{\mathrm{inj}_P(o), \pi/2\sqrt{b}\}$ where $\mathrm{inj}_P(o)$ is the injectivity radius at o and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Then, the mean curvature of the geodesic sphere $S_P(r_0) = \partial B_P(r_0)$ can be estimated from below by the mean curvature of a geodesic sphere of a space form of curvature b, namely,

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if} \quad b > 0, \\ 1/t & \text{if} \quad b = 0, \\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if} \quad b < 0. \end{cases}$$

By a *cylinder* in the warped space $N^{n+\ell}$ we mean a closed subset of the form

$$\mathcal{C}_{r_0} = \{(x, y) \in N^{n+\ell} : x \in L^\ell \text{ and } y \in B_P(r_0)\}.$$

Since the submanifolds $L^\ell \times \{p_0\} \subset N^{n+\ell}$ are totally geodesic, we have that

$$|\rho H_{\mathcal{C}_{r_0}}| \geq \frac{n-1}{\ell + n - 1} C_b(r_0)$$

where $H_{\mathcal{C}_{r_0}}$ is the mean curvature vector field of the hypersurface $L^\ell \times S_p(r_0)$.

The following theorem extends the result in [1] where the cylinders under consideration are contained in product spaces $\mathbb{R}^\ell \times P^n$. After the statement, we recall from [2] the concept of an Omori-Yau pair on a Riemannian manifold and discuss some implications of its existence.

Theorem 1. *Let $f\colon M^m \to L^\ell \times_\rho P^n$ be an isometric immersion where L^ℓ carries an Omori-Yau pair for the Hessian and the functions ρ, $|\mathrm{grad}\log\rho|$ are bounded. If f is proper and $f(M) \subset \mathcal{C}_{r_0}$, then $\sup_M |H| = +\infty$ or*

$$\sup_M \rho|H| \geq \frac{m-\ell}{m} C_b(r_0) \tag{1}$$

where H is the mean curvature vector field of f.

We see in the proof that the existence in L^ℓ of a Omori-Yau pair for the Hessian provides conditions, in a function theoretic form, that guarantee the validity of the Omori-Yau Maximum Principle on M^m in terms of the corresponding property of L^ℓ and the geometry of the immersion.

Definition 2. The pair of functions (h, γ) for $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma\colon M \to \mathbb{R}_+$ form an *Omori-Yau pair for the Hessian* in M if they satisfy:

(a) $h(0) > 0$ and $h'(t) \geq 0$ for all $t \in \mathbb{R}_+$,

(b) $\limsup_{t\to+\infty} th(\sqrt{t})/h(t) < +\infty$,

(c) $\int_0^{+\infty} \mathrm{d}t/\sqrt{h(t)} = +\infty$,

(d) The function γ is proper,

(e) $|\mathrm{grad}\,\gamma| \leq c\sqrt{\gamma}$ for some $c > 0$ outside a compact subset of M,

(f) $\mathrm{Hess}\,\gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

Similarly, the pair (h, γ) forms an *Omori-Yau pair for the Laplacian* in M if they satisfy conditions (a) to (e) and

(f') $\Delta\gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

The following fundamental result due to Pigola, Rigoli and Setti [3] gives sufficient conditions for an Omori-Yau Maximum Principle to hold for a Riemannian manifold.

Theorem 3. *Assume that a Riemannian manifold M carries an Omori-Yau pair for the Hessian (respec., Laplacian). Then, the Omori–Yau Maximum Principle for the Hessian (respec., Laplacian) holds in M.*

Recall that the *Omori–Yau Maximum Principle for the Hessian* holds for M if for any function $g \in C^\infty(M)$ bounded from above there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in M such that

(a) $\lim_{k\to\infty} g(p_k) = \sup_M g$,

(b) $|\operatorname{grad} g(p_k)| \leq 1/k$,

(c) Hess $g(p_k)(X,X) \leq (1/k)g_M(X,X)$ for all $X \in T_{p_k}M$.

Similarly, the *Omori–Yau Maximum Principle for the Laplacian* holds for M if the above properties are satisfied with (c) replaced by

(c') $\Delta g(p_k) \leq 1/k$.

Example 4. Let M^m be a complete but non compact Riemannian manifold and denote $r(y) = \operatorname{dist}_M(y,o)$ for some reference point $o \in M^m$. Assume that the radial sectional curvature of M^m satisfies $K^{\mathrm{rad}} \geq -h(r)$, where the smooth function h satisfies (a) to (c) in Definition 2 and is even at the origin, that is, $h^{(2k+1)}(0) = 0$ for $k \in \mathbb{N}$. Then, it was shown in [3] that the functions (h, r^2) form an Omori-Yau pair for the Hessian. As for the function h, one can choose

$$h(t) = t^2 \prod_{j=1}^{N} (\log^{(j)}(t))^2, \ \ t \gg 1,$$

where $\log^{(j)}$ stands for the j-th iterated logarithm.

To conclude this section, we first observe that Theorem 1 is sharp. This is clear from (1) by taking as P^n a space-form and as M the hypersurface $L^\ell \times S_P(r_0)$ in $N^{n+\ell}$. Moreover, in view of Example 4 it follows taking $L^\ell = \mathbb{R}^\ell$ and constant ρ that we recover the result in [1].

1 The proof

We first introduce some additional notations and then we recall a few basic facts on warped product manifolds.

Let $\langle\, ,\, \rangle$ denote the metrics in $N^{n+\ell}$, L^ℓ and M^m whereas $(\, ,\,)$ stands for the metric in P^n. The corresponding norms are $|\ \ |$ and $\|\ \ \|$. In addition, let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections in M^m and $N^{n+\ell}$, respectively, and ∇^L and ∇^P the ones in L^ℓ and P^n.

We always denote vector fields in TL by T, S and in TP by X, Y. In addition, we identify vector fields in TL and TP with *basic* vector fields in TN by taking $T(x, y) = T(x)$ and $X(x, y) = X(y)$.

For the Lie-brackets of basic vector fields, we have that $[T, S] \in TL$ and $[X, Y] \in TP$ are basic and that $[X, T] = 0$. Then, we have

$$\widetilde{\nabla}_S T = \nabla^L_S T,$$

$$\widetilde{\nabla}_X T = \widetilde{\nabla}_T X = T(\varrho)X$$

and

$$\widetilde{\nabla}_X Y = \nabla^P_X Y - \langle X, Y\rangle \mathrm{grad}^L \varrho$$

where the vector fields X, Y and T are basic and $\varrho = \log \rho$.

Our proof follows the main steps in [2]. In fact, a substantial part of the argument is to show that the Omori-Yau pair for the Hessian in L^ℓ induces an Omori-Yau pair for the Laplacian for a non compact M^m when $|H|$ is bounded. Thus, the Omori–Yau Maximum Principle for the Laplacian holds in M^m, and the proof follows from a application of the latter.

Suppose that M^m is non compact and let (h, Γ) be an Omori-Yau pair for the Hessian in L^ℓ. For $p \in M^m$ denote $f(p) = (x(p), y(p))$. Set $\tilde{\Gamma}(x, y) = \Gamma(x)$ for $(x, y) \in N^{n+\ell}$ and

$$\gamma(p) = \tilde{\Gamma}(f(p)) = \Gamma(x(p)).$$

We show next that (h, γ) is an Omori-Yau pair for the Laplacian in M^m. First, we argue that the function γ is proper. To see this, let $p_k \in M^m$ be a divergent sequence, i.e., $p_k \to \infty$ in M^m as $k \to +\infty$. Thus, $f(p_k) \to \infty$ in $N^{n+\ell}$ since f is proper. Since $f(M)$ lies inside a cylinder, then $x(p_k) \to \infty$ in L^ℓ. Hence, $\gamma(p_k) \to +\infty$ as $k \to +\infty$ since Γ is proper, and thus γ is proper.

It remains to verify conditions (e) and (f') in Definition 2. We have from $\tilde{\Gamma}(x,y)=\Gamma(x)$ that

$$\langle \mathrm{grad}^N\tilde{\Gamma}(x,y),X\rangle = 0.$$

Thus,

$$\mathrm{grad}^N\tilde{\Gamma}(x,y)=\mathrm{grad}^L\Gamma(x).$$

Since $\gamma=\tilde{\Gamma}\circ f$, we obtain

$$\mathrm{grad}^N\tilde{\Gamma}(f(p))=\mathrm{grad}^M\gamma(p)+\mathrm{grad}^N\tilde{\Gamma}(f(p))^{\perp} \tag{2}$$

where $(\ )^{\perp}$ denotes taking the normal component to f. Then,

$$|\mathrm{grad}^M\gamma(p)|\leq|\mathrm{grad}^N\tilde{\Gamma}(f(p))|=|\mathrm{grad}^L\Gamma(x(p))|\leq c\sqrt{\Gamma(x(p))}=c\sqrt{\gamma(p)}$$

outside a compact subset of M^m, and thus (e) holds.

We have that

$$\tilde{\nabla}_T\mathrm{grad}^N\tilde{\Gamma}=\nabla^L_T\mathrm{grad}^L\Gamma.$$

Hence,

$$\mathrm{Hess}\ \tilde{\Gamma}(T,S)=\mathrm{Hess}\ \Gamma(T,S)$$

and

$$\mathrm{Hess}\ \tilde{\Gamma}(T,X)=0.$$

Moreover,

$$\tilde{\nabla}_X\mathrm{grad}^N\tilde{\Gamma}=\tilde{\nabla}_X\mathrm{grad}^L\Gamma=\mathrm{grad}^L\Gamma(\varrho)X.$$

Hence,

$$\mathrm{Hess}\ \tilde{\Gamma}(X,Y)=\langle\mathrm{grad}^L\Gamma,\mathrm{grad}^L\varrho\rangle\langle X,Y\rangle.$$

For a unit vector $e\in T_pM$, set $e=e^L+e^P$ where $e^L\in T_{x(p)}L$ and $e^P\in T_{y(p)}P$. Then,

$$\mathrm{Hess}\ \tilde{\Gamma}(f(p))(e,e)=\mathrm{Hess}\ \Gamma(x(p))(e^L,e^L)+\langle\mathrm{grad}^L\Gamma(x(p)),\mathrm{grad}^L\varrho(x(p))\rangle|e^P|^2.$$

Moreover, an easy computation using (2) yields

$$\mathrm{Hess}\ \gamma(p)(e,e)=\mathrm{Hess}\ \tilde{\Gamma}(f(p))(e,e)+\langle\mathrm{grad}^L\Gamma(x(p)),\alpha(p)(e,e)\rangle$$

where α denotes the second fundamental of f with values in the normal bundle. Thus,

$$\begin{aligned}\mathrm{Hess}\ \gamma(p)(e,e) &= \mathrm{Hess}\ \Gamma(x(p))(e^L,e^L)+\langle\mathrm{grad}^L\Gamma(x(p)),\mathrm{grad}^L\varrho(x(p))\rangle|e^P|^2\\ &\quad+\langle\mathrm{grad}^L\Gamma(x(p)),\alpha(p)(e,e)\rangle.\end{aligned}$$

Since Hess $\Gamma \leq d\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some positive constant d outside a compact subset of L^ℓ and the immersion is proper, then

$$\text{Hess }\Gamma(x(p))(e^L, e^L) \leq d\sqrt{\gamma(p)h(\sqrt{\gamma(p)})}|e^L|^2 \leq d\sqrt{\gamma(p)h(\sqrt{\gamma(p)})}$$

outside a compact subset of M^m. From $|\text{grad}^L\Gamma| \leq c\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some c outside a compact subset of L^ℓ and $\sup_L |\text{grad}^L\varrho| < +\infty$, we have

$$\langle \text{grad}^L\Gamma(x(p)), \text{grad}^L\varrho(x(p))\rangle |e^P|^2 \leq c'\sqrt{\gamma(p)}$$

for some positive constant c' outside a compact subset of M^m. Being γ proper and h unbounded from (a) and (b) in Definition 2, then

$$\sqrt{\gamma} \leq \sqrt{\gamma h(\sqrt{\gamma})}$$

outside a compact subset of M^m. Thus, we obtain

$$\text{Hess }\gamma(e,e) \leq d_1\sqrt{\gamma h(\sqrt{\gamma})} + \langle \text{grad}^L\Gamma(x), \alpha(e,e)\rangle \tag{3}$$

for same constant $d_1 > 0$, outside a compact subset of M^m.

On the other hand, we may assume that

$$|H| \leq c\sqrt{h(\sqrt{\gamma})} \tag{4}$$

for some constant $c > 0$, outside a compact subset of M^m. Otherwise, there exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that $p_k \to \infty$ as $k \to +\infty$ and

$$|H(p_k)| > k\sqrt{h(\sqrt{\gamma(p_k)})}.$$

Being γ proper and h unbounded from (a) and (b) in Definition 2, we conclude that $\sup_M |H| = +\infty$, in which case we are done with the proof of the theorem.

We obtain from (3) using (4) that

$$\Delta\gamma \leq c_1\sqrt{\gamma h(\sqrt{\gamma})}$$

for some constant $c_1 > 0$ outside a compact subset of M^m, and thus (f') has been proved.

Consider the distance function $r(y) = \mathrm{dist}_P(y, o)$ in $B_P(r_0)$ and define $\tilde{r} \in C^\infty(N)$ by $\tilde{r}(x, y) = r(y)$. Then,

$$\langle \mathrm{grad}^N \tilde{r}(x, y), T\rangle = 0.$$

Thus,

$$\rho^2(x)\mathrm{grad}^N \tilde{r}(x, y) = \mathrm{grad}^P r(y).$$

We obtain that

$$\widetilde{\nabla}_T \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_T(\rho^{-2}\mathrm{grad}^P r) = -\rho^{-2}T(\varrho)\mathrm{grad}^P r.$$

Therefore,

$$\mathrm{Hess}\ \tilde{r}(T, S) = 0$$

and

$$\mathrm{Hess}\ \tilde{r}(T, X) = -\rho^{-2}T(\varrho)\langle \mathrm{grad}^P r, X\rangle = -T(\varrho)(\mathrm{grad}^P r, X).$$

Moreover,

$$\widetilde{\nabla}_X \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_X(\rho^{-2}\mathrm{grad}^P r) = \rho^{-2}\left(\nabla^P_X \mathrm{grad}^P r - \langle X, \mathrm{grad}^P r\rangle \mathrm{grad}^L \varrho\right).$$

Hence,

$$\mathrm{Hess}\ \tilde{r}(X, Y) = \rho^{-2}\langle \nabla^P_X \mathrm{grad}^P r, Y\rangle = (\nabla^P_X \mathrm{grad}^P r, Y) = \mathrm{Hess}\ r(X, Y).$$

For $e \in TM$, we have

$$\mathrm{Hess}\ \tilde{r}(e, e) = -2\langle \mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + \mathrm{Hess}\ r(e^P, e^P).$$

From the Hessian comparison theorem, we obtain

$$\mathrm{Hess}\ r(e^P, e^P) \geq C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2).$$

Therefore,

$$\mathrm{Hess}\ \tilde{r}(e, e) \geq -2\langle \mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2). \quad (5)$$

We define $u \in C^\infty(M)$ by

$$u(p) = r(y(p)).$$

Thus, $u = \tilde{r} \circ f$ and

$$\mathrm{grad}^N \tilde{r}(f(p)) = \mathrm{grad}^M u(p) + \mathrm{grad}^N \tilde{r}(f(p))^\perp. \quad (6)$$

Using (6) gives

$$\text{Hess } u(e_i, e_j) = \text{Hess } \tilde{r}(e_i, e_j) + \langle \text{grad}^N \tilde{r}, \alpha(e_i, e_j) \rangle$$

where $e_1, \ldots, e_m$ an orthonormal frame of TM. Thus,

$$\Delta u = \sum_{j=1}^{m} \text{Hess } \tilde{r}(e_j, e_j) + m \langle \text{grad}^N \tilde{r}, H \rangle. \tag{7}$$

We have from $e_j = e_j^L + e_j^P$ that

$$1 = \langle e_j, e_j \rangle = \rho^2 \|e_j^P\|^2 + \sum_{k=1}^{\ell} \langle e_j, T_k \rangle^2$$

where $T_1, \ldots, T_\ell$ is an orthonormal frame for TL. Hence,

$$m = \rho^2 \sum_{j=1}^{m} \|e_j^P\|^2 + \sum_{k=1}^{\ell} |T_k^\top|^2,$$

where $T^\top$ is the tangent component of T. We obtain that

$$\sum_{j=1}^{m} \|e_j^P\|^2 \geq (m - \ell)\rho^{-2}. \tag{8}$$

We obtain from (5) and

$$(\text{grad}^P r, e_j^P) = \langle \text{grad}^N \tilde{r}, e_j^P \rangle = \langle \text{grad}^N \tilde{r}, e_j \rangle = \langle \text{grad}^M u, e_j \rangle$$

that

$$\text{Hess } \tilde{r}(e_j, e_j) \geq -2 \langle \text{grad}^L \varrho, e_j \rangle \langle \text{grad}^M u, e_j \rangle + C_b(u)(\|e_j^P\|^2 - \langle \text{grad}^M u, e_j \rangle^2).$$

Taking trace and using (8) gives

$$\sum_{j=1}^{m} \text{Hess } \tilde{r}(e_j, e_j) \geq -2 \langle \text{grad}^L \varrho, \text{grad}^M u \rangle + C_b(u) \left((m - \ell)\rho^{-2} - |\text{grad}^M u|^2 \right).$$

Since

$$\langle \text{grad}^N \tilde{r}, \text{grad}^N \tilde{r} \rangle = \rho^2 (\rho^{-2} \text{grad}^P r, \rho^{-2} \text{grad}^P r) = \rho^{-2},$$

we have

$$\langle \mathrm{grad}^N \tilde{r}, H\rangle \geq -\rho^{-1}|H|.$$

We conclude using (7) that

$$\Delta u \geq -2\langle \mathrm{grad}^L \varrho, \mathrm{grad}^M u\rangle + C_b(u)\left((m-\ell)\rho^{-2} - |\mathrm{grad}^M u|^2\right) - m\rho^{-1}|H|.$$

Thus,

$$\rho|H| \geq \frac{m-\ell}{m} C_b(u) - \frac{\rho^2}{m}\left(\Delta u + 2|\mathrm{grad}^L \varrho||\mathrm{grad}^M u| + C_b(u)|\mathrm{grad}^M u|^2\right).$$

If M^m is compact, the proof follows easily by computing the inequality at a point of maximum of u. Thus, we may now assume that M^m is non compact and that (4) holds.

Since $f(M) \subset \mathcal{C}_{r_0}$, we have $u^* = \sup_M u \leq r_0 < +\infty$. By the Omori-Yau maximum principle there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that

$$u(p_k) > u^* - 1/k, \quad |\mathrm{grad}^M u(p_k)| < 1/k \;\text{ and }\; \Delta u(p_k) < 1/k.$$

By assumption, we have $\sup_L \rho = K_1 < +\infty$ and $\sup_L |\mathrm{grad}^L \varrho| = K_2 < +\infty$. Hence,

$$\sup_M \rho|H| \geq \rho(p_k)|H(p_k)| \geq \frac{m-\ell}{m} C_b(u(p_k)) - \frac{K_1^2}{m}\left(\frac{1+2K_2}{k} + \frac{1}{k^2} C_b(u(p_k))\right).$$

Letting $k \to +\infty$, we obtain

$$\sup_M \rho|H| \geq \frac{m-\ell}{m} C_b(u^*) \geq \frac{m-\ell}{m} C_b(r_0),$$

and this concludes the proof of the theorem.

References

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