



UNIVERSITY OF SOUTHAMPTON  
FACULTY OF SOCIAL AND HUMAN SCIENCES  
School of Mathematics

# ON BREDON (CO-)HOMOLOGICAL DIMENSIONS OF GROUPS

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ABSTRACT

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The objects of interest in this thesis are classifying spaces  $E_{\mathfrak{F}}G$  for discrete groups  $G$  with stabilisers in a given family  $\mathfrak{F}$  of subgroups of  $G$ . The main focus of this thesis lies in the family  $\mathfrak{F}_{\text{vc}}(G)$  of virtually cyclic subgroups of  $G$ . A classifying space for this specific family is denoted by  $\underline{E}G$ . It has a prominent appearance in the Farrell–Jones Conjecture. Understanding the finiteness properties of  $\underline{E}G$  is important for solving the conjecture.

This thesis aims to contribute to answering the following question for a group  $G$ : what is the minimal dimension a model for  $\underline{E}G$  can have? One way to attack this question is using methods in homological algebra. The natural choice for a cohomology theory to study  $G$ -CW-complexes with stabilisers in a given family  $\mathfrak{F}$  is known as Bredon cohomology. It is the study of cohomology in the category of  $\mathcal{O}_{\mathfrak{F}}G$ -modules. This category relates to models for  $E_{\mathfrak{F}}G$  in the same way as the category of  $G$ -modules relates to the study of universal covers of Eilenberg–Mac Lane spaces  $K(G, 1)$ .

In this thesis we study Bredon (co-)homological dimensions of groups. A major part of this thesis is devoted to collect existing homological machinery needed to study these dimensions for arbitrary families  $\mathfrak{F}$ . We contribute to this collection.

After this we turn our attention to the specific case of  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$ . We derive a geometric method for obtaining a lower bound for the Bredon (co-)homological dimension of a group  $G$  for a general family  $\mathfrak{F}$ , and subsequently show how to exploit this method in various cases for  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$ .

Furthermore we construct model for  $\underline{E}G$  in the case that  $G$  belongs to a certain class of infinite cyclic extensions of a group  $B$  and that a model for  $\underline{E}B$  is known. We give bounds on the dimensions of these models. Moreover, we use this construction to give a concrete model for  $\underline{E}G$ , where  $G$  is a soluble Baumslag–Solitar group. Using this model we are able to determine the exact Bredon (co-)homological dimensions of these groups.

The thesis concludes with the study of groups  $G$  of low Bredon dimension for the family  $\mathfrak{F}_{\text{vc}}(G)$  and we give a classification of countable, torsion-free, soluble groups which admit a tree as a model for  $\underline{E}G$ .

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## Declaration of Authorship

I, Martin Fluch, declare that the thesis entitled “*On Bredon (Co-)Homological Dimensions of Groups*” and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly while in candidature for a research degree at this university;
- no part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: “Classifying Spaces with Virtually Cyclic Stabilisers for Certain Infinite Cyclic Extensions”, 2010, arXiv:1005.1281v2.

Signed: 

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# Introduction

## 1. Classifying Spaces and Bredon (Co-)Homology of Groups

Classifying spaces and their finiteness conditions form an important part of various areas in pure mathematics such as group theory, algebraic topology and geometric topology.

Given a group  $G$  and a non-empty family  $\mathfrak{F}$  of subgroups of  $G$  which is closed under conjugation and finite intersections, one can consider the homotopy category of  $G$ -CW-complexes with stabilisers in  $\mathfrak{F}$ . This category is known to have terminal objects, see for example [Lüc05, p. 275]. A terminal object in this category is called a *classifying space* of  $G$  for the family  $\mathfrak{F}$  or alternatively, a *model for  $E_{\mathfrak{F}}G$* .

If  $\mathfrak{F} = \{1\}$  is the trivial family of subgroups, then the universal cover  $EG$  of an Eilenberg–Mac Lane space  $K(G, 1)$  is a model for  $E_{\mathfrak{F}}G$ . If  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$  is the family of finite subgroups of  $G$ , then a model for  $E_{\mathfrak{F}}G$  is also known as the *universal space for proper actions*. This space is commonly denoted by  $\underline{EG}$  and it has a prominent appearance as the geometric object in the Baum–Connes Conjecture.

Recently the classifying space  $\underline{\underline{EG}}$  of  $G$  for the family  $\mathfrak{F}_{\text{vc}}(G)$  of virtually cyclic subgroups of  $G$  has caught the interest of the mathematical community (recall that a group is called *virtually cyclic* if it contains a cyclic subgroup of finite index). The reason for this is that the classifying space  $\underline{\underline{EG}}$  appears on the geometric side of the *Farrell–Jones Conjecture* for Algebraic  $K$ - and  $L$ -Theory. This conjecture has originally been stated by Thomas Farrell and Lowell Jones in 1993 in their famous paper [FJ93].

Let  $R$  be a ring with unit and involution. There exists  $G$ -homology theories

$$H_n^G(?; \mathbf{K}_R) \quad \text{and} \quad H_n^G(?; \mathbf{L}_R^{\langle -\infty \rangle})$$

in the sense of [LR05, pp. 738f.] such that, if evaluated at a singleton

space  $\{*\}$ , we recover the algebraic  $K$ - and  $L$ -groups of the group ring  $RG$ . That is

$$H_n^G(\{*\}; \mathbf{K}_R) \cong K_n(RG) \quad \text{and} \quad H_n^G(\{*\}; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RG)$$

for all  $n \in \mathbb{Z}$  [LR05, p. 735]. Now the Farrell–Jones Conjecture makes the following prediction.

**Farrell–Jones Conjecture.** [LR05, p. 736] *The assembly maps*

$$\begin{aligned} A_{vc}: H_n^G(\underline{EG}; \mathbf{K}_R) &\rightarrow H_n^G(\{*\}; \mathbf{K}_R) \\ A_{vc}: H_n^G(\underline{EG}; \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow H_n^G(\{*\}; \mathbf{L}_R^{\langle -\infty \rangle}) \end{aligned}$$

*induced by the projection  $\underline{EG} \rightarrow \{*\}$  are isomorphisms for all  $n \in \mathbb{Z}$ .*

The codomains of the assembly maps are the groups which we want to compute but whose computation is known to be difficult. On the other hand, the domains of the assembly maps are easier to calculate as one can apply methods from Algebraic Topology such as spectral sequences and Chern characters to it [LR05].

The Farrell–Jones Conjecture is known to imply numerous other famous conjectures from different fields of pure mathematics, including the *Bass Conjecture* in Algebraic  $K$ -Theory, the *Borel Conjecture* in Geometric Topology, the *Kaplansky Conjecture* in Group Theory and the *Novikov Conjecture* in Topology [LR05].

Progress in studying the Farrell–Jones Conjecture relies much on understanding finiteness conditions of the classifying space  $\underline{EG}$ . Models for  $EG$  and  $\underline{EG}$  have been studied extensively, see for example [Lüc05]. However, there is not much known yet about the classifying space for the family of virtually cyclic subgroups. Classes of groups that are understood are word hyperbolic groups [JPL06], virtually polycyclic groups [LW07], relative hyperbolic groups [LO07] and CAT(0)-groups [Lüc09, Far09]. Furthermore, there exist general constructions for finite index extensions [Lüc00] and direct limits of groups [LW07]. Some more specific constructions can also be found in [CFH06] and [MPP08].

The focus in this thesis lies on groups  $G$  which admit a finite dimensional model for  $\underline{EG}$ . This leads to the study of the Bredon geometric dimension of



a group  $G$  with respect to the family  $\mathfrak{F}_{\text{vc}}(G)$ , which by definition is the least integer  $n$  (or  $\infty$ ) such that there exists an  $n$ -dimensional model for  $\underline{EG}$ .

Homological methods provide suitable tools to study finiteness conditions of classifying spaces. The natural choice of a homology theory for  $G$ -CW-complexes with stabilisers in a given family  $\mathfrak{F}$  is the Bredon cohomology of groups. This homology theory has been introduced for finite groups by Glen Bredon in [Bre67] and it has been generalised to arbitrary groups and arbitrary families of subgroups by Lück [Lüc89]. Related to the Bredon geometric dimension of a group is the concept of the Bredon homological and cohomological dimension of a group which is defined in a purely algebraic way.

We aim in this thesis to utilise the algebraic Bredon machinery as far as possible in order to study the Bredon geometric dimensions of groups  $G$  with respect to the family  $\mathfrak{F}_{\text{vc}}(G)$ .

## 2. Structure of this Thesis

The first three chapters in this thesis do not specialise to the family of virtually cyclic subgroups but introduce the theory in a more general setting.

In Chapter 1 the category of Bredon modules over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  is introduced. Free and projective Bredon modules are constructed. It is explained how the categorical tensor product gives rise to a tensor product over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  which is the Bredon analogue to the tensor product over the group ring  $\mathbb{Z}G$  in the category of  $G$ -modules. This tensor product is used to define flat Bredon modules. The chapter is finished with the definition of the restriction, induction and coinduction functors and a summary of their basic properties.

In Chapter 2,  $G$ -CW-complexes and classifying spaces with stabilisers in a given family  $\mathfrak{F}$  are defined. It is explained how one derives from the categorical definition the homotopy characterisation of a classifying space. Geometric finiteness conditions are discussed and their relationship to algebraic properties in the corresponding category of Bredon modules.

Chapter 3 introduces the notion of Bredon (co-)homological dimension. The relationship between the algebraic and geometric Bredon dimensions is studied as well as how the algebraic Bredon dimensions depend on the

family of subgroups. We deduce the algebraic analogue to a result from Lück and Weiermann [LW07] which gives a lower bound for the dimension when passing to a larger group, see Theorem 3.37 and Theorem 3.38. In the same chapter we construct a standard resolution and derive an algebraic analogue to a result in [LW07] which gives upper bounds on the Bredon dimensions of direct unions of groups, see Proposition 3.5 and Theorem 3.42; these results are a generalisation of work by Nucinkis [Nuc04] which she has carried out for the family of finite subgroups. In Section 12 we study the tensor product of projective resolutions which gives us the possibility to derive an upper bound for the Bredon cohomological dimension of direct products of groups, see Theorem 3.61. Finally we derive a Künneth formula for Bredon homology, see Theorem 3.67.

In Chapter 4 we begin to specialise to the family of virtually cyclic subgroups. Using geometric methods we derive a lower bound for the Bredon (co-)homological dimension of a group  $G$  (still with respect to a general family of subgroups). Using this tool we use known classifying spaces for the family of virtually cyclic groups in order to calculate the Bredon (co-)homological dimensions  $\underline{\text{hd}} G$  and  $\underline{\text{cd}} G$  for various groups. The results include the dimensions for  $\mathbb{Z}^2$ , free groups and the fundamental groups of finite graphs of finite groups. We also study of the Bredon cohomological dimension  $\underline{\text{cd}} G$  for nilpotent groups. The chapter concludes by investigating under which conditions an elementary amenable group  $G$  admits a finite dimensional model for  $\underline{EG}$ .

In the next chapter we turn our attention to the construction of a concrete model for  $\underline{EG}$  where  $G = B \rtimes \mathbb{Z}$  is an infinite cyclic extension of a group  $B$ . Under certain conditions on the action of  $\mathbb{Z}$  on  $B$ , we can make a classifying space of  $G$  from a model for  $\underline{EB}$ . The construction relies on a generalisation of a result by Juan-Pineda and Leary [JPL06], see Proposition 5.9. The class of groups for which this result is applicable include certain HNN-extensions with abelian or free base group and standard wreath products by  $\mathbb{Z}$ , see Section 5. We calculate the algebraic and geometric Bredon dimensions of the soluble Baumslag–Solitar groups  $BS(1, m)$ ,  $m \in \mathbb{Z} \setminus \{0\}$ , with respect to the family of virtually cyclic subgroups, see Theorem 5.20. We end this chapter by showing that some of the key ideas of this chapter can be applied

successfully in other settings than infinite cyclic extensions. Namely, we use them to calculate the least dimension a model for  $\underline{EG}$  can have when  $G$  is a free product.

The final chapter of this thesis is an attempt to study and classify groups with low Bredon dimension with respect to the family of virtually cyclic subgroups. Using the result of Theorem 5.20 and a classification result by Gildenhuys [Gil79] we classify countable, torsion-free, soluble groups  $G$  which have Bredon geometric dimension 1 with respect to the family of virtually cyclic subgroups, see Theorem 6.6.

### 3. Notation, Conventions and Preliminaries

The set of natural numbers is denoted by  $\mathbb{N}$  and 0 is considered to be a natural number. The group of integers is  $\mathbb{Z}$ , the field of rational numbers is denoted by  $\mathbb{Q}$ , the field of real numbers is denoted by  $\mathbb{R}$  and the field of complex numbers is denoted by  $\mathbb{C}$ . Rings are always assumed to have a unit. If  $G$  is a group and  $R$  a ring, then  $RG$  denotes the *group ring* which consists of all formal  $R$ -linear combinations of elements in  $G$ .

If  $a, b \in \mathbb{R} \cup \{\pm\infty\}$ , then  $[a, b]$  denotes the closed interval

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

As a topological space  $\mathbb{R}$  is considered to have the standard topology obtained from the Euclidian metric. Similarly  $\mathbb{C}$  has the topology of the underlying Euclidian space  $\mathbb{R}^2$ .

If  $n \in \mathbb{N}$ , then the  $(n-1)$ -sphere  $S^{n-1}$  and the  $n$ -disc  $D^n$  are the subspaces

$$\begin{aligned} S^{n-1} &:= \{x \in \mathbb{R}^n : |x| = 1\}, \\ D^n &:= \{x \in \mathbb{R}^n : |x| \leq 1\}. \end{aligned}$$

We set  $S^{-1} := \emptyset$ . The 1-sphere  $S^1$  can be identified with multiplicative group of complex numbers  $z$  with  $|z| = 1$  and this multiplicative structure makes  $S^1$  into a topological group.

Throughout this thesis we are working in the convenient category of *compactly generated topological spaces* in the sense of [Ste67]. By definition a subset  $A \subset X$  of a compactly generated space  $X$  is closed in  $X$  if and only if  $A \cap K$  is closed in  $X$  for every compact subset  $K$  of  $X$ . Locally compact spaces are compactly generated.

We use the following notation for categories:  $\mathfrak{Set}$  denotes the category of sets and  $\mathfrak{Ab}$  denotes the category of abelian groups. If  $R$  is a ring, then  $\text{Mod-}R$  ( $R\text{-Mod}$ ) denotes the category of right (left)  $R$ -modules. In the special case that  $R$  is the group ring  $\mathbb{Z}G$  we denote this category by  $\text{Mod-}G$  ( $G\text{-Mod}$ ); the objects in this category are called  $G$ -modules.

We use the symbols  $\amalg$  and  $\coprod$  to denote the product and coproduct in a category. In particular  $\amalg$  denotes the cartesian product and  $\coprod$  denotes the disjoint union in the category of sets.

We assume in this thesis that the reader is familiar with the basic concepts of transformation groups [Kaw91], category theory [ML98], and homological algebra [Wei94].

Furthermore we assume that the reader is familiar with the classical cohomology of groups and classical and cohomological finiteness conditions of groups [Bro82, Bie81]. In particular, we denote by  $\text{hd } G$  the *homological dimension* of a group  $G$ , by  $\text{cd } G$  its *cohomological dimension* and by  $\text{gd } G$  its *geometric dimension*. For virtually torsion-free groups  $G$  we have the notion of *virtual cohomological dimension* and this dimension is denoted by  $\text{vcd } G$ , see [Bro82, pp. 225f.].

## The Category of Bredon Modules

### 1. Families of Subgroups

**Definition 1.1.** Let  $G$  be a group. A set  $\mathfrak{F}$  of subgroups of  $G$  is called a *family* if it is non-empty and closed under conjugation. We say that  $\mathfrak{F}$  is a *semi-full* family if  $H \cap K \in \mathfrak{F}$  for any  $H, K \in \mathfrak{F}$ . We say that  $\mathfrak{F}$  is a *full* family if  $\mathfrak{F}$  is closed under taking subgroups.

**Example 1.2.** Commonly used families are the following:

- (1) the *trivial* family  $\{1\}$  which consists of the trivial subgroup only;
- (2) the family  $\mathfrak{F}_{\text{fin}}(G)$  of finite subgroups of  $G$ ;
- (3) the family  $\mathfrak{F}_{\text{vc}}(G)$  of virtually cyclic subgroups of  $G$ ;
- (4) the family  $\mathfrak{F}_{\text{all}}(G)$  of all subgroups of  $G$ ;
- (5) given a non-empty  $G$ -set  $X$  we have the family

$$\mathfrak{F}(X) := \{G_x : x \in X\}$$

of stabilisers of  $X$ .

Note that the examples (1) to (4) are full families of subgroups of  $G$ . However, the family  $\mathfrak{F}(X)$  is in general neither subgroup closed or even intersection closed.

There are different common constructions how to obtain a new family of subgroups from a given one. In what follows we list those which appear in this thesis.

If  $\mathfrak{F}$  is a family of subgroups of  $G$  and  $K$  a subgroup of  $G$  then

$$\mathfrak{F} \cap K := \{H \cap K : H \in \mathfrak{F}\}$$

is a family of subgroups of  $K$  provided that  $\mathfrak{F} \cap K$  is not empty. The latter is ensured if  $\mathfrak{F}$  is a full family of subgroups of  $G$ . In this case  $\mathfrak{F} \cap K$  is a full family, too.

Given two groups  $G_1$  and  $G_2$  and families  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  of subgroups of  $G_1$  and  $G_2$  respectively we define their *cartesian product*  $\mathfrak{F}_1 \times \mathfrak{F}_2$  to be the set

$$\mathfrak{F}_1 \times \mathfrak{F}_2 := \{H_1 \times H_2 : H_1 \in \mathfrak{F}_1 \text{ and } H_2 \in \mathfrak{F}_2\}.$$

This is a family of subgroups of the group  $G_1 \times G_2$ . If  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are semi-full families of subgroups, then so is  $\mathfrak{F}_1 \times \mathfrak{F}_2$ . But in general it is not true that the cartesian product of two full families is again a full family: not every subgroup  $K$  of  $H_1 \times H_2 \in \mathfrak{F}_1 \times \mathfrak{F}_2$  is equal to  $K_1 \times K_2$  for some  $K_i \in \mathfrak{F}_i$ .

Given an arbitrary family  $\mathfrak{F}$  of a group  $G$  we can always *complete* it to a full family of subgroups of  $G$ . This *completion* is denoted by  $\tilde{\mathfrak{F}}$  and is by definition

$$\tilde{\mathfrak{F}} := \{H \leq G : H \leq K \text{ for some } K \in \mathfrak{F}\}.$$

This is by construction the smallest full family of subgroups of  $G$  which contains the family  $\mathfrak{F}$ .

**Definition 1.3.** A *pair*  $(\mathfrak{G}, \mathfrak{F})$  of families of subgroups of  $G$  consists of two families  $\mathfrak{F}$  and  $\mathfrak{G}$  of subgroups of  $G$  with  $\mathfrak{F} \subset \mathfrak{G}$ . A pair  $(\mathfrak{G}, \mathfrak{F})$  of families of subgroups is called *semi-full* (*full*) if both  $\mathfrak{F}$  and  $\mathfrak{G}$  are semi-full (full).

## 2. The Orbit Category

**Definition 1.4.** Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Then the *orbit category*  $\mathcal{O}_{\mathfrak{F}}G$  is the following small category. The objects of  $\mathcal{O}_{\mathfrak{F}}G$  are homogeneous  $G$ -spaces  $G/H$  with  $H \in \mathfrak{F}$  and the morphisms of  $\mathcal{O}_{\mathfrak{F}}G$  are  $G$ -maps. In the case that  $\mathfrak{F} = \mathfrak{F}_{\text{all}}(G)$  we write  $\mathcal{O}G$  for the orbit category.

Given two subgroups  $H$  and  $K$  of  $G$  we denote the set of all  $G$ -maps from  $G/H$  to  $G/K$  by  $[G/H, G/K]_G$ . The set  $[G/H, G/H]_G$  is a monoid in general and we denote its identity element either by  $\text{id}$  or  $1$ .

A  $G$ -map  $f: G/H \rightarrow G/K$  is characterised by its value on the coset  $H$ . If  $f(H) = xK$  for some  $x \in G$ , then the condition that  $f$  is a  $G$ -map implies

$$\begin{aligned} xK \in (G/K)^H &= \{xK \in G/K : hxK = xK \text{ for all } h \in H\} \\ &= \{xK \in G/K : H^x \leq K\}. \end{aligned}$$

Conversely, given any  $xK \in (G/K)^H$ , there exists a unique  $G$ -map  $f: G/H \rightarrow G/K$  with  $f(H) = xK$ . Therefore we have a bijective correspondence

$$[G/H, G/K]_G \cong (G/K)^H. \quad (1.1)$$

given by  $f \mapsto f(H)$ .

Therefore we can label any  $G$ -map  $f$  between homogeneous  $G$ -spaces as follows: we denote by  $f_{x,H,K}$  the unique  $G$ -map  $f: G/H \rightarrow G/K$  which maps  $H$  to  $xK$ . With this notation two  $G$ -maps  $f_{x,H,K}$  and  $f_{x',H',K'}$  are the same if and only if  $H = H'$ ,  $K = K'$  and  $x^{-1}x' \in K$ . In particular  $f_{x,H,H}$  is the identity map on  $G/H$  if and only if  $x \in H$ .

If we are given two  $G$ -maps  $f_{x,H,K}$  and  $f'_{y,K,L}$ , then the composite map  $f_{y,K,L} \circ f_{x,H,K}$  is a  $G$ -map  $G/H \rightarrow G/L$  and we have

$$(f_{y,K,L} \circ f_{x,H,K})(H) = f_{y,K,L}(xK) = xf_{y,K,L}(K) = xyL.$$

In other words we have the following simple rule to calculate the composite of two  $G$ -maps between homogeneous  $G$ -spaces:

$$f_{y,K,L} \circ f_{x,H,K} = f_{xy,H,L}$$

The structure of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  depends not only on the group  $G$  but also very much on the family  $\mathfrak{F}$  of subgroups of  $G$ . We list a few standard facts from the theory of topological transformation groups which illustrate this situation.

- (1) If  $\mathfrak{F} = \{1\}$ , then the orbit category has only one object  $G/1$ . Clearly every element of  $[G/1, G/1]_G$  is invertible, that is  $[G/1, G/1]_G = \text{Aut}(G/1)$ . We have an isomorphism of groups  $G \rightarrow \text{Aut}(G/1)$  which sends an element  $g$  to the automorphism

$$l_g: G/1 \mapsto G/1,$$

$$x \mapsto gx.$$

In particular, every morphism in the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  is invertible.

- (2) If  $\mathfrak{F} \subset \mathfrak{F}_{\text{fin}}(G)$ , then still every endomorphism in  $\mathcal{O}_{\mathfrak{F}}G$  is invertible, that is  $[G/H, G/H]_G = \text{Aut}(G/H)$  for every  $H \in \mathfrak{F}$ . This is because if  $f_{g,H,H}$  is an endomorphism of  $\mathcal{O}_{\mathfrak{F}}G$ , then  $H^g \leq H$  and since  $H$  is finite it follows that  $H^g = H$ . Therefore also  $H^{g^{-1}} \leq H$  and  $f_{g^{-1},H,H}$  is a morphism of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ . Necessarily  $f_{g^{-1},H,H}$  is the inverse to  $\varphi$ .
- (3) In general one has that  $\text{Aut}(G/H)$  is isomorphic to the Weyl-group  $W_G(H) := N_G(H)/H$  of  $H$  in  $G$ . This is, because an endomorphism

$f_{g,H,H}$  of  $\mathcal{O}_{\mathfrak{F}}G$  is invertible if and only if  $g \in N_G(H)$  and two endomorphism  $f_{g,H,H}$  and  $f_{g',H,H}$  are the same if  $g'g^{-1} \in H$ . However, if  $H$  is not finite then there may exist elements in  $\text{mor}(G/H, G/H)$  which are not invertible and therefore do not belong to the automorphism group  $\text{Aut}(G/H)$ .

Thus broadly speaking, the larger the family  $\mathfrak{F}$  becomes the more the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  loses structure.

### 3. The Category of Bredon Modules

**Definition 1.5.** Let  $\mathfrak{F}$  be a family of subgroups of a group  $G$ . A functor

$$M: \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{Ab}$$

from the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  to the category  $\mathfrak{Ab}$  of abelian groups is called a *Bredon module*  $M$  over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  (or an  $\mathcal{O}_{\mathfrak{F}}G$ -module). If the functor  $M$  is contravariant (covariant) then we call  $M$  a *right (left)*  $\mathcal{O}_{\mathfrak{F}}G$ -module.

Let  $M$  and  $N$  be two  $\mathcal{O}_{\mathfrak{F}}G$ -modules of the same variance. A *morphism*  $f: M \rightarrow N$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules is a natural transformation from the functor  $M$  to the functor  $N$ .

Let  $M$  be a right (left)  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $\varphi$  a morphism of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ . If there is no danger of confusion, then we may abbreviate the homomorphism  $M(\varphi)$  by  $\varphi^*$  ( $\varphi_*$  respectively). In order to avoid complicating the language we shall understand a statement about Bredon modules without specified variance to be true for both left and right Bredon modules.

**Examples 1.6.** The following are simple but yet important standard examples of some Bredon modules:

- (1) Let  $A$  be an abelian group. Then  $\underline{A}$  denotes the *constant*  $\mathcal{O}_{\mathfrak{F}}G$ -module given by  $\underline{A}(G/H) := A$  and  $\underline{A}(\varphi) := \text{id}$  for any object  $G/H$  and any morphism  $\varphi$  of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ . It is both a left and a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. If we want to emphasise the dependency on the family  $\mathfrak{F}$  then we may write  $\underline{A}_{\mathfrak{F}}$  for the constant  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{A}$ .
- (2) An important special case of the above example is the *trivial*  $\mathcal{O}_{\mathfrak{F}}G$ -module which is the constant  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ .



- (3) Let  $K$  be a fixed subgroup of  $G$ . We construct a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[?, G/K]_G$  as follows: Given an object  $G/H$  of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  we let  $\mathbb{Z}[G/H, G/K]_G$  be the free abelian group with basis the set  $[G/H, G/K]_G$ . If  $\varphi: G/H \rightarrow G/L$  is a morphism in  $\mathcal{O}_{\mathfrak{F}}G$ , then  $\varphi^*: \mathbb{Z}[G/L, G/K]_G \rightarrow \mathbb{Z}[G/H, G/K]_G$  is the unique homomorphism of abelian groups which maps the basis element  $f \in [G/L, G/H]_G$  to  $f \circ \varphi \in [G/H, G/K]_G$ .
- (4) In a similar way as above we can construct a left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[G/K, ?]_G$ . Given a morphism  $\varphi$  of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  the homomorphism  $\varphi_*$  is defined by pre-composition instead of post-composition.

The class of all right  $\mathcal{O}_{\mathfrak{F}}G$ -modules together with the morphisms of  $\mathcal{O}_{\mathfrak{F}}G$ -modules form a category which we denote by  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$ . Similar we have the category  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  of all left  $\mathcal{O}_{\mathfrak{F}}G$ -modules. By construction these categories are the functor categories  $[\mathcal{O}_{\mathfrak{F}}G^{\text{op}}, \mathfrak{Ab}]$  and  $[\mathcal{O}_{\mathfrak{F}}G, \mathfrak{Ab}]$  respectively [Mit65, pp. 63ff.]. It follows from standard arguments in category theory that the functor categories  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  and  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$  inherit many properties from the abelian category  $\mathfrak{Ab}$  [Fre64, ML98, Wei94]. In what follows we collect some of those results for  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ .

The category  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  is abelian, complete and cocomplete (that is arbitrary limits and colimits exist) since the category  $\mathfrak{Ab}$  is. Limits and colimits are calculated componentwise. This includes: products, coproducts, direct limits, kernels, images and intersections. In particular, filtered limits (which include direct limits) are exact in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  as they are exact in  $\mathfrak{Ab}$  [Wei94, p. 57]. Furthermore, since kernels and images are calculated component wise we have that a sequence

$$M' \rightarrow M \rightarrow M''$$

of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules is exact at  $M$  if and only if the corresponding sequences

$$M''(G/H) \rightarrow M(G/H) \rightarrow M'(G/H)$$

of abelian groups are exact at  $M(G/H)$  for every  $H \in \mathfrak{F}$ .

Finally, we remark that the category  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  has enough projectives because the category  $\mathfrak{Ab}$  is cocomplete and has enough projectives [Wei94, p. 43]. Since  $\mathfrak{Ab}$  is complete and has enough injectives, it follows by a similar

$$\begin{array}{ccc}
\mathbb{Z}[G/H, G/K]_G & \xrightarrow{\mathbb{Z}[G/H, \varphi]_G} & \mathbb{Z}[G/H, G/K']_G \\
\downarrow \mathbb{Z}[\psi, G/K]_G & \searrow \mathbb{Z}[\varphi, \psi]_G & \downarrow \mathbb{Z}[\psi, G/K']_G \\
\mathbb{Z}[G/H', G/K]_G & \xrightarrow{\mathbb{Z}[G/H, \varphi]_G} & \mathbb{Z}[G/H', G/K']_G
\end{array}$$

FIGURE 1

argument that  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  has enough injectives, too. Therefore we can define left and right derived functors and take advantage of homological methods in the study of the category of Bredon modules over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ .

**Definition 1.7.** Let  $G_1$  and  $G_2$  be two groups and  $\mathfrak{F}$  and  $\mathfrak{G}$  families of subgroups of  $G_1$  and  $G_2$  respectively. A  $\mathcal{O}_{\mathfrak{F}}G_1$ - $\mathcal{O}_{\mathfrak{G}}G_2$ -bimodule  $M$  is a bifunctor

$$M: \mathcal{O}_{\mathfrak{F}}G_1 \times \mathcal{O}_{\mathfrak{G}}G_2 \rightarrow \mathfrak{Ab}$$

that is covariant in the first variable and contravariant in the second variable.

**Example 1.8.** Given a group  $G$  and family  $\mathfrak{F}$  of subgroups of  $G$ , then we have a  $\mathcal{O}_{\mathfrak{F}}G$ - $\mathcal{O}_{\mathfrak{F}}G$ -bifunctor

$$\mathbb{Z}[?, ??]_G: \mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{Ab}.$$

which is defined as follows. Given a pair  $G/K$  and  $G/H$  of objects in  $\mathcal{O}_{\mathfrak{F}}G$ , its value is defined to be the free abelian group  $\mathbb{Z}[G/H, G/K]_G$ . Given any pair  $\psi: G/K \rightarrow G/K'$  and  $\varphi: G/H' \rightarrow G/H$  of morphisms in  $\mathcal{O}_{\mathfrak{F}}G$ , the group homomorphism

$$\mathbb{Z}[\varphi, \psi]_G: \mathbb{Z}[G/H, G/K]_G \rightarrow \mathbb{Z}[G/H', G/K']_G$$

is defined to be the unique group homomorphism which sends a basis element  $f \in [G/H, G/K]_G$  to  $\psi \circ f \circ \varphi \in [G/H', G/K']_G$ . This is precisely the necessary definition needed in order to combine the constructions (3) and (4) in Example 1.6 into a  $\mathcal{O}_{\mathfrak{F}}G$ - $\mathcal{O}_{\mathfrak{F}}G$ -bimodule, see the diagram in Figure 1 and [ML98, p. 37].

#### 4. Bredon Modules and $G$ -Modules

Recall that a right  $G$ -module  $M$  is an abelian group  $M$  with an action of  $G$  on the right. The action of  $G$  is extended linearly to a homomorphism from the group ring  $\mathbb{Z}G$  into the endomorphism ring of  $M$ . The category of all right  $G$ -modules is denoted by  $\text{Mod-}G$ .

In what follows we consider the special case that  $\mathfrak{F} = \{1\}$  is the trivial family of subgroups. In Section 2 we have already noted that  $\text{mor}(G/1, G/1) = \text{Aut}(G/1)$  is isomorphic to the group  $G$ . This isomorphism is given by  $\varphi_{g,1,1} \mapsto g^{-1}$ . Its inverse is given by  $g \mapsto \varphi_{g^{-1},1,1}$ .

Now a functor from  $\mathcal{O}_{\mathfrak{F}}G$  to  $\mathfrak{Ab}$  determines an abelian group  $M' = M(G/1)$  and a homomorphism  $\text{mor}(G/1, G/1) \rightarrow \text{End}(M')$ . Since all endomorphisms of  $G/1$  are invertible it follows that this homomorphism is actually a homomorphism  $G \rightarrow \text{Aut}(M')$ . It is given by  $g \mapsto M(\varphi_{g^{-1},1,1})$ . If  $M$  is a contravariant functor, that is a right  $\mathcal{O}_{\mathfrak{F}}G$ -module, then we have

$$\varphi_{(gh)^{-1},1,1}^* = \varphi_{h^{-1}g^{-1},1,1}^* = (\varphi_{g^{-1},1,1} \circ \varphi_{h^{-1},1,1})^* = \varphi_{h^{-1},1,1}^* \circ \varphi_{g^{-1},1,1}^*$$

for all  $g, h \in G$ . Therefore  $xg := \varphi_{g^{-1},1,1}^*$  defines a right action of  $G$  on  $M'$  and this makes  $M'$  into a right  $G$ -module.

In the case that  $\mathfrak{F} = \{1\}$  we can reverse this construction. Given any right  $G$ -module  $M'$  we can construct a right  $\mathcal{O}_{\mathfrak{F}}G$ -module in the obvious way as follows. We set  $M(G/1) := M'$  and if  $\varphi_{g,1,1}$  is a morphism of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ , then we let  $\varphi_{g,1,1}^*$  be the morphism given by  $x \mapsto xg^{-1}$  for all  $x \in M'$ . Then

$$\begin{aligned} (\varphi_{g,1,1} \circ \varphi_{h,1,1})^* &= \varphi_{hg,1,1}^* = x \mapsto x(hg)^{-1} \\ &= x \mapsto (xg^{-1})h^{-1} = \varphi_{h,1,1}^* \circ \varphi_{g,1,1}^* \end{aligned}$$

which shows that  $M$  is indeed a contravariant functor.

Thus in the case that  $\mathfrak{F} = \{1\}$  we have a one-to-one correspondence between right  $\mathcal{O}_{\mathfrak{F}}G$  modules and right  $G$ -modules given by the above construction. Furthermore a morphism  $f: M \rightarrow N$  between two right  $\mathcal{O}_{\mathfrak{F}}G$ -modules is given by a single homomorphism  $f': M' \rightarrow N'$  of abelian groups. It follows from the fact that  $f$  is a natural transformation that  $f'$  is a homomorphism of  $G$ -modules. It follows that the assignment  $M \mapsto M'$  and  $f \mapsto f'$  is functorial.

Therefore one has the known result that the categories  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  and  $\text{Mod-}G$  are naturally isomorphic if  $\mathfrak{F} = \{1\}$  is the trivial family of subgroups of  $G$ . Of course one has the dual result that the category  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$  of left Bredon modules over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  and the category  $G\text{-Mod}$  of left  $G$ -modules are naturally isomorphic in the case that  $\mathfrak{F} = \{1\}$ . In other words the theory of Bredon modules is a generalisation of the theory of modules over group rings.

## 5. $\mathfrak{F}$ -Sets and Free Bredon Modules

Free objects are usually defined as left adjoint to a suitable forgetful functor. In the case of Bredon modules, the target category of this forgetful functor is not the category  $\mathfrak{S}\mathfrak{e}\mathfrak{t}$  of sets but the category of  $\mathfrak{F}$ -sets, which we denote by  $\mathfrak{F}\text{-}\mathfrak{S}\mathfrak{e}\mathfrak{t}$ . There are several ways to see and describe this category.

**Definition 1.9.** An  $\mathfrak{F}$ -set  $\Delta = (\Delta, \varphi)$  is a pair consisting of a set  $\Delta$  and a function  $\varphi: \Delta \rightarrow \mathfrak{F}$ . For  $H \in \mathfrak{F}$  we denote by  $\Delta_H$  the pre-image  $\varphi^{-1}(\{H\})$  and call it the  $H$ -component of the  $\mathfrak{F}$ -set  $\Delta$ . A map  $f: (\Delta, \varphi) \rightarrow (\Delta', \varphi')$  of  $\mathfrak{F}$ -sets is a function  $f: \Delta \rightarrow \Delta'$  of sets such that the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{f} & \Delta' \\ & \searrow \varphi & \swarrow \varphi' \\ & \mathfrak{F} & \end{array}$$

commutes.

Note that by definition the class of all  $\mathfrak{F}$ -sets, together with maps of  $\mathfrak{F}$ -sets, forms a *comma category over  $\mathfrak{F}$*  in the sense of [ML98, p. 45]. We denote this category by  $\mathfrak{F}\text{-}\mathfrak{S}\mathfrak{e}\mathfrak{t}$ .

**Lemma 1.10.** *Consider the set  $\mathfrak{F}$  as a discrete category. Then the functor category  $[\mathfrak{F}, \mathfrak{S}\mathfrak{e}\mathfrak{t}]$  is isomorphic to  $\mathfrak{F}\text{-}\mathfrak{S}\mathfrak{e}\mathfrak{t}$ .*

PROOF. Note, that since  $\mathfrak{F}$  is considered as a discrete category a functor  $\mathfrak{F} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}$  is characterised by its values on the objects of  $\mathfrak{F}$ . Given a  $\mathfrak{F}$ -set  $\Delta$ , there exists precisely one functor  $\Delta: \mathfrak{F} \rightarrow \mathfrak{S}\mathfrak{e}\mathfrak{t}$  that maps  $H$  to  $\Delta_H$  for every  $H \in \mathfrak{F}$ . This gives a bijection between the objects of  $\mathfrak{F}\text{-}\mathfrak{S}\mathfrak{e}\mathfrak{t}$  and the objects of  $[\mathfrak{F}, \mathfrak{S}\mathfrak{e}\mathfrak{t}]$ . Moreover, any morphism  $f: \Delta \rightarrow \Delta'$  in  $\mathfrak{F}\text{-}\mathfrak{S}\mathfrak{e}\mathfrak{t}$  induces a collection of functions  $f_H: \Delta_H \rightarrow \Delta'_H$  indexed by the elements  $\mathfrak{F}$ . Since  $\mathfrak{F}$  is a discrete

category this gives rise to a natural transformation between the corresponding functors  $\Delta: \mathfrak{F} \rightarrow \mathfrak{Set}$  and  $\Delta': \mathfrak{F} \rightarrow \mathfrak{Set}$  and thus a morphism in  $[\mathfrak{F}, \mathfrak{Set}]$ . It follows that we get a bijection between the corresponding morphism sets in  $\mathfrak{F}\text{-Set}$  and  $[\mathfrak{F}, \mathfrak{Set}]$ . Thus the two categories are isomorphic.  $\square$

There exists the obvious forgetful functor from the category  $\mathfrak{F}\text{-Set}$  to the category  $\mathfrak{Set}$  which sends a  $\mathfrak{F}$ -set  $\Delta$  to the underlying set  $\Delta$ . Using this functor we can pull back much of the terminology for sets to the category of  $\mathfrak{F}$ -sets. In particular we speak of a *finite (countable)  $\mathfrak{F}$ -set* if the underlying set is finite (countable). Only with categorical statements we have to be careful: for example the  $\mathfrak{F}$ -set  $\Delta'$  is a *subset* of the  $\mathfrak{F}$ -set  $\Delta$  if  $\Delta'_H \subset \Delta_H$  for every  $H \in \mathfrak{F}$ . As in functor categories limits and colimits are calculated component wise. In particular this is true for the product (cartesian product) and coproduct (disjoint union) of  $\mathfrak{F}$ -sets. In detail, if  $\Delta_i$  are  $\mathfrak{F}$ -sets indexed by some index set  $I$  then their product and coproduct are given by

$$\left(\prod_{i \in I} \Delta_i\right)_H = \prod_{i \in I} \Delta_{i,H} \quad \text{and} \quad \left(\coprod_{i \in I} \Delta_i\right)_H = \coprod_{i \in I} \Delta_{i,H}$$

for every  $H \in \mathfrak{F}$ .

Given a  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  we denote the *underlying  $\mathfrak{F}$ -set* also by  $M$ , which is given by

$$M_H := M(G/H)$$

for all  $H \in \mathfrak{F}$ . A morphism of  $\mathcal{O}_{\mathfrak{F}}G$ -modules gives in an obvious way rise to a map of the underlying  $\mathfrak{F}$ -sets. In this way we get a forgetful functor

$$U: \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{F}\text{-Set}$$

(and likewise we have a forgetful functor from  $\mathcal{O}_{\mathfrak{F}}G\text{-Mod}$  to  $\mathfrak{F}\text{-Set}$ ). We say that a  $\mathfrak{F}$ -set  $X$  is a *subset* of an  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  if  $X$  is a subset of the  $\mathfrak{F}$ -set  $UM$ . Any subset of an  $\mathcal{O}_{\mathfrak{F}}G$ -module is implicitly considered as a  $\mathfrak{F}$ -set.

**Definition 1.11.** Let  $M$  be an  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $X$  a subset of  $M$ . Then the smallest submodule of  $M$  containing  $X$  is denoted by  $\langle X \rangle$  and is called the submodule of  $M$  *generated* by the  $\mathfrak{F}$ -set  $X$ . If  $M = \langle X \rangle$  then we say that  $M$  is *generated* by  $X$  and that  $X$  is a  *$\mathfrak{F}$ -set of generators* of  $M$ . We say that  $M$  is a *finitely generated  $\mathcal{O}_{\mathfrak{F}}G$ -module* if there exists a finite  $\mathfrak{F}$ -set of generators of  $M$ .

**Lemma 1.12.** *Let  $K \in \mathfrak{F}$  and consider the right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[?, G/K]_G$  of Example 1.6. Then the subset  $\Delta$  of  $\mathbb{Z}[?, G/K]_G$  given by*

$$\Delta_H := \begin{cases} \{\text{id}\} & \text{if } H = K, \\ \emptyset & \text{otherwise} \end{cases} \quad (1.2)$$

*is a generating set of  $\mathbb{Z}[?, G/K]_G$ .*

PROOF. Denote by  $M$  the submodule of  $\mathbb{Z}[?, G/K]_G$  generated by  $\Delta$ . We know that  $M(G/H)$  is a subgroup of  $\mathbb{Z}[G/H, G/K]_G$  for any  $H \in \mathfrak{F}$  and we want to show that actually equality holds in every case.

Therefore let  $\varphi \in [G/H, G/K]_G$  be a generator of  $\mathbb{Z}[G/H, G/K]_G$ . Since  $\Delta$  generates  $M$  we know that  $\text{id} \in M(G/K)$ . Then  $\varphi^*(\text{id}) = \text{id} \circ \varphi = \varphi \in M(G/H)$ . Since this is true for any generator  $\varphi$  of the group  $\mathbb{Z}[G/H, G/K]_G$  we must have that  $M(G/H) = \mathbb{Z}[G/H, G/K]_G$  and the claim follows.  $\square$

**Proposition 1.13.** *The forgetful functor  $U: \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{F}\text{-Set}$  has a left adjoint  $F: \mathfrak{F}\text{-Set} \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ .*

PROOF. First we define the functor  $F$  for singleton  $\mathfrak{F}$ -sets. Let  $K \in \mathfrak{F}$  and consider the singleton  $\mathfrak{F}$ -set  $\Delta$  with  $\Delta_K := \{*\}$  and  $\Delta_H := \emptyset$  for  $H \neq K$ . We set

$$F\Delta := \mathbb{Z}[?, G/K]_G$$

and identify  $\Delta$  with the singleton subset of  $\mathbb{Z}[?, G/K]_K$  as given in (1.2) in the previous lemma. We have to show that for any (right)  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  the adjoint relation

$$\text{mor}_{\mathfrak{F}}(F\Delta, M) \cong \text{mor}(\Delta, UM) \quad (1.3)$$

is satisfied, where the morphism set on the left is in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  and the morphism set on the right is in  $\mathfrak{F}\text{-Set}$ . But this follows from the Yoneda type formula in the next lemma.

A general  $\mathfrak{F}$ -set  $\Delta$  can always be written as the coproduct

$$\Delta = \coprod_{x \in \Delta} \Delta_x$$

of its singleton subsets  $\Delta_x$ . The natural way to extend the definition of the functor  $F$  to arbitrary  $\mathfrak{F}$ -sets is to set

$$F\Delta := \coprod_{x \in \Delta} F\Delta_x.$$

Then we have isomorphisms

$$\begin{aligned} \text{mor}_{\mathfrak{F}}(F\Delta, M) &\cong \prod_{x \in \Delta} \text{mor}_{\mathfrak{F}}(F\Delta_x, M) \\ &\cong \prod_{x \in \Delta} \text{mor}(\Delta_x, UM) \cong \text{mor}(\Delta, UM) \end{aligned}$$

which are natural, both in  $\Delta$  and  $M$ . Thus  $F$  is a left adjoint functor to the forgetful functor  $U$ .  $\square$

Note that there is a canonical inclusion of the  $\mathfrak{F}$ -set  $\Delta = (\Delta, \varphi)$  into  $F\Delta$  given by

$$x \mapsto \text{id} \in (F\Delta_x)(G/\varphi(x)).$$

Using this inclusion we have a canonical way to identify the  $\mathfrak{F}$ -set  $\Delta$  as a subset of the right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $F\Delta$ . Note that under this identification  $\Delta$  becomes a generating  $\mathfrak{F}$ -set of  $F\Delta$ .

**Lemma 1.14** (Yoneda Type Formula). *Let  $K \in \mathfrak{F}$  and let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then there exists an isomorphism*

$$e_K: \text{mor}_{\mathfrak{F}}(\mathbb{Z}[?, G/K]_G, M) \cong M(G/K)$$

*of abelian groups given by the evaluation map  $e_K(f) := f_K(\text{id})$ . This isomorphism is natural in  $M$ .*

PROOF. For the proof of the first part see for example [MV03, p. 9]. The naturality claim states that for any morphism  $\eta: M \rightarrow N$  of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules, the diagram

$$\begin{array}{ccc} \text{mor}_{\mathfrak{F}}(\mathbb{Z}[?, G/K], M) & \xrightarrow{e_K} & M(G/K) \\ \eta_* \downarrow & & \downarrow \eta_K \\ \text{mor}_{\mathfrak{F}}(\mathbb{Z}[?, G/K], N) & \xrightarrow{e_K} & N(G/K) \end{array}$$

commutes, where  $\eta_*$  is the homomorphism which maps any morphism  $f \in \text{mor}_{\mathfrak{F}}(\mathbb{Z}[?, G/K], M)$  to  $\eta \circ f \in \text{mor}_{\mathfrak{F}}(\mathbb{Z}[?, G/K], N)$ . But this follows immediately from

$$(e_K \circ \eta_*)(f) = e_K(\eta \circ f) = (\eta \circ f)_K(\text{id}) = \eta_K(f_K(\text{id})) = (\eta_K \circ e_K)(f). \quad \square$$

**Definition 1.15.** Let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and let  $B$  a subset of  $M$ . We say that  $M$  is *free with basis  $B$*  if there exists an isomorphism

$$FB \cong M$$

that maps  $B$  seen as a subset of  $FB$  to  $B$  as a subset of  $M$ .

In the terms of the adjoint relation (1.3) the above definition can be interpreted in the following familiar way: a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is free with basis  $B$  if  $B$  is a subset of  $M$  such that for any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and any morphism  $f_0: B \rightarrow N$  of  $\mathfrak{F}$ -sets there exists a unique extension of  $f_0$  to a morphism  $f: M \rightarrow N$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules.

Note that from the proof of Proposition 1.13 follows that the  $\mathcal{O}_{\mathfrak{F}}G$ -modules of the form  $\mathbb{Z}[?, G/K]_G$ ,  $K \in \mathfrak{F}$ , are the building blocks for free right Bredon modules and if  $\Delta = (\Delta, \varphi)$  is a  $\mathfrak{F}$ -set, then

$$F\Delta = \coprod_{\delta \in \Delta} \mathbb{Z}[?, G/\varphi(\delta)]_G. \quad (1.4)$$

**Lemma 1.16.** *A  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is finitely generated if and only if there exists a short exact sequence of  $\mathcal{O}_{\mathfrak{F}}G$ -modules*

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

where  $F$  is a finitely generated free  $\mathcal{O}_{\mathfrak{F}}G$ -module.

PROOF. If  $M$  is finitely generated, then there exists a finite generating  $\mathfrak{F}$ -set  $X$  of  $M$ . Set  $F := FX$ . Then  $F$  is finitely generated and surjects onto  $M$ . If one lets  $K$  be the kernel of this surjection one obtains the above short exact sequence.

On the other hand, if  $F$  is free with a finite basis  $X$ , then the image of  $X$  under the surjection  $F \rightarrow M$  is a finite  $\mathfrak{F}$ -subset of  $M$  which generates  $M$ . Therefore  $M$  is finitely generated.  $\square$

**Definition 1.17.** A  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is called *finitely presented* if there exists a short exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

where  $F$  is a finitely generated free  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $N$  is a finitely generated  $\mathcal{O}_{\mathfrak{F}}G$ -module.

Note that the definitions, results and their proofs in this section carry word for word over to left  $\mathcal{O}_{\mathfrak{F}}G$ -modules with right  $\mathcal{O}_{\mathfrak{F}}G$ -modules of the form  $\mathbb{Z}[?, G/K]_G$  replaced by corresponding left  $\mathcal{O}_{\mathfrak{F}}G$ -modules of the form  $\mathbb{Z}[G/K, ?]_G$ .



## 6. From $G$ -Sets to Bredon Modules

There is an alternative way to see the construction of the previous section, namely as a functor from the category of  $G$ -sets to the category of right Bredon modules.

Recall that given subgroups  $H$  and  $K$  of  $G$  there exists the identification

$$\eta: [G/H, G/K]_G \cong (G/K)^H$$

which sends a  $G$ -map  $\psi$  to the image  $\psi(H)$ . Now  $[?, ??]_G$  is a bifunctor

$$[?, ??]_G: \mathcal{O}G \times \mathcal{O}G \rightarrow \mathfrak{Set}$$

contravariant in the first and covariant in the second variable. Restricting this functor to  $\mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}G$  and composed with the functor  $\mathbb{Z}[?]: \mathfrak{Set} \rightarrow \mathfrak{Ab}$  which sends a set  $X$  to the free abelian group  $\mathbb{Z}[X]$  with basis  $X$  this gives the functor of Example 1.8.

The functor  $[?, ??]_G$  extends to a bi-functor

$$[?, ??]_G: \mathcal{O}G \times G\text{-}\mathfrak{Set} \rightarrow \mathfrak{Set},$$

contravariant in the first and contravariant in the second variable, which sends a transitive  $G$ -set  $G/H$  and a  $G$ -set  $X$  to the set  $[G/H, X]_G$  of all  $G$ -maps from  $G/H$  to  $X$ . Note that as before there is an identification

$$\eta: [G/H, X]_G \cong X^H \tag{1.5}$$

which sends a  $G$ -map  $\psi: G/H \rightarrow X$  to  $\psi(H)$ .

If  $f: X \rightarrow Y$  is a  $G$ -map, then this gives a map

$$\begin{aligned} f_*: [G/H, X]_G &\rightarrow [G/H, Y]_G, \\ \psi &\mapsto f \circ \psi. \end{aligned}$$

Under the identification (1.5) is just the restriction of  $f$  to a map  $X^H \rightarrow Y^H$ .

On the other hand, if  $\varphi: G/H \rightarrow G/K$  is a morphism of  $\mathcal{O}G$  then this gives a map

$$\begin{aligned} \varphi^*: [G/K, X]_G &\rightarrow [G/H, X]_G, \\ \psi &\mapsto \psi \circ \varphi. \end{aligned}$$

To see what  $\varphi_*$  becomes under the identification assume that  $\varphi = f_{g,H,K}$ . Then  $g$  is uniquely determined up to right multiplication by an element of  $K$

and  $H^g \leq K$ . If  $x \in X^K$  and  $h \in H$ , then

$$hgx = g(g^{-1}hg)x = gx$$

and therefore  $gx \in X^H$ . If we set  $\psi := \eta^{-1}(x) \in [G/K, X]_G$ , then

$$\varphi^*(\psi)(H) = (\psi \circ \varphi)(H) = \psi(gK) = g\psi(K) = gx.$$

Thus  $\varphi^*$  becomes under the identification (1.5) the map  $X^K \rightarrow X^H$  which sends  $x$  to  $gx$ .

Let  $\Delta = (\Delta, \varphi)$  be a  $\mathfrak{F}$ -set and consider the  $G$ -set

$$X := \coprod_{x \in \Delta} G/\varphi(x)$$

which is the disjoint union of homogeneous  $G$ -spaces  $G/\varphi(x)$  with  $\varphi(x) \in \mathfrak{F}$ .

Then

$$[?, X]_G = \coprod_{x \in \Delta} [?, G/\varphi(x)]_G$$

and since the functor  $\mathbb{Z}[?]: \mathfrak{Set} \rightarrow \mathfrak{Ab}$  commutes with coproducts it follows that

$$\mathbb{Z}[?, X]_G = \coprod_{x \in \Delta} \mathbb{Z}[?, G/\varphi(x)]_G$$

is the free right  $\mathcal{O}_{\mathfrak{F}}G$ -module with basis the  $\mathfrak{F}$ -set  $\Delta$  as introduced in the previous section. On the other hand it is clear that if  $X$  is a  $G$ -set with  $\mathfrak{F}(X) \subset \mathfrak{F}$ , then  $\mathbb{Z}[?, X]_G$  is a free  $\mathcal{O}_{\mathfrak{F}}G$ -module. Therefore we obtain the following result.

**Proposition 1.18.** *Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Then we have a covariant functor*

$$\mathbb{Z}[?, ??]_G: G\text{-}\mathfrak{Set} \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G.$$

*This functor sends disjoint unions of  $G$ -sets to coproducts in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ . The free right  $\mathcal{O}_{\mathfrak{F}}G$ -modules are precisely all the Bredon modules of the form  $\mathbb{Z}[?, X]_G$  where  $X$  is a  $G$ -set with  $\mathfrak{F}(X) \subset \mathfrak{F}$ .  $\square$*

Let  $X$  be a set and let  $X_\lambda$ ,  $\lambda \in \Lambda$ , be a collection of subsets of  $X$  indexed by an abstract index set  $\Lambda$ . We say that  $X$  is the *directed union of the sets*  $X_\lambda$  if the following two conditions hold:

- (1) for every  $x \in X$  there exists a  $\lambda \in \Lambda$  such that  $x \in X_\lambda$ ;
- (2) for every  $\lambda_1, \lambda_2 \in \Lambda$  there exists a  $\mu \in \Lambda$  such that  $X_{\lambda_1} \subset X_\mu$  and  $X_{\lambda_2} \subset X_\mu$ .

Since a directed union is a special case of a colimit in the category of sets (or  $G$ -sets) we may identify  $X = \varinjlim X_\lambda$ .

**Lemma 1.19.** *Assume that the  $G$ -set  $X$  is the directed union of  $G$ -invariant subsets  $X_\lambda$ ,  $\lambda \in \Lambda$ . Let  $H$  be a subgroup  $G$ . Then  $X^H$  is the directed union of the subsets  $X_\lambda^H$ ,  $\lambda \in \Lambda$ . That is*

$$X^H = \varinjlim X_\lambda^H.$$

PROOF. This follows immediately from the fact that  $X_\lambda^H = X^H \cap X_\lambda$ .  $\square$

**Proposition 1.20.** *The homomorphism of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules*

$$\varinjlim \mathbb{Z}[?, X_\lambda]_G \rightarrow \mathbb{Z}[?, X]_G \tag{1.6}$$

*induced by the canonical monomorphisms  $\mathbb{Z}[?, X_\lambda]_G \hookrightarrow \mathbb{Z}[?, X]_G$  is an isomorphism.*

PROOF. We have

$$\begin{aligned} \mathbb{Z}[?, X]_G &\cong \mathbb{Z}[?, \varinjlim X_\lambda]_G \\ &\cong \mathbb{Z}[\varinjlim [?, X_\lambda]_G] && \text{(Lemma 1.19)} \\ &\cong \varinjlim \mathbb{Z}[?, X_\lambda]_G \end{aligned}$$

where the last isomorphism is due to the fact that the functor  $\mathbb{Z}[?]: \mathfrak{Set} \rightarrow \mathfrak{Ab}$  commutes with arbitrary colimits since it is the left adjoint to the forgetful functor  $U: \mathfrak{Ab} \rightarrow \mathfrak{Set}$ , see [ML98, pp. 118f.]. Now the composite of this sequence of isomorphism is precisely the homomorphism (1.6).  $\square$

**Lemma 1.21.** *Let  $X$  be a  $G$ -set such that the orbit space  $X/G$  is countable. Then  $\mathbb{Z}[?, X]_G$  is countably generated if one of the following conditions holds:*

- (1)  $\mathfrak{F}(X) \subset \mathfrak{F}$ ;
- (2)  $G$  and  $\mathfrak{F}$  are countable.

PROOF. Let  $R$  be a complete system of representatives for the orbits  $X/G$ . Then the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[?, X]_G$  is a countable coproduct

$$\mathbb{Z}[?, X]_G \cong \coprod_{x \in R} \mathbb{Z}[?, G/G_x]_G.$$

If  $\mathfrak{F}(X) \subset \mathfrak{F}$  then the right hand side is free and thus  $\mathbb{Z}[?, X]_G$  is countably generated. This proves the first case.

Thus assume that  $G$  and  $\mathfrak{F}$  are countable. We construct for any  $x \in R$  a countably generated free  $\mathcal{O}_{\mathfrak{F}}G$  module  $F_x$  that surjects onto  $\mathbb{Z}[\?, G/G_x]_G$ . The construction is dual to the construction given in [Wei94, p. 43]. Since  $G$  is countable the set  $[G/H, G/G_x]_G$  is countable for any  $H \in \mathfrak{F}$ . Set  $F_{\varphi, H} := \mathbb{Z}[\?, G/H]_G$  for each  $\varphi \in [G/H, G/G_x]_G$  and  $H \in \mathfrak{F}$ . Then

$$F_x := \coprod_{H \in \mathfrak{F}} \coprod_{\varphi \in [G/H, G/G_x]_G} F_{\varphi, H}$$

is a countable free  $\mathcal{O}_{\mathfrak{F}}G$ -module that surjects onto  $\mathbb{Z}[\?, G/G_x]_G$ . This surjection can be constructed as follows. For each  $\varphi \in [G/H, G/G_x]_G$  let  $f_{\varphi, H}: F_{\varphi, H} \rightarrow \mathbb{Z}[\?, G/G_x]_G$  be the unique morphism of  $\mathcal{O}_{\mathfrak{F}}G$ -modules that maps the generator of  $F_{\varphi, H}$  to the generator  $\varphi$  of  $\mathbb{Z}[G/H, G/G_x]_G$ . Then

$$f_x := \coprod_{H \in \mathfrak{F}} \coprod_{\varphi \in [G/H, G/G_x]_G} f_{\varphi, H}$$

is a surjection of  $F_x$  onto  $\mathbb{Z}[\?, G/G_x]$ . It follows that

$$F := \coprod_{x \in R} F_x$$

is a countably generated free  $\mathcal{O}_{\mathfrak{F}}G$ -module that surjects onto  $\mathbb{Z}[\?, X]_G$ .  $\square$

## 7. Projective Bredon Modules

It follows from a categorical argument that free objects share the following universal property: any morphism  $f: P \rightarrow M$  from a free  $\mathcal{O}_{\mathfrak{F}}G$ -module  $P$  to an arbitrary  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  factors through any epimorphism  $p: M' \rightarrow M$ . That is we can always find a morphism  $P \rightarrow M'$  making the following diagram, with the row exact, commute:

$$\begin{array}{ccc} & P & \\ & \ddots & \downarrow f \\ M' & \xrightarrow{p} & M \longrightarrow 0 \end{array} \quad (1.7)$$

Projective objects are the usual generalisation of free objects. We recall the definition of a projective object in the category of Bredon modules and the following result, which is a standard result for abelian categories.

**Definition 1.22.** An  $\mathcal{O}_{\mathfrak{F}}G$ -module  $P$  is called *projective* if for every diagram of the form (1.7) with exact row, there exists a morphism  $P \rightarrow M'$  that makes the diagram commute.

**Proposition 1.23.** *Let  $P$  be a Bredon module over the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ . Then the following statements for  $P$  are equivalent:*

- (1)  $P$  is projective;
- (2) every exact sequence  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  splits;
- (3)  $\text{mor}_{\mathfrak{F}}(P, ?)$  is an exact functor;
- (4)  $P$  is a direct summand of a free  $\mathcal{O}_{\mathfrak{F}}G$ -module.

PROOF. The result can be found in any homological algebra book, for example [Wei94, pp. 33ff.].  $\square$

Again the above definitions and results are valid in the category of left  $\mathcal{O}_{\mathfrak{F}}G$ -modules as they are valid in the category of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules.

## 8. Two Tensor Products for Bredon Modules

There are two possible ways to define a tensor product for Bredon modules. The first one generalises the tensor product over the group ring  $\mathbb{Z}G$  for  $G$ -modules. The second tensor product is the generalisation of the tensor product over  $\mathbb{Z}$  in the category of  $G$ -modules with the diagonal action of  $\mathbb{Z}G$  on the tensor product.

**8.1. The Tensor Product over  $\mathfrak{F}$ .** The definition of the tensor product over  $\mathfrak{F}$  involves the categorical tensor product as described in [Sch70b, pp. 45ff.]. Given a small category  $\mathcal{B}$ , the categorical tensor product is a bifunctor

$$? \otimes_{\mathcal{B}} ?? : [\mathcal{B}^{\text{op}}, \mathfrak{Ab}] \times [\mathcal{B}, \mathfrak{Ab}] \rightarrow \mathfrak{Ab}$$

with properties expected from a tensor product.

In the case that  $\mathcal{B} = \mathcal{O}_{\mathfrak{F}}G$  a concrete model for this tensor product is given in [Lüc89, p. 166]: if  $M$  is a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $N$  is a left  $\mathcal{O}_{\mathfrak{F}}G$ -module, then let  $P$  be the abelian group

$$P := \coprod_{H \in \mathfrak{F}} M(G/H) \otimes N(G/H) \tag{1.8}$$

where the tensor product is taken over  $\mathbb{Z}$ . Let  $Q$  be the subgroup of  $P$  generated by all elements of the form  $\varphi^*(m) \otimes n - m \otimes \varphi_*(n)$  with  $m \in M(G/H)$ ,  $n \in N(G/K)$ ,  $\varphi \in [G/K, G/H]_G$ ,  $H, K \in \mathfrak{F}$ . Then the *tensor product*  $M \otimes_{\mathfrak{F}} N$  of  $M$  and  $N$  over  $\mathfrak{F}$  is defined as the abelian group

$$M \otimes_{\mathfrak{F}} N := P/Q.$$

If  $f: M \rightarrow M'$  and  $g: N \rightarrow N'$  are morphisms of right and left  $\mathcal{O}_{\mathfrak{F}}G$ -modules respectively, then  $f \otimes_{\mathfrak{F}} g: M \otimes_{\mathfrak{F}} N \rightarrow M' \otimes_{\mathfrak{F}} N'$  is defined in the obvious way.

Altogether the tensor product over  $\mathfrak{F}$  becomes an additive bifunctor

$$? \otimes_{\mathfrak{F}} ?? : \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \times \mathcal{O}_{\mathfrak{F}}G\text{-Mod} \rightarrow \mathfrak{Ab}. \quad (1.9)$$

**Proposition 1.24.** *Let  $M$  be a fixed right  $\mathcal{O}_{\mathfrak{F}}G$ -module and let  $N$  be a fixed left  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then the functors*

$$M \otimes_{\mathfrak{F}} ?? : \mathcal{O}_{\mathfrak{F}}G\text{-Mod} \rightarrow \mathfrak{Ab}$$

and

$$? \otimes_{\mathfrak{F}} N : \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \rightarrow \mathfrak{Ab}$$

preserve arbitrary colimits.

PROOF. See [Sch70b, pp. 46f.]. □

The fact that the functor  $? \otimes_{\mathfrak{F}} N$  preserves colimits is not a surprise because functors that have right adjoints preserve colimits [ML98, pp. 118f.] and the functor  $? \otimes_{\mathfrak{F}} N$  has a right adjoint, namely  $\text{Mor}(N, ?)$  [Sch70b, p. 46]. This is the functor that assigns to each abelian group  $A$  the right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\text{Mor}(N, A)$ , which is defined on the objects  $G/H$  of  $\mathcal{O}_{\mathfrak{F}}G$  by  $\text{Mor}(N, A)(G/H) := \text{mor}(N(G/H), A)$ , where the morphism set on the defining side is in  $\mathfrak{Ab}$ . Explicitly, for an abelian group  $A$  and a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  the adjoint relation is

$$\text{mor}(M \otimes_{\mathfrak{F}} N, A) \cong \text{mor}(M, \text{Mor}(N, A)). \quad (1.10)$$

**Lemma 1.25.** *Let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $N$  a left  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then for every  $K \in \mathfrak{F}$  we have isomorphisms*

$$M(?) \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, ?]_G \cong M(G/K)$$

and

$$\mathbb{Z}[?, G/K]_G \otimes_{\mathfrak{F}} N(?) \cong N(G/K)$$

which are natural in  $M$  and  $N$ .

PROOF. These are known results, see for example [Lüc89, p. 166] or [MV03, p. 14]. We carry out the details for the first isomorphism in order

to exhibit the precise definition of the isomorphism. The second isomorphism is constructed in the same way.

Let  $P$  and  $Q$  be the abelian groups as in the definition of the tensor product over  $\mathfrak{F}$ . That is we have that  $M(?) \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, ?]_G$  is a quotient of the group

$$P = \coprod_{H \in \mathfrak{F}} M(G/H) \otimes \mathbb{Z}[G/K, G/H]_G.$$

Observe that each element of the abelian group  $M(G/H) \otimes \mathbb{Z}[G/K, G/H]_G$  can be written uniquely as a finite sum

$$m_1 \otimes \varphi_1 + \dots + m_n \otimes \varphi_n$$

with  $m_i \in M(G/H)$  and  $\varphi_i \in [G/H, G/K]_G$ . If  $m \in M(G/H)$  and  $\varphi \in [G/K, G/H]_G$  then  $\varphi^*(m) \in M(G/K)$ . It follows that there exists a unique well defined homomorphism

$$\eta_H: M(G/K) \otimes \mathbb{Z}[G/K, G/H]_G \rightarrow M(G/K)$$

of abelian groups for which  $\eta_H(m \otimes \varphi) = \varphi^*(m)$  for every  $m \in M(G/H)$  and  $\varphi \in [G/K, G/H]_G$ . The collection  $\{\eta_H : H \in \mathfrak{F}\}$  defines then a homomorphism

$$\eta: P \rightarrow M(G/K).$$

This homomorphism is surjective, since for every  $m \in M(G/K)$  we have  $\eta_K(m \otimes \text{id}) = \text{id}^*(m) = m$ . Furthermore, elements of the form

$$\varphi_1^*(m) \otimes \varphi_2 - m \otimes (\varphi_1)_*(\varphi_2) = \varphi_1^*(m) \otimes \varphi_2 - m \otimes (\varphi_1 \circ \varphi_2)$$

are in the kernel of  $\eta$ , since

$$\begin{aligned} \eta(\varphi_1^*(m) \otimes \varphi_2 - m \otimes (\varphi_1 \circ \varphi_2)) &= \eta(\varphi_1^*(m) \otimes \varphi_2) - \eta(m \otimes (\varphi_1 \circ \varphi_2)) \\ &= \varphi_2^*(\varphi_1^*(m)) - (\varphi_1 \circ \varphi_2)^*(m) \\ &= (\varphi_1 \circ \varphi_2)^*(m) - (\varphi_1 \circ \varphi_2)^*(m) \\ &= 0. \end{aligned}$$

It follows that  $Q \subset \ker \eta$ .

On the other hand, assume that  $m \otimes \varphi \in \ker \eta$ . Then  $\varphi^*(m) = \eta(m \otimes \varphi) = 0$  and we get

$$m \otimes \varphi = -(\varphi^*(m) \otimes \text{id} - m \otimes \varphi_*(\text{id})) \in Q.$$

Hence  $\ker \eta = Q$  and  $\eta$  induces an isomorphism

$$M(?) \otimes_{\mathfrak{F}} \mathbb{Z}[G/K, ?]_G = P/Q \cong M(G/K).$$

The naturality of this isomorphism is evident.  $\square$

A priori the tensor product  $M \otimes_{\mathfrak{F}} N$  of two  $\mathcal{O}_{\mathfrak{F}}G$ -modules is only an abelian group. But if either  $M$  or  $N$  is a Bredon bimodule, then the tensor product can be made into a Bredon module as well. More precisely, assume we are given two groups  $G_1$  and  $G_2$ , a family  $\mathfrak{F}$  of subgroups of  $G_1$  and a family  $\mathfrak{G}$  of subgroups of  $G_2$ . If  $M$  is an  $\mathcal{O}_{\mathfrak{G}}G_2$ - $\mathcal{O}_{\mathfrak{F}}G_1$ -bimodule and  $N$  a left  $\mathcal{O}_{\mathfrak{F}}G_1$ -module, then

$$M(?, ??) \otimes_{\mathfrak{F}} N(??)$$

is a left  $\mathcal{O}_{\mathfrak{G}}G_2$ -module. Similarly, if  $M$  is a right  $\mathcal{O}_{\mathfrak{F}}G_1$ -module and  $N$  an  $\mathcal{O}_{\mathfrak{F}}G_1$ - $\mathcal{O}_{\mathfrak{G}}G_2$ -bimodule, then

$$M(?) \otimes_{\mathfrak{F}} N(?, ??)$$

is a right  $\mathcal{O}_{\mathfrak{G}}G_2$ -module.

For a fixed  $\mathcal{O}_{\mathfrak{F}}G_1$ - $\mathcal{O}_{\mathfrak{G}}G_2$ -bimodule we get an adjoint relation for the tensor product similar to (1.10):

**Lemma 1.26.** *Let  $B$  be a  $\mathcal{O}_{\mathfrak{F}}G_1$ - $\mathcal{O}_{\mathfrak{G}}G_2$ -bimodule. Then the functor*

$$? \otimes_{\mathfrak{F}} B: \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1 \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2$$

*is left adjoint to the functor*

$$\text{mor}_{\mathfrak{G}}(B, ?): \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2 \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1.$$

*Explicitly we have for every right  $\mathcal{O}_{\mathfrak{F}}G_1$ -module  $M$  and every right  $\mathcal{O}_{\mathfrak{G}}G_2$ -module  $N$  the adjoint relation*

$$\text{mor}_{\mathfrak{G}}(M \otimes_{\mathfrak{F}} B, N) \cong \text{mor}_{\mathfrak{F}}(M, \text{mor}_{\mathfrak{G}}(B, N)). \quad (1.11)$$

PROOF. See [Lüc89, p. 169].  $\square$

**8.2. The Tensor Product over  $\mathbb{Z}$ .** The second tensor product for Bredon modules is the *tensor product over  $\mathbb{Z}$*  [Lüc89, p. 166]. Given two right  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $M$  and  $N$  define a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M \otimes N$  as follows.

For  $H \in \mathfrak{F}$  let  $(M \otimes N)(G/H) := M(G/H) \otimes N(G/H)$  where the tensor product on the defining side is taken over  $\mathbb{Z}$ . If  $\varphi: G/H \rightarrow G/K$  is a morphism



in the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ , then define  $(M \otimes N)(\varphi) := \varphi \otimes \varphi$ , which is a homomorphism  $(M \otimes N)(G/K) \rightarrow (M \otimes N)(G/H)$ .

If  $f: M \rightarrow M'$  and  $g: N \rightarrow N'$  are morphisms in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ , then  $f \otimes g$  is defined to be the morphism

$$f \otimes g: M \otimes N \rightarrow M' \otimes N'$$

which is given by  $(f \otimes g)_H := f_H \otimes g_H$  for every  $H \in \mathfrak{F}$ .

In this way the tensor product of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules over  $\mathbb{Z}$  is an additive bifunctor

$$? \otimes ?? : \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \times \text{Mod-}\mathcal{O}_{\mathfrak{F}}G \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G.$$

The tensor product over  $\mathbb{Z}$  for left  $\mathcal{O}_{\mathfrak{F}}G$ -modules is defined in a similar way.

## 9. Flat Bredon Modules

The tensor product functor over  $\mathfrak{F}$  maps epimorphisms to epimorphisms and thus this functor is right exact. But in general the tensor product over  $\mathfrak{F}$  is not exact.

**Definition 1.27.** A right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is called *flat* if the functor  $M \otimes_{\mathfrak{F}} ??$  is exact. Dually a left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  is called *flat* if the functor  $? \otimes_{\mathfrak{F}} N$  is exact.

**Proposition 1.28.** *Projective  $\mathcal{O}_{\mathfrak{F}}G$ -modules are flat.*

PROOF. This is true in general in abelian categories. But the result follows also from Proposition 1.24 and Lemma 1.25.  $\square$

Under mild conditions on the family  $\mathfrak{F}$  of subgroups Nucinkis has given a characterisation of flat  $\mathcal{O}_{\mathfrak{F}}G$ -modules in [Nuc04, p. 38], which is the Bredon equivalent to Lazard's Theorem in [Laz69, p. 84].

**Proposition 1.29.** [Nuc04, Theorem 3.2] *Assume that  $\mathfrak{F}$  is a full family of subgroups of  $G$ . Then the following statements are equivalent for a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ :*

- (1)  $M$  is flat;
- (2) any morphism  $\varphi: P \rightarrow M$  from a finitely presented  $\mathcal{O}_{\mathfrak{F}}G$ -module  $P$  to  $M$  factors through some finitely generated free  $\mathcal{O}_{\mathfrak{F}}G$ -module  $F$ ;
- (3)  $M$  is the direct limit of finitely generated free  $\mathcal{O}_{\mathfrak{F}}G$ -modules.  $\square$

## 10. Restriction, Induction and Coinduction

The concept of restriction, induction and coinduction as known for modules over group rings generalizes to Bredon modules. Roughly speaking in the case of group rings these functors are defined using a ring homomorphism induced by an inclusion  $H \hookrightarrow G$  where  $H$  is a subgroup of  $G$ . In the case of Bredon cohomology the role of this ring homomorphism is replaced by a functor between orbit categories. The following definition is due to Lück [Lüc89, pp. 166f.].

**Definition 1.30.** Let  $\mathfrak{F}$  be a family of subgroups of a group  $G_1$  and let  $\mathfrak{G}$  be a family of subgroups of a group  $G_2$ . Furthermore let  $F: \mathcal{O}_{\mathfrak{F}}G_1 \rightarrow \mathcal{O}_{\mathfrak{G}}G_2$  be a functor between the corresponding orbit categories. Associated with the functor  $F$  we have the following three additive functors:

$$\begin{aligned} \text{res}_F: \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2 &\rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1, \\ M &\mapsto M(??) \otimes_{\mathfrak{G}} \mathbb{Z}[F(?), ??]_{G_2} \quad (\text{restriction with } F), \end{aligned}$$

$$\begin{aligned} \text{ind}_F: \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1 &\rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2, \\ M &\mapsto M(?) \otimes_{\mathfrak{F}} \mathbb{Z}[??, F(?)]_{G_2} \quad (\text{induction with } F) \end{aligned}$$

and

$$\begin{aligned} \text{coind}_F: \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1 &\rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2, \\ M &\mapsto \text{mor}_{\mathfrak{F}}(\mathbb{Z}[F(?), ??]_{G_2}, M(?)) \quad (\text{coinduction with } F). \end{aligned}$$

There are two other ways to interpret the restriction functor. Namely we have natural equivalences of  $\text{res}_F$  with the following two functors (see [Lüc89, pp. 116f.]):

$$\begin{aligned} \text{res}'_F: \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2 &\rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1, \\ M &\mapsto M \circ F \end{aligned}$$

and

$$\begin{aligned} \text{res}''_F: \text{Mod-}\mathcal{O}_{\mathfrak{G}}G_2 &\rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}}G_1, \\ M &\mapsto \text{mor}_{\mathfrak{G}}(\mathbb{Z}[??, F(?)]_{G_2}, M(?)). \end{aligned}$$

Here the first natural equivalence is essentially due to Lemma 1.25. The second natural equivalence is due to the Yoneda-style isomorphism

$$\mathrm{mor}_{\mathfrak{G}}(\mathbb{Z}[??, F(?) ]_{G_2}, M(??)) \cong (M \circ F)(?)$$

which gives a natural equivalence of  $\mathrm{res}''_F$  with  $\mathrm{res}'_F$ .

As in the ordinary case, restriction, induction and coinduction with  $F$  are closely related functors. From the adjunction relation (1.11) we get the following result:

**Proposition 1.31.** *Induction with  $F$  is a left adjoint to restriction with  $F$ . Coinduction with  $F$  is a right adjoint to restriction with  $F$ .*

PROOF. Due to (1.11) we have the following sequences of natural isomorphisms for any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  and right  $\mathcal{O}_{\mathfrak{G}}G$ -module  $N$ :

$$\begin{aligned} \mathrm{mor}_{\mathfrak{G}}(\mathrm{ind}_F M, N) &\cong \mathrm{mor}_{\mathfrak{G}}(M(?) \otimes_{\mathfrak{F}} \mathbb{Z}[??, F(?) ]_{G_2}, N(??)) \\ &\cong \mathrm{mor}_{\mathfrak{F}}(M(?), \mathrm{mor}_{\mathfrak{G}}(\mathbb{Z}[??, F(?) ]_{G_2}, N(??))) \\ &\cong \mathrm{mor}_{\mathfrak{F}}(M, \mathrm{res}'_F N) \\ &\cong \mathrm{mor}_{\mathfrak{F}}(M, \mathrm{res}_F N) \end{aligned}$$

and

$$\begin{aligned} \mathrm{mor}_{\mathfrak{F}}(\mathrm{res}_F N, M) &\cong \mathrm{mor}_{\mathfrak{F}}(N(??) \otimes_{\mathfrak{G}} \mathbb{Z}[F(?), ??]_{G_2}, M(?)) \\ &\cong \mathrm{mor}_{\mathfrak{G}}(N(??), \mathrm{mor}_{\mathfrak{F}}(\mathbb{Z}[F(?), ??]_{G_2}, M(?))) \\ &\cong \mathrm{mor}_{\mathfrak{G}}(N, \mathrm{coind}_F M). \quad \square \end{aligned}$$

In the following we list some further properties for the above functors, though not all of them will be needed. Most of the following results are direct consequences of the adjunction result above.

**Proposition 1.32.** *Restriction with  $F$  is an exact functor, induction with  $F$  is a right exact functor and coinduction with  $F$  is a left exact functor.*

PROOF. This is a direct application of Theorem 2.6.1 in [Wei94, pp. 51f.]. It states that if  $L$  and  $R$  are additive functors and  $L$  is a left adjoint to  $R$  (and therefore  $R$  is a right adjoint to  $L$ ), then  $L$  is right exact and  $R$  is left exact.  $\square$

**Proposition 1.33.** *Induction and restriction with  $F$  preserve arbitrary colimits. Coinduction and restriction with  $F$  preserve arbitrary limits.*

PROOF. The result follows from the fact that left adjoints preserve all colimits and dually that right adjoints preserve all limits, see [ML98, pp. 118f.].  $\square$

**Proposition 1.34.** *Induction with  $F$  preserves frees and projectives. If  $M$  is a finitely generated  $\mathcal{O}_{\mathfrak{F}}G_1$ -module, then so is  $\text{ind}_F M$ . If both families  $\mathfrak{F}$  and  $\mathfrak{G}$  are full families of subgroups, then induction with  $F$  preserves flats.*

PROOF. All the statements except the last one can be found in [Lüc89, p. 169]. If  $M$  is a flat right  $\mathcal{O}_{\mathfrak{F}}G_1$ -module and  $\mathfrak{F}$  is a full family of subgroups of  $G_1$ , then from Proposition 1.29 it follows that  $M$  is the direct limit of finitely generated free  $\mathcal{O}_{\mathfrak{F}}G_1$ -modules  $M_\lambda$ . Then since induction with  $F$  preserves colimits, we get

$$\text{ind}_F M \cong \text{ind}_F(\varinjlim M_\lambda) \cong \varinjlim(\text{ind}_F M_\lambda).$$

Since the  $M_\lambda$  are finitely generated free Bredon modules so are the  $\text{ind}_F M_\lambda$ . Thus  $\text{ind}_F M$  is the direct limit of finitely generated free  $\mathcal{O}_{\mathfrak{G}}G_2$ -modules and since the family  $\mathfrak{G}$  of subgroups of  $G_2$  is full we can apply again Proposition 1.29 from which it then follows that the  $\mathcal{O}_{\mathfrak{G}}G_2$ -module  $\text{ind}_F M$  is flat.  $\square$

**Proposition 1.35.** *Coinduction with  $F$  preserves injectives.*

PROOF. This is a direct application of Theorem 2.3.10 in [Wei94, p. 41], since coinduction with  $F$  is an additive functor that is right adjoint to the exact restriction functor.  $\square$

## CHAPTER 2

# Classifying Spaces

### 1. $G$ -CW-Complexes

$G$ -CW-complexes have been introduced by J. H. C. Whitehead in [Whi49] and are widely known by now. The concept has been generalised to the equivariant case in [Mat71], [Ill72] and [tD87]. In this thesis we use the definition described in [Lüc89, pp. 6f.]. Even though we are concerned with the study of classifying spaces for discrete groups we will state the definition of a  $G$ -CW-complex and of a classifying space first for topological groups before we pass to discrete groups in the subsequent studies.

By a *topological group*  $G$  we understand a group  $G$  which is at the same time a Hausdorff space such that the map

$$\begin{aligned} G \times G &\rightarrow G, \\ (g, h) &\mapsto gh^{-1} \end{aligned}$$

is continuous.

**Definition 2.1.** [Lüc89, pp. 6f.] Let  $G$  be a topological group acting continuously on a topological space  $X$ . A  *$G$ -CW-complex structure on  $X$*  consists of

- (1) a filtration  $X_0 \subset X_1 \subset X_2 \subset \dots$  of  $X$  which exhausts  $X$ , and
- (2) a collection  $\{e_i^n : i \in I_n\}$  of  $G$ -subspaces  $e_i^n \subset X_n$  for each  $n \in \mathbb{N}$ ,

with the properties

- (1)  $X$  has the weak topology with respect to the filtration  $\{X_n : n \in \mathbb{N}\}$  (that is  $B \subset X$  is closed in  $X$  if and only if  $B \cap X_n$  is closed in  $X_n$  for every  $n \in \mathbb{N}$ );
- (2) for each  $n \geq 1$  there exists a  $G$ -pushout as in Figure 2 such that  $e_i^n = Q_i^n(G/H_i \times \text{Int } D^n)$ . Here the  $H_i$  are closed subgroups of  $G$ , the  $q_i: G/H_i \times S^{n-1} \rightarrow X_{n-1}$  are continuous maps and the  $Q_i: G/H_i \times D^n \rightarrow X_n$  are continuous maps corresponding to the  $q_i$ .

$$\begin{array}{ccc}
\coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod q_i} & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod Q_i} & X_n
\end{array}$$

FIGURE 2. Attaching equivariant  $n$ -cells to the  $(n - 1)$ -skeleton of  $X$ ,  $n \geq 1$ .

The  $G$ -subspace  $X_n$  is called the  $n$ -skeleton of  $X$ . The  $e_i^n$  are called the *open equivariant  $n$ -cells* of  $X$ . The *(closed) equivariant  $n$ -cells* are the  $G$ -subspaces  $\bar{e}_i^n := Q_i(G/H_i \times D^n)$ .

Note that if  $G = 1$  is the trivial group, one recovers from the above definition the non-equivariant CW-complex in the sense of [Whi49].

If  $G$  is a discrete group, then one can express the above definition also in the following way: a CW-complex  $X$  with a  $G$ -action is a  $G$ -CW-complex if the action of  $G$  on  $X$  is cellular and the cell stabilisers are the point stabilisers [Lüc89, p. 8]. That is, the action of  $G$  on  $X$  permutes the cells and any  $g \in G$  which fixes a cell fixes this cell pointwise.

There are various finiteness properties for  $G$ -CW-complexes which are generalisations of the corresponding finiteness properties of CW-complexes. The following is a list of some common finiteness properties.

**Definition 2.2.** Let  $X$  be a  $G$ -CW-complex as in the definition before.

- (1) If there exists a integer  $n \geq -1$  such that  $X = X_n$  (with the convention that  $X_{-1} := \emptyset$ ), then the least such integer is called the dimension of  $X$  and we denote this fact by  $\dim X = n$ . If no such integer exists, then we say that  $X$  is an *infinite dimensional  $G$ -CW-complex* and we denote this fact by  $\dim X = \infty$ .
- (2) We say that  $X$  is of *finite type* if it has only finitely many equivariant cells in each dimension.
- (3) We say that  $X$  is *finite* if  $X$  consists of only finitely many equivariant cells.

Note that a  $G$ -CW-complex is finite if and only if it is of finite type and finite dimension. Moreover, a  $G$ -CW-complex  $X$  is finite if and only if the quotient space  $X/G$  is compact.

## 2. Classifying Spaces

In the literature there are several variations of the concept of a universal  $G$ -space (also known as a classifying space of  $G$ ) for the family  $\mathfrak{F}$ , see for example the survey article [Lüc05], which is also the source of the following definition.

**Definition 2.3.** Let  $G$  be a topological group and let  $\mathfrak{F}$  be a semi-full family of closed subgroups of  $G$ . A  $G$ -CW-complex  $X$  is a *classifying space of  $G$  for the family  $\mathfrak{F}$*  or a *model for  $E_{\mathfrak{F}}G$* , if it satisfies the following two conditions:

- (1)  $\mathfrak{F}(X) \subset \mathfrak{F}$ ;
- (2) if  $Y$  is a  $G$ -CW-complex with  $\mathfrak{F}(Y) \subset \mathfrak{F}$ , then there exists a  $G$ -map  $f: Y \rightarrow X$  which is unique up to  $G$ -homotopy.

In other words, a model for  $E_{\mathfrak{F}}G$  is a terminal object in the homotopy category of  $G$ -CW-complexes with isotropy groups in the family  $\mathfrak{F}$ . In particular, a model for  $E_{\mathfrak{F}}G$  is only unique up to  $G$ -homotopy and the  $G$ -homotopy class of a classifying space of  $G$  for the family  $\mathfrak{F}$  can be seen as an invariant of the group  $G$ .

For any given group  $G$  and semi-full family  $\mathfrak{F}$  of subgroups  $G$  there exists a classifying space of  $G$  for the family  $\mathfrak{F}$  [Lüc05, p. 275]. Furthermore it has been shown in [Lüc05, p. 275], that a  $G$ -CW-complex  $X$  with  $\mathfrak{F}(X) \subset \mathfrak{F}$  is a model for  $E_{\mathfrak{F}}G$  if and only if the fixed point set  $X^H$  is weakly contractible for every  $H \in \mathfrak{F}$ . A space  $X$  is called *weakly contractible* if the homotopy groups  $\pi_n(X, x)$  are trivial for all  $n \in \mathbb{N}$  and all  $x \in X$ .

A contractible space is always weakly contractible. However, in general a weakly contractible space does not need to be contractible. But if  $G$  is discrete and  $X$  is a model for  $E_{\mathfrak{F}}G$ , then for every  $H \in \mathfrak{F}$  the fixed point space  $X^H$  has the homotopy type of a CW-complex and is therefore contractible [Whi78, pp. 219ff.]. Thus we obtain the following known characterisation result, see for example [LM00, p. 295].

**Proposition 2.4.** *Let  $G$  be a discrete group and let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . A  $G$ -CW-complex  $X$  is a model for  $E_{\mathfrak{F}}G$  if and only if the following two conditions are satisfied:*

- (1)  $\mathfrak{F}(X) \subset \mathfrak{F}$ ;
- (2)  $X^H$  is contractible for every  $H \in \mathfrak{F}$ . □

For full families of subgroups the the above result has the following corollary, which is often used as the definition of a classifying space of discrete groups for full families of subgroups.

**Corollary 2.5.** *Let  $G$  be a discrete group and let  $\mathfrak{F}$  be a full family of subgroups of  $G$ . A  $G$ -CW-complex  $X$  is a model for  $E_{\mathfrak{F}}G$  if and only if the following two conditions are satisfied:*

- (1)  $X^H = \emptyset$  for every subgroup  $H$  of  $G$  which is not in  $\mathfrak{F}$ ;
- (2)  $X^H$  is contractible for every  $H \in \mathfrak{F}$ .

PROOF. We only need to show that for full families  $\mathfrak{F}$  the assumption (1) in Proposition 2.4 is equivalent to the assumption (1) in this corollary.

“ $\Rightarrow$ ”: Let  $H$  be a subgroup of  $G$  which is not in the family  $\mathfrak{F}$ . Assume towards a contradiction that  $X^H \neq \emptyset$  and let  $x \in X^H$ . Then  $H$  is a subgroup of  $G_x$ . Now  $G_x \in \mathfrak{F}(X) \subset \mathfrak{F}$  and since  $\mathfrak{F}$  is a full family of subgroups of  $G$  we get  $H \in \mathfrak{F}$  which is a contradiction! Therefore  $X^H = \emptyset$ .

“ $\Leftarrow$ ”: Let  $H \in \mathfrak{F}(X)$ . Then  $X^H \neq \emptyset$  and thus  $H \in \mathfrak{F}$ . □

**Examples 2.6.** (1) If  $\mathfrak{F} = \{1\}$  is the trivial family of subgroups then a contractible  $G$ -CW-complex  $X$  is a model for  $E_{\mathfrak{F}}G$  if the action of  $G$  on  $X$  is free. In particular the universal cover of an Eilenberg–Mac Lane space  $K(G, 1)$  is a model for  $E_{\mathfrak{F}}G$ . It is customary to abbreviate  $EG := E_{\mathfrak{F}}G$ .

(2) If  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$  a model for  $E_{\mathfrak{F}}G$  is also known as the universal space of proper actions of  $G$ . In literature the abbreviation  $\underline{EG} := E_{\mathfrak{F}}G$  is commonly used.

(3) In the case that  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$ , which is the family of subgroups on which focus of study of this thesis lies, the abbreviation  $\underline{\underline{EG}} := E_{\mathfrak{F}}G$  is used.



### 3. Free Resolutions Obtained from Classifying Spaces

If  $\mathfrak{F}$  is a semi-full family of subgroups of  $G$ , then a model for  $E_{\mathfrak{F}}G$  can be used to construct a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$ . The construction is as follows (see for example [Lüc89, pp. 151f.] or [MV03, pp. 10ff.]).

Let  $X$  be a  $G$ -CW-complex. Consider the *cellular chain complex*  $C(X) = (C_*(X), d_*)$ . This chain complex is defined by

$$C_n(X) := H_n^{\Delta}(X_n, X_{n-1})$$

with  $H_n^{\Delta}$  denoting the singular homology functor. The differentials

$$d_n: C_n(X) \rightarrow C_{n-1}(X) \tag{2.1}$$

of the cellular chain complex are the connecting homomorphisms of the triple  $(X_n, X_{n-1}, X_{n-2})$ , see for example [Geo08, pp. 40ff.].

Let  $\Delta_n$  be the set of all  $n$ -cells of the  $G$ -CW-complex  $X$ . Since  $G$  acts on  $X$  by permuting the cells of  $X$  the set  $\Delta_n$  is in a natural way a  $G$ -set. Note that  $\Delta_n^H$  is the set of all  $n$ -cells of the CW-complex  $X^H$  for any group subgroup  $H$  of  $G$ .

We define the right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{C}_n(X)$  to be

$$\underline{C}_n(X) := \mathbb{Z}[\Delta_n]_G.$$

For each  $n \geq 1$  we define homomorphisms

$$d_n: \underline{C}_n(X) \rightarrow \underline{C}_{n-1}(X)$$

of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules as follows. First note that for every  $H \in \mathfrak{F}$  and  $n \geq 1$  we have  $\underline{C}_n(X)(G/H) = C_n(X^H)$ . Let

$$d_{n,H}: C_n(X)(G/H) \rightarrow C_{n-1}(X)(G/H)$$

be the differential  $d_n: C_n(X^H) \rightarrow C_{n-1}(X^H)$  of the cellular chain complex  $C(X^H)$ . If  $\varphi \in [G/H, G/K]_G$  is a morphism of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$ , say  $\varphi = f_{g,H,K}$ , then  $\varphi^*: C_n(X)(G/K) \rightarrow C_n(X)(G/H)$  is for each  $n \in \mathbb{N}$  the homomorphism induced by the map  $X^K \rightarrow X^H$  which sends  $x$  to  $gx$ . Since this map defines a chain map  $C(X^K) \rightarrow C(X^H)$  this implies that the

diagram

$$\begin{array}{ccc}
C_n(X^K) & \xrightarrow{d_{n,K}} & C_{n-1}(X^K) \\
\downarrow \varphi^* & & \downarrow \varphi^* \\
C_n(X^H) & \xrightarrow{d_{n,H}} & C_{n-1}(X^H)
\end{array}$$

commutes for every  $n \geq 1$ . In particular this implies that the homomorphism  $d_{n,H}$  define a homomorphism  $d_n: \underline{C}_n(X) \rightarrow \underline{C}_{n-1}(X)$  of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules.

Furthermore, for every  $H \in \mathfrak{F}$  there exists an augmentation homomorphism  $\varepsilon_H: C_0(X^H) \rightarrow \mathbb{Z}$  which sends every 0-cell of the CW-complex  $X^H$  to 1. It follows that these homomorphism define an *augmentation homomorphism*

$$\varepsilon: \underline{C}_0(X) \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}}.$$

**Lemma 2.7.** *The sequence*

$$\dots \rightarrow \underline{C}_2(X) \xrightarrow{d_2} \underline{C}_1(X) \xrightarrow{d_1} \underline{C}_0(X) \xrightarrow{\varepsilon} \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0 \quad (2.2)$$

*is a chain complex of  $\mathcal{O}_{\mathfrak{F}}G$ -modules.*

PROOF. Let  $H \in \mathfrak{F}$ . Then the sequence (2.2) evaluated at  $G/H$  is the augmented cellular chain complex of the CW-complex  $X^H$  and the claim follows.  $\square$

**Lemma 2.8.** *Assume that  $X^H$  is contractible for every  $H \in \mathfrak{F}$ . Then the sequence (2.2) is exact.*

PROOF. If  $X^H$  is contractible then the augmented cellular chain complex of the CW-complex  $X^H$  is exact. Thus the claim follows by evaluating the sequence (2.2) at  $G/H$  for any  $H \in \mathfrak{F}$ .  $\square$

The results of this section yield the following conclusion.

**Proposition 2.9.** *Let  $\mathfrak{F}$  be a semi-full family of subgroups and let  $X$  be a model for  $E_{\mathfrak{F}}G$ . Then the sequence (2.2) of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules is a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ .*

PROOF. Since  $X$  is a model for  $E_{\mathfrak{F}}G$ , we have that  $\mathfrak{F}(X) \subset \mathfrak{F}$  and so the  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $\underline{C}_n(X)$  are free by Proposition 1.18. The fixed point sets  $X^H$  are contractible for any  $H \in \mathfrak{F}$  and therefore the sequence (2.2) is exact by Lemma 2.8.  $\square$

#### 4. Geometric Finiteness Conditions in Terms of Algebraic Properties

It follows from the construction of the previous section that the finiteness conditions of Definition 2.2 on a model  $X$  for  $E_{\mathfrak{F}}G$  imply the following statements:

- (1) if  $\dim X = n$  then there exists a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  of length  $n$  in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ ;
- (2) if  $X$  is of finite type then there exists a resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  by finitely generated free Bredon modules in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ ;
- (3) if  $X$  is finite then there exists a finite length resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  by finitely generated free Bredon modules in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ .

In [LM00] it has been shown that the above statements are nearly reversible. The relevant part of the main result in this article is the following

**Proposition 2.10.** [LM00, Theorem 0.1] *Let  $G$  be a discrete group, let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$  and let  $n \geq 3$ . Then we have:*

- (1) *there is a  $n$ -dimensional model for  $E_{\mathfrak{F}}G$  if and only if there exists a projective resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  of length  $n$  in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ .*
- (2) *there exists a finite type model for  $E_{\mathfrak{F}}G$  if and only if there exists a model for  $E_{\mathfrak{F}}G$  with finite equivariant 2-skeleton and the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  has a resolution by finitely generated projective Bredon modules in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ ;*
- (3) *there exists a finite model for  $E_{\mathfrak{F}}G$  if and only if there exists a model for  $E_{\mathfrak{F}}G$  with finite equivariant 2-skeleton and the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  has a resolution of finite length by finitely generated free Bredon modules in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$ . □*

In this thesis we focus on the question whether for a group  $G$  and full family  $\mathfrak{F}$  of subgroups of  $G$ , there exists a finite dimensional model for  $E_{\mathfrak{F}}G$ . This leads to the following definition.

**Definition 2.11.** Let  $G$  be a group and  $\mathfrak{F}$  semi-full family of subgroups. Assume that there exists a finite dimensional model for  $E_{\mathfrak{F}}G$ . Then the least

integer  $n \geq 0$  for which there exists an  $n$ -dimensional model for  $E_{\mathfrak{F}}G$  is called the *Bredon geometric dimension of  $G$  for the family  $\mathfrak{F}$*  and we denote this by  $\text{gd}_{\mathfrak{F}} G := n$ . If there exist no finite dimensional model for  $E_{\mathfrak{F}}G$ , then we set  $\text{gd}_{\mathfrak{F}} G := \infty$ .

Following the notation introduced at the end of Section 2 we abbreviate  $\text{gd}_{\mathfrak{F}} G$  by  $\text{gd } G$  if  $\mathfrak{F} = \{1\}$ , by  $\underline{\text{gd}} G$  if  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$  and by  $\underline{\underline{\text{gd}}} G$  if  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$ .

## Bredon (Co-)Homological Dimensions

### 1. Bredon (Co-)Homology

Since the category  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  has enough projectives we can define derived functors and do homological algebra [Wei94, pp. 30ff.]. We are interested in the derived functors of the morphism functor  $\text{mor}_{\mathfrak{F}}(?, ?)$  and the tensor product functor  $? \otimes_{\mathfrak{F}} ?$ . Therefore, for each right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ , we choose a projective resolution  $P_*(M)$  of  $M$ .

**Definition 3.1.** Let  $N$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then  $\text{Ext}_{\mathfrak{F}}^n(?, N)$  is the  $n$ -th right derived functor of  $\text{mor}_{\mathfrak{F}}(?, N)$ , that is

$$\text{Ext}_{\mathfrak{F}}^n(M, N) := H_n(\text{mor}_{\mathfrak{F}}(P_*(M), N)).$$

for any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  and all  $n \in \mathbb{N}$ .

Likewise, if  $N$  is a left  $\mathcal{O}_{\mathfrak{F}}G$ -module, then  $\text{Tor}_n^{\mathfrak{F}}(?, N)$  is the  $n$ -th left derived functor of  $? \otimes_{\mathfrak{F}} N$ , that is

$$\text{Tor}_n^{\mathfrak{F}}(M, N) := H_n(P_*(M) \otimes_{\mathfrak{F}} N)$$

for all right  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $M$  and all  $n \in \mathbb{N}$ .

It is a standard fact in homological algebra that this definition is – up to natural isomorphism – independent of the choice of the projective resolutions. Furthermore it is a standard fact that the  $\text{Ext}_{\mathfrak{F}}^*$  and  $\text{Tor}_{\mathfrak{F}}^*$  functors are also functorial in the second variable.

**Proposition 3.2.** *The following statements about a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  are equivalent:*

- (1)  $M$  is projective;
- (2)  $\text{mor}_{\mathfrak{F}}(M, ?)$  is an exact functor;
- (3)  $\text{Ext}_{\mathfrak{F}}^n(M, N) = 0$  for every right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and every  $n \geq 1$ ;
- (4)  $\text{Ext}_{\mathfrak{F}}^1(M, N) = 0$  for every right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ .

**Proposition 3.3.** *The following statements for a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  are equivalent:*

- (1)  $M$  is flat;
- (2)  $M \otimes_{\mathfrak{F}} ?$  is an exact functor;
- (3)  $\mathrm{Tor}_n^{\mathfrak{F}}(M, N) = 0$  for every left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and every  $n \geq 1$ ;
- (4)  $\mathrm{Tor}_1^{\mathfrak{F}}(M, N) = 0$  for every left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ .

PROOF OF PROPOSITION 3.2 AND 3.3. These are standard results in homological algebra, see for example [Wei94, p. 50] and [Wei94, p. 69].  $\square$

Let  $N$  be a left  $\mathcal{O}_{\mathfrak{F}}G$ -module. We say that a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  is  $? \otimes_{\mathfrak{F}} N$ -acyclic if the groups  $\mathrm{Tor}_n^{\mathfrak{F}}(M, N)$  vanish for every  $n \geq 1$ . Thus  $M$  is flat if and only if it is  $? \otimes_{\mathfrak{F}} N$ -acyclic for any left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ .

Note that from the theory of derived functors, it follows that we can relax the requirement on the resolution of  $M$  used to calculate the Tor groups. In fact any  $? \otimes_{\mathfrak{F}} N$ -acyclic resolution of  $M$  will be sufficient [Wei94, p. 47]. As this requirement is satisfied by flat  $\mathcal{O}_{\mathfrak{F}}G$ -modules this means we can calculate the Tor groups using flat resolutions.

**Definition 3.4.** Let  $G$  be a group,  $\mathfrak{F}$  a family of subgroups of  $G$  and let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then the *Bredon cohomology groups*  $H_{\mathfrak{F}}^n(G; M)$  of  $G$  with coefficients in  $M$  are defined as the Ext groups of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  with coefficients in  $M$ , that is

$$H_{\mathfrak{F}}^n(G; M) := \mathrm{Ext}_{\mathfrak{F}}^n(\mathbb{Z}_{\mathfrak{F}}, M).$$

Similarly, if  $N$  is a left  $\mathcal{O}_{\mathfrak{F}}G$ -module, then the *Bredon homology groups*  $H_n^{\mathfrak{F}}(G; N)$  of  $G$  with coefficients in  $N$  are defined to be the Tor groups of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  with coefficients in  $N$ , that is

$$H_n^{\mathfrak{F}}(G; N) := \mathrm{Tor}_n^{\mathfrak{F}}(\mathbb{Z}_{\mathfrak{F}}, N).$$

## 2. The Standard Resolution

Concrete examples of projective resolutions of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  are useful in order to calculate the Bredon (co-)homology groups of a group  $G$ . In Section 3 we have already seen how to obtain a free resolution of  $\mathbb{Z}_{\mathfrak{F}}$  from a model for  $E_{\mathfrak{F}}G$ . Another example of a very specific free resolution of  $\mathbb{Z}_{\mathfrak{F}}$  is the standard resolution.

Recall that in classical group (co-)homology there exists the standard resolution

$$\dots \rightarrow \mathbb{Z}[G \times G \times G] \rightarrow \mathbb{Z}[G \times G] \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0 \quad (3.1)$$

of the trivial  $G$ -module  $\mathbb{Z}$  by free  $G$ -modules, see for example [Bro82, pp. 18f.]. This resolution is also known as the bar resolution. From the view point of category theory, standard resolutions arise from simplicial objects constructed from comonads (also known as triples), see [ML98, pp. 180ff.] and [Wei94, pp. 278ff.].

Nucinkis has shown in [Nuc04, pp. 41f.] how the construction (3.1) generalises to the Bredon setting for the family  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$  of finite subgroups of  $G$ . That is, there exists a free resolution

$$\dots \rightarrow \mathbb{Z}[?, \Delta_2]_G \rightarrow \mathbb{Z}[?, \Delta_1]_G \rightarrow \mathbb{Z}[?, \Delta_0]_G \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  where  $\Delta_n$  is the  $G$ -set

$$\Delta_n := \{(g_0K_0, \dots, g_nK_n) : g_i \in G \text{ and } K_i \in \mathfrak{F}\}.$$

It turns out that we can construct a resolution of this form of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  for an arbitrary non-empty family  $\mathfrak{F}$  of subgroups of  $G$ . The details are as follows.

For  $n \geq 1$  and  $0 \leq i \leq n$  define  $G$ -maps  $\partial_i: \Delta_n \rightarrow \Delta_{n-1}$  by

$$\partial_i(g_0K_0, \dots, g_nK_n) := (g_0K_0, \dots, \widehat{g_iK_i}, \dots, g_nK_n)$$

where  $(g_0K_0, \dots, \widehat{g_iK_i}, \dots, g_nK_n)$  denotes the  $(n-1)$ -tuple obtained from the  $n$ -tuple  $(g_0K_0, \dots, g_nK_n)$  by deleting the  $i$ -th component. With these maps the collection  $\Delta_* := \{\Delta_n : n \in \mathbb{N}\}$  of  $G$ -sets becomes a semi-simplicial complex.

Let  $\Delta_{-1}$  be the singleton  $G$ -set  $\Delta_{-1} := \{*\}$ . We get an augmentation  $G$ -map  $\varepsilon: \Delta_0 \rightarrow \Delta_{-1}$  if we set  $\varepsilon(g_0K_0) := *$  for every  $g_0K_0 \in \Delta_0$ , that is  $\varepsilon \circ \partial_0 = \varepsilon \circ \partial_1$ .

Applying the functor  $\mathbb{Z}[?, ??]_G$  to the semi-simplicial  $G$ -set  $\Delta$  gives a semi-simplicial  $\mathcal{O}_{\mathfrak{F}}G$ -module

$$\mathbb{Z}[?, \Delta_*]_G := \{\mathbb{Z}[?, \Delta_n]_G : n \in \mathbb{N}\}$$

with augmentation  $\varepsilon: \mathbb{Z}[\?, \Delta_0]_G \rightarrow \mathbb{Z}[\?, \{*\}]_G = \underline{\mathbb{Z}}_{\mathfrak{F}}$ . The associated augmented chain complex  $C_*(\Delta_*)$  is given by

$$C_n(\Delta_*) := \begin{cases} \mathbb{Z}[\?, \Delta_*]_G & n \geq -1 \\ 0 & \text{otherwise} \end{cases}$$

with the differentials given by

$$d_n := \begin{cases} \sum_{i=0}^n (-1)^i \partial_i & n > 0 \\ \varepsilon & n = 0 \\ 0 & n < 0 \end{cases}$$

It follows that  $C_*(\Delta_*)$  is necessarily a complex of  $\mathcal{O}_{\mathfrak{F}}G$ -modules, that is  $d_{n-1} \circ d_n = 0$  for every  $n \in \mathbb{Z}$  [Wei94, pp. 259ff.].

**Proposition 3.5.** *The sequence*

$$\dots \rightarrow \mathbb{Z}[\?, \Delta_2]_G \xrightarrow{d_2} \mathbb{Z}[\?, \Delta_1]_G \xrightarrow{d_1} \mathbb{Z}[\?, \Delta_0]_G \xrightarrow{\varepsilon} \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0 \quad (3.2)$$

*is a resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ . If  $\mathfrak{F}$  is a semi-full family of subgroups then this resolution is free.*

PROOF. First observe that  $\mathbb{Z}[\?, \Delta_{-1}]_G = \underline{\mathbb{Z}}_{\mathfrak{F}}$ . Thus the sequence (3.2) is nothing else than the associated augmented chain complex  $C_*(\Delta_*)$ .

We need to show that the sequence (3.2) evaluated at any object  $G/H$  of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  is an exact sequence of abelian groups. We know already that

$$\dots \rightarrow \mathbb{Z}[G/H, \Delta_2]_G \xrightarrow{d_{2,H}} \mathbb{Z}[G/H, \Delta_1]_G \xrightarrow{d_{1,H}} \mathbb{Z}[G/H, \Delta_0]_G \xrightarrow{\varepsilon_H} \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

is a chain complex of abelian groups. Thus it remains to show that there exists a contracting homotopy  $h: \text{id} \simeq 0$ . But such a contracting homotopy is known to be given by

$$h_n(g_0K_0, \dots, g_nK_n) := (H, g_0K_0, \dots, g_nK_n)$$

for  $n \in \mathbb{N}$ ,  $h_{-1}(\ast) := (H)$  and  $h_n := 0$  for  $n > -1$ .

Given  $\sigma := (g_0K_0, \dots, g_nK_n) \in \Delta_n$  its stabiliser  $G_\sigma$  is

$$G_\sigma = K_0^{g_0^{-1}} \cap \dots \cap K_n^{g_n^{-1}}.$$

If  $\mathfrak{F}$  is a semi-full family of subgroups of  $G$ , then  $G_\sigma \in \mathfrak{F}$  for any  $\sigma \in \Delta_n$ . That is  $\mathfrak{F}(\Delta_n) \subset \mathfrak{F}$ . Hence the  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $\mathbb{Z}[\?, \Delta_n]_G$  are free by Proposition 1.18.  $\square$



**Definition 3.6.** We call the resolution (3.2) the *standard resolution* of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ .

**Lemma 3.7.** *Let  $G$  be a group and let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . If both  $G$  and  $\mathfrak{F}$  are countable then the standard resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  is countably generated.*

PROOF. If  $G$  and  $\mathfrak{F}$  are countable then  $\Delta_n$  is countable and thus  $\Delta_n/G$  is countable. Now the claim follows from Lemma 1.21.  $\square$

### 3. Bredon (Co-)Homological Dimensions

In Section 4 in the previous chapter we have introduced the Bredon geometric dimension of a group  $G$  with respect to a family  $\mathfrak{F}$  of subgroups of  $G$ . It has two closely related algebraic invariants, the Bredon cohomological and Bredon homological dimension. They are the obvious generalisations of the classical (co-)homological dimensions.

**Definition 3.8.** Let  $G$  be a group and let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Assume that there exists an integer  $n \in \mathbb{N}$  such that the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  has a projective resolution

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  of length  $n$  but not one of length  $n - 1$ . We say that  $G$  has *Bredon cohomological dimension  $n$  with respect to the family  $\mathfrak{F}$* , which we denote by  $\text{cd}_{\mathfrak{F}}G := n$ . If no finite length projective resolution exists, then we say that  $G$  has *infinite Bredon cohomological dimension with respect to  $\mathfrak{F}$* , which we denote by  $\text{cd}_{\mathfrak{F}}G := \infty$ .

We abbreviate  $\text{cd}_{\mathfrak{F}}G$  by  $\text{cd}G$ ,  $\underline{\text{cd}}G$  or  $\underline{\underline{\text{cd}}}G$  in the case that  $\mathfrak{F}$  is the trivial family, the family of finite or the family of virtually cyclic subgroups respectively.

The definition of the Bredon homological dimension follows the same idea, except that projective  $\mathcal{O}_{\mathfrak{F}}G$ -modules are replaced by flat  $\mathcal{O}_{\mathfrak{F}}G$ -modules:

**Definition 3.9.** Let  $G$  be a group and let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Assume that there exists an integer  $n \in \mathbb{N}$  such that the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  has a flat resolution

$$0 \rightarrow Q_n \rightarrow \dots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

in  $\text{Mod-}\mathcal{O}_{\mathfrak{F}}G$  of length  $n$  but not one of length  $n - 1$ . We say that  $G$  has *Bredon homological dimension  $n$  with respect to the family  $\mathfrak{F}$* , which we denote by  $\text{hd}_{\mathfrak{F}} G := n$ . If no finite length flat resolution exists then we say that  $G$  has *infinite Bredon homological dimension with respect to  $\mathfrak{F}$* , which we denote by  $\text{hd}_{\mathfrak{F}} G := \infty$ .

Analogous to before we abbreviate  $\text{hd}_{\mathfrak{F}} G$  by  $\text{hd } G$ ,  $\underline{\text{hd}} G$  or  $\underline{\underline{\text{hd}}} G$  in the case that  $\mathfrak{F}$  is the trivial family, the family of finite or the family of virtually cyclic subgroups respectively.

The Bredon (co-)homological dimension is a special case of the projective and flat dimension of a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ . These dimensions are defined in a similar spirit as the minimal length of a projective (or respectively flat) resolution of the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ . We denote the *projective dimension* of  $M$  by  $\text{pd}_{\mathfrak{F}} M$  and the *flat dimension* of  $M$  by  $\text{fld}_{\mathfrak{F}} M$ . With this notation, the (co-)homological dimension of a group  $G$  is the projective and flat dimension of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ , that is

$$\text{cd}_{\mathfrak{F}} G = \text{pd}_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}} \quad \text{and} \quad \text{hd}_{\mathfrak{F}} G = \text{fld}_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}}.$$

The following two propositions are standard results in homological algebra in abelian categories; their proof can be found in [Wei94, pp. 93ff.], for example.

**Proposition 3.10.** *Let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then the following statements are equivalent:*

- (1)  $\text{pd}_{\mathfrak{F}} M \leq n$ ;
- (2)  $\text{Ext}_{\mathfrak{F}}^d(M, N) = 0$  for every right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and every  $d > n$ ;
- (3)  $\text{Ext}_{\mathfrak{F}}^{n+1}(M, N) = 0$  for every right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ ;
- (4) given any projective resolution of  $M$ ,

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

the kernel of  $P_n \rightarrow P_{n-1}$  is projective. □

There are two immediate applications of this result. The first is that if we can show that  $\text{Ext}_{\mathfrak{F}}^d(M, N) \neq 0$  for some right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ , then  $\text{pd}_{\mathfrak{F}} M \geq d$ . The second application is that, given a projective resolution  $P_*$  of a right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  with  $\text{pd}_{\mathfrak{F}} M \leq n$ , we obtain a projective resolution

$$0 \rightarrow K \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$$

of length  $n$ , where  $K$  is the kernel of  $d_n: P_n \rightarrow P_{n-1}$  and  $K \rightarrow P_{n-1}$  is the restriction of  $d_n$  to  $K$ . That is, any projective resolution of  $M$  can be truncated by inserting a suitable projective kernel as soon as the length of the resolution exceeds the projective dimension of  $M$ .

**Proposition 3.11.** *Let  $M$  be a right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then the following statements are equivalent:*

- (1)  $\text{fd}_{\mathfrak{F}} M \leq n$ ;
- (2)  $\text{Tor}_d^{\mathfrak{F}}(M, N) = 0$  for every left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$  and every  $d > n$ ;
- (3)  $\text{Tor}_{n+1}^{\mathfrak{F}}(M, N) = 0$  for every left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $N$ ;
- (4) given any flat resolution of  $M$ ,

$$\dots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0,$$

the kernel of  $Q_n \rightarrow Q_{n-1}$  is flat. □

Of course this proposition has two analogous immediate applications, just as the previous proposition had. Firstly, a non-trivial  $\text{Tor}_d^{\mathfrak{F}}(M, N)$  gives rise to a lower bound for the flat dimension of  $M$ . Secondly, any flat resolution of  $M$  can be truncated by inserting a suitable flat kernel as soon as the length of the resolution exceeds the flat dimension of  $M$ .

#### 4. Cohomological vs. Homological vs. Geometric Dimension

In this section, we will compare the three Bredon dimensions we have introduced in the previous section for a fixed family  $\mathfrak{F}$  of subgroups of  $G$ . The first result is just a direct consequence of the fact that projective Bredon modules are flat.

**Lemma 3.12.** *For any family  $\mathfrak{F}$  of subgroups of a group  $G$  we have*

$$\text{hd}_{\mathfrak{F}} G \leq \text{cd}_{\mathfrak{F}} G. \quad \square$$

The next result has been proven by Nucinkis in [Nuc04, p. 42] for the family of finite subgroups of  $G$ . The proof also works without modification for more general families of subgroups.

**Theorem 3.13.** *Let  $G$  be a countable group and let  $\mathfrak{F}$  be a full family of subgroups of  $G$ . If  $\mathfrak{F}$  is countable then*

$$\text{cd}_{\mathfrak{F}} G \leq \text{hd}_{\mathfrak{F}} G + 1.$$

To prove this theorem, we need a result by Nucinkis that gives an upper bound on the cohomological dimension of countably presented flat modules.

**Proposition 3.14.** [Nuc04, Proposition 3.5] *Let  $\mathfrak{F}$  be a full family of subgroups. Then every countably presented flat right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  has  $\text{pd}_{\mathfrak{F}} M \leq 1$ .  $\square$*

PROOF OF THEOREM 3.13. In order to avoid triviality, we assume that  $\text{hd}_{\mathfrak{F}} G$  is finite. Consider the standard resolution

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  as defined in section 2. Since  $G$  and  $\mathfrak{F}$  are countable, this resolution is countably generated by Lemma 3.7.

Let  $K$  be the  $n$ -th kernel of the above resolution, which is flat. It follows that  $K$  is countably presented. Since the family  $\mathfrak{F}$  of subgroups is assumed to be full we can apply Proposition 3.14, which tells us that  $\text{pd}_{\mathfrak{F}} K \leq 1$ . From this it follows that there exists a projective resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  of length  $n + 1$  and hence  $\text{cd}_{\mathfrak{F}} G \leq n + 1$ .  $\square$

**Remark 3.15.** Let  $G$  be a countable group. Then  $G$  has only countably many finitely generated subgroups.

Hence in the case  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$ , Theorem 3.13 and Lemma 3.12 combine to recover the statement of Theorem 4.1 in [Nuc04]. Moreover, since virtual cyclic groups are finitely generated we get that Theorem 4.1 in [Nuc04] also holds for the family of virtual cyclic subgroups of  $G$ , that is to say we have the following result.

**Theorem 3.16.** *Let  $G$  be a countable group. Then*

$$\underline{\text{cd}} G \leq \underline{\text{hd}} G + 1. \quad \square$$

Next we compare the Bredon cohomological dimension to the Bredon geometric dimension of a group in the case that  $\mathfrak{F}$  is a full family of subgroups of  $G$ . In Section 3 we have seen that a model for  $E_{\mathfrak{F}}G$  gives rise to a projective resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ . If the model of  $E_{\mathfrak{F}}G$  has finite dimension  $n$ , then the projective resolution of  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  has length  $n$ . Thus we have the following result.

**Lemma 3.17.** *For any semi-full family  $\mathfrak{F}$  of subgroups of  $G$  we have*

$$\mathrm{cd}_{\mathfrak{F}} G \leq \mathrm{gd}_{\mathfrak{F}} G. \quad \square$$

As a consequence of the first part of Proposition 2.10 we get the following statement about the geometric and cohomological Bredon dimension of a group  $G$ .

**Proposition 3.18.** *Let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . If  $\mathrm{cd}_{\mathfrak{F}} G \geq 3$  or  $\mathrm{gd}_{\mathfrak{F}} G \geq 4$  then  $\mathrm{cd}_{\mathfrak{F}} G = \mathrm{gd}_{\mathfrak{F}} G$ .*

PROOF. If  $\mathrm{cd}_{\mathfrak{F}} \geq 3$  then there exists a model  $E_{\mathfrak{F}}G$  of dimension  $\mathrm{cd}_{\mathfrak{F}} G$  by Proposition 2.10, that is  $\mathrm{gd}_{\mathfrak{F}} G \leq \mathrm{cd}_{\mathfrak{F}} G$  and thus equality holds by the previous Lemma. If  $\mathrm{gd}_{\mathfrak{F}} G \geq 4$  then there exists no projective resolution of length  $\mathrm{gd}_{\mathfrak{F}} G - 1$  by Proposition 2.10 and therefore  $\mathrm{cd}_{\mathfrak{F}} G \geq \mathrm{gd}_{\mathfrak{F}} G$ . Again equality holds by the previous lemma.  $\square$

In [EG57] it has been shown that  $\mathrm{cd} G = \mathrm{gd} G$  whenever  $\mathrm{cd} G \geq 3$ . In the same paper it has been asked whether  $\mathrm{cd} G = \mathrm{gd} G$  in general. The statement that this is true is known as the Eilenberg–Ganea Conjecture. Groups with  $\mathrm{cd} G \leq 1$  cannot give counter examples to this conjecture since  $\mathrm{cd} G = 0$  implies that  $G$  is trivial and  $\mathrm{cd} G = 1$  implies that  $G$  is free by a famous work of Stallings [Sta68] and Swan [Swa69]. The trivial group has geometric dimension 0 and since free groups can act freely on a tree it follows that  $\mathrm{gd} G = 1$  for free groups. Therefore a possible counter example to the Eilenberg–Ganea Conjecture needs to be a torsion free group  $G$  with  $\mathrm{cd} G = 2$  and  $\mathrm{gd} G = 3$ . Until the present day, neither such counter example has been found nor has the conjecture been proven.

The conjecture generalises to the Bredon setting as follows.

**Eilenberg–Ganea Conjecture** (for Bredon Cohomology). *Let  $\mathfrak{F}$  be a semi-full of subgroups of a group  $G$ . Then  $\mathrm{cd}_{\mathfrak{F}} G = \mathrm{gd}_{\mathfrak{F}} G$ .*

Let  $G$  be a group and consider  $\mathfrak{F} = \mathfrak{F}_{\mathrm{fin}}(G)$ . We see in the next section that the implication  $\mathrm{gd} G = 0 \Rightarrow \underline{\mathrm{cd}} G = 0$  is actually an equivalence. Therefore  $\mathrm{gd} G = 1$  implies  $\underline{\mathrm{cd}} G = 1$ . On the other hand it is known that  $\underline{\mathrm{cd}} G = 1$  implies that the rational cohomological dimension  $\mathrm{cd}_{\mathbb{Q}} G = 1$ , see for example [BLN01, p. 493], which in turn implies that  $\mathrm{gd} G = 1$  by a result by Dunwoody [Dun79]. Altogether, this and Proposition 3.18 imply

that  $\underline{\text{cd}} G = \underline{\text{gd}} G$  for all groups  $G$  with  $\underline{\text{cd}} G \neq 2$ . Thus a possible counter example for the above conjecture for the family of finite subgroups must satisfy  $\underline{\text{cd}} G = 2$  and  $\underline{\text{gd}} G = 3$ . Brady, Leary and Nucinkis have shown in [BLN01] that there exist certain right-angled Coxeter groups which have precisely this property. That is, the Eilenberg–Ganea Conjecture is false for the family of finite subgroups. This also implies that the statement of Proposition 3.18 is the best possible if one does not impose any further restriction on the family  $\mathfrak{F}$ .

A natural question is to ask whether the Eilenberg–Ganea Conjecture is true for the family  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$  of virtually cyclic subgroups. It is unknown, whether  $\underline{\text{cd}} G = 1$  is equivalent to  $\underline{\text{gd}} G = 1$  (we will show in the end of this thesis that this is true for countable, torsion-free soluble groups, see Theorem 6.6). Therefore possible counter examples to the conjecture must fall – similar to [EG57] – into one of the following three cases.

- (1)  $\underline{\text{cd}} G = 1$  and  $\underline{\text{gd}} G = 2$ ;
- (2)  $\underline{\text{cd}} G = 1$  and  $\underline{\text{gd}} G = 3$ ;
- (3)  $\underline{\text{cd}} G = 2$  and  $\underline{\text{gd}} G = 3$ ;

There is not much known about groups  $G$  with  $\underline{\text{gd}} G = 2$  or  $\underline{\text{gd}} G = 3$ . Juan-Pineda and Leary have shown in [JPL06] that for Gromov-hyperbolic groups  $\underline{\text{gd}} G \leq 2$  implies that  $\underline{\text{gd}} G = 2$  provided that  $G$  is not virtually cyclic. Lück and Weiermann have shown that  $\text{vcd} G = 2$  implies  $\underline{\text{gd}} G = 3$  for virtually polycyclic groups. In the next chapter we will show that the Eilenberg–Ganea Conjecture for the family  $\mathfrak{F}_{\text{vc}}$  holds for these groups. Moreover, we will show in Chapter 5 that the soluble Baumslag–Solitar groups  $BS(1, m)$  satisfy the Eilenberg–Ganea Conjecture for the family  $\mathfrak{F}_{\text{vc}}$ .

## 5. Groups of Bredon Dimension Zero

**Proposition 3.19.** *Let  $G$  be a group and  $\mathfrak{F}$  a semi-full family of subgroups of  $G$ . Then  $\text{gd}_{\mathfrak{F}} G = 0$  if and only if  $G \in \mathfrak{F}$ .*

PROOF. Again the proof is a standard argument. If  $G \in \mathfrak{F}$  then any singleton space  $\{*\}$  is a model for  $E_{\mathfrak{F}} G$  and thus  $\text{gd}_{\mathfrak{F}} G = 0$ . On the other hand, if  $\text{gd}_{\mathfrak{F}} G = 0$ , then  $G$  has a singleton space  $\{*\}$  as a model for  $E_{\mathfrak{F}} G$

as this is the only 0-dimensional contractible  $G$ -CW-complex which exists. Clearly  $\{*\}^G \neq \emptyset$  and thus  $G \in \mathfrak{F}$ .  $\square$

**Proposition 3.20.** *Let  $G$  be a group and  $\mathfrak{F}$  a family of subgroups of  $G$ . If  $G \in \mathfrak{F}$ , then the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  is free and in particular  $\text{cd}_{\mathfrak{F}} G = 0$ . If the family  $\mathfrak{F}$  is semi-full, then  $\text{cd}_{\mathfrak{F}} G = 0$  implies that  $G \in \mathfrak{F}$ .*

In order to prove this statement we need a result from Symonds. For a family  $\mathfrak{F}$  of subgroups of  $G$ , he defines a *component* of  $\mathfrak{F}$  to be an equivalence class under the equivalence relation generated by inclusion [Sym05, p. 265]. Note that if  $\mathfrak{F}$  is a semi-full family of subgroups of  $G$ , then  $\mathfrak{F}$  has only one component. This is because for any  $H_1, H_2 \in \mathfrak{F}$ , we have that  $H_1 \cap H_2$  is contained in  $\mathfrak{F}$  and is a common subgroup of  $H_1$  and  $H_2$ .

**Lemma 3.21.** [Sym05, Lemma 2.5] *Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Then the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  is projective if and only if each component of  $\mathfrak{F}$  has a unique maximal element  $M$  and this  $M$  is equal to its normaliser  $N_G(M)$  in  $G$ .*  $\square$

PROOF OF PROPOSITION 3.20. Assume first that  $G \in \mathfrak{F}$ , then a standard argument shows that  $\text{cd}_{\mathfrak{F}} G = 0$  as follows. Since  $[G/H, G/G]_G$  contains only one map (namely the trivial map) we have that  $\mathbb{Z}[G/H, G/G]_G = \mathbb{Z}$  for every  $H \in \mathfrak{F}$ . Furthermore, every morphism of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  is mapped to the identity map. Thus it follows that the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  is equal to the free right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}[?, G/G]_G$ . Hence  $\text{cd}_{\mathfrak{F}} G = 0$ .

Next, assume that the family  $\mathfrak{F}$  is semi-full and that  $\text{cd}_{\mathfrak{F}} G = 0$ . Let  $H_1, H_2 \in \mathfrak{F}$  be two arbitrary subgroups. Since  $\mathfrak{F}$  is closed under finite intersections it has only one component. Then by Lemma 3.21 the family  $\mathfrak{F}$  has a unique maximal element  $M$  with  $M = N_G(M)$ . Assume towards a contradiction that  $N_G(M)$  (and therefore  $M$ ) is a proper subgroup of  $G$ . Then there exists a  $g \in G \setminus M$  such that  $M^g \neq M$ . Since  $\mathfrak{F}$  is closed under conjugation, it follows that  $M^g \in \mathfrak{F}$ . Now let  $N \in \mathfrak{F}$  with  $M^g \leq N$ . Then  $M \leq N^{g^{-1}}$  and so  $M = N^{g^{-1}}$  by the maximality of  $M$ . Therefore  $M^g = N$  and since  $N$  was an arbitrary element of  $\mathfrak{F}$  with  $M^g \geq N$  it follows that  $M^g$  is maximal in  $\mathfrak{F}$ . Thus by the uniqueness of a maximal element in  $\mathfrak{F}$  we have that  $M^g = M$ , which is a contradiction. Therefore  $M = G$  and so  $G \in \mathfrak{F}$ .  $\square$

For completeness, we include the statement about groups  $G$  with  $\text{hd}_{\mathfrak{F}} G$  equal to zero. This result is just the statement of Theorem 3.13 in the case  $\text{hd}_{\mathfrak{F}} G = 0$ .

**Proposition 3.22.** *Let  $G$  be a group and let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . If both  $G$  and  $\mathfrak{F}$  are countable then  $\text{hd}_{\mathfrak{F}} G = 0$  implies  $\text{cd}_{\mathfrak{F}} G \leq 1$ .  $\square$*

Note that the estimation  $\text{cd}_{\mathfrak{F}} G \leq 1$  is sharp. For example,  $\mathbb{Q}$  is the direct union of its cyclic subgroups. It follows by two results which we will prove later in Section 11, namely Theorem 3.42 and Corollary 3.44, that  $\underline{\text{hd}} \mathbb{Q} = 0$  and  $\underline{\text{cd}} \mathbb{Q} \leq 1$ . On the other hand  $\mathbb{Q}$  is not virtually cyclic and so  $\underline{\text{cd}} \mathbb{Q} \neq 0$  by Proposition 3.20. Therefore  $\mathbb{Q}$  is group with  $\underline{\text{hd}} \mathbb{Q} = 0$  and  $\underline{\text{cd}} \mathbb{Q} = 1$ .

## 6. Induction with $I_K$ and Preservation of Exactness

In Section 4 we have compared different types of Bredon dimensions for a given group and with respect to a given family of subgroups. In order to make comparisons of the same kind of Bredon dimensions, but for different groups, or if we are interested how the the choice of the family of subgroups affects a certain Bredon dimension, then the restriction and induction functors are an important tool. The exactness of these functors is an important property and we already know that the restriction functor is always exact. The result we present in this section is due to Symonds [Sym05].

In this section we consider induction with the following functor. Let  $K$  be a fixed subgroup of  $G$ . If  $\emptyset \neq \mathfrak{F} \cap K \subset \mathfrak{F}$  then we obtain a well defined functor

$$I_K: \mathcal{O}_{\mathfrak{F} \cap K} K \rightarrow \mathcal{O}_{\mathfrak{F}} G$$

of orbit categories as follows: on objects of  $\mathcal{O}_{\mathfrak{F} \cap K} K$  we set  $I_K(K/H) := G/H$  and if  $f: K/H \rightarrow K/L$  is a  $G$ -map, then  $I_K(f)$  is the obvious extension of  $f$  to a  $G$ -map  $G/H \rightarrow G/L$ , which by abuse of notation we will also denote by  $f$ .

**Lemma 3.23.** *Let  $H \in \mathfrak{F}$  and  $L \in \mathfrak{F} \cap K$ . Let  $R$  be a complete set of representatives for the left cosets in  $(G/K)^H$ . Then there exists a bijection*

$$\eta_L: [G/H, G/L]_G \rightarrow \coprod_{x \in R} [K/H^x, K/L]_K \quad (3.3)$$



of sets.

PROOF. If  $x \in R$  then  $H^x \leq K$  and thus the right hand side of (3.3) is well defined. Let  $f_{g,H,L} \in [G/H, G/L]_G$ . Then  $H^g \leq L$  and since  $L \in \mathfrak{F} \cap K$  we have that  $H^g \leq K$ . This implies that  $gK \in (G/K)^H$  and thus there exists a unique  $x \in R$  and  $y \in K$  such that  $g = xy$ .

Assume that  $f_{g',H,L} = f_{g,H,L}$ . Then  $g' = gl$  for some  $l \in L \leq K$ . Furthermore there exists a unique  $x' \in R$  and  $y' \in K$  such that  $g' = x'y'$ . Thus  $gl = x'y'$  and from this follows that  $x^{-1}x' = yl(y')^{-1} \in K$ . This means that  $x$  and  $x'$  are in the same left coset of  $K$  in  $G$ . Therefore  $x = x'$  and  $yl = y'$ . That is  $x$  is uniquely determined by  $f_{g,H,L}$  and  $y$  is uniquely determined by  $f_{g,H,L}$  up to right multiplication by an element of  $L$ .

Since  $(H^x)^y = H^{xy} = H^g \leq L$  we get that  $f_{y,H^x,L} \in [K/H^x, K/L]_K$  and we define  $\eta_L(f_{g,H,L}) := f_{y,H^x,L}$ . Since  $x$  is uniquely determined by the map  $f_{g,H,L}$ , and since  $y$  is uniquely determined by the map  $f_{g,H,L}$  up to right multiplication by an element of  $L$ , this definition of  $\eta_L(f_{g,H,L})$  is well defined and ensures that  $\eta_L$  is an injective map.

It remains to show that the map  $\eta_L$  is surjective. Therefore choose an arbitrary  $x \in R$  and let  $f_{y,H^x,L} \in [K/H^x, K/L]_K$ . Then

$$H^{xy} = (H^x)^y \leq L$$

and thus  $f_{xy,H,L} \in [G/H, G/L]_G$  and  $\eta_L(f_{xy,H,L}) = f_{y,H^x,L}$ . Therefore, we conclude that  $\eta$  is surjective.  $\square$

**Lemma 3.24.** *Let  $H \in \mathfrak{F}$ . Then the left  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module*

$$\mathbb{Z}[K/H^x, ?]_K$$

*is free for every  $x \in G$  for which  $xK \in (G/K)^H$ .*

PROOF. We only need to show that  $H^x \in \mathfrak{F} \cap K$ . Since  $\mathfrak{F}$  is closed under conjugation we have  $H^x \in \mathfrak{F}$ . Since  $xK \in (G/H)^H$  we have  $H^x \leq K$  and thus  $H^x = H^x \cap K \in \mathfrak{F} \cap K$ .  $\square$

**Lemma 3.25.** *Assume that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then the collection of isomorphisms  $\{\eta_L : L \in \mathfrak{F} \cap K\}$  defined in Lemma 3.23 give an isomorphism*

$$\eta: \mathbb{Z}[G/H, I_K(?)]_G \rightarrow \prod_{x \in R} \mathbb{Z}[K/H^x, ?]_K \quad (3.4)$$

of left  $\mathcal{O}_{\mathfrak{F} \cap K} G$ -modules. In particular,  $\mathbb{Z}[G/H, I_K(?)]_G$  is a free left  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -module.

PROOF. The assumption on  $\mathfrak{F} \cap K$  ensures that both the functor  $I_K$  and the category of left  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules are defined. We have to show that the isomorphisms  $\eta_L$  form a natural transformation of covariant functors. We do this by chasing generators around the necessary diagrams.

Let  $L_1, L_2 \in \mathfrak{F} \cap K$ ,  $\varphi \in [K/L_1, K/L_2]_K$  and  $f \in [G/H, I_K(K/L_1)]_G = [G/H, G/L_1]_G$ . Then there exists a  $z \in K$  such that  $\varphi = f_{z, L_1, L_2}$  and a  $g \in G$  such that  $f = f_{g, H, L_1}$ . Furthermore there exists a unique  $x \in R$  such that we can write  $f_{g, H, L_1} = f_{xy, H, L_1}$  for some  $y \in K$ . This  $y$  is unique up to right multiplication by an element of  $L_1$ . We need to chase  $f$  around the following diagram.

$$\begin{array}{ccc} [G/H, G/L_1]_G & \xrightarrow{\eta_{L_1}} & [K/H^x, K/L_1]_K \\ \varphi_* \downarrow & & \downarrow \varphi_* \\ [G/H, G/L_2]_G & \xrightarrow{\eta_{L_2}} & [K/H^x, K/L_2]_K \end{array}$$

On the one hand we have

$$\begin{aligned} (\varphi_* \circ \eta_{L_1})(f_{g, H, L_1}) &= \varphi_*(f_{y, H^x, L_1}) \\ &= f_{z, L_1, L_2} \circ f_{y, H^x, L_1} \\ &= f_{yz, H^x, L_2} \in \mathbb{Z}[K/H^x, K/L_2]_K. \end{aligned}$$

On the other hand we have

$$\begin{aligned} (\eta_{L_2} \circ \varphi_*)(f_{g, H, L_1}) &= \eta_{L_2}(f_{z, L_1, L_2} \circ f_{xy, H, L_1}) \\ &= \eta_{L_2}(f_{xyz, H, L_2}) \\ &= f_{yz, H^x, L_2} \in \mathbb{Z}[K/H^x, K/L_2]_K \end{aligned}$$

where the last equality follows from  $yz \in K$ . Therefore the maps  $\{\eta_L : L \in \mathfrak{F} \cap K\}$  define a homomorphism of left  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules. Since each  $\eta_L$  is an isomorphism it follows that  $\eta$  is an isomorphism, too.

The Bredon modules in the coproduct on the right hand side of (3.4) are all free by Lemma 3.24. We can conclude that the right hand side of (3.4) is free and the claim follows.  $\square$

Note that in Symonds' article [Sym05, p. 226] the above result together with Lemma 1.25 is used as the *definition* of the induction functor  $\text{ind}_{I_K} : \text{Mod-}\mathcal{O}_{\mathfrak{F} \cap K} K \rightarrow \text{Mod-}\mathcal{O}_{\mathfrak{F}} G$ .

**Proposition 3.26.** *Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Let  $K \leq G$  be some subgroup for which  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then induction with  $I_K: \mathcal{O}_{\mathfrak{F} \cap K} K \rightarrow \mathcal{O}_{\mathfrak{F}} G$  is an exact functor.*

PROOF. Let  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  be an exact sequence of right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules. Applying the functor  $\text{ind}_{I_K}$  to this sequence yields the sequence

$$0 \rightarrow \text{ind}_{I_K} L \rightarrow \text{ind}_{I_K} M \rightarrow \text{ind}_{I_K} N \rightarrow 0 \quad (3.5)$$

of  $\mathcal{O}_{\mathfrak{F}} G$ -modules. We evaluate this sequence at  $H \in \mathfrak{F}$  and obtain

$$\begin{aligned} 0 \rightarrow L(?) \otimes_{\mathfrak{F} \cap K} \mathbb{Z}[G/H, I_K(?)]_G \rightarrow M(?) \otimes_{\mathfrak{F} \cap K} \mathbb{Z}[G/H, I_K(?)]_G \\ \rightarrow N(?) \otimes_{\mathfrak{F} \cap K} \mathbb{Z}[G/H, I_K(?)]_G \rightarrow 0 \end{aligned} \quad (3.6)$$

By the previous lemma the left  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -module  $\mathbb{Z}[G/H, I_K(?)]_G$  is free and hence flat. Thus the sequence (3.6) is exact. Since this holds for every  $H \in \mathfrak{F}$  we have that the sequence (3.5) of  $\mathcal{O}_{\mathfrak{F}} G$ -modules is exact.  $\square$

## 7. Restriction with $I_K$ and Preservation of Projectives

Let  $G$  be a group and  $\mathfrak{F}$  a family of subgroups of  $G$ . Let  $K$  be a subgroup of  $G$  such that  $\emptyset \neq \mathfrak{F} \cap K \subset \mathfrak{F}$ . Lemma 3.25 states that restriction with  $I_K$  preserves free *left* Bredon modules. It turns out that restriction with  $I_K$  preserves free *right* Bredon modules, too. However, we get a different answer to how the restricted free right Bredon modules look like. The following statements together with their proofs are due to Martínez-Pérez [MP02, p. 167].

**Lemma 3.27.** *Let  $\mathfrak{F}$  be a family of subgroups of  $G$  and let  $K$  be a subgroup of  $G$  such that  $\mathfrak{F} \cap K \neq \emptyset$ . Then for any  $H \in \mathfrak{F}$  and any complete set  $R$  of representatives for the double cosets  $K \backslash G / H$  we have an isomorphism*

$$\eta: \mathbb{Z}[I_K(?), G/H]_G \rightarrow \coprod_{x \in R} \mathbb{Z}[?, K/(K \cap H^{x^{-1}})]_K$$

*of right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules. In particular if  $\mathfrak{F} \cap K \subset \mathfrak{F}$  then  $\mathbb{Z}[I_K(?), G/H]_G$  is a free right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -module.*

PROOF. Since  $\mathfrak{F} \cap K \neq \emptyset$  the category of right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules is defined and the claim of the lemma makes sense.

Let  $L \in \mathfrak{F} \cap K$  and let  $f_{g,L,H} \in [G/L, K/H]_G$ . Then there exists a unique  $x \in R$  such that we can write  $g = yxh$  for some  $y \in K$  and  $h \in H$ . Since  $g$  is uniquely determined by  $f$ , up to right multiplication by an element of  $H$ , it follows that  $x$  is uniquely determined by  $f$ . Assume that we have  $y_1 \in K$  and  $h_1 \in H$  such that  $yxh = y_1xh_1$ . Then  $y_1^{-1}y = xh_1h^{-1}x^{-1} \in H^{x^{-1}}$  and since  $y_1^{-1}y \in K$  we get even that  $y$  and  $y_1$  lie in the same left coset of  $K \cap H^{x^{-1}}$  in  $K$ . Furthermore from  $L^{yxh} = L^g \leq H$  follows that  $L^y \leq H^{h^{-1}x^{-1}} = H^{x^{-1}}$ . Since  $L \leq K$  and  $y \in K$  we get  $L^y \leq K^y = K$  and thus altogether that  $L^y \leq K \cap H^{x^{-1}}$ . Hence  $f_{y,L,K \cap H^{x^{-1}}} \in [K/L, K/(K \cap H^{x^{-1}})]_K$  and we obtain a well defined map

$$\eta_L: \mathbb{Z}[G/L, G/H]_G \rightarrow \prod_{x \in R} \mathbb{Z}[K/L, K/(K \cap H^{x^{-1}})]_K$$

by  $\eta_L(f_{yxh,L,G}) := f_{y,L,K \cap H^{x^{-1}}} \in [K/L, K/(K \cap H^{x^{-1}})]_K$ .

It follows that the above defined map is bijective. By chasing generators around the usual diagrams it follows that the collection  $\{\eta_L : L \in \mathfrak{F} \cap L\}$  of maps defines an isomorphism  $\eta$  of right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules as required in the statement of the lemma.

Let  $H \in \mathfrak{F}$ . Since  $\mathfrak{F}$  is closed under conjugation  $H^{x^{-1}} \in \mathfrak{F}$  and thus  $K \cap H^{x^{-1}} \in \mathfrak{F} \cap H^{x^{-1}}$ . Therefore all the summands of the codomain of  $\eta$  are free right  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -modules and therefore the domain of  $\eta$  must be a free  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -module, too.  $\square$

**Proposition 3.28.** [MP02, Lemma 3.7] *Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Let  $K \leq G$  be some subgroup of  $G$  such that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then restriction with  $I_K$  preserves free Bredon modules and consequently also projective Bredon modules.*

PROOF. The first statement is Lemma 3.27 (for right Bredon modules) and 3.25 (for left Bredon modules). The statement about projective right Bredon modules follows from the first statement since restriction is an additive functor.  $\square$

## 8. Shapiro's Lemma in the Bredon Setting

**Proposition 3.29.** *Let  $\mathfrak{F}$  be a family of subgroups of  $G$  and let  $K$  be a subgroup of  $G$  such that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then*

- (1) *For any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  and any right  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $N$  we have*

$$\mathrm{Ext}_{\mathfrak{F} \cap K}^*(\mathrm{res}_{I_K} M, N) \cong \mathrm{Ext}_{\mathfrak{F}}^*(M, \mathrm{coind}_{I_K} N);$$

- (2) *For any right  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  and any left  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $N$  we have*

$$\mathrm{Tor}_*^{\mathfrak{F} \cap K}(\mathrm{res}_{I_K} M, N) \cong \mathrm{Tor}_*^{\mathfrak{F}}(M, \mathrm{ind}_{I_K} N).$$

*In both cases the isomorphism is natural in both  $M$  and  $N$ .*

PROOF. From a projective resolution

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

of the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$  we obtain a sequence

$$\dots \rightarrow \mathrm{res}_{I_K} P_2 \rightarrow \mathrm{res}_{I_K} P_1 \rightarrow \mathrm{res}_{I_K} P_0 \rightarrow \mathrm{res}_{I_K} M \rightarrow 0$$

of  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -modules. This sequence is exact (since restriction preserves exactness) and each  $\mathrm{res}_{I_K} P_i$  is projective by Proposition 3.28.

Now consider the first case, that is  $M$  is a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $N$  is a right  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module. Then

$$\begin{aligned} \mathrm{Ext}_{\mathfrak{F} \cap K}^n(M, N) &= H_n(\mathrm{mor}_{\mathfrak{F} \cap K}(\mathrm{res}_{I_K} P_*, N)) \cong \\ &H_n(\mathrm{mor}_{\mathfrak{F}}(P_*, \mathrm{coind}_{I_K} N)) = \mathrm{Ext}_{\mathfrak{F}}^n(M, \mathrm{coind}_{I_K} N) \end{aligned}$$

for any  $n \in \mathbb{N}$ . Here the middle isomorphism is due to the fact that the restriction functor  $\mathrm{res}_{I_K}$  is left adjoint to coinduction functor  $\mathrm{coind}_{I_K}$ . In particular the isomorphism is natural in  $M$  and  $N$ . This proves the first isomorphism of the statement of the proposition.

Consider the second part of the statement. That is,  $M$  is a right  $\mathcal{O}_{\mathfrak{F}}G$ -module and  $N$  is a left  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module. Then we have for any  $n \in \mathbb{N}$  natural

isomorphisms

$$\begin{aligned} \operatorname{res}_{I_K} P_n(??) \otimes_{\mathfrak{F} \cap K} N(??) &\cong (P_n(?) \otimes_{\mathfrak{F}} \mathbb{Z}[I_K(??), ?]_G) \otimes_{\mathfrak{F} \cap K} N(??) \\ &\cong P_n(?) \otimes_{\mathfrak{F}} (\mathbb{Z}[I_K(??), ?]_G \otimes_{\mathfrak{F} \cap K} N(??)) \\ &\cong P_n(?) \otimes_{\mathfrak{F}} \operatorname{ind}_{I_K} N(?), \end{aligned}$$

where the middle isomorphism is due to the fact that the categorical tensor product is associative [MP02, p. 163]. Using this we obtain

$$\begin{aligned} \operatorname{Tor}_n^{\mathfrak{F} \cap K}(M, N) &= H_n(\operatorname{res}_{I_K} P_* \otimes_{\mathfrak{F} \cap K} N) \cong \\ &H_n(P_* \otimes_{\mathfrak{F}} \operatorname{ind}_{I_K} N) = \operatorname{Tor}_n^{\mathfrak{F}}(M, \operatorname{ind}_{I_K} N) \end{aligned}$$

which again is natural in both  $M$  and  $N$ .  $\square$

**Corollary 3.30.** *Let  $\mathfrak{F}$  be a family of subgroups of  $G$ . Furthermore, let  $K$  be a subgroup of  $G$  such that  $\emptyset \neq \mathfrak{F} \cap K \subset \mathfrak{F}$ . Then restriction with  $I_K$  preserves flat right Bredon modules.*

PROOF. Let  $M$  be a flat right  $\mathcal{O}_{\mathfrak{F}}G$ -module. Then

$$\operatorname{Tor}_1^{\mathfrak{F} \cap K}(\operatorname{res}_{I_K} M, N) \cong \operatorname{Tor}_1^{\mathfrak{F}}(M, \operatorname{ind}_{I_K} N) = 0$$

for any left  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $N$ . Therefore  $\operatorname{res}_{I_K} M$  is flat.  $\square$

**Proposition 3.31** (Shapiro's Lemma). *Let  $\mathfrak{F}$  be a family of subgroups of  $G$  and let  $K$  be a subgroup of  $G$  such that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then for any right  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $M$  and any left  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $N$ , there exist isomorphisms*

$$H_{\mathfrak{F} \cap K}^*(K; M) \cong H_{\mathfrak{F}}^*(G; \operatorname{coind}_{I_K} M)$$

and

$$H_*^{\mathfrak{F} \cap K}(K; N) \cong H_*^{\mathfrak{F}}(G; \operatorname{ind}_{I_K} N)$$

which are natural in  $M$  and  $N$ .

PROOF. Note that  $\operatorname{res}_{I_K} \mathbb{Z}_{\mathfrak{F}} \cong \mathbb{Z}_{\mathfrak{F} \cap K}$ . Then Proposition 3.29 says that we have isomorphisms, natural in  $M$  and  $N$ , such that

$$H_{\mathfrak{F} \cap K}^*(K; M) = \operatorname{Ext}_{\mathfrak{F} \cap K}^*(\mathbb{Z}_{\mathfrak{F} \cap K}, M) \cong \operatorname{Ext}_{\mathfrak{F}}^*(\mathbb{Z}_{\mathfrak{F}}, \operatorname{coind}_{I_K} M) = H_{\mathfrak{F}}^*(G; M)$$

and likewise

$$H_*^{\mathfrak{F} \cap K}(K; N) = \operatorname{Tor}_*^{\mathfrak{F} \cap K}(\mathbb{Z}_{\mathfrak{F} \cap K}, N) \cong \operatorname{Tor}_*^{\mathfrak{F}}(\mathbb{Z}_{\mathfrak{F}}, \operatorname{ind}_{I_K} N) = H_*^{\mathfrak{F}}(G; N). \quad \square$$

## 9. Bredon Dimensions for Subgroups

The following results about the dimension of subgroups are generalisations of the corresponding results in classical (co-)homology of groups. The proofs for the classical statements work without structural modifications.

**Proposition 3.32.** *Let  $G$  be a group and  $\mathfrak{F}$  a family of subgroups of  $G$ . Then for any subgroup  $K$  of  $G$  such that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$  we have inequalities*

$$\text{cd}_{\mathfrak{F} \cap K} K \leq \text{cd}_{\mathfrak{F}} G \quad \text{and} \quad \text{hd}_{\mathfrak{F} \cap K} K \leq \text{hd}_{\mathfrak{F}} G.$$

PROOF. Since restriction is exact in general and since restriction with  $I_K$  preserves projectives (Proposition 3.28) we get from a projective resolution

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_{\mathfrak{F}} \rightarrow 0,$$

where  $n = \text{cd}_{\mathfrak{F}} G$ , a projective resolution

$$0 \rightarrow \text{res}_{I_K} P_n \rightarrow \dots \rightarrow \text{res}_{I_K} P_1 \rightarrow \text{res}_{I_K} P_0 \rightarrow \text{res}_{I_K} \mathbb{Z}_{\mathfrak{F}} \rightarrow 0.$$

Now the first claim follows from  $\text{res}_{I_K} \mathbb{Z}_{\mathfrak{F}} = \mathbb{Z}_{\mathfrak{F} \cap K}$ .

Similarly, since restriction with  $I_K$  preserves flats (Corollary 3.30), one obtains from a flat resolution

$$0 \rightarrow Q_m \rightarrow \dots \rightarrow Q_1 \rightarrow Q_0 \rightarrow \mathbb{Z}_{\mathfrak{F}} \rightarrow 0$$

with  $m = \text{hd}_{\mathfrak{F}} G$  a flat resolution of the trivial  $\mathcal{O}_{\mathfrak{F} \cap K} K$ -module of length  $m$ . Therefore the second statement is true, too.  $\square$

Note that if  $\mathfrak{F}$  is a family of subgroups of  $G$  and if we are given a chain of subgroups  $H \leq K \leq G$  such that  $\emptyset \neq \mathfrak{F} \cap H \subset \mathfrak{F} \cap K \subset \mathfrak{F}$ , then we get the inequalities

$$\text{cd}_{\mathfrak{F} \cap H} H \leq \text{cd}_{\mathfrak{F} \cap K} K \leq \text{cd}_{\mathfrak{F}} G$$

and

$$\text{hd}_{\mathfrak{F} \cap H} H \leq \text{hd}_{\mathfrak{F} \cap K} K \leq \text{hd}_{\mathfrak{F}} G$$

as in the case of classical group (co-)homology. In particular, if  $\mathfrak{F}$  is a full family of subgroups of  $G$  then we get the above inequalities for any chain of subgroups  $H \leq K \leq G$ .

For the geometric Bredon dimension we have the analogous result to Proposition 3.32.

**Proposition 3.33.** *Let  $G$  be a group and let  $\mathfrak{F}$  be a full family of subgroups of  $G$ . Then for any subgroup  $K$  of  $G$  we have that*

$$\mathrm{gd}_{\mathfrak{F} \cap K} K \leq \mathrm{gd}_{\mathfrak{F}} G.$$

PROOF. The proof is standard. First of all observe that  $\mathfrak{F} \cap K$  is a full family of subgroups of  $K$ , since  $\mathfrak{F}$  is a full family of subgroups of  $G$ . Thus the geometric dimension  $\mathrm{gd}_{\mathfrak{F} \cap K} K$  is defined. In order to avoid triviality, assume that  $n := \mathrm{gd}_{\mathfrak{F}} G$  is finite. Then there exists an  $n$ -dimensional model  $X$  for  $E_{\mathfrak{F}} G$  which is also a  $K$ -space when restricting the action of  $G$  to  $K$ . Since  $\mathfrak{F}$  is closed under taking subgroups, it follows that  $\mathfrak{F} \cap K \subset \mathfrak{F}$  and as a consequence  $X$  is an  $n$ -dimensional model for  $E_{\mathfrak{F} \cap K} K$ . Thus  $\mathrm{gd}_{\mathfrak{F} \cap K} K \leq n$  and the proposition follows.  $\square$

As before, if  $\mathfrak{F}$  satisfies the conditions of Proposition 3.33 then so does  $\mathfrak{F} \cap K$  and we get for any subgroup  $H$  of  $K$  the sequence of inequalities

$$\mathrm{gd}_{\mathfrak{F} \cap H} H \leq \mathrm{gd}_{\mathfrak{F} \cap K} K \leq \mathrm{gd}_{\mathfrak{F}} G$$

as in the case of classical group (co-)homology.

## 10. (Co-)Homological Dimension when Passing to Larger Families of Subgroups

In this section we consider the following setup: Let  $(\mathfrak{G}, \mathfrak{F})$  be a pair of families of subgroups of  $G$  and we denote by  $I$  the inclusion functor  $I: \mathcal{O}_{\mathfrak{F}} G \hookrightarrow \mathcal{O}_{\mathfrak{G}} G$  of the corresponding orbit categories.

**Proposition 3.34.** (1) *If there exists a  $k \in \mathbb{N}$  such that  $\mathrm{pd}_{\mathfrak{F}}(\mathrm{res}_I P) \leq k$  for every projective  $\mathcal{O}_{\mathfrak{G}} G$ -module  $P$  then*

$$\mathrm{cd}_{\mathfrak{F}} G \leq \mathrm{cd}_{\mathfrak{G}} G + k.$$

(2) *If there exists a  $k \in \mathbb{N}$  such that  $\mathrm{fld}_{\mathfrak{F}}(\mathrm{res}_I Q) \leq k$  for every flat  $\mathcal{O}_{\mathfrak{G}} G$ -module  $Q$  then*

$$\mathrm{hd}_{\mathfrak{F}} G \leq \mathrm{hd}_{\mathfrak{G}} G + k.$$

To prove this proposition we need the following standard result from homological algebra.

**Lemma 3.35.** *Assume that we have a resolution*

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_0 \rightarrow M \rightarrow 0$$



of the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $M$ . If there exists a  $k \in \mathbb{N}$  such that  $\mathrm{pd}_{\mathfrak{F}} X_i \leq k$  for all  $0 \leq i \leq n$ , then  $\mathrm{pd}_{\mathfrak{F}} M \leq n + k$ . Similarly, if there exists a  $k \in \mathbb{N}$  such that  $\mathrm{fld}_{\mathfrak{F}} X_i \leq k$  for all  $0 \leq i \leq n$ , then  $\mathrm{fld}_{\mathfrak{F}} M \leq n + k$ .

PROOF. We prove the first claim by induction on  $n$ . If  $n = 0$  then  $X_0 \cong M$  and  $\mathrm{pd}_{\mathfrak{F}} M = \mathrm{pd}_{\mathfrak{F}} X_0 \leq k$  and we are done.

Thus assume that  $n' \geq 1$  and that the statement of the lemma is true for  $n \leq n' - 1$ . Then we have a short exact sequence

$$0 \rightarrow X_{n'} \rightarrow X_{n'-1} \rightarrow \mathrm{im} d_{n'} \rightarrow 0$$

with  $\mathrm{pd}_{\mathfrak{F}} X_{n'} \leq k$  and  $\mathrm{pd}_{\mathfrak{F}} X_{n'-1} \leq k$ . Then a standard argument in homological algebra, see for example [Wei94, p. 95], implies that  $\mathrm{pd}_{\mathfrak{F}}(\mathrm{im} d_{n'}) \leq k + 1$ . Since  $\mathrm{im} d_{n'} = \ker d_{n'-1}$  we get a resolution

$$0 \rightarrow \ker d_{n'-1} \rightarrow X_{n'-2} \rightarrow \dots \rightarrow X_0 \rightarrow M \rightarrow 0$$

of  $M$  by  $\mathcal{O}_{\mathfrak{F}}G$ -modules of projective dimension at most  $k + 1$ . This resolution has length  $n' - 1$  and we can apply the induction hypothesis. It follows that  $\mathrm{pd}_{\mathfrak{F}} M \leq (n' - 1) + (k + 1) = n' + k$ . Therefore the statement of the lemma is also true in the case  $n = n'$ .

Finally, the remaining statement of the lemma for the flat dimension of  $M$  is verified in exactly the same way.  $\square$

PROOF OF PROPOSITION 3.34. We prove the cohomological statement first: In order to avoid triviality assume that  $n := \mathrm{cd}_{\mathfrak{G}} G$  is finite. Then there exists a projective resolution

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z}_{\mathfrak{G}} \rightarrow 0$$

of length  $n$  of the trivial  $\mathcal{O}_{\mathfrak{G}}G$ -module  $\mathbb{Z}_{\mathfrak{G}}$ . Applying the restriction functor  $\mathrm{res}_I$  to this sequence yields a sequence

$$0 \rightarrow \mathrm{res}_I P_n \rightarrow \dots \rightarrow \mathrm{res}_I P_0 \rightarrow \mathbb{Z}_{\mathfrak{F}} \rightarrow 0$$

of  $\mathcal{O}_{\mathfrak{F}}G$ -modules that is exact, since restriction preserves exactness in general. By assumption  $\mathrm{pd}_{\mathfrak{F}}(\mathrm{res}_I P_i) \leq k$  for all  $0 \leq i \leq n$  and so the first claim of the proposition follows now from Lemma 3.35.

Now the homological statement is proven in exactly the same way and this concludes the proof.  $\square$

The following result gives an upper bound for the dimensions  $\text{pd}_{\mathfrak{F}}(\text{res}_I P)$ , which appear in the first part of Proposition 3.34, in terms of Bredon cohomological dimensions of the subgroups in  $\mathfrak{G}$ .

**Proposition 3.36.** *Assume that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$  for every  $K \in \mathfrak{G}$ . Then the following two statements are true:*

(1) *If there exists a  $k \in \mathbb{N}$  such that  $\text{cd}_{\mathfrak{F} \cap K} K \leq k$  for every  $K \in \mathfrak{G}$  then*

$$\text{pd}_{\mathfrak{F}}(\text{res}_I P) \leq k$$

*for every projective  $\mathcal{O}_{\mathfrak{G}}G$ -module  $P$ .*

(2) *Assume further that  $\mathfrak{F}$  is a full family of subgroups of  $G$ . If there exists a  $k \in \mathbb{N}$  such that  $\text{hd}_{\mathfrak{F} \cap K} K \leq k$  for every  $K \in \mathfrak{G}$  then*

$$\text{fld}_{\mathfrak{F}}(\text{res}_I P) \leq k$$

*for every projective  $\mathcal{O}_{\mathfrak{G}}G$ -module  $P$ .*

PROOF. Since restriction is an additive functor it is enough to carry out the proof for projective  $\mathcal{O}_{\mathfrak{G}}G$ -modules of the form  $P = \mathbb{Z}[?, G/K]_G$  for  $K \in \mathfrak{G}$ .

We prove the cohomological statement first. By assumption  $\text{cd}_{\mathfrak{F} \cap K} K \leq k$  and therefore there exists a projective resolution

$$0 \rightarrow P_k \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z}_{\mathfrak{F} \cap K} \rightarrow 0$$

of the trivial  $\mathcal{O}_{\mathfrak{F} \cap K}K$ -module  $\mathbb{Z}_{\mathfrak{F} \cap K}$ . Since  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ , the inclusion functor  $I_K: \mathcal{O}_{\mathfrak{F} \cap K}K \rightarrow \mathfrak{F}$  is defined. We have by Proposition 3.26 that induction with  $I_K$  is exact. Therefore we get an exact sequence

$$0 \rightarrow \text{ind}_{I_K} P_k \rightarrow \dots \rightarrow \text{ind}_{I_K} P_0 \rightarrow \text{ind}_{I_K} \mathbb{Z}_{\mathfrak{F} \cap K} \rightarrow 0.$$

Induction preserves projectives and thus we have obtained a projective resolution of the  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\text{ind}_{I_K} \mathbb{Z}_{\mathfrak{F} \cap K}$  of length  $k$ . By Lemma 2.7 in [Sym05, p. 268] we have that  $\mathbb{Z}[?, G/K]_G \cong \text{ind}_{I_K} \mathbb{Z}_{\mathfrak{F} \cap K}$  as  $\mathcal{O}_{\mathfrak{F}}G$ -modules. On the other hand we have the equality  $\mathbb{Z}[?, G/K]_G = \text{res}_I P$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules and therefore  $\text{pd}_{\mathfrak{F}}(\text{res}_I P) \leq k$ .

Next consider the assumptions of the homological statement of the proposition. Since  $\text{hd}_{\mathfrak{F} \cap K} K \leq k$  we have a flat resolution

$$0 \rightarrow Q_k \rightarrow \dots \rightarrow Q_0 \rightarrow \mathbb{Z}_{\mathfrak{F} \cap K} \rightarrow 0$$

of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$ . As before we can apply the induction functor  $I_K$  to obtain an exact sequence

$$0 \rightarrow \operatorname{ind}_{I_K} Q_k \rightarrow \dots \rightarrow \operatorname{ind}_{I_K} Q_0 \rightarrow \mathbb{Z}[?, G/K]_G \rightarrow 0.$$

Since  $\mathfrak{F}$  is assumed to be a full family of subgroups of  $G$  it follows that  $\mathfrak{F} \cap K$  is a full family of subgroups of  $K$ . It follows from Proposition 1.34 that induction preserves flats and hence the above resolution is a flat resolution of  $\mathbb{Z}[?, G/K]_G$  of length  $k$ . Now the claim follows from the fact  $\mathbb{Z}[?, G/K]_G = \operatorname{res}_I(P)$ .  $\square$

The next two theorems are the algebraic counterparts to Proposition 5.1 (i) in [LW07].

**Theorem 3.37.** *Let  $(\mathfrak{G}, \mathfrak{F})$  be a pair of families of subgroups of  $G$ . Assume that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$  for every  $K \in \mathfrak{G}$ . If there exists a  $k \in \mathbb{N}$  such that  $\operatorname{cd}_{\mathfrak{F} \cap K} K \leq k$  for every  $K \in \mathfrak{G}$ , then*

$$\operatorname{cd}_{\mathfrak{F}} G \leq \operatorname{cd}_{\mathfrak{G}} G + k.$$

PROOF. This is an immediate consequence of Proposition 3.34 and Proposition 3.36.  $\square$

**Theorem 3.38.** *Let  $(\mathfrak{G}, \mathfrak{F})$  be a pair of families of subgroups of  $G$ . Assume that  $\mathfrak{F}$  is a full family. If there exists a  $k \in \mathbb{N}$  such that  $\operatorname{hd}_{\mathfrak{F} \cap K} K \leq k$  for every  $K \in \mathfrak{G}$ , then*

$$\operatorname{hd}_{\mathfrak{F}} G \leq \operatorname{hd}_{\mathfrak{G}} G + k.$$

PROOF. Let  $Q$  be a flat  $\mathcal{O}_{\mathfrak{G}}G$ -module. By Proposition 1.29 we have that  $Q$  is the filtered colimit of finitely generated free  $\mathcal{O}_{\mathfrak{G}}G$ -modules  $Q_\lambda$ . Since  $\operatorname{Mod}\text{-}\mathcal{O}_{\mathfrak{G}}G$  is an AB5-category with enough projectives, and  $?\otimes_{\mathfrak{F}}B$  is a left adjoint for any left  $\mathcal{O}_{\mathfrak{F}}G$ -module, we have by Corollary 2.6.16 in [Wei94, p. 58] that

$$\operatorname{Tor}_{k+1}^{\mathfrak{F}}(\varinjlim \operatorname{res}_I Q_\lambda, B) \cong \varinjlim \operatorname{Tor}_{k+1}^{\mathfrak{F}}(\operatorname{res}_I Q_\lambda, B) \quad (*)$$

for any left  $\mathcal{O}_{\mathfrak{F}}G$ -module  $B$ . By Proposition 3.36, part (2), it follows that  $\operatorname{fd}_{\mathfrak{F}}(\operatorname{res}_I Q_\lambda) \leq k$ . Therefore  $\operatorname{Tor}_{k+1}^{\mathfrak{F}}(\operatorname{res}_I Q_\lambda, B) = 0$  for every  $\lambda$  and thus the right hand side of (\*) is equal to 0, too. Hence we get that

$$\operatorname{Tor}_{k+1}^{\mathfrak{F}}(\operatorname{res}_I Q, B) \cong \operatorname{Tor}_{k+1}^{\mathfrak{F}}(\varinjlim \operatorname{res}_I Q_\lambda, B) = 0$$

for any left  $\mathcal{O}_{\mathfrak{F}}G$  module  $B$  since the restriction functor commutes with colimits. But this implies that  $\text{fld}_{\mathfrak{F}}(\text{res}_I Q) \leq k$ . Since  $Q$  was an arbitrary flat  $\mathcal{O}_{\mathfrak{G}}G$ -module, we can apply Proposition 3.34 to get  $\text{hd}_{\mathfrak{F}} G \leq \text{hd}_{\mathfrak{G}} G + k$ .  $\square$

The following example is the algebraic equivalent to the first part of Corollary 5.4 in [LW07, p. 18].

**Example 3.39.** Consider the pair  $(\mathfrak{F}_{\text{vc}}(G), \mathfrak{F}_{\text{fin}}(G))$  of subgroups of  $G$ . The family  $\mathfrak{F}_{\text{fin}}(G)$  is closed under conjugation. For any  $K \in \mathfrak{F}_{\text{vc}}(G)$  we have that  $\mathfrak{F}_{\text{fin}}(G) \cap K$  is a non-empty subset of  $\mathfrak{F}_{\text{fin}}(G)$  and it is known that  $\underline{\text{hd}} K \leq 1$  and  $\underline{\text{cd}} K \leq 1$ , see for example [LW07, p. 22]. Hence we have by Theorem 3.38 and Theorem 3.37 the two inequalities

$$\underline{\text{hd}} G \leq \underline{\text{hd}} G + 1 \quad \text{and} \quad \underline{\text{cd}} G \leq \underline{\text{cd}} G + 1.$$

## 11. (Co-)Homological Dimension for Direct Unions

This section is motivated by Theorem 4.2 in [Nuc04, pp. 42f.]. In this theorem upper bounds for the Bredon (co-)homological dimension for direct unions of groups are given for the family of finite subgroups. The proof given by Nucinkis extends to a more general setting.

A direct union of groups is a special case of direct limits of groups, a constructive description for the latter can for example be found in [Rob96, pp. 22ff.].

**Definition 3.40.** Let  $\{G_\lambda : \lambda \in \Lambda\}$  be a family of subgroups of a group  $G$ , indexed by an abstract indexing set  $\Lambda$ . We say that  $G$  is the *direct union of the groups  $G_\lambda$*  if the following two conditions hold:

- (1) for every  $\lambda, \mu \in \Lambda$  there exists a  $\nu \in \Lambda$  such that  $G_\lambda \leq G_\nu$  and  $G_\mu \leq G_\nu$ ;
- (2) for every  $g \in G$  there exists a  $\lambda \in \Lambda$  such that  $g \in G_\lambda$ .

Note that we can recover  $G$  from this definition as a direct limit of the subgroups  $G_\lambda$  in the sense of [Rob96, pp. 22ff.] as follows. We define a relation “ $\leq$ ” on  $\Lambda$  by

$$\lambda \leq \mu :\iff G_\lambda \leq G_\mu.$$

In this way  $\Lambda$  becomes a directed set. Whenever  $G_\lambda \leq G_\mu$  there exists an inclusion map  $\varphi_\lambda^\mu: G_\lambda \rightarrow G_\mu$ . It is clear that

$$\{G_\lambda, \varphi_\lambda^\mu : \lambda, \mu \in \Lambda \text{ and } \lambda \leq \mu\}$$

forms a directed system of groups and that the obvious inclusion homomorphisms  $\iota_\lambda: G_\lambda \hookrightarrow G$  give an isomorphism

$$\iota: \varinjlim G_\lambda \rightarrow G.$$

We use this isomorphism to identify the direct limit  $\varinjlim G_\lambda$  with  $G$ .

**Definition 3.41.** Let  $G$  be the direct union of a family  $\{G_\lambda\}$  of its subgroups indexed by the set  $\Lambda$ . Assume that we are given a family  $\mathfrak{F}$  of subgroups of  $G$  and for each  $\lambda \in \Lambda$  a family  $\mathfrak{F}_\lambda$  of subgroups of  $G_\lambda$ . We say that these families of subgroups are *compatible with the direct union* if the following four conditions are satisfied:

- (1)  $\mathfrak{F}_\lambda \subset \mathfrak{F}_\mu$  for every  $\lambda, \mu \in \Lambda$  with  $\lambda \leq \mu$ ,
- (2)  $\mathfrak{F}_\lambda \subset \mathfrak{F}$  for every  $\lambda \in \Lambda$ ,
- (3)  $\mathfrak{F} \subset \bigcup_{\lambda \in \Lambda} \mathfrak{F}_\lambda$ ,
- (4)  $\mathfrak{F}_\lambda = \mathfrak{F} \cap G_\lambda$  for all  $\lambda \in \Lambda$ .

The main result of this section will be a generalised version of Theorem 4.2 in [Nuc04]:

**Theorem 3.42.** *Assume that a group  $G$  is the direct union of a family  $\{G_\lambda : \lambda \in \Lambda\}$  of its subgroups. Assume that we are given full families  $\mathfrak{F}$  and  $\mathfrak{F}_\lambda$ ,  $\lambda \in \Lambda$ , which are all non-empty and that are compatible with the direct union. Then*

- (1)  $\text{hd}_{\mathfrak{F}} G \leq \sup\{\text{hd}_{\mathfrak{F}_\lambda} G_\lambda\}$  in general, and
- (2)  $\text{cd}_{\mathfrak{F}} G \leq \sup\{\text{cd}_{\mathfrak{F}_\lambda} G_\lambda\} + 1$  if the index set  $\Lambda$  is countable.

Note that due to Proposition 3.32 we always have the inequalities

$$\sup\{\text{hd}_{\mathfrak{F}_\lambda} G_\lambda\} \leq \text{hd}_{\mathfrak{F}} G \quad \text{and} \quad \sup\{\text{cd}_{\mathfrak{F}_\lambda} G_\lambda\} \leq \text{cd}_{\mathfrak{F}} G.$$

In particular the inequality for the homological dimension in Theorem 3.42 is always an equality.

Before we prove the above theorem let us first give a criterion for the families  $\mathfrak{F}$  and  $\mathfrak{F}_\lambda$ ,  $\lambda \in \Lambda$ , to be compatible with the direct union.

**Proposition 3.43.** *Let  $G$  be the direct union of a family  $\{G_\lambda : \lambda \in \Lambda\}$  of its subgroups. Let  $\mathfrak{F}$  be a family of subgroups of  $G$  and  $\mathfrak{F}_\lambda$  families of subgroups of  $G_\lambda$ . Assume that  $\mathfrak{F}$  is closed under forming subgroups, every  $K \in \mathfrak{F}$  is finitely generated and  $\mathfrak{F}_\lambda = \mathfrak{F} \cap G_\lambda$  for every  $\lambda \in \Lambda$ . Then the families of subgroups  $\mathfrak{F}$  and  $\mathfrak{F}_\lambda$ ,  $\lambda \in \Lambda$ , are compatible with the direct union.*

PROOF. Condition (4) of Definition 3.41 is satisfied by assumption. Since  $\mathfrak{F}$  is closed under forming subgroups condition (2) is satisfied. If  $\lambda \leq \mu$  then  $G_\lambda \leq G_\mu$  and it follows that  $\mathfrak{F}_\lambda \subset \mathfrak{F}_\mu$ . Hence condition (1) is satisfied. Now let  $K \in \mathfrak{F}$  be an arbitrary group. Then there exists a finite set of generators  $\{k_1, \dots, k_n\}$  of  $K$ . For each  $1 \leq i \leq n$  there exists a  $\lambda_i \in \Lambda$  such that  $k_i \in G_{\lambda_i}$ . So there exists a  $\lambda \in \Lambda$  such that  $\lambda_i \leq \lambda$  for every  $1 \leq i \leq n$ . It follows that  $\{k_1, \dots, k_n\} \subset G_\lambda$  and therefore  $K \subset G_\lambda$ . Hence  $K \in \mathfrak{F} \cap G_\lambda = \mathfrak{F}_\lambda$  and condition (3) is satisfied.  $\square$

**Corollary 3.44.** *The families  $\mathfrak{F}_{\text{fin}}(G)$  and  $\mathfrak{F}_{\text{fin}}(G_\lambda)$  are compatible with the direct union. Likewise the families  $\mathfrak{F}_{\text{vc}}(G)$  and  $\mathfrak{F}_{\text{vc}}(G_\lambda)$  are compatible with the direct union.*  $\square$

Note that from the first case of the Corollary 3.44 and together with Theorem 3.42 we recover Theorem 4.2 in [Nuc04].

Before proving Theorem 3.42 we require some auxiliary results.

First of all, if  $\emptyset \neq \mathfrak{F} \cap K \subset \mathfrak{F}$  then we can extend the functor  $I_K: \mathcal{O}_{\mathfrak{F} \cap K} K \rightarrow \mathcal{O}_{\mathfrak{F}} G$  (see Section 6) to  $K$ -sets  $X$  with  $\mathfrak{F}(X) \subset \mathfrak{F} \cap K$  by applying it to each orbit separately. That is, if  $X = \coprod_{x \in R} K/K_x$  where  $R$  is a complete system of representatives for the  $K$ -orbits of  $X$ , then

$$I_K(X) := \coprod_{x \in R} G/K_x.$$

The set  $I_K(X)$  is then a  $G$ -set with  $\mathfrak{F}(I_K(X)) = \mathfrak{F}(X) \subset \mathfrak{F} \cap K \subset \mathfrak{F}$ .

**Lemma 3.45.** *Let  $K \leq G$  be a subgroup such that  $\mathfrak{F} \cap K$  is a non-empty subset of  $\mathfrak{F}$ . Then*

$$\text{ind}_{I_K} \mathbb{Z}[?, X]_G \cong \mathbb{Z}[?, I_K(X)]_G$$

for any  $K$ -set  $X$  with  $\mathfrak{F}(X) \subset \mathfrak{F} \cap K$ .

PROOF. Let  $R$  be a complete set of representatives of the orbit space  $X/K$ . Then we have the following sequence of isomorphisms of  $\mathcal{O}_{\mathfrak{F}}G$ -modules.

$$\begin{aligned}
\operatorname{ind}_{I_K} \mathbb{Z}[?, X]_G &\cong \prod_{x \in R} \operatorname{ind}_{I_K} \mathbb{Z}[?, K/K_x]_K \\
&\cong \prod_{x \in R} (\mathbb{Z}[??, K/K_x]_K \otimes_{\mathfrak{F} \cap K} \mathbb{Z}[?, I_K(??)]_G) \\
&\cong \prod_{x \in R} \mathbb{Z}[?, I_K(K/K_x)]_G \\
&\cong \prod_{x \in R} \mathbb{Z}[?, G/K_x]_G \\
&\cong \mathbb{Z}[?, I_K(X)]_G \quad \square
\end{aligned}$$

**Lemma 3.46.** *Consider the direct limiting system*

$$\{\mathbb{Z}[?, G/G_\lambda]_G, \varphi_\lambda^\mu : \lambda, \mu \in \Lambda \text{ and } \lambda \leq \mu\}$$

directed by  $\Lambda$  where the morphisms  $\varphi_\lambda^\mu: \mathbb{Z}[?, G/G_\lambda]_G \rightarrow \mathbb{Z}[?, G/G_\mu]_G$  are induced by the projections  $G/G_\lambda \rightarrow G/G_\mu$ . Then

$$\varinjlim \mathbb{Z}[?, G/G_\lambda]_G \cong \mathbb{Z}_{\mathfrak{F}}.$$

PROOF. For each  $\lambda \in \Lambda$  we have a homomorphism

$$\eta_\lambda: \mathbb{Z}[?, G/G_\lambda]_G \rightarrow \mathbb{Z}[?, G/G]_G = \mathbb{Z}_{\mathfrak{F}}$$

induced by the projection  $p_\lambda G/G_\lambda \rightarrow G/G$ . Clearly  $\eta_\mu = \varphi_\lambda^\mu \circ \eta_\lambda$  for all  $\lambda \leq \mu$ . Thus the  $\eta_\lambda$  define a homomorphism

$$\eta: \varinjlim \mathbb{Z}[?, G/G_\lambda]_G \rightarrow \mathbb{Z}_{\mathfrak{F}}$$

We need to show that

$$\eta_H: \varinjlim \mathbb{Z}[G/G_\lambda^H] \rightarrow \mathbb{Z}$$

is an isomorphism for every  $H \in \mathfrak{F}$ . Since the functor  $\mathbb{Z}[?]: \mathfrak{Set} \rightarrow \mathfrak{Ab}$  commutes with arbitrary colimits it is enough to show that  $\varinjlim (G/G_\lambda)^H = (G/G)^H = G/G$  where the last equality is due to the trivial action of  $H$  on the singleton set  $G/G$ .

We verify this by showing that  $G/G$  satisfies the universal property of a colimit. Therefore assume that we are given a set  $X$  and a collection of maps  $f_\lambda: G/G_\lambda \rightarrow X$  such that  $f_\lambda = f_\mu \circ \varphi_\lambda^\mu$  for all  $\lambda \leq \mu$ .

First of all, observe that each  $f_\lambda$  is a constant function. To see this, observe that if  $g \in G$  there exists  $\mu \in \Lambda$  such that  $g \in G_\mu$  and  $G_\lambda \leq G_\mu$ .

Then

$$f_\lambda(gG_\lambda) = f_\mu(\varphi_\lambda^\mu(gG_\lambda)) = f_\mu(G_\mu) = f_\mu(\varphi_\lambda^\mu(G_\lambda)) = f_\lambda(G_\lambda)$$

which is independent of the choice of  $g \in G$ . Furthermore, the value  $f_\lambda(G_\lambda)$  is independent of  $\lambda$ . This is because given any  $\lambda_1, \lambda_2 \in \Lambda$  there exists a  $\mu$  such that  $G_{\lambda_1} \leq G_\mu$  and  $G_{\lambda_2} \leq G_\mu$ . Then

$$f_{\lambda_1}(G_{\lambda_1}) = f_\mu(\varphi_{\lambda_1}^\mu(G_{\lambda_1})) = f_\mu(G_\mu) = f_\mu(\varphi_{\lambda_2}^\mu(G_{\lambda_2})) = f_{\lambda_2}(G_{\lambda_2}).$$

Now it follows that there exists a function  $f: G/G \rightarrow X$  such that  $f_\lambda = f \circ \varphi_\lambda$  for all  $\lambda \in \Lambda$  and that this function must be unique. Thus  $G/G$  has the universal property of a colimit and this concludes the proof.  $\square$

PROOF OF THEOREM 3.42. The main part of the proof consists of constructing a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\mathbb{Z}_{\mathfrak{F}}$  using the standard resolutions of the trivial  $\mathcal{O}_{\mathfrak{F}_\lambda}G_\lambda$ -modules  $\mathbb{Z}_{\mathfrak{F}_\lambda}$  for  $\lambda \in \Lambda$ .

For every  $\lambda \in \Lambda$  let

$$\dots \longrightarrow F_{\lambda,2} \xrightarrow{d_{\lambda,2}} F_{\lambda,1} \xrightarrow{d_{\lambda,1}} F_{\lambda,0} \xrightarrow{\varepsilon_\lambda} Z_\lambda \longrightarrow 0 \quad (3.7)$$

be the sequence of  $\mathcal{O}_{\mathfrak{F}}G$ -modules obtained from the standard resolution (3.2) of the trivial  $\mathcal{O}_{\mathfrak{F}_\lambda}G_\lambda$ -module  $\mathbb{Z}_{\mathfrak{F}_\lambda}$  by applying the functor  $\text{ind}_{I_{G_\lambda}}$ . Induction by  $I_{G_\lambda}$  is exact and thus the sequence (3.7) is exact. From the construction of the standard resolution and from Lemma 3.45 we know that

$$F_{\lambda,n} \cong \mathbb{Z}[\?, I_{G_\lambda}(\Delta_{\lambda,n})]_G.$$

Furthermore  $Z_\lambda \cong \mathbb{Z}[\?, G/G_\lambda]_G$ .

Observe that

$$I_{G_\lambda}(\Delta_{\lambda,n}) = \{(g_0K_0, \dots, g_nK_n) : g_i \in G \text{ and } K_i \in \mathfrak{F}_\lambda\}.$$

In particular for any  $\lambda \leq \mu$  we have  $\mathfrak{F}_\lambda \subset \mathfrak{F}_\mu$  and therefore  $I_{G_\lambda}(\Delta_{\lambda,n}) \subset I_{G_\mu}(\Delta_{\mu,n})$ . We denote this inclusion by  $\varphi_\lambda^\mu$ . Clearly this inclusion is  $G$ -equivariant. Therefore we get a morphism

$$\varphi_\lambda^\mu: F_{\lambda,n} \rightarrow F_{\mu,n}$$

of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules. The collection  $\{F_{\lambda,n}, \varphi_\lambda^\mu : \lambda, \mu \in \Lambda \text{ and } \lambda \leq \mu\}$  forms a direct limiting system directed by  $\Lambda$ . For each  $n \in \mathbb{N}$  we denote its limit by  $F_n$ .

Similarly, for any  $\lambda \leq \mu$  there exists a unique  $G$ -map  $\varphi: G/G_\lambda \rightarrow G/G_\mu$  which sends  $G_\lambda$  to  $G_\mu$ . These  $G$ -maps give rise to morphisms  $\varphi_\lambda^\mu: Z_\lambda \rightarrow Z_\mu$ .



It follows that the collection  $\{Z_\lambda, \varphi_\lambda^\mu : \lambda, \mu \in \Lambda \text{ and } \lambda \leq \mu\}$  forms a direct limiting system directed by  $\Lambda$ . We denote its limit by  $Z$ .

It follows that for every  $\lambda \leq \mu$  and  $n \geq 1$  we have that the diagrams

$$\begin{array}{ccc}
F_{\lambda,n} & \xrightarrow{d_{\lambda,n}} & F_{\lambda,n-1} \\
\varphi_\lambda^\mu \downarrow & & \downarrow \varphi_\lambda^\mu \\
F_{\mu,n} & \xrightarrow{d_{\mu,n}} & F_{\mu,n-1}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
F_{\lambda,0} & \xrightarrow{\varepsilon_\lambda} & Z_\lambda \\
\varphi_\lambda^\mu \downarrow & & \downarrow \varphi_\lambda^\mu \\
F_{\mu,0} & \xrightarrow{\varepsilon_\mu} & Z_\mu
\end{array}$$

commute. Therefore we obtain a sequence

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} Z \longrightarrow 0. \quad (3.8)$$

This sequence is exact, since direct limits preserve exactness.

It follows that the collection  $\{I_{G_\lambda}(\Delta_{\lambda,n}), \varphi_\lambda^\mu \text{ and } \lambda \leq \mu\}$  of  $G$ -sets is a direct limiting system directed by  $\Lambda$ . Its limit is

$$\Delta_n := \{(g_0 K_0, \dots, g_n K_n) : g_i \in G \text{ and } K \in \mathfrak{F}\}$$

and this  $G$ -set is actually the direct union of the  $I_{G_\lambda}(\Delta_{\lambda,n})$ . As a consequence  $F_n \cong \mathbb{Z}[\Delta_n]_G$  by Proposition 1.20. Furthermore, it follows that  $Z \cong \underline{\mathbb{Z}}_{\mathfrak{F}}$  by Lemma 3.46. Thus the sequence (3.8) is nothing else but the free standard resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ . We are now ready to prove the two claims of the theorem:

We first show that

$$\text{hd}_{\mathfrak{F}} G \leq \sup\{\text{hd}_{\mathfrak{F}_\lambda} G_\lambda\}.$$

In order to avoid triviality we assume that  $n := \sup\{\text{hd}_{\mathfrak{F}_\lambda} G_\lambda\}$  is finite. From the resolution (3.8) of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module we obtain the resolution

$$0 \rightarrow K \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0 \quad (3.9)$$

where  $K$  is the  $(n-1)$ -th kernel of the resolution (3.8). We claim that  $K$  is flat. Since direct limits and induction with  $I_{G_\lambda}$  are exact, it follows that

$$K \cong \varinjlim (\text{ind}_{I_{G_\lambda}} K_\lambda)$$

where the  $K_\lambda$  are the  $(n-1)$ -th kernels of the free standard resolution of the trivial  $\mathcal{O}_{\mathfrak{F}_\lambda}G_\lambda$ -modules  $\underline{\mathbb{Z}}_{\mathfrak{F}_\lambda}$ . But these are flat because  $\text{hd}_{\mathfrak{F}_\lambda} G_\lambda \leq n$ . Since the functor  $\text{ind}_{I_{G_\lambda}}$  preserves flats we have that  $K$  is the direct limit of flats

and hence is flat as well. Therefore (3.9) is a flat resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  of length  $n$  and so  $\mathrm{hd}_{\mathfrak{F}} G \leq \sup\{\mathrm{hd}_{\mathfrak{F}_\lambda} G_\lambda\}$ .

Next, we assume that the set  $\Lambda$  is countable and we want to verify the second claim of the theorem, that

$$\mathrm{cd}_{\mathfrak{F}} G \leq \sup\{\mathrm{cd}_{\mathfrak{F}_\lambda} G_\lambda\} + 1.$$

Again, in order to avoid triviality we assume that  $n := \sup\{\mathrm{cd}_{\mathfrak{F}_\lambda} G_\lambda\}$  is finite. As before, let  $K$  be the  $(n - 1)$ -th kernel of the standard resolution (3.8). Similarly, it follows that  $K$  is the direct limit

$$K \cong \varinjlim (\mathrm{ind}_{G_\lambda} K_\lambda)$$

of projectives. Since  $\Lambda$  is assumed to be countable we can apply Lemma 3.4 in [Nuc04, p. 40]. This lemma states that the limit of a countable directed system of projective right  $\mathcal{O}_{\mathfrak{F}}G$ -modules has projective dimension at most 1. Hence  $\mathrm{pd}_{\mathfrak{F}} K = 1$  and there exists a projective resolution

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow K \rightarrow 0$$

of  $K$ . We can splice this sequence together with (3.9) to get a projective resolution

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ . Hence  $\mathrm{cd}_{\mathfrak{F}} G \leq \sup\{\mathrm{cd}_{\mathfrak{F}_\lambda} G_\lambda\} + 1$ .  $\square$

**Proposition 3.47.** *Let  $G$  be a group and  $\mathfrak{F}$  be a full family of finitely generated subgroups of  $G$ . If  $G$  is locally  $\mathfrak{F}$ , that is every finitely generated subgroup of  $G$  is contained in the family  $\mathfrak{F}$ , then  $\mathrm{hd}_{\mathfrak{F}} G = 0$ . If in addition  $G$  is countable, then  $\mathrm{cd}_{\mathfrak{F}} G \leq 1$ .*

PROOF. Every group  $G$  is the direct union of its finitely generated subgroups  $G_\lambda$ . Set  $\mathfrak{F}_\lambda := \mathfrak{F} \cap G_\lambda$ . By assumption  $G_\lambda \in \mathfrak{F}$ , and so by Proposition 3.20,  $\underline{\mathbb{Z}}_{\mathfrak{F}_\lambda}$  is projective and in particular a flat  $\mathcal{O}_{\mathfrak{F}_\lambda}G_\lambda$ -module, and thus  $\mathrm{cd}_{\mathfrak{F}} G_\lambda = \mathrm{hd}_{\mathfrak{F}} G_\lambda = 0$ . Since  $\mathfrak{F}$  is closed under forming subgroups, we have that the families  $\mathfrak{F}$  and  $\mathfrak{F}_\lambda$  are compatible with the limit by Proposition 3.43. Now the first part of Theorem 3.42 gives  $\mathrm{hd}_{\mathfrak{F}} G = 0$ . If  $G$  is countable then  $\mathfrak{F}$  is countable by Remark 3.15 and thus the second part of Theorem 3.42 gives the estimation  $\mathrm{cd}_{\mathfrak{F}} G \leq 1$ .  $\square$

Examples of families which satisfy the conditions of Proposition 3.47 are  $\mathfrak{F}_{\text{fin}}(G)$  and  $\mathfrak{F}_{\text{vc}}(G)$  in case that  $G$  is locally finite and  $\mathfrak{F}_{\text{vc}}(G)$  in the case that  $G$  is locally virtually cyclic. Thus we obtain the following corollary to Proposition 3.47.

**Corollary 3.48.** *Assume that  $G$  is a locally finite group. Then  $\underline{\text{hd}} G = 0$  in general and if in addition  $G$  is countable then  $\underline{\text{cd}} G \leq 1$ . Similarly, if  $G$  is locally virtually cyclic then  $\underline{\text{hd}} G = 0$  in general, and if in addition  $G$  is countable, then  $\underline{\text{cd}} G \leq 1$ .  $\square$*

## 12. Tensor Product of Projective Resolutions

Let  $G$  and  $H$  be groups and  $f: G \rightarrow H$  a group homomorphism. Assume that we are given families  $\mathfrak{F}$  and  $\mathfrak{G}$  of subgroups of  $G$  and  $H$  respectively, such that  $f(\mathfrak{F}) \subset \mathfrak{G}$ .

Then we can construct a functor  $f: \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathcal{O}_{\mathfrak{G}}H$  as follows. Given an object  $G/K$  in  $\mathcal{O}_{\mathfrak{F}}G$  we set  $f(G/K) := H/f(K)$  which is an object of  $\mathcal{O}_{\mathfrak{G}}H$ . If  $\varphi: G/K \rightarrow G/L$  is a morphism in  $\mathcal{O}_{\mathfrak{F}}G$  which maps  $K \rightarrow gL$ ,  $g \in G$ , then  $K^g \leq L$  and therefore

$$f(K)^{f(g)} = f(K^g) \leq f(L).$$

Hence there exists a unique  $H$ -map  $f(G/K) \rightarrow f(G/L)$  which maps the coset  $f(K)$  to the coset  $f(g)f(L)$ . We denote this  $H$ -map by  $f(\varphi)$ . This way we get a map  $f: \text{mor}(G/K, G/L) \rightarrow \text{mor}(f(G/K), f(G/L))$  for each pair  $G/K, G/L$  of objects in  $\mathcal{O}_{\mathfrak{F}}G$ .

**Lemma 3.49.** *The above construction gives a functor  $f: \mathcal{O}_{\mathfrak{F}}G \rightarrow \mathcal{O}_{\mathfrak{G}}H$ .*

PROOF. If  $\text{id}$  is the identity on  $G/K$ , that is  $\text{id} = \varphi_{e,K,K}$ , then

$$f(\text{id}) = f(\varphi_{e,K,K}) = \varphi_{f(e),f(K),f(K)} = \varphi_{e,f(K),f(K)} = \text{id}$$

is the identity on  $f(G/K)$ .

Assume we are given two morphisms  $\varphi_{g_1,K_1,K_2}$  and  $\varphi_{g_2,K_2,K_3}$  in  $\mathcal{O}_{\mathfrak{F}}G$ , then we know that

$$\varphi_{g_2,K_2,K_3} \circ \varphi_{g_1,K_1,K_2} = \varphi_{g_2g_1,K_1,K_3}.$$

Therefore

$$\begin{aligned} f(\varphi_{g_2, K_2, K_3} \circ \varphi_{g_1, K_1, K_3}) &= f(\varphi_{g_2 g_1, K_1, K_3}) = \varphi_{f(g_2 g_1), f(K_1), f(K_3)} \\ &= \varphi_{f(g_2) f(g_1), f(K_1), f(K_3)} = \varphi_{f(g_1), f(K_1), f(K_2)} \circ \varphi_{f(g_2), f(K_2), f(K_3)}. \quad \square \end{aligned}$$

We apply the above result to the following setting. Given two groups  $G_1$  and  $G_2$  we consider the group  $G := G_1 \times G_2$ . Assume we are given two semi-full families of subgroups  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  of  $G_1$  and  $G_2$  respectively. The natural choice for a family  $\mathfrak{F}$  of subgroups of  $G$  is the cartesian product of the families  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$ . Recall that by definition this family is

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = \{H_1 \times H_2 : H_1 \in \mathfrak{F}_1 \text{ and } H_2 \in \mathfrak{F}_2\},$$

see Section 1 in Chapter 1. The projection homomorphisms  $p_i: G \rightarrow G_i$  satisfy  $p_i(\mathfrak{F}) = \mathfrak{F}_i$ . Therefore we obtain projection functors

$$p_i: \mathcal{O}_{\mathfrak{F}} G \rightarrow \mathcal{O}_{\mathfrak{F}_i} G_i$$

Given right  $\mathcal{O}_{\mathfrak{F}_i} G_i$ -modules  $M_i$ ,  $i = 1, 2$ , we can now form a tensor product of these modules over  $\mathbb{Z}$  as follows. Applying the restriction functor  $\text{res}_{p_i}$  to  $M_i$  we obtain a  $\mathcal{O}_{\mathfrak{F}} G$ -module  $\text{res}_{p_i} M_i$  and we can form the  $\mathcal{O}_{\mathfrak{F}} G$ -module

$$\text{res}_{p_1} M_1 \otimes \text{res}_{p_2} M_2.$$

Essentially this module is the tensor product of  $M_1$  and  $M_2$  with the orbit category  $\mathcal{O}_{\mathfrak{F}_1} G_1$  acting on the first factor and the orbit category  $\mathcal{O}_{\mathfrak{F}_2} G_2$  acting on the second factor of the tensor product. This construction is the generalisation of the classical construction of the tensor product  $M_1 \otimes M_2$  of a  $G_1$ -module  $M_1$  and a  $G_2$ -module  $M_2$  with  $G_1$  acting on the first factor and  $G_2$  acting on the second factor. This in turn makes  $M_1 \otimes M_2$  a  $G$ -module.

**Lemma 3.50.** *For  $i = 1, 2$  we have the equality  $\text{res}_{p_i} \underline{\mathbb{Z}}_{\mathfrak{F}_i} = \underline{\mathbb{Z}}_{\mathfrak{F}}$ .*

PROOF. Given a  $H = H_1 \times H_2 \in \mathfrak{F}$  we have

$$(\text{res}_{p_i} \underline{\mathbb{Z}}_{\mathfrak{F}_i})(G/H) = \underline{\mathbb{Z}}_{\mathfrak{F}_i}(G_i/H_i) = \mathbb{Z} = \underline{\mathbb{Z}}_{\mathfrak{F}}(G/H),$$

and similarly if  $\varphi$  is a morphism in  $\mathcal{O}_{\mathfrak{F}} G$ , then

$$(\text{res}_{p_i} \underline{\mathbb{Z}}_{\mathfrak{F}_i})(\varphi) = (\underline{\mathbb{Z}}_{\mathfrak{F}_i} \circ p_i)(\varphi) = \text{id} = \underline{\mathbb{Z}}_{\mathfrak{F}}(\varphi). \quad \square$$

**Corollary 3.51.** *Let  $M_i$  be an  $\mathcal{O}_{\mathfrak{F}_i} G_i$ -module,  $i = 1, 2$ . Then*

$$\text{res}_{p_1} M_1 \otimes \text{res}_{p_2} \underline{\mathbb{Z}}_{\mathfrak{F}_2} = \text{res}_{p_1} M_1 \quad \text{and} \quad \text{res}_{p_1} \underline{\mathbb{Z}}_{\mathfrak{F}_1} \otimes \text{res}_{p_2} M_2 = \text{res}_{p_2} M_2.$$

In particular we have the equality  $\text{res}_{p_1} \mathbb{Z}_{\mathfrak{F}_1} \otimes \text{res}_{p_2} \mathbb{Z}_{\mathfrak{F}_2} = \mathbb{Z}_{\mathfrak{F}}$ .  $\square$

**Lemma 3.52.** *Let  $F_i = \mathbb{Z}[?, X_i]_{G_i}$  be free right  $\mathcal{O}_{\mathfrak{F}_i} G_i$ -modules,  $i = 1, 2$ . Then  $\text{res}_{p_1} F_1 \otimes \text{res}_{p_2} F_2$  is the free right  $\mathcal{O}_{\mathfrak{F}} G$ -module  $\mathbb{Z}[?, X]_G$  with the diagonal action of  $G = G_1 \times G_2$  on  $X := X_1 \times X_2$ .*

PROOF. For each  $H = H_1 \times H_2 \in \mathfrak{F}$  we have

$$\begin{aligned} (\text{res}_{p_1} F_1 \otimes \text{res}_{p_2} F_2)(G/H) &= \mathbb{Z}[G_1/H_1, X_1]_{G_1} \otimes \mathbb{Z}[G_2/H_2, X_2]_{G_2} \\ &= \mathbb{Z}[[G_1/H_1, X_1]_{G_1} \times [G_2/H_2, X_2]_{G_2}] \\ &\cong \mathbb{Z}[G/H, X]_G \end{aligned}$$

and this isomorphism is given by

$$\psi_1 \otimes \psi_2 \mapsto \psi$$

where  $\varphi: G/H \rightarrow X$  is the  $G$ -map given by

$$H \mapsto (\psi_1(H_1), \psi_2(H_2)).$$

Denote this isomorphism by  $\eta_H$ . We claim that these isomorphisms form isomorphism

$$\eta: \mathbb{Z}[?, X_1]_{G_1} \otimes \mathbb{Z}[?, X_2]_{G_2} \rightarrow \mathbb{Z}[?, X]_G$$

of  $\mathcal{O}_{\mathfrak{F}} G$ -modules. This can be verified by a straightforward diagram chase as follows.

Let  $H = H_1 \times H_2, K = K_1 \times K_2 \in \mathfrak{F}$  and  $\varphi = (\varphi_1, \varphi_2) \in [G/H, G/K]_G$ . Furthermore, let  $\psi_1 \otimes \psi_2$  a basis element of  $\mathbb{Z}[G_1/K_1, X_1]_{G_1} \otimes \mathbb{Z}[G_2/K_2, X_2]_{G_2}$ .

Then

$$\begin{aligned} (\eta_H \circ \varphi^*)(\psi_1 \otimes \psi_2) &= \eta_H((\psi_1 \otimes \psi_2) \circ \varphi) \\ &= \eta_H((\psi_1 \circ \varphi_1) \otimes (\psi_2 \circ \varphi_2)) \\ &= (\psi_1 \circ \varphi_1, \psi_2 \circ \varphi_2) \\ &= \varphi^*(\psi_1, \psi_2) \\ &= (\varphi^* \circ \eta_K)(\psi_1 \otimes \psi_2). \end{aligned}$$

Therefore the  $\eta_{G/H}$  form an homomorphism of right  $\mathcal{O}_{\mathfrak{F}} G$ -modules and since each  $\eta_{G/H}$  is an isomorphism the homomorphism  $\eta$  is an isomorphism, too.  $\square$

**Corollary 3.53.** *Let  $P_i$  be projective right  $\mathcal{O}_{\mathfrak{F}_i} G_i$ -modules,  $i = 1, 2$ . Then  $\text{res}_{p_1} P_1 \otimes \text{res}_{p_2} P_2$  is a projective  $\mathcal{O}_{\mathfrak{F}} G$ -module.*

PROOF. Since  $P_i$  is projective it is a direct summand of a free  $\mathcal{O}_{\mathfrak{F}_i}G_i$ -module  $F_i$ , say  $F_i = P_i \oplus Q_i$  for some projective  $Q_i$ . Since restriction and the tensor product over  $\mathbb{Z}$  are additive functors we get by the previous lemma that

$$\begin{aligned} \operatorname{res}_{p_1} F_1 \otimes \operatorname{res}_{p_2} F_2 &= (\operatorname{res}_{p_1} P_1 \otimes \operatorname{res}_{p_2} P_2) \oplus (\operatorname{res}_{p_1} P_1 \otimes \operatorname{res}_{p_2} Q_2) \\ &\quad \oplus (\operatorname{res}_{p_1} Q_1 \otimes \operatorname{res}_{p_2} P_2) \oplus (\operatorname{res}_{p_1} Q_1 \otimes \operatorname{res}_{p_2} Q_2) \end{aligned}$$

is a free  $\mathcal{O}_{\mathfrak{F}}G$ -module. Therefore  $\operatorname{res}_{p_1} P_1 \otimes \operatorname{res}_{p_2} P_2$  is a direct summand of a free  $\mathcal{O}_{\mathfrak{F}}G$ -module and thus projective.  $\square$

**Lemma 3.54.** *Let  $F_i = \mathbb{Z}[?, X_i]_{G_i}$  be a free right  $\mathcal{O}_{\mathfrak{F}_i}G_i$ -modules,  $i = 1, 2$ . Then  $\operatorname{res}_{p_i} F_i$  is a free right  $\mathcal{O}_{\mathfrak{F}}G$ -module.*

PROOF. This follows essentially from the observation that

$$\operatorname{res}_{p_i} \mathbb{Z}[?, X_i]_{G_i} = \mathbb{Z}[?, X_i]_{G_i},$$

where the  $G_i$  set  $X_i$  on the right hand side of the equation is seen as a  $G$ -set by  $gx := p_i(g)x$ .  $\square$

**Corollary 3.55.** *Let  $P_i$  be a projective right  $\mathcal{O}_{\mathfrak{F}_i}G_i$ -module. Then  $\operatorname{res}_{p_i} P_i$  is a projective  $\mathcal{O}_{\mathfrak{F}}G$ -module.*

PROOF. This follows again from the fact that  $\operatorname{res}_{p_i}$  is an additive functor and hence a direct summand of free  $\mathcal{O}_{\mathfrak{F}_i}G_i$ -modules is mapped to direct summand of a  $\mathcal{O}_{\mathfrak{F}}G$ -module which is free by the previous lemma.  $\square$

Let  $P_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  be a resolution of right  $\mathcal{O}_{\mathfrak{F}_1}G_1$ -modules and let  $Q_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  be a resolution of right  $\mathcal{O}_{\mathfrak{F}_2}G_2$ -modules. Then we can form the double complex

$$C_{p,q} := \operatorname{res}_{p_1} P_p \otimes \operatorname{res}_{p_2} Q_q \tag{3.10}$$

of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules where  $p, q \in \mathbb{N}$ . Taking the total complex of this double complex gives the chain complex

$$\dots \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0.$$

Denote the epimorphisms  $P_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  and  $Q_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  by  $\varepsilon_1$  and  $\varepsilon_2$ . Then we obtain a morphism  $\varepsilon := \operatorname{res}_{p_1} \varepsilon_1 \otimes \operatorname{res}_{p_2} \varepsilon_2$  from  $C_0$  onto  $\operatorname{res}_{p_1} \underline{\mathbb{Z}}_{\mathfrak{F}_1} \otimes \operatorname{res}_{p_2} \underline{\mathbb{Z}}_{\mathfrak{F}_2}$ .

**Proposition 3.56.** *Let  $P_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  and  $Q_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  be free (projective) resolutions. Then*

$$\dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \xrightarrow{\varepsilon} \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0 \quad (3.11)$$

*is a free (projective) resolution of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ .*

PROOF. It follows straight from Lemma 3.52 (Corollary 3.53) that the  $\mathcal{O}_{\mathfrak{F}}G$ -modules  $C_i$  are free (projective). The domain of  $\varepsilon$  is by construction  $C_0$  and its codomain is  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  by Corollary 3.51. Thus it remains to show that the sequence (3.11) is exact. For this we need to show that (3.11) evaluated at any object  $G/H$  of the orbit category  $\mathcal{O}_{\mathfrak{F}}G$  is exact.

Therefore let  $H = H_1 \times H_2 \in \mathfrak{F}$ . Set  $P'_* := P_*(G_1/H_1) = (\text{res}_{p_1} P_*)(G/H)$  and  $Q'_* := Q_*(G_2/H_2) = (\text{res}_{p_2} Q_*)(G/H)$ . Since restriction is an exact functor we obtain two exact resolutions  $P'_* \xrightarrow{\varepsilon'_1} \mathbb{Z}$  and  $Q'_* \xrightarrow{\varepsilon'_2} \mathbb{Z}$  of abelian groups with  $\varepsilon'_i := (\text{res}_{p_i} \varepsilon_i)_H$ . Since  $P_k$  is projective it is a direct summand of a free  $\mathcal{O}_{\mathfrak{F}_1}G_1$ -module  $F_k$ . By definition  $F_k(G_1/H_1)$  is a free abelian group. Since colimits are calculated componentwise it follows that  $P_k(G_1/H_1)$  is a direct summand of  $F_k(G_1/H_1)$  and hence a free abelian group. Thus  $P'_* \rightarrow \mathbb{Z}$  is a free resolution of  $\mathbb{Z}$  and likewise is  $Q'_* \rightarrow \mathbb{Z}$ . Let  $C'_*$  be the total complex of the double complex  $P'_* \otimes Q'_*$ . It follows that this complex gives a free resolution of  $\mathbb{Z}$  where the augmentation map  $\varepsilon': C'_0 \rightarrow \mathbb{Z}$  is given by  $\varepsilon' := \varepsilon'_1 \otimes \varepsilon'_2$ , see for example [Bro82, p. 107]. We claim that this resolution of  $\mathbb{Z}$  is identical with the resolution obtained from evaluating

$$\dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

at  $G/H$ . This claim is verified by straightforward calculation as follows.

First, for any  $n \in \mathbb{N}$  we have

$$\begin{aligned} C_n(G/H) &= \prod_{k=0}^n C_{k,n-k}(G/H) \\ &= \prod_{k=0}^n (\text{res}_{p_1} P_k)(G/H) \otimes (\text{res}_{p_2} Q_{n-k})(G/H) \\ &= \prod_{k=0}^n P'_k \otimes Q'_{n-k} = C'_n \end{aligned} \quad (3.12)$$

and  $\underline{\mathbb{Z}}_{\mathfrak{F}}(G/H) = \mathbb{Z}$  by definition. Hence the sequences agree on objects and it remains to show that the homomorphisms agree as well.

For the homomorphism  $\varepsilon_{G/H}: C_0(G/H) \rightarrow \mathbb{Z}$  we have the following sequence of equalities

$$\varepsilon_H = (\text{res}_{p_1} \varepsilon_1 \otimes \text{res}_{p_2} \varepsilon_2)_H = (\text{res}_{p_1} \varepsilon_1)_H \otimes (\text{res}_{p_2} \varepsilon_2)_H = \varepsilon'_1 \otimes \varepsilon'_2 = \varepsilon'.$$

A similar kind of argument shows that, for any  $n \geq 1$ , the differentials  $d_{n,H}: C_n(G/H) \rightarrow C_{n-1}(G/H)$  agree with the differentials  $d'_n: C'_n \rightarrow C'_{n-1}$  under the identification (3.12).  $\square$

**Corollary 3.57.** *If  $\text{cd}_{\mathfrak{F}_1} G_1 \leq m$  and  $\text{cd}_{\mathfrak{F}_2} G_2 \leq n$  then  $\text{cd}_{\mathfrak{F}} G \leq m + n$ .*  $\square$

In order to simplify the notation we set  $\tilde{H}_i := p_i(H)$  and  $\tilde{H} := \tilde{H}_1 \times \tilde{H}_2$  for any subgroup  $H$  of  $G$ . That is,  $\tilde{H}$  is the smallest subgroup of  $G$  which contains  $H$  and which is of the form  $H_1 \times H_2$  with  $H_1 \leq G_1$  and  $H_2 \leq G_2$ .

**Lemma 3.58.** *Let  $X_1$  be a  $G_1$ -set and  $X_2$  a  $G_2$ -set. Consider the set  $X = X_1 \times X_2$  with the diagonal  $G$ -action given by  $(g_1, g_2)(x_1, x_2) := (g_1 x_1, g_2 x_2)$ . Then*

$$X^H = X^{\tilde{H}} = X_1^{\tilde{H}_1} \times X_2^{\tilde{H}_2}.$$

PROOF. The second equality is clear by construction of  $\tilde{H}$  and  $X$ . It remains to show that the first equality is true. Since  $H \leq \tilde{H}$  we have that  $X^{\tilde{H}} \subset X^H$ . Thus it remains to show that  $X^H \subset X^{\tilde{H}}$ . Let  $x = (x_1, x_2) \in X^H$  and let  $h = (h_1, h_2) \in \tilde{H}$ . There exist  $h'_1 \in \tilde{H}_1$  and  $h'_2 \in \tilde{H}_2$  such that  $(h_1, h'_2), (h'_1, h_2) \in H$ . Then  $(h_1 x_1, h'_2 x_2) = (h_1, h'_2)(x_1, x_2) = (x_1, x_2)$  and likewise  $(h'_1 x_1, h_2 x_2) = (h'_1, h_2)(x_1, x_2) = (x_1, x_2)$ . Hence  $(h_1, h_2)(x_1, x_2) = (x_1, x_2)$  and this implies that  $x \in X^{\tilde{H}}$ .  $\square$

Recall that given a family  $\mathfrak{F}$  of subgroups of a group  $G$ , we defined its subgroup completion  $\bar{\mathfrak{F}}$  to be the smallest *full* family of subgroups of  $G$  which contains  $\mathfrak{F}$  (see Section 1 in Chapter 1).

**Lemma 3.59.** *Assume that  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are full families of subgroups. Let  $X_1, Y_1$  be  $G_1$ -sets with stabilisers in  $\mathfrak{F}_1$  and let  $X_2, Y_2$  be  $G_2$ -sets with stabilisers in  $\mathfrak{F}_2$ . Consider the sets  $X := X_1 \times X_2$  and  $Y := Y_1 \times Y_2$  with the diagonal action of  $G = G_1 \times G_2$ . Then any morphism*

$$f: \mathbb{Z}[?, X]_G \rightarrow \mathbb{Z}[?, Y]_G$$

*of right  $\mathcal{O}_{\mathfrak{F}}G$ -modules can be extended to a morphism of right  $\mathcal{O}_{\bar{\mathfrak{F}}}G$ -modules by  $f_H := f_{\tilde{H}}$ .*



PROOF. Since  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  are full families of subgroups of  $G_1$  and  $G_2$  it follows that  $\tilde{H} \in \mathfrak{F}$  for any  $H \in \tilde{\mathfrak{F}}$ . Furthermore it follows from the previous lemma that  $\mathbb{Z}[G/H, X]_G = \mathbb{Z}[G/\tilde{H}, X]_G$  and  $\mathbb{Z}[G/H, Y]_G = \mathbb{Z}[G/\tilde{H}, Y]_G$ . Therefore  $f_H$  is defined for any  $H \in \tilde{\mathfrak{F}}$ . We need to show that this extension of  $f$  indeed gives a morphism of  $\mathcal{O}_{\tilde{\mathfrak{F}}}G$ -modules.

Let  $H, K \in \tilde{\mathfrak{F}}$  and let  $\varphi := \varphi_{g, H, K}: G/H \rightarrow G/K$  be a morphism in  $\mathcal{O}_{\tilde{\mathfrak{F}}}G$ . We need to show that the diagram

$$\begin{array}{ccc}
\mathbb{Z}[G/K, X]_G & \xrightarrow{f_K} & \mathbb{Z}[G/K, Y]_G \\
\varphi^* \downarrow & & \downarrow \varphi^* \\
\mathbb{Z}[G/H, X]_G & \xrightarrow{f_H} & \mathbb{Z}[G/H, Y]_G
\end{array} \tag{3.13}$$

commutes.

Since  $H^g \leq K$  it follows that  $\tilde{H}^g \leq \tilde{K}$ . Let  $\tilde{\varphi} := \varphi_{g, \tilde{H}, \tilde{K}}$  which is a morphism in  $\mathcal{O}_{\tilde{\mathfrak{F}}}G$  as well as a morphism in  $\mathcal{O}_{\mathfrak{F}}G$ . By the definition of the Bredon modules  $\mathbb{Z}[?, X]_G$  and  $\mathbb{Z}[?, Y]_G$  and from Lemma 3.58 it follows that  $\varphi^* \circ \text{id} = \text{id} \circ \tilde{\varphi}^*$ . Now in order to show that the diagram (3.13) commutes we imbed it into the following larger diagram.

$$\begin{array}{ccccc}
\mathbb{Z}[G/\tilde{K}, X]_G & \xrightarrow{f_{\tilde{K}}} & & \xrightarrow{f_{\tilde{K}}} & \mathbb{Z}[G/\tilde{K}, Y]_G \\
\downarrow \tilde{\varphi}^* & \swarrow \text{id} & \mathbb{Z}[G/K, X]_G & \xrightarrow{f_K} & \mathbb{Z}[G/K, Y]_G & \searrow \text{id} \\
& & \downarrow \varphi^* & & \downarrow \varphi^* & \\
& & \mathbb{Z}[G/H, X]_G & \xrightarrow{f_H} & \mathbb{Z}[G/H, Y]_G & \\
& \swarrow \text{id} & & & & \searrow \text{id} \\
\mathbb{Z}[G/\tilde{H}, X]_G & \xrightarrow{f_{\tilde{H}}} & & \xrightarrow{f_{\tilde{H}}} & \mathbb{Z}[G/\tilde{H}, Y]_G & \downarrow \tilde{\varphi}^*
\end{array}$$

The large outer square of this diagram commutes by assumption. The upper and lower trapezoid commute by definition and the above said means that the left and right trapezoid commute. It follows that the inner small square commutes, that is, we have shown that the diagram (3.13) commutes.  $\square$

Under the same assumptions as in the previous lemma we can extend the morphism  $\varepsilon: \mathbb{Z}[\?, X]_G \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}}$  of  $\mathcal{O}_{\mathfrak{F}}G$ -modules to a morphism

$$\varepsilon: \mathbb{Z}[\?, X]_G \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}}$$

of  $\mathcal{O}_{\mathfrak{F}}G$ -modules by setting  $\varepsilon_H := \varepsilon_{\tilde{H}}$  for every  $H \in \tilde{\mathfrak{F}}$ .

Assume that  $\text{cd}_{\mathfrak{F}_1} G_1 \leq m$  and  $\text{cd}_{\mathfrak{F}_2} G_2 \leq n$ . Then there exists projective resolutions  $P_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  and  $Q_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  of length  $m$  and  $n$  respectively. By an Eilenberg swindle we may assume the resolutions are free. Then by Lemma 3.52 the double chain complex (3.10) of  $\mathcal{O}_{\mathfrak{F}}G$ -modules satisfies the assumptions of Lemma 3.59. Hence we may extend it to a double chain complex of  $\mathcal{O}_{\mathfrak{F}}G$ -modules. Passing to the total complex we obtain a sequence

$$\dots \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \xrightarrow{\varepsilon} \underline{\mathbb{Z}}_{\mathfrak{F}} \rightarrow 0$$

of  $\mathcal{O}_{\mathfrak{F}}G$ -modules. By construction the exactness of this sequence follows from the exactness of the sequence (3.11) which was proven in Proposition 3.56. This sequence has length  $n + m$ . This proves the following result.

**Proposition 3.60.** *Let  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  be full families of subgroups. Then*

$$\text{cd}_{\mathfrak{F}} G \leq \text{cd}_{\mathfrak{F}_1} G_1 + \text{cd}_{\mathfrak{F}_2} G_2. \quad \square$$

**Theorem 3.61.** *Let  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  be a full families of subgroups of  $G_1$  and  $G_2$  respectively. Let  $\mathfrak{F} := \mathfrak{F}_1 \times \mathfrak{F}_2$ . Assume that  $\mathfrak{G}$  is a full family of subgroups of  $G = G_1 \times G_2$  with  $\mathfrak{G} \subset \mathfrak{F}$  and such that  $\mathfrak{G} \cap K \neq \emptyset$  for every  $K \in \tilde{\mathfrak{F}}$ . If there exists  $k \in \mathbb{N}$  such that  $\text{cd}_{\mathfrak{G} \cap K} K \leq k$  for every  $K \in \tilde{\mathfrak{F}}$  then*

$$\text{cd}_{\mathfrak{G}} G \leq \text{cd}_{\mathfrak{F}_1} G_1 + \text{cd}_{\mathfrak{F}_2} G_2 + k.$$

PROOF. Let  $K \in \tilde{\mathfrak{F}}$ . Then  $K \leq \tilde{K} \in \tilde{\mathfrak{F}}$  and  $\mathfrak{G} \cap K$  is a non-empty subset of  $\mathfrak{G} \cap \tilde{K}$ . Thus  $\text{cd}_{\mathfrak{G} \cap K} K \leq \text{cd}_{\mathfrak{G} \cap \tilde{K}} \tilde{K}$  which by the assumption of the theorem is less or equal to  $k$ . Then  $\text{cd}_{\mathfrak{G}} G \leq \text{cd}_{\mathfrak{F}} G + k$  by Theorem 3.37. Now the statement follows from the previous proposition.  $\square$

**Corollary 3.62.** *Let  $G := G_1 \times G_2$ . Then*

$$\underline{\text{cd}} G \leq \underline{\text{cd}} G_1 + \underline{\text{cd}} G_2 \quad \text{and} \quad \underline{\underline{\text{cd}}} G \leq \underline{\underline{\text{cd}}} G_1 + \underline{\underline{\text{cd}}} G_2 + 3.$$

PROOF. The cartesian product  $K_1 \times K_2$  of two finite groups is finite and therefore  $\underline{\text{cd}}(K_1 \times K_2) = 0$ . Thus  $\underline{\text{cd}} G \leq \underline{\text{cd}} G_1 + \underline{\text{cd}} G_2$  by Theorem 3.61.

On the other hand, the cartesian product  $K_1 \times K_2$  of two virtually cyclic groups is a virtually polycyclic group with virtually cohomological

dimension  $\text{vcd}(K_1 \times K_2) \leq 2$ . In [LW07] it has been shown that this implies  $\underline{\text{gd}}(K_1 \times K_2) \leq 3$ . Then  $\underline{\text{cd}} G \leq \underline{\text{cd}} G_1 + \underline{\text{cd}} G_2 + 3$  follows from Theorem 3.61.  $\square$

Note that the inequality in Corollary 3.62 is sharp. For example, take  $G_1 := \mathbb{Z}$  and  $G_2 := \mathbb{Z}$ . Then  $\underline{\text{cd}} G_i = 1$  and  $\underline{\text{cd}} G_i = 0$ . Moreover  $\underline{\text{cd}} G = 2$  and in Proposition 4.5 we will see that  $\underline{\text{cd}} G = 3$ . Therefore we have in this case

$$\underline{\text{cd}} G = \underline{\text{cd}} G_1 + \underline{\text{cd}} G_2 \quad \text{and} \quad \underline{\text{cd}} G = \underline{\text{cd}} G_1 + \underline{\text{cd}} G_2 + 3.$$

### 13. Crossproduct and Künneth Formula for Bredon Homology

In this section we consider  $G := G_1 \times G_2$  with the notation as in the section before. Moreover, throughout this section  $M$  and  $M'$  will be a left  $\mathcal{O}_{\mathfrak{F}_1} G_1$ -module and  $\mathcal{O}_{\mathfrak{F}_2} G_2$ -module respectively.

**Lemma 3.63.** *The map*

$$\begin{aligned} [G/H, G/K]_G \times M(G_1/\tilde{H}_1) \times M'(G_2/\tilde{H}_2) \\ \rightarrow [G_1/H_1, G_1/\tilde{K}_1]_{G_1} \times M(G_1/H_1) \\ \times [G_2/H_2, G_2/\tilde{K}_2]_{G_2} \times M'(G_2/H_2) \\ (gK, m, m') \mapsto (\tilde{g}_1 \tilde{K}_1, m, \tilde{g}_2 \tilde{K}_2, m'), \end{aligned} \quad (3.14)$$

where  $H, K \in \mathfrak{F}$ , defines for every  $K \in \mathfrak{F}$  an isomorphism of groups

$$\begin{aligned} \theta: \coprod_{H \in \mathfrak{F}} \mathbb{Z}[G/H, G/K]_G \otimes M(G_1/\tilde{H}_1) \otimes M'(G_2/\tilde{H}_2) \\ \rightarrow \left( \coprod_{H_1 \in \mathfrak{F}_1} \mathbb{Z}[G_1/H_1, G_1/\tilde{K}_1]_{G_1} \otimes M(G_1/H_1) \right) \otimes \\ \left( \coprod_{H_2 \in \mathfrak{F}_2} \mathbb{Z}[G_2/H_2, G_2/\tilde{K}_2]_{G_2} \otimes M'(G_2/H_2) \right). \end{aligned} \quad (3.15)$$

PROOF. First note that the right hand side of (3.15) can be rewritten to

$$\coprod_{H \in \mathfrak{F}} \left( \mathbb{Z}[G_1/\tilde{H}_1, G_1/\tilde{K}_1]_{G_1} \otimes M(G_1/\tilde{H}_1) \otimes \mathbb{Z}[G_2/\tilde{H}_2, G_2/\tilde{K}_2]_{G_2} \otimes M'(G_2/\tilde{H}_2) \right).$$

Thus it is enough to show that the rule (3.14) gives rise to an isomorphism for each fixed  $H \in \mathfrak{F}$ . In turn, for this it is enough to show that the obvious

restriction of the map (3.14) gives rise to an isomorphism

$$\mathbb{Z}[G/H, G/K]_G \rightarrow \mathbb{Z}[G_1/H_1, G_1/\tilde{K}_1]_{G_1} \otimes \mathbb{Z}[G_2/H_2, G_2/\tilde{K}_2]_{G_2}. \quad (3.16)$$

Note that any  $G$ -map  $f: G/H \rightarrow G/K$  is in fact a  $G$ -map  $f: G_1/\tilde{H}_1 \times G_2/\tilde{H}_2 \rightarrow G_1/\tilde{K}_1 \times G_2/\tilde{K}_2$  which is uniquely characterised by  $G_i$ -maps  $\tilde{f}_i: G_i/\tilde{H}_i \rightarrow G_i/\tilde{K}_i$  which map  $\tilde{H}_i \mapsto \tilde{g}_i\tilde{K}_i$ ,  $i = 1, 2$ . In other words, the map  $f \mapsto (\tilde{f}_1, \tilde{f}_2)$  gives an isomorphism

$$[G/H, G/K]_G \rightarrow [G_1/\tilde{H}_1, G_1/\tilde{K}_1]_{G_1} \times [G_2/\tilde{H}_2, G_2/\tilde{K}_2]_{G_2} \quad (3.17)$$

of sets. In turn this gives rise to an isomorphism of groups as in (3.16) and by construction this isomorphism agrees with the isomorphism obtained by the obvious restriction of the map (3.14).  $\square$

**Lemma 3.64.** *Let  $K \in \mathfrak{F}$ . Then the map (3.14) defines (this definition is made precise in the proof) an isomorphism*

$$\begin{aligned} \eta: (\text{res}_{p_1} \mathbb{Z}[\cdot, G_1/\tilde{K}_1]_{G_1} \otimes \text{res}_{p_2} \mathbb{Z}[\cdot, G_2/\tilde{K}_2]_{G_2}) \otimes_{\mathfrak{F}} (\text{res}_{p_1} M \otimes \text{res}_{p_2} M') \\ \longrightarrow (\mathbb{Z}[\cdot, G_1/\tilde{K}_1]_{G_1} \otimes_{\mathfrak{F}_1} M) \otimes (\mathbb{Z}[\cdot, G_2/\tilde{K}_2]_{G_2} \otimes_{\mathfrak{F}_2} M') \end{aligned} \quad (3.18)$$

of abelian groups.

PROOF. For each  $H \in \mathfrak{F}$  we have

$$\begin{aligned} (\text{res}_{p_1} \mathbb{Z}[\cdot, G_1/\tilde{K}_1]_{G_1} \otimes \text{res}_{p_2} \mathbb{Z}[\cdot, G_2/\tilde{K}_2]_{G_2})(G/H) \\ = \mathbb{Z}[G_1/\tilde{H}_1, G_1/\tilde{K}_1]_{G_1} \otimes \mathbb{Z}[G_2/\tilde{H}_2, G_2/\tilde{K}_2]_{G_2} = \mathbb{Z}[G/H, G/K]_G \end{aligned}$$

using the identification given by (3.17).

Now the domain of the isomorphism (3.18) is by definition the abelian group  $P/Q$  with

$$\begin{aligned} P = \prod_{H \in \mathfrak{F}} ((\text{res}_{p_1} \mathbb{Z}[\cdot, G_1/\tilde{K}_1]_{G_1} \otimes \text{res}_{p_2} \mathbb{Z}[\cdot, G_2/\tilde{K}_2]_{G_2})(G/H) \\ \otimes (\text{res}_{p_1} M \otimes \text{res}_{p_2} M')(G/H)) \\ = \prod_{H \in \mathfrak{F}} \mathbb{Z}[G/H, G/K]_G \otimes M(G_1/\tilde{H}_1) \otimes M'(G_2/\tilde{H}_2) \end{aligned}$$

where  $Q$  is the subgroup of  $P$  generated by all elements of the form

$$\varphi^*(f) \otimes (m \otimes m') - f \otimes \varphi_*(m \otimes m') \quad (3.19)$$

where  $f: G/H \rightarrow G/K$  is a  $G$ -map (that is a generator of the abelian group  $\mathbb{Z}[G/H, G/K]_G$ ),  $m \otimes m' \in M(G_1/\tilde{L}_1) \otimes M'(G_2/\tilde{L}_2)$ ,  $\varphi \in [G/L, G/H]_G$  and  $H, L \in \mathfrak{F}$ .

Likewise  $\mathbb{Z}[?, G_1/\tilde{K}_1]_{G_1} \otimes_{\mathfrak{F}_1} M$  is the abelian group  $R/S$  with

$$R := \coprod_{H_1 \in \mathfrak{F}_1} \mathbb{Z}[G_1/H_1, G_1/\tilde{K}_1]_{G_1} \otimes M(G_1/H_1)$$

where  $S$  is the subgroup of  $R$  generated by all elements of the form

$$\varphi_1^*(f_1) \otimes m - f_1 \otimes \varphi_{1*}(m) \quad (3.20)$$

where  $f_1: G_1/H_1 \rightarrow G_1/\tilde{K}_1$  is a  $G_1$ -map (that is a generator of the abelian group  $\mathbb{Z}[G_1/H_1, G_1/\tilde{K}_1]_{G_1}$ ),  $m \in M(G_1/L_1)$ ,  $\varphi_1 \in [G_1/L_1, G_1/H_1]_{G_1}$  and  $H_1, L_1 \in \mathfrak{F}_1$ . And in the very same fashion we express the abelian group  $\mathbb{Z}[?, G_2/\tilde{K}_2]_{G_2} \otimes_{\mathfrak{F}_2} M'$  as the quotient  $R'/S'$ .

Since the tensor product is right exact the projection homomorphism  $\pi_1: R \rightarrow R/S$  and  $\pi_2: R' \rightarrow R'/S'$  give an epimorphism  $\pi_1 \otimes \pi_2: R \otimes R' \rightarrow R/S \otimes R'/S'$ . If we precompose this epimorphism with the isomorphism  $\theta: P \rightarrow R \otimes R'$  from Lemma 3.63 we get an epimorphism

$$P \rightarrow R/S \otimes R'/S'$$

The first claim of Lemma 3.64 is now that this epimorphism factors through  $P/Q$ , that is there exists a homomorphism  $\eta$  making the diagram

$$\begin{array}{ccc} P & \xrightarrow{\theta} & R \otimes R' \\ \pi \downarrow & & \downarrow \pi_1 \otimes \pi_2 \\ P/Q & \xrightarrow{\eta} & R/S \otimes R'/S' \end{array}$$

commute. Since  $\pi$  and  $(\pi_1 \otimes \pi_2) \circ \theta$  are both epimorphisms it follows that this  $\eta$  is necessarily unique and also an epimorphism. In order to see that the epimorphism  $\eta$  exists, we must show that  $Q \subset \ker((\pi_1 \otimes \pi_2) \circ \theta)$ . Since  $\theta$  is an isomorphism this is equivalent to  $\theta(Q) \subset \ker(\pi_1 \otimes \pi_2)$ . By Proposition 6, [Bou98, p. 252] the kernel of  $\pi_1 \otimes \pi_2$  has the following simple description

$$\ker(\pi_1 \otimes \pi_2) = \langle s \otimes r', r \otimes s' : r \in R, r' \in R', s \in S, s' \in S' \rangle.$$

Thus let  $x$  be a generator of  $Q$  as in (3.19), then

$$\begin{aligned} \theta(x) &= \theta(\varphi^*(f) \otimes (m \otimes m')) - \theta(f \otimes \varphi_*(m \otimes m')) \\ &= \theta((f \circ \varphi) \otimes (m \otimes m')) - \theta(f \otimes \varphi_*(m) \otimes \varphi_*(m')) \end{aligned}$$

$$\begin{aligned}
&= ((\tilde{f}_1 \circ \tilde{\varphi}_1) \otimes m \otimes (\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m') - (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes \tilde{f}_2 \otimes \tilde{\varphi}_{2*}(m')) \\
&= ((\tilde{f}_1 \circ \tilde{\varphi}_1) \otimes m \otimes (\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m') - (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes (\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m') \\
&\quad + (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes (\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m') - (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes \tilde{f}_2 \otimes \tilde{\varphi}_{2*}(m')) \\
&= (((\tilde{f}_1 \circ \tilde{\varphi}_1) \otimes m - \tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m)) \otimes (\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m') \\
&\quad + (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes ((\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m' - \tilde{f}_2 \otimes \tilde{\varphi}_{2*}(m'))) \\
&= \left( \underbrace{((\tilde{\varphi}_1^*(\tilde{f}_1) \otimes m - \tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m))}_{\in S} \otimes \underbrace{((\tilde{f}_2 \circ \tilde{\varphi}_2) \otimes m')}_{\in R'} \right) \\
&\quad + \left( \underbrace{(\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m))}_{\in R} \otimes \underbrace{((\tilde{\varphi}_2^*(\tilde{f}_2) \otimes m' - \tilde{f}_2 \otimes \tilde{\varphi}_{2*}(m'))}_{\in S'} \right) \\
&\in \ker(\pi_1 \otimes \pi_2).
\end{aligned}$$

This concludes the first part of the claim of the lemma.

It remains to show that the epimorphism  $\eta$  is actually an isomorphism. For this we need to show that  $\theta$  maps  $Q$  epimorphically onto  $\ker(\pi_1 \otimes \pi_2)$ .

Let  $s$  be a generator of  $S$  as in (3.20). Let  $f_2: G_2/H_2 \rightarrow G_1/\tilde{K}_2$  be an arbitrary  $G_2$ -map and  $m' \in M'(G_2/H_2)$ , that is  $r := f_2 \otimes m'$  is a generator of  $R'$ . Let  $\varphi: G_1/L_1 \times G_2/H_2 \rightarrow G_1/H_1 \times G_2/H_2$  be the  $G$ -map given by  $\varphi(L_1 \times H_2) := \varphi_1(H_1) \times H_2$ . Then  $\tilde{\varphi}_1 = \varphi_1$  and  $\tilde{\varphi}_2 = \text{id}$ . Set  $L := L_1 \times H_2$  and  $H := H_1 \times H_2$ . Furthermore set  $f := f_1 \times f_2$ . Then  $\tilde{f}_1 = f_1$  and  $\tilde{f}_2 = f_2$  by construction. With these definitions it follows that  $x := \varphi^*(f) \otimes (m \otimes m') - f \otimes \varphi_*(m \otimes m')$  is a generator of  $Q$  and

$$\begin{aligned}
\theta(x) &= \theta(\varphi^*(f) \otimes (m \otimes m')) - \theta(f \otimes \varphi_*(m \otimes m')) \\
&= (\tilde{\varphi}_1^*(\tilde{f}_1) \otimes m \otimes \tilde{\varphi}_2^*(\tilde{f}_2) \otimes m') - (\tilde{f}_1 \otimes \tilde{\varphi}_{1*}(m) \otimes \tilde{f}_2 \otimes \tilde{\varphi}_{2*}(m')) \\
&= (\varphi_1^*(f_1) \otimes m \otimes f_2 \otimes m') - (f_1 \otimes \varphi_1(m) \otimes f_2 \otimes m') \\
&= (\varphi_1(f_1) \otimes m - f_1 \otimes \varphi_1(m)) \otimes (f_2 \otimes m') \\
&= s \otimes r'.
\end{aligned}$$

Similarly to (3.20) we can show that any element  $r \otimes s'$  with  $r$  a generator of  $R$  and  $s'$  a generator of  $S'$  is contained in  $\theta(Q)$ . Thus  $\ker(\pi_1 \otimes \pi_2) \subset \theta(Q)$  and equality holds.  $\square$

**Lemma 3.65.** *The isomorphism  $\eta$  in Lemma 3.64 is natural in each of its factors. That is, given morphisms  $f_1: \mathbb{Z}[?, G_1/\tilde{K}_1]_{G_1} \rightarrow \mathbb{Z}[?, G_1/\tilde{L}_1]_{G_1}$ ,*

$f_2: \mathbb{Z}[?, G_2/\tilde{K}_2]_{G_2} \rightarrow \mathbb{Z}[?, G_2/\tilde{L}_2]_{G_2}$ ,  $f_3: M \rightarrow N$ ,  $f_4: M' \rightarrow N'$  we have

$$\eta \circ (f_1, f_2, f_3, f_4) = ((f_1 \otimes f_2) \otimes_{\mathfrak{F}} (f_3 \otimes f_4)) \circ \eta.$$

PROOF. This follows immediately from the simple form of the map (3.14).  $\square$

**Proposition 3.66.** *Let  $F = \mathbb{Z}[?, X_1]_{G_1}$  and  $F' = \mathbb{Z}[?, X_2]_{G_2}$  be a free  $\mathcal{O}_{\mathfrak{F}_1} G_1$ -module and  $\mathcal{O}_{\mathfrak{F}_2} G_2$ -module. Then the isomorphism  $\eta$  defined in Lemma 3.64 induces in a canonical way an isomorphism*

$$\begin{aligned} \eta: (\text{res}_{p_1} F \otimes \text{res}_{p_2} F') \otimes_{\mathfrak{F}} (\text{res}_{p_1} M \otimes \text{res}_{p_2} M') \\ \longrightarrow (F \otimes_{\mathfrak{F}_1} M) \otimes (F' \otimes_{\mathfrak{F}_2} M'). \end{aligned}$$

PROOF. The main work has already been done in Lemma 3.64 and the remaining claim follows from the additivity of the setup. For completeness, the purely technical details are as follows. Write

$$X_1 = \coprod_{\alpha} G_1/H_{1,\alpha} \quad \text{and} \quad X_2 = \coprod_{\beta} G_2/H_{2,\beta}$$

as the disjoint union of transitive  $G_1$ -sets and  $G_2$ -sets with stabilisers in  $\mathfrak{F}_1$  and  $\mathfrak{F}_2$  respectively. Then  $F = \coprod F_{\alpha}$  with  $F_{\alpha} := \mathbb{Z}[?, G_1/H_{1,\alpha}]_{G_1}$  and similarly  $F' = \coprod F'_{\beta}$  with  $F'_{\beta} := \mathbb{Z}[?, G_2/H_{2,\beta}]_{G_2}$ . We get

$$\begin{aligned} (\text{res}_{p_1} F \otimes \text{res}_{p_2} F') \otimes_{\mathfrak{F}} (\text{res}_{p_1} M \otimes \text{res}_{p_2} M') \\ = \coprod_{\alpha, \beta} ((\text{res}_{p_1} F_{\alpha} \otimes \text{res}_{p_2} F'_{\beta}) \otimes_{\mathfrak{F}} (\text{res}_{p_1} M \otimes \text{res}_{p_2} M')) \end{aligned}$$

which is mapped by  $\eta := \coprod \eta_{\alpha, \beta}$  summand wise isomorphically onto

$$\coprod_{\alpha, \beta} ((F_{\alpha} \otimes_{\mathfrak{F}_1} M) \otimes (F'_{\beta} \otimes_{\mathfrak{F}_2} M')) = (F \otimes_{\mathfrak{F}_1} M) \otimes (F' \otimes_{\mathfrak{F}_2} M'). \quad \square$$

Now let  $F_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  be a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}_1} G_1$ -module and let  $F'_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  be a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}_2} G_2$ -module. From the previous section we know that the total complex of  $\text{res}_{p_1} F_* \otimes \text{res}_{p_2} F'_*$  gives a free resolution of the trivial  $\mathcal{O}_{\mathfrak{F}} G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$ . Tensoring this total complex over the orbit category  $\mathcal{O}_{\mathfrak{F}} G$  with  $\text{res}_{p_1} M \otimes \text{res}_{p_2} M'$  gives a chain complex whose objects are given by

$$C_n := \coprod_{k=0}^n ((\text{res}_{p_1} F_k \otimes \text{res}_{p_2} F'_{n-k}) \otimes_{\mathfrak{F}} (\text{res}_{p_1} M \otimes \text{res}_{p_2} M'))$$

The isomorphism from Proposition 3.66 maps these groups isomorphically onto

$$C'_n := \prod_{k=0}^n ((F_k \otimes_{\mathfrak{F}_1} M) \otimes (F'_{n-k} \otimes_{\mathfrak{F}_2} M')).$$

We denote these isomorphisms by  $\eta_n: C_n \rightarrow C'_n$ ,  $n \in \mathbb{N}$ . Using Lemma 3.65 one can conclude that this collection of isomorphisms defines a chain map

$$\eta: C_* \rightarrow C'_*.$$

Now  $C'_*$  is nothing else than the chain complex obtained as a total complex of  $(F_* \otimes_{\mathfrak{F}_1} M) \otimes (F'_* \otimes_{\mathfrak{F}_2} M')$ . Following [Bro82, pp. 108f.], if  $z$  is a  $p$ -cycle of  $F_* \otimes_{\mathfrak{F}_1} M$  and  $z'$  a  $q$ -cycle of  $F'_* \otimes_{\mathfrak{F}_2} M'$ , then

$$z \times z' := \eta^{-1}(z \otimes z')$$

is a  $(p+q)$ -cycle of  $C_*$ . Its homology class depends only on the homology class of  $z$  and  $z'$ . Thus we obtain a *homology cross product*

$$\times: H_p^{\mathfrak{F}_1}(G_1; M) \otimes H_q^{\mathfrak{F}_2}(G_2; M') \rightarrow H_{p+q}^{\mathfrak{F}_1 \times \mathfrak{F}_2}(G_1 \times G_2; \text{res}_{p_1} M \otimes \text{res}_{p_2} M')$$

which maps  $[z] \otimes [z'] \mapsto [z] \times [z'] := [z \times z']$  as in the classical case.

**Theorem 3.67** (Künneth Formula for Bredon Homology). *Assume that there exists free resolutions  $F_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_1}$  and  $F'_* \rightarrow \underline{\mathbb{Z}}_{\mathfrak{F}_2}$  such that the chain complex  $F_* \otimes_{\mathfrak{F}_1} M$  or the chain complex  $F'_* \otimes_{\mathfrak{F}_2} M'$  is a free chain complex. Then for every  $n \in \mathbb{N}$  there exists a short exact sequence*

$$\begin{aligned} 0 \rightarrow \prod_{k=0}^n H_k^{\mathfrak{F}_1}(G_1; M) \otimes H_{n-k}^{\mathfrak{F}_2}(G_2; M') \\ \xrightarrow{\alpha} H_n^{\mathfrak{F}_1 \times \mathfrak{F}_2}(G_1 \times G_2; \text{res}_{p_1} M \otimes \text{res}_{p_2} M') \\ \rightarrow \prod_{k=0}^{n-1} \text{Tor}_1(H_k^{\mathfrak{F}_1}(G_1; M), H_{n-k-1}^{\mathfrak{F}_2}(G_2; M')) \rightarrow 0 \end{aligned}$$

of abelian groups. The homomorphism  $\alpha$  is given by the homological cross product, that is,  $\alpha(z \otimes z') := z \times z'$ . If both chain complexes  $F_* \otimes_{\mathfrak{F}_1} M$  and  $F'_* \otimes_{\mathfrak{F}_2} M'$  are dimension wise free, then this sequence is split, but this splitting is not natural.

PROOF. This is essentially the Künneth Formula for chain complexes of abelian groups applied to chain complexes  $F_* \otimes_{\mathfrak{F}_1} M$  and  $F'_* \otimes_{\mathfrak{F}_2} M'$  (see for example [Spa66, pp. 227ff.]) together with Proposition 3.66.  $\square$



## Bredon Dimensions for the Family $\mathfrak{F}_{\text{vc}}$

### 1. A Geometric Lower Bound for $\text{hd}_{\mathfrak{F}} G$ and $\text{cd}_{\mathfrak{F}} G$

Assume that  $\mathfrak{F}$  is a semi-full family of subgroups of  $G$ . Then the geometric dimension  $\text{gd}_{\mathfrak{F}} G$  is defined and gives an upper bound for the dimension  $\text{cd}_{\mathfrak{F}} G$ , and therefore also for  $\text{hd}_{\mathfrak{F}} G$ . In this section, we will prove a result that can be used to obtain a lower bound for the cohomological and homological Bredon dimension of  $G$  using geometrical methods.

**Definition 4.1.** Let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . We say that  $Y$  is a model for  $B_{\mathfrak{F}}G$  if there exists a model  $X$  for  $E_{\mathfrak{F}}G$  such that  $Y = X/G$ , that is,  $Y$  is the orbit space of some classifying space for the family  $\mathfrak{F}$ .

The main result of this section will be the following theorem. It is essentially the generalisation of the classical fact that

$$H_n(G; \mathbb{Z}) \cong H_n(Y) \quad \text{and} \quad H^n(G; \mathbb{Z}) \cong H^n(Y)$$

where  $Y$  is an Eilenberg–Mac Lane space  $K(G, 1)$ , see for example [Bro82, pp. 36ff.].

**Theorem 4.2.** *Let  $G$  be a group and let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . Then for every  $n \in \mathbb{N}$  we have isomorphisms*

$$H_n^{\mathfrak{F}}(G; \mathbb{Z}_{\mathfrak{F}}) \cong H_n(B_{\mathfrak{F}}G) \quad \text{and} \quad H_{\mathfrak{F}}^n(G; \mathbb{Z}_{\mathfrak{F}}) \cong H^n(B_{\mathfrak{F}}G)$$

*of abelian groups.*

Now this result together with Proposition 3.10 and Proposition 3.11 implies the following immediate result.

**Corollary 4.3.** *If  $H_n(B_{\mathfrak{F}}G) \neq 0$  then  $\text{hd}_{\mathfrak{F}} G \geq n$ . Likewise  $H^m(B_{\mathfrak{F}}G) \neq 0$  implies  $\text{cd}_{\mathfrak{F}} G \geq m$ .  $\square$*

Before we can prove this theorem we need the following auxiliary result. This result is the main reason for the Theorem 4.2 to be true.

**Proposition 4.4.** *Let  $\mathfrak{F}$  be a semi-full family of subgroups of  $G$ . Then we have an isomorphism  $\underline{C}_*(E_{\mathfrak{F}}G) \otimes_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}} \cong C_*(B_{\mathfrak{F}}G)$  of chain complexes.*

PROOF. To simplify the notation, denote by  $X$  a model for  $E_{\mathfrak{F}}G$  and by  $\pi: X \rightarrow X/G$  the canonical projection onto the orbit space. For each  $H \in \mathfrak{F}$  this map restricts to the map  $\pi: X^H \rightarrow X/G$  which in turn induces a chain map

$$\pi_H: C_*(X^H) \rightarrow C_*(X/G).$$

Then the coproduct of these chain maps is a chain map

$$P_* := \coprod_{H \in \mathfrak{F}} C_*(X^H) \rightarrow C_*(X/G)$$

which we will denote (with abuse of notation) by  $\pi$ .

Let  $\tau$  be an  $n$ -cell of the CW-complex  $X/G$ . Then there exists an  $n$ -cell  $\sigma$  of the  $G$ -CW-complex  $X$  such that  $\pi(\sigma) = \tau$ . Let  $H$  be the isotropy group  $G_\sigma$  of  $\sigma$ . Then  $H \in \mathfrak{F}$  and  $\tau$  lies in the image of  $\pi_H$ . Therefore  $\pi$  is surjective.

Denote by  $Q_*$  the kernel of the map  $\pi$ . We claim that

$$\underline{C}_*(X) \otimes_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}} = P_*/Q_*. \quad (4.1)$$

Recall that by definition

$$\underline{C}_*(X) \otimes_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}} = P'_*/Q'_*$$

where

$$P'_* = \coprod_{H \in \mathfrak{F}} C_*(X^H) \otimes \mathbb{Z} = \coprod_{H \in \mathfrak{F}} C_*(X^H) = P_*$$

and  $Q'_*$  is the subcomplex generated by the elements of the form  $\varphi^*(\sigma) - \sigma$ . Note that in the case of the trivial  $\mathcal{O}_{\mathfrak{F}}G$ -module  $\underline{\mathbb{Z}}_{\mathfrak{F}}$  we have  $\varphi_* = \text{id}$ . Now (4.1) follows from the following claim.

**Claim 1.**  $Q'_n = Q_n$  for all  $n \in \mathbb{N}$ .

“ $Q'_n \subset Q_n$ ”: This inclusion follows immediately from

$$\pi(\varphi^*(\sigma) - \sigma) = \pi(g\sigma) - \pi(\sigma) = 0.$$

“ $Q'_n \supset Q_n$ ”: Let  $x \in Q_n$ . Then there exists pairwise distinct orbits  $A_1, \dots, A_n$  of orbits of  $n$ -cells of  $X$  such that we can write

$$x = x_1 + \dots + x_r$$

with each  $x_s$ ,  $1 \leq s \leq r$  satisfying the following: there exists  $k_s \geq 1$  and for  $1 \leq i \leq k_s$  there exist unique  $H_{s,i} \in \mathfrak{F}$ ,  $\sigma_{s,i} \in X^{H_{s,i}}$  and  $a_{s,i} \in \mathbb{Z} \setminus \{0\}$  such that  $\sigma_{s,i} \in A_s$  and

$$x_s = \sum_{i=1}^{k_s} a_{s,i} \sigma_{s,i}.$$

Now  $\pi(x) = 0$  if and only if  $\pi(x_s) = 0$  for each  $1 \leq s \leq r$ . In particular each  $x_s \in Q_n$  and it is enough to show that each  $x_s \in Q'_n$ .

Thus assume that  $r = 1$ . We omit the variable  $s$  from the notation, that is

$$x = \sum_{i=1}^k a_i \sigma_i \tag{4.2}$$

where the  $\sigma_i$  are  $n$ -cell of  $X^{H_i}$  and the  $a_i$  are all non-zero integers and the  $\sigma_i$  belong all to the same orbit.

We can find elements  $g_i \in G$  such that  $\sigma_{i+1} = g_i \sigma_i$  for  $i = 1, \dots, k-1$ . For each  $i$  set  $K_i := H_i \cap g_i H_i g_i^{-1}$ . Since  $\mathfrak{F}$  is assumed to be closed under conjugation and taking finite intersections, it follows that the  $K_i$  are all elements of  $\mathfrak{F}$ . By construction we have that  $g_i^{-1} K_i g_i = g_i^{-1} H_i g_i \cap H_i \leq H_i$  and thus there exists a  $G$ -map  $\varphi_i: G/K_i \rightarrow G/H_i$  that maps  $K_i$  to  $g_i H_i$ . Then  $\varphi_i^*(\sigma_i) = g_i \sigma_i = \sigma_{i+1}$ .

Using this equality together with  $s_i := a_1 + \dots + a_i$ , we can successively rewrite the right hand side of (4.2) in the following way:

$$\begin{aligned} x &= a_1(\sigma_1 - \sigma_2) + (a_1 + a_2)\sigma_2 + a_3\sigma_3 + \dots + a_k\sigma_k \\ &= s_1(\sigma_1 - \varphi_1^*(\sigma_1)) + s_2(\sigma_2 - \sigma_3) + (s_2 + a_3)\sigma_3 + a_4\sigma_4 + \dots + a_k\sigma_k \\ &= s_1(\sigma_1 - \varphi_1^*(\sigma_1)) + s_2(\sigma_2 - \varphi_2^*(\sigma_2)) + \dots + \\ &\quad + s_{k-1}(\sigma_{k-1} - \varphi_{k-1}^*(\sigma_{k-1})) + s_k\sigma_k. \end{aligned}$$

On the other hand, by assumption we have  $\pi(\sigma_1) = \dots = \pi(\sigma_k)$  and thus  $\pi(x) = a_1\pi(\sigma_1) + \dots + a_k\pi(\sigma_k) = s_k\pi(\sigma_1)$  and this is equal to 0 if and only if  $s_k = 0$ . Thus

$$x = \sum_{i=1}^{k-1} s_i(\sigma_i - \varphi_i^*(\sigma_i)) \in Q'_n.$$

Alltogether  $Q'_n = Q_n$  for every  $n \in \mathbb{N}$  and this proves Claim 1. On the other hand, Claim 1 implies the claim of the proposition and this concludes the proof.  $\square$

PROOF OF THEOREM 4.2. First of all, we have

$$\begin{aligned}
H_n(B_{\mathfrak{F}}G) &\cong H_n(C_*(B_{\mathfrak{F}}G)) \\
&\cong H_n(\underline{C}_*(E_{\mathfrak{F}}G) \otimes_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}}) && \text{(by Proposition 4.4)} \\
&\cong H_n^{\mathfrak{F}}(G; \underline{\mathbb{Z}}_{\mathfrak{F}})
\end{aligned}$$

and this proves the first isomorphism.

In order to verify the second isomorphism, we first observe that we have

$$\begin{aligned}
\text{hom}(C_*(B_{\mathfrak{F}}G), \mathbb{Z}) &\cong \text{mor}_{\mathfrak{F}}(C_*(B_{\mathfrak{F}}G), \underline{\mathbb{Z}}_{\mathfrak{F}}) \\
&\cong \text{mor}_{\mathfrak{F}}(\underline{C}_*(E_{\mathfrak{F}}G) \otimes_{\mathfrak{F}} \underline{\mathbb{Z}}_{\mathfrak{F}}, \underline{\mathbb{Z}}_{\mathfrak{F}}) && \text{(by Proposition 4.4)} \\
&\cong \text{mor}_{\mathfrak{F}}(\underline{C}_*(E_{\mathfrak{F}}G), \text{mor}_{\mathfrak{F}}(\underline{\mathbb{Z}}_{\mathfrak{F}}, \underline{\mathbb{Z}}_{\mathfrak{F}})) && \text{(adjoint isomorphism)} \\
&\cong \text{mor}_{\mathfrak{F}}(\underline{C}_*(E_{\mathfrak{F}}G), \underline{\mathbb{Z}}_{\mathfrak{F}}).
\end{aligned}$$

From this it follows

$$\begin{aligned}
H^n(B_{\mathfrak{F}}G) &\cong H_n(\text{hom}(C_*(B_{\mathfrak{F}}G), \mathbb{Z})) \\
&= H_n(\text{mor}_{\mathfrak{F}}(\underline{C}_*(E_{\mathfrak{F}}G), \underline{\mathbb{Z}}_{\mathfrak{F}})) \\
&\cong H_{\mathfrak{F}}^n(G; \underline{\mathbb{Z}}_{\mathfrak{F}})
\end{aligned}$$

which is the second isomorphism.  $\square$

## 2. Applications of Theorem 4.2

In order to apply the result of Theorem 4.2 we need groups  $G$  for which we know nice enough models  $X$  for  $\underline{EG}$  so that we can determine the homology or cohomology groups of  $X/G$ . In this section we present examples where this is the case. Juan-Pineda and Leary have described in [JPL06, p. 138] a model for  $\underline{\mathbb{Z}\mathbb{Z}^2}$ . The construction goes back to Lück and it is as follows.

Label the maximal infinite cyclic subgroups of  $\mathbb{Z}^2$  by  $H_i$ ,  $i \in \mathbb{Z}$ . We have  $\mathbb{Z}^2/H_i \cong \mathbb{Z}$  and there exists a 1-dimensional model  $X_i$  for  $E(\mathbb{Z}^2/H_i)$ . It is a line on which  $\mathbb{Z}^2/H_i$  acts by translation. Let  $\pi_i: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2/H_i$  be the canonical projection. Then  $\mathbb{Z}^2$  acts on  $X_i$  by  $gx := \pi_i(g)x$ .

For each  $i \in \mathbb{Z}$  we consider the join  $X_i * X_{i+1}$ . There exists canonical embeddings  $\varphi_i: X_i \hookrightarrow X_i * X_{i+1}$  and  $\psi_i: X_i \hookrightarrow X_{i-1} * X_i$ . We let  $X$  be the  $\mathbb{Z}^2$ -space

$$X := \left( \prod_{i \in \mathbb{Z}} X_i * X_{i+1} \right) / \sim$$

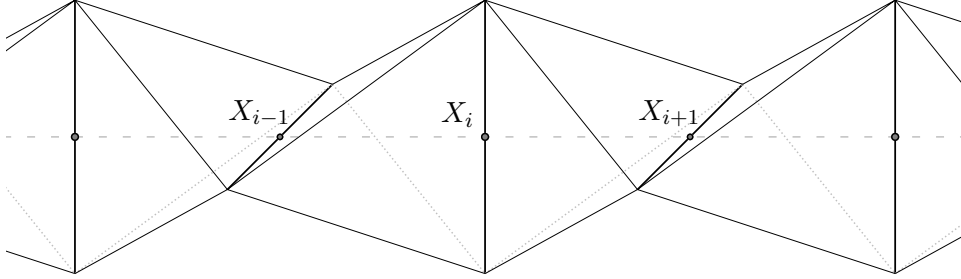


FIGURE 3. A model for  $\underline{E}\mathbb{Z}^2$ . The thick lines represents the models  $X_i \cong \mathbb{R}$  for  $E\mathbb{Z}^2/H_i$ .

where the equivalence relation “ $\sim$ ” is given by

$$y_1 \sim y_2 :\iff \exists x \in X_i: \varphi_i(x) = y_1 \text{ and } \psi_i(x) = y_2.$$

Note that the embeddings  $\varphi_i: X_i \hookrightarrow X_i * X_{i+1}$  induce embeddings  $\varphi_i: X_i \hookrightarrow X$  and we use them to identify the  $X_i$  as subspaces of  $X$ . See Figure 3 for a schematic picture of the space  $X$ .

It follows that  $X$  is a 3-dimensional model for  $\underline{E}\mathbb{Z}^2$ . This is because of the following observations:

- (1)  $X$  is contractible by construction;
- (2) if  $H$  is an infinite cyclic subgroup of  $\mathbb{Z}^2$ , then  $H \leq H_i$  for some unique  $i \in \mathbb{Z}$  and therefore  $X^H = X^{H_i} = X_i$  which is contractible;
- (3) if  $H$  is not cyclic, then  $H = K_1 \times K_2$  with  $K_1$  and  $K_2$  infinite cyclic subgroups of  $\mathbb{Z}^2$  and

$$X^H \subset X^{K_1} \cap X^{K_2} = X^{H_{i_1}} \cap X^{H_{i_2}} = \emptyset$$

for some  $i_1, i_2 \in \mathbb{Z}$ ,  $i_1 \neq i_2$ .

We have that  $(X_i * X_{i+1})/G \cong S^3$  and  $X_i/G \cong S^1$  for every  $i \in \mathbb{Z}$  and thus  $H_3(X/G)$  is free abelian of infinite rank [JPL06, p. 138]. In particular  $X$  is a model for  $\underline{E}\mathbb{Z}^2$  of minimal dimension, that is  $\underline{\text{gd}}\mathbb{Z}^2 = 3$ .

Now Theorem 4.2 states that  $\underline{\text{hd}}\mathbb{Z}^2 \geq 3$  and therefore we get the following complete statement about the Bredon dimensions with respect to the family of virtually cyclic subgroups.

**Proposition 4.5.**  $\underline{\text{hd}} \mathbb{Z}^2 = \underline{\text{cd}} \mathbb{Z}^2 = \underline{\text{gd}} \mathbb{Z}^2 = 3.$  □

From this result we obtain immediately two interesting consequences regarding virtually polycyclic groups.

**Proposition 4.6.** *Let  $G$  be a virtually polycyclic group. Then*

$$\underline{\text{cd}} G = \underline{\text{gd}} G.$$

PROOF. To avoid triviality we assume that  $G$  is not virtually cyclic. By Proposition 2 in [Seg83, p. 2], the group  $G$  is virtually poly- $\mathbb{Z}$ . Since  $G$  is not virtually cyclic we have  $\text{vcd } G \geq 2$ . In particular  $G$  contains a subgroup that is an extension of  $\mathbb{Z}$  by  $\mathbb{Z}$ , which in turn contains a subgroup isomorphic to  $\mathbb{Z}^2$ . Thus  $\underline{\text{cd}} G \geq \underline{\text{cd}} \mathbb{Z}^2 = 3$ . Now the claim follows from Proposition 3.18. □

Note that in Theorem 5.12 in [LW07, p. 22] a complete description of  $\underline{\text{gd}} G$  for virtually polycyclic groups are given. The above result states that the same theorem gives a complete description of  $\underline{\text{cd}} G$  for virtually polycyclic groups, too.

**Proposition 4.7.** *Let  $G$  be a virtually polycyclic group with  $\text{vcd } G = 2$ . Then*

$$\underline{\text{hd}} G = \underline{\text{cd}} G = \underline{\text{gd}} G = 3.$$

PROOF. Since  $G$  has a subgroup isomorphic to  $\mathbb{Z}^2$  we have  $\underline{\text{hd}} G \geq \underline{\text{hd}} \mathbb{Z}^2 = 3$ . On the other hand Theorem 5.12 in [LW07, p. 22] gives  $\underline{\text{gd}} G = 3$  and thus all three Bredon dimensions agree and are equal to 3. □

The next two applications of Theorem 4.2 rely on Juan-Pineda and Leary's construction of a model for  $\underline{E}G$  for a class of group which includes Gromov-hyperbolic groups, see Proposition 9 and Corollary 10 in [JPL06].

The class of groups considered in [JPL06] is characterised by the following condition: Every infinite virtually cyclic subgroup  $H$  of  $G$  is contained in a unique maximal virtually cyclic subgroup  $H_{\max}$  of  $G$  which is equal to its own normaliser. This class is known to contain all Gromov-hyperbolic groups [GdlH90, Theorem 8.37].

Recall that if  $H$  is a virtually cyclic subgroup it is known (see for example [JPL06]) that  $H$  has a unique maximal normal finite subgroup  $N$  and one of the following three cases is true:  $H$  is finite,  $H/N$  is infinite cyclic (in

this case we call  $H$  orientable) or  $H/N$  is infinite dihedral (in this case we call  $H$  non-orientable).

**Proposition 4.8.** [JPL06, Proposition 9 and Corollary 10] *Let  $G$  be a group satisfying the above condition on the set of infinite virtually cyclic subgroups. Let  $\mathcal{C}$  be a complete system of representatives for the conjugacy classes of maximal infinite virtually cyclic subgroups of  $G$ . Denote by  $\mathcal{C}_o$  and the set of orientable elements of  $\mathcal{C}$  and denote by  $\mathcal{C}_n$  the set of non-orientable elements of  $\mathcal{C}$ . Then a model for  $\underline{\underline{E}}G$  can be obtained from model for  $\underline{E}G$  by attaching*

- (1) orbits of 0-cells indexed by  $\mathcal{C}$ ;
- (2) orbits of 1-cells indexed by  $\mathcal{C}_o \cup \{1, 2\} \times \mathcal{C}_n$ ;
- (3) orbits of 2-cells indexed by  $\mathcal{C}$ .

Furthermore, a model for  $\underline{B}G$  can be obtained from a model for  $\underline{B}G$  by attaching 2-cells indexed by  $\mathcal{C}_o$ .  $\square$

**Proposition 4.9.** *Let  $G$  be a Gromov-hyperbolic group which is not virtually cyclic. Then  $\underline{\underline{hd}}G \geq 2$ . Moreover, if  $\underline{\underline{gd}}G \leq 2$ , then*

$$\underline{\underline{hd}}G = \underline{\underline{cd}}G = \underline{\underline{gd}}G = 2.$$

PROOF. It has been shown in [JPL06, p. 141], that if  $G$  is Gromov-hyperbolic group which not virtually cyclic, then  $H_2(\underline{B}G) \neq 0$ . This follows from the following two facts:

- (1) for large enough integers  $d$  the Ribs complex  $R_d(G)$  is a *finite* model for  $\underline{E}G$  [BCH94, MS02];
- (2)  $G$  has infinitely many conjugacy classes of maximal infinite virtually cyclic subgroups [JPL06, p. 141, Theorem 13].

Thus it follows from Theorem 4.2 that  $\underline{\underline{hd}}G \geq 2$  if  $G$  is a Gromov-hyperbolic group which is not virtually cyclic.

If moreover  $\underline{\underline{gd}}G \leq 2$ , then there exists a 2-dimensional model for  $\underline{E}G$  by Proposition 4.8, that is  $\underline{\underline{gd}}G \leq 2$ . Therefore we have altogether

$$2 \leq \underline{\underline{hd}}G \leq \underline{\underline{cd}}G \leq \underline{\underline{gd}}G \leq 2$$

and equality holds.  $\square$

**Corollary 4.10.** *Let  $F$  be a free group of rank at least 2. Then*

$$\underline{\underline{hd}}F = \underline{\underline{cd}}F = \underline{\underline{gd}}F = 2.$$

PROOF. Free groups are characterised by the fact that they can act freely on a tree, that is  $\text{gd } F = 1$ . Since  $F$  is torsion free it follows that  $\text{gd } F = \text{gd } F = 1$ . Free groups are Gromov-hyperbolic and thus the claim follows from Proposition 4.9.  $\square$

Another example of Gromov-hyperbolic groups are fundamental groups of finite graphs of finite groups. We use the notation as introduced by Serre in [Ser80, pp. 41ff.]. Given a connected, non-empty, orientated graph  $Y$ , a graph  $(G, Y)$  of groups consists of

- (1) a collection of groups  $G_P$  indexed by the vertices  $P \in \text{vert } Y$  of  $Y$ ;
- (2) a collection of groups  $G_y$  indexed by the edges  $y \in \text{edge } Y$  of  $Y$  subject to the condition  $G_y = G_{\bar{y}}$  where  $\bar{y}$  denotes the inverse edge of  $y$ ;
- (3) for each edge  $y \in \text{edge } Y$ , a monomorphism  $G_y \hookrightarrow G_{t(y)}$  where  $t(y)$  denotes the terminal vertex of the edge  $y$ .

From this data one can construct the *fundamental group*  $\pi_1(G, Y)$  of the graph of groups  $(G, Y)$ , see [Ser80]. By abuse of notation we say that a group is the *fundamental group* of the graph of groups  $(G, Y)$  if  $G \cong \pi_1(G, Y)$ .

The  $G$  is the fundamental group of a graph of groups, then there exists canonical inclusions of the vertex groups  $G_P$ ,  $P \in \text{vert } Y$ , into  $G$ . Furthermore, one can construct a tree  $T$  with an action of  $G$  with vertex stabilisers being precisely the conjugates of the groups  $G_P$  and with the edge stabilisers being precisely the conjugates of the edge groups  $G_y$ ,  $y \in \text{edge } Y$ , and such that  $T/G = Y$ . This tree is called the *universal cover of  $(G, Y)$*  or the *Bass–Serre tree* associated with the fundamental group of the graph of groups  $(G, Y)$ . Conversely, every group  $G$  which admits an action on a tree  $T$  is the fundamental group of some graph of groups  $(G, Y)$  such that  $T$  is the associated Bass–Serre tree of the graph of groups.

- Examples 4.11.**
- (1) Let  $(1, Y)$  be a graph of groups whose vertex and edge groups are all trivial. Then  $\pi_1(1, Y) \cong \pi_1(Y)$  is the fundamental group of the graph  $Y$ .
  - (2) Let  $Y$  be the graph with two edges (that is non-oriented edge) and two vertices, that is a line segment. Let  $(G, Y)$  be the graph of groups with vertex groups  $A$  and  $B$  and with edge group  $C$ , see



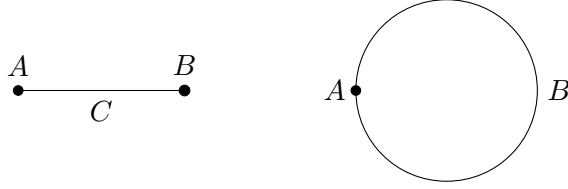


FIGURE 4. A segment of groups and loop of groups [Ser80, p. 41].

Figure 4. Then  $\pi_1(G, Y) \cong A *_C B$  is the free product of  $A$  and  $B$  with the common subgroup  $C$  amalgamated.

- (3) Let  $Y$  be the graph with two edges and one vertex, that is a loop. Let  $(G, Y)$  be the graph of groups with vertex group  $A$  and with edge group  $B$ , see Figure 4. Then  $\pi_1(G, Y)$  is an HNN-extension of the group  $A$ .

**Proposition 4.12.** *Let  $G$  be the fundamental group of a finite graph of finite groups  $(G, Y)$ , and assume that  $G$  is not virtually cyclic. Then*

$$\underline{\text{hd}} G = \underline{\text{cd}} G = \underline{\text{gd}} G = 2.$$

PROOF. Fundamental groups of a finite graph of finite groups have a free subgroup of finite index [DD89, p. 104]. Free groups are Gromov-hyperbolic and the property of being Gromov-hyperbolic is preserved by finite extensions. Therefore the group  $G$  is Gromov-hyperbolic.

Fundamental groups of finite graphs of groups are known to admit a 1-dimensional model for  $\underline{EG}$ . To see this, let  $T$  be the Bass–Serre tree associated with the graph of groups  $(G, Y)$ . Then  $\mathfrak{F}(T) \subset \mathfrak{F}_{\text{fin}}(G)$ . On the other hand, it is a well known fact that a finite group cannot act fixed point free on a tree and therefore  $T^H \neq \emptyset$  for all  $H \in \mathfrak{F}_{\text{fin}}(G)$ . Thus  $T$  is a one-dimensional model for  $\underline{EG}$  and we have  $\underline{\text{gd}} G \leq 1$ .

Altogether the conditions of Proposition 4.9 are satisfied and the claim follows since  $G$  is not virtually cyclic by assumption.  $\square$

### 3. Dimensions of Extensions of $C_\infty$

The results of this section rely on a result which Martínez-Pérez has obtained using a spectral sequence that she has constructed for Bredon (co-)homology in [MP02].

**Proposition 4.13** (Martínez-Pérez). *Let  $G$  be an extension*

$$0 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 0$$

*of a group  $N$  by a group  $Q = G/N$ . Let*

$$\mathfrak{H} := \{H \leq G : N \leq H \text{ and } H/N \in \mathfrak{F}_{\text{vc}}(Q)\}$$

*and set*

$$m := \sup\{\underline{\text{hd}} H : H \in \mathfrak{H}\} \quad \text{and} \quad n := \sup\{\underline{\text{cd}} H : H \in \mathfrak{H}\}.$$

*Then we have the estimates:*

$$m \leq \underline{\text{hd}} G \leq \underline{\text{hd}} Q + m \quad \text{and} \quad n \leq \underline{\text{cd}} G \leq \underline{\text{cd}} Q + n.$$

PROOF. The lower bound is a direct consequence of Proposition 3.32. The upper bound is due to Martínez-Pérez's result in [MP02, pp. 171f.].  $\square$

For future reference we state the following corollary to this result.

**Corollary 4.14.** *Let  $G$  be an extension of a group  $N$  by a group  $Q$ . Assume that  $\underline{\text{hd}} Q < \infty$ . Then  $\underline{\text{hd}} G < \infty$  if and only if there exists an integer  $k$  such that  $\underline{\text{hd}} H \leq k$  for every  $N \leq H \leq G$  such that  $H/N$  is virtually cyclic.*

*The statement remains true if “ $\underline{\text{hd}}$ ” is replaced by “ $\underline{\text{cd}}$ ” or “ $\underline{\text{gd}}$ ”.*

PROOF. The statements about  $\underline{\text{hd}} G$  and  $\underline{\text{cd}} G$  are direct consequences of Proposition 4.13. The last statements follows from the second since the Bredon cohomological and Bredon geometric dimension agree for values greater than 3 by Proposition 3.18.  $\square$

In this section we apply Proposition 4.13 to the special case where  $N = C_\infty$ , that is  $G$  is an extension

$$0 \rightarrow C_\infty \rightarrow G \rightarrow Q \rightarrow 0$$

of the infinite cyclic group  $C_\infty$  by an arbitrary group  $Q$ .

**Proposition 4.15.** *Let  $G$  be an extension of the infinite cyclic group  $C_\infty$  by an arbitrary group  $Q$ . Then precisely one of the following cases occurs:*

- (1)  $Q$  has no element of infinite order. Then all  $H \in \mathfrak{H}$  are virtually cyclic and

$$\underline{\underline{\text{hd}}} G \leq \underline{\underline{\text{hd}}} Q \quad \text{and} \quad \underline{\underline{\text{cd}}} G \leq \underline{\underline{\text{cd}}} Q;$$

- (2)  $Q$  has elements of infinite order. Then

$$3 \leq \underline{\underline{\text{hd}}} G \leq \underline{\underline{\text{hd}}} Q + 3 \quad \text{and} \quad 3 \leq \underline{\underline{\text{cd}}} G \leq \underline{\underline{\text{cd}}} Q + 3.$$

PROOF. If  $Q$  does not have elements of infinite order then every  $C_\infty \leq H \leq G$  with  $H/C_\infty$  virtually cyclic is itself virtually cyclic and therefore  $\underline{\underline{\text{hd}}} G = \underline{\underline{\text{cd}}} G = 0$ . Now the first claim follows from Proposition 4.13.

If  $Q$  does have an element of infinite order then there exists  $C_\infty \leq H \leq G$  with  $H/C_\infty$  infinite virtually cyclic. In this case  $\text{vcd } H = 2$  and therefore  $\underline{\underline{\text{hd}}} H = \underline{\underline{\text{gd}}} H = 3$  by Proposition 4.7. The second claim follows now from 4.13.  $\square$

Note that we can replace  $C_\infty$  by an infinite virtually cyclic group  $N$  and still get the same result.

**Proposition 4.16.** *For the braid group  $B_3$  we have the estimate*

$$3 \leq \underline{\underline{\text{hd}}} B_3 \leq \underline{\underline{\text{cd}}} B_3 = \underline{\underline{\text{gd}}} B_3 \leq 5.$$

PROOF. The braid group  $B_3$  is an extension of the infinite cyclic group  $C_\infty$  by the modular group  $C_2 * C_3$ . Now  $C_2 * C_3$  is not virtually cyclic and therefore  $\underline{\underline{\text{hd}}}(C_2 * C_3) = \underline{\underline{\text{cd}}}(C_2 * C_3) = 2$  by Proposition 4.12. Moreover it has an element of infinite order and thus  $3 \leq \underline{\underline{\text{hd}}} B_3 \leq \underline{\underline{\text{cd}}} B_3 \leq 5$  by Proposition 4.15. Furthermore  $\underline{\underline{\text{cd}}} B_3 \geq 3$  and therefore it must be equal to  $\underline{\underline{\text{gd}}} B_3$  by Proposition 3.18.  $\square$

#### 4. Nilpotent Groups

The *Hirsch length*  $\text{h}G$  of a group  $G$  is an invariant of groups which has originally been defined for polycyclic groups. For polycyclic groups it is the number of infinite cyclic factors in a infinite cyclic series of  $G$  [Rob96, p. 152].

The notion of Hirsch length can be extended to elementary amenable groups, which is a class of groups which contains all locally nilpotent groups

and also all soluble groups (more detail will follow in the next section). The extension can be done in such a way that the following holds.

**Proposition 4.17.** [Hil91, Theorem 1] *Let  $G$  be an elementary amenable group. Then*

- (1) *if  $H$  is a subgroup of  $G$  then  $\mathrm{h}H \leq \mathrm{h}G$ ;*
- (2) *if  $G$  is the direct union of subgroups  $G_\lambda$ ,  $\lambda \in \Lambda$ , then*

$$\mathrm{h}G = \sup\{\mathrm{h}G_\lambda\};$$

- (3) *if  $H$  is a normal subgroup of  $G$ , then  $\mathrm{h}G = \mathrm{h}H + \mathrm{h}(G/H)$ .  $\square$*

A finitely generated nilpotent group is known to be polycyclic [Rob96, p. 137]. It is shown in [LW07] that

$$\mathrm{vcd} G - 1 \leq \underline{\underline{\mathrm{gd}}} G \leq \mathrm{vcd} G + 1$$

for virtually polycyclic groups  $G$ . For virtually polycyclic groups the virtually cohomological dimension  $\mathrm{vcd} G$  is equal to the Hirsch length  $\mathrm{h}G$ . Therefore Proposition 4.6 states that we have for finitely generated nilpotent groups the estimate

$$\mathrm{h}G - 1 \leq \underline{\underline{\mathrm{cd}}} G \leq \mathrm{h}G + 1.$$

If  $G$  is a countable group, then  $G$  is the countable direct union of its finitely generated subgroups  $G_\lambda$  and we have the estimate

$$k \leq \underline{\underline{\mathrm{cd}}} G \leq k + 1$$

where  $k := \sup\{\underline{\underline{\mathrm{cd}}} G_\lambda\}$ . On the other hand, if  $G$  is locally nilpotent group, then

$$\mathrm{h}G = \sup\{\mathrm{h}G_\lambda\},$$

by Proposition 4.17. Since  $\mathrm{h}G_\lambda = \underline{\underline{\mathrm{cd}}} G_\lambda$  for all  $\lambda$  it follows that  $k = \mathrm{h}G$ . We get the following estimate for  $\underline{\underline{\mathrm{cd}}} G$  for locally nilpotent groups.

**Proposition 4.18.** *Let  $G$  be a countable locally nilpotent group with finite Hirsch length  $\mathrm{h}G$ . Then*

$$\mathrm{h}G - 1 \leq \underline{\underline{\mathrm{cd}}} G \leq \mathrm{h}G + 2.$$

*In particular this estimate is true for countable nilpotent groups.*

Note that if  $G$  locally virtually cyclic, then  $\underline{\text{cd}} G \leq 1$ . Otherwise  $G$  contains a subgroup isomorphic to  $\mathbb{Z}^2$  and it follows that  $\underline{\text{cd}} G \geq 3$  and  $\text{h}G \geq 2$ .

The proof of Proposition 4.18 relies heavily on Lück and Weiermann's geometric result for virtually polycyclic groups [LW07]. In what follows we will give an algebraic proof which avoids the use of geometric results as far as possible. The only geometric input we need in the following result is that  $\underline{\text{gd}} \mathbb{Z}^n \leq n + 1$ . In the case of  $n = 2$  this follows from the very simple model for  $\underline{\text{E}}\mathbb{Z}^2$  explained in [JPL06] and for general  $n \in \mathbb{N}$  it follows from a construction in [CFH06]. In both cases the results are obtained with a much simpler machinery than the general result in [LW07]. Note that the geometric results enter in the proof of Theorem 4.22 at the following two places: we need  $\underline{\text{gd}} \mathbb{Z}^n \leq n + 1$  for general  $n$  in the proof of Lemma 4.20 and  $\underline{\text{gd}} \mathbb{Z}^2 = 3$  is implicitly used in the inequality (4.3) in the proof of Proposition 4.21.

If  $G$  is a nilpotent group, then the set  $\tau(G)$  consisting of all elements of  $G$  which have finite order is a fully invariant subgroup of  $G$  and the quotient  $G/\tau(G)$  is torsionfree [Rob96, p. 132]. The subgroup  $\tau(G)$  is called the *torsion-subgroup* of  $G$ .

**Lemma 4.19.** *Let  $G$  be a countable nilpotent group. Then  $\underline{\text{hd}} G \leq \underline{\text{hd}} G/\tau(G)$  and  $\underline{\text{cd}} G \leq \underline{\text{gd}} G/\tau(G) + 1$ .*

PROOF. Let  $S$  be a subgroup of  $G$  such that  $\tau(G) \leq S$  and  $S/\tau(G)$  is a virtually cyclic subgroup of  $G/\tau(G)$ . We claim that  $\underline{\text{hd}} S = 0$  and  $\underline{\text{cd}} S \leq 1$ .

Since  $G/\tau(G)$  is torsion-free it follows that  $S/\tau(G)$  is infinite cyclic. Thus we have a short exact sequence

$$1 \rightarrow \tau(G) \rightarrow S \rightarrow S/\tau(G) \rightarrow 1$$

The group  $S$  is the countable and hence it is the countable union of its finitely generated subgroups  $S_\lambda$ . Since  $S_\lambda$  is a finitely generated nilpotent group it follows that  $S_\lambda$  is a polycyclic group. In particular  $S_\lambda$  satisfies the maximal condition on subgroups [Rob96, p. 152]. As a consequence this implies that  $\tau(G) \cap S_\lambda$  cannot have an infinite strictly ascending sequence of finite groups and hence it is finite. Therefore  $S_\lambda$  is virtually cyclic. It follows that  $S$  is locally virtually cyclic. Therefore  $\underline{\text{hd}} S = 0$  and  $\underline{\text{cd}} S \leq 1$  and this proves the claim.

We can apply Martínez-Pérez's spectral sequence, that is the result of Proposition 4.13, and we get the desired inequalities

$$\underline{\text{hd}} G \leq \underline{\text{hd}} Q \quad \text{and} \quad \underline{\text{cd}} G \leq \underline{\text{cd}} Q + 1. \quad \square$$

**Lemma 4.20.** *Let  $G$  be a torsion-free abelian group. Then*

$$\underline{\text{cd}} G \leq \text{h}G + 2.$$

PROOF. In order to avoid triviality we assume that the Hirsch length of  $G$  is finite. In general, torsion-free soluble groups of finite Hirsch length are countable [Bie81, p. 100]. In particular  $G$  is countable. The group  $G$  is the direct union of its finitely generated subgroups  $G_\lambda$ . Since  $G$  is torsion-free and abelian, it follows that  $G_\lambda \cong \mathbb{Z}^{n_\lambda}$ . Then  $\underline{\text{cd}} \mathbb{Z}^{n_\lambda} \leq \underline{\text{gd}} \mathbb{Z}^{n_\lambda} \leq n_\lambda + 1 = \text{h}G_\lambda + 1$  where the last inequality is due to [CFH06]. Therefore

$$\underline{\text{cd}} G \leq \sup\{\underline{\text{cd}} G_\lambda\} + 1 \leq \sup\{\text{h}G_\lambda\} + 2 = \text{h}G + 2. \quad \square$$

**Proposition 4.21.** *Let  $G$  be a torsion-free nilpotent group. Then*

$$\underline{\text{cd}} G \leq \text{h}G + 5(c - 1) + 2$$

where  $c$  is the nilpotency class of  $G$ .

PROOF. If  $c = 1$  then  $G$  is abelian and the claim is the statement of Lemma 4.20. Therefore assume that  $c \geq 2$  and that the statement is true for groups with nilpotency class strictly less than  $c$ .

Let  $N := \zeta(G)$  be the centre of  $G$  and consider the central extension

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1.$$

Since  $G$  is torsion-free it follows by a theorem of Mal'cev that  $Q$  is torsion-free [Rob96, p. 137]. The nilpotency class of  $Q$  is  $c - 1$  and therefore we have by induction the inequality

$$\underline{\text{cd}} Q \leq \text{h}Q + 5(c - 2) + 2.$$

Let  $H$  be a subgroup of  $G$  with  $N \leq H$  and  $H/N$  a virtually cyclic subgroup of  $Q$ . Since  $Q$  is torsion-free it follows that  $H/N$  is infinite cyclic. Therefore we have a short exact sequence

$$1 \rightarrow N \rightarrow H \rightarrow H/N \rightarrow 1$$

and this sequence is split since  $H/N$  is free. Every element of  $H$  commutes with every element of  $N = \zeta(G)$  and thus  $N \leq \zeta(H)$ . Therefore the above extension is central and hence  $H \cong N \times H/N$ . Then

$$\begin{aligned} \underline{\text{cd}} H &\leq \underline{\text{cd}} N + \underbrace{\underline{\text{cd}} H/N}_{=0} + 3 & (4.3) \\ &\leq \text{h}N + 5 \end{aligned}$$

where the first inequality is due to Corollary 3.62 and the second inequality is due to Lemma 4.20. We can apply Proposition 4.13 and get altogether

$$\begin{aligned} \underline{\text{cd}} G &\leq \text{h}N + 5 + \underline{\text{cd}} Q \\ &\leq \text{h}N + \text{h}Q + 5(c-1) + 2 \\ &\leq \text{h}G + 5(c-1) + 2 \quad \square \end{aligned}$$

**Theorem 4.22.** *Let  $G$  be a countable nilpotent group. Then*

$$\underline{\text{cd}} G \leq \text{h}G + 5(c-1) + 3$$

where  $c$  is the nilpotency class of  $G$ .

PROOF. Let  $Q := G/\tau(G)$  and consider the short exact sequence

$$1 \rightarrow \tau(G) \rightarrow G \rightarrow Q \rightarrow 1.$$

Since  $\tau(G)$  is locally finite we have  $\text{h}(\tau(G)) = 0$  and it follows that  $\text{h}Q = \text{h}G$ . Furthermore the nilpotency class of  $Q$  is at most  $c$ . We can apply Proposition 4.21 to  $Q$  and we get

$$\begin{aligned} \underline{\text{cd}} Q &\leq \text{h}Q + 5(c-1) + 2 \\ &= \text{h}G + 5(c-1) + 2. \end{aligned}$$

The claim follows now from Lemma 4.19. □

## 5. Elementary Amenable Groups

In the previous section we have already made a vague reference to elementary amenable groups. This class of groups was first introduced by von Neumann in [Neu29]. It is the smallest class of groups which contains all finite groups and the infinite cyclic group and which is closed under forming extensions, increasing unions, see for example [HL92]. This class is closed under forming subgroups and quotients. It follows that the class of elementary

amenable groups includes all locally finite groups, all locally nilpotent groups and all virtually soluble groups.

In the previous section we have shown that locally nilpotent groups  $G$  with finite Hirsch length have finite Bredon cohomological dimension with respect to the family  $\mathfrak{F}_{\text{vc}}(G)$  and thus admit a finite dimensional model for  $\underline{EG}$ .

The natural question is to ask, if the same is true for elementary amenable groups. However, since the class of elementary amenable groups contains arbitrary large locally finite groups (which have Hirsch length 0 by Proposition 4.17) one can quickly see that this cannot be the case.

**Lemma 4.23.** [LW07] *Let  $G$  be a locally finite group. If  $|G| = \aleph_n$  for some  $n \in \mathbb{N}$  then  $\underline{\text{gd}} G = n + 1$ . If  $|G| = \aleph_\omega$  then  $\underline{\text{gd}} G = \infty$ .*

PROOF. Since  $G$  is locally finite it follows that  $\mathfrak{F}_{\text{vc}}(G) = \mathfrak{F}_{\text{fin}}(G)$  and therefore  $\underline{\text{gd}} G = \underline{\text{gd}} G$ . Thus the first claim is directly the statement of Example 5.32 in [LW07, p. 36]. The remaining claim needs a simple intermediate step. Assume  $|G| = \aleph_\omega$ . Let  $n \in \mathbb{N}$  and let  $X$  be a subset of  $G$  with  $|X| = \aleph_n$ . Then the subgroup  $H$  generated by  $X$  has cardinality  $\aleph_n$ , too. Thus  $\underline{\text{gd}} H = n + 1$  by the first part. Since  $n$  was chosen arbitrary we deduce from  $\underline{\text{gd}} G \geq \underline{\text{gd}} H = n + 1$  that  $\underline{\text{gd}} G = \infty$ .  $\square$

In what will follow in this section, two special subgroups of a group  $G$  will be of importance. Given an arbitrary group  $G$  there exists a unique maximal normal locally finite subgroup which we denote by  $\Lambda(G)$  [Rob96, p. 436]. There exists also a unique maximal normal torsion subgroup which we denote by  $\tau(G)$ , see for example [LR04, p. 90]. The subgroup  $\tau(G)$  is called the *torsion radical* of  $G$ . Note that for nilpotent groups  $\tau(G)$  agrees with the definition on page 95. Clearly  $\Lambda(G) \leq \tau(G)$  but equality does not hold in general [Rob96, pp. 422ff.]. However, equality holds in case that  $G$  is soluble [LR04, p. 90].

**Lemma 4.24.** *Let  $G$  be an elementary amenable group with finite Hirsch length. Then*

$$|G| < \aleph_\omega \iff |\Lambda(G)| < \aleph_\omega.$$



PROOF. “ $\Rightarrow$ ”: Clear.

“ $\Leftarrow$ ”: The quotient  $G/\Lambda(G)$  has a finite index torsion-free soluble subgroup  $H$ , see [Weh95]. Since  $G/\Lambda(G)$  has finite Hirsch length it follows that  $H$  has finite Hirsch length and thus it is countable. Then also  $G/\Lambda(G)$  is countable. Assume that  $|\Lambda(G)| \leq \aleph_n$  for some  $n \in \mathbb{N}$ . Then an extension of  $|\Lambda(G)|$  by a countable group has still cardinality at most  $\aleph_n$ . Thus  $|G| < \aleph_\omega$ .  $\square$

**Lemma 4.25.** *Let  $G$  be an elementary amenable group with  $\text{h}G = \infty$ . Then  $\underline{\text{gd}} G = \infty$ .*

PROOF. We have  $\underline{\text{gd}} G \geq \underline{\text{cd}} G \geq \text{cd} G - 1 \geq \text{cd}_{\mathbb{Q}} G - 1$  where the last inequality is due to the following standard argument [BLN01]: evaluating any projective resolution  $P_* \rightarrow \mathbb{Z}$  of the trivial  $\mathcal{O}_{\mathfrak{F}_{\text{fin}}(G)}G$ -module at  $G/1$  and tensoring it with  $\mathbb{Q}$  gives a projective resolution of the trivial  $\mathbb{Q}G$ -module  $\mathbb{Q}$ .

However,  $\text{cd}_{\mathbb{Q}} G \geq \text{h}G$  is true for any elementary amenable group by [Hil91, pp. 167f]. Therefore, if  $\text{h}G = \infty$  it follows that  $\text{cd}_{\mathbb{Q}} G = \infty$  and thus  $\underline{\text{gd}} G = \infty$ , too.  $\square$

Combining Lemma 4.23, 4.24 and 4.25 gives immediately the following

**Lemma 4.26.** *Let  $G$  be an elementary amenable group. Then*

$$\underline{\text{gd}} G < \infty \Rightarrow |G| < \aleph_\omega \text{ and } \text{h}G < \infty. \quad \square$$

In other words, the question, which elementary amenable groups do admit a finite dimensional model for  $\underline{EG}$  reduces to the following question: what conditions on a elementary amenable group  $G$  in addition to  $|G| < \aleph_\omega$  and  $\text{h}G < \infty$  need to be satisfied in order to ensure  $\underline{\text{gd}} G < \infty$ ? One may also ask if those conditions are actually sufficient?

The structure of elementary amenable groups  $G$  with finite Hirsch length has been studied by Hillman and Linnell in [HL92] and Wehrfritz in [Weh95].

**Proposition 4.27.** [HL92, Weh95] *Let  $G$  be an elementary amenable group with  $\text{h}G < \infty$ . Then  $G/\Lambda(G)$  is a finite extension of a torsion-free soluble group.*  $\square$

In what follows we will need frequently the following result proven by Lück.

**Lemma 4.28.** [Lüc00, p. 191]. *Let  $G$  be a group and let  $H$  be a finite index subgroup of  $G$ . Then*

$$\underline{\underline{\text{gd}}} G \leq |G : H| \cdot \underline{\underline{\text{gd}}} H.$$

*In particular  $\underline{\underline{\text{gd}}} G < \infty$  if and only if  $\underline{\underline{\text{gd}}} H < \infty$ .* □

Note that since torsion-free soluble groups of finite Hirsch length are countable [Bie81, p. 100] it follows that an elementary amenable group  $G$  of finite Hirsch length is countable modulo  $\Lambda(G)$  [HL92]. Therefore an elementary amenable group which admits a finite dimensional model for  $\underline{EG}$  are not far away from being countable torsion-free soluble with finite Hirsch length.

Therefore we will restrict first to the following question: does a countable soluble group  $G$  with finite Hirsch length admit a finite dimensional model for  $\underline{EG}$ ? In the following we try to describe a possible way to obtain a positive answer to this question. However, the proposed strategy has still gaps which remain open at the time of writing.

We collect two results about soluble groups and linear groups which will be needed in what follows.

**Proposition 4.29.** [Weh74, Corollary 1.2] *Let  $G$  be a finite extension of a torsion-free, soluble group. Then  $\text{h}G < \infty$  implies that  $G$  is  $\mathbb{Q}$ -linear.* □

**Proposition 4.30.** (Gruneberg, see for example [Weh73, p. 102]) *Let  $G$  be a linear group. Then its Fitting subgroup  $\text{Fit}(G)$  is nilpotent.* □

Furthermore, for what follows, we need the next two assumptions to be satisfied. We return our attention to these assumptions in a moment.

**4.31.** *Let  $G$  be a countable torsion-free soluble group with  $\text{h}G < \infty$ . Let  $N := \text{Fit}(G)$  and denote by  $\zeta(N)$  the centre of  $N$ . Then there exists an integer  $k \geq 0$  such that  $\underline{\underline{\text{gd}}} K \leq k$  for every  $\zeta(N) \leq K \leq G$  with  $K/\zeta(N)$  virtually cyclic.*

**4.32.** *Let  $G$  be a countable soluble group with  $\text{h}G < \infty$ . Then*

$$\underline{\underline{\text{gd}}}(G/\Lambda(G)) < \infty \Rightarrow \underline{\underline{\text{gd}}}(G) < \infty.$$

**Proposition 4.33.** *Assume that the assumptions 4.31 and 4.32 are satisfied. Then any countable soluble group  $G$  with finite Hirsch length admits a finite dimensional model for  $\underline{\underline{EG}}$ .*

PROOF. We proof by induction on the Hirsch length  $h(G)$ . If  $h(G) = 0$  then  $G$  is locally finite which in turn implies  $\underline{\underline{gd}} G \leq 1$ .

Thus we may assume that  $h(G) \geq 1$ . Then  $G/\Lambda(G)$  is a finite extension of a torsion free soluble group  $H$  by Proposition 4.27. Since  $G$  is not locally finite it follows that  $H$  is non-trivial. Since  $H$  is soluble it follows that  $N := \text{Fit}(H)$  must be non-trivial as it contains the smallest non-trivial term of the derived series [Rob96, p. 133]. Moreover  $H$  is  $\mathbb{Q}$ -linear by Proposition 4.29 and therefore  $N$  must be nilpotent by Proposition 4.30. Therefore the centre  $\zeta(N)$  of  $N$  must be non-trivial and since  $H$  is torsion-free it follows that  $h(\zeta(N)) \geq 1$ . Since  $\zeta(N)$  is characteristic in  $N$  and since  $N$  is normal in  $H$  it follows that  $\zeta(N)$  is normal in  $H$ . Therefore we can form the quotient group  $H/\zeta(N)$  and  $h(H/\zeta(N)) \leq h(G/\Lambda(G)) - 1 = h(G) - 1$ . It follows by induction that  $\underline{\underline{gd}}(H/\zeta(N)) < \infty$ .

Since we assume that assumption 4.31 is satisfied for the group  $H$  it follows that there exists an integer  $k$  such that

$$\underline{\underline{gd}} K \leq k$$

for every  $\zeta(N) \leq K \leq H$  for which  $K/\zeta(N)$  is virtually cyclic. Therefore it follows by Corollary 4.14 that  $\underline{\underline{gd}}(H/\zeta(N)) < \infty$  implies  $\underline{\underline{gd}} H < \infty$ . Since  $H$  has finite index in  $G/\Lambda(G)$  this implies that  $\underline{\underline{gd}}(G/\Lambda(G)) < \infty$  by Lemma 4.28. Finally, assumption 4.32 implies that  $\underline{\underline{gd}} G < \infty$ .  $\square$

We return our attention to the assumption 4.31 which is clearly a necessary condition in Proposition 4.33. The centre  $\zeta(N)$  of the Fitting subgroup  $N = \text{Fit}(G)$  of a soluble group  $G$  is known to be the centraliser  $C_G(N)$  of  $N$  in  $G$  [Rob96, p. 149]. This is another information in addition to the many constraints we know from the setup in the assumption 4.31. By Proposition 4.18 we know that  $\underline{\underline{gd}}(\zeta(N))$  is finite and one may hope that with all the additional information provided one can conclude that virtually cyclic extensions

$$1 \rightarrow \zeta(N) \rightarrow K \rightarrow K/\zeta(N) \rightarrow 1$$

within the given countable torsion-free soluble  $\mathbb{Q}$ -linear group  $G$  do have a bound on  $\underline{\text{gd}} K$ .

**Proposition 4.34.** *Let  $G$  be a torsion-free soluble group with  $\text{h}G < \infty$ . Let  $N := \text{Fit}(G)$ . Assume there exists an integer  $k \geq 0$  such that  $\underline{\text{gd}} K \leq k$  for every infinite cyclic extension  $K$  of  $\zeta(N)$  within  $G$ . Then the assumption 4.31 is satisfied.*

In order to prove this proposition we need three auxiliary results.

**Lemma 4.35.** *Let  $G$  be a torsion-free soluble group with  $\text{h}G < \infty$ . Let  $N := \text{Fit}(G)$ . Then  $G/N$  has a bound on the order of its finite subgroups.*

PROOF. Let  $Q := G/N$ . By [BK01, pp. 29f.] we have that  $\Lambda(Q)$  is finite and  $Q/\Lambda(Q)$  is an Euclidean crystallographic group. Since  $Q/\Lambda(Q)$  is crystallographic it follows that  $Q/\Lambda(Q)$  has a bound on the order of its finite subgroup, say  $|K| \leq k$  for any finite  $K \leq Q/\Lambda(Q)$ .

Now let  $H$  be an arbitrary finite subgroup of  $Q$ . Then

$$\begin{aligned} |H| &= |H : H \cap \Lambda(Q)| \cdot |H \cap \Lambda(Q)| \\ &= \underbrace{|H\Lambda(Q) : \Lambda(Q)|}_{\leq k} \cdot \underbrace{|H \cap \Lambda(Q)|}_{\leq |\Lambda(Q)|} \\ &\leq k \cdot |\Lambda(Q)| < \infty \end{aligned}$$

which is a bound independent of  $H$ . □

**Lemma 4.36.** *Let  $G$  and  $N$  as in the previous lemma. Then  $G/\zeta(N)$  has a bound on the orders of its finite subgroups.*

PROOF. The group  $G$  is linear (Proposition 4.29) and therefore  $N$  is nilpotent (Proposition 4.30).

Now  $\zeta(N)$  is torsion-free and therefore each upper central factor of  $N$  is torsion-free by a result of Mal'cev [Rob96, p. 137]. In particular  $\zeta(N/\zeta(N))$  is torsion-free. Thus every upper central factor of  $N/\zeta(N)$  is torsion-free. Since  $N/\zeta(N)$  is nilpotent its upper central series reaches  $N/\zeta(N)$ . It follows that  $N/\zeta(N)$  is torsion-free, too.

Consider the short exact sequence

$$1 \rightarrow N/\zeta(N) \rightarrow G/\zeta(N) \rightarrow Q \rightarrow 1.$$

Since  $(G/\zeta(N))/(N/\zeta(N)) \cong G/N \cong Q$  there exists a upper bound  $k \geq 1$  on the order of the finite subgroups of  $Q$  by Lemma 4.35. Let  $H$  be a finite subgroup of  $G/\zeta(N)$ . Since  $N/\zeta(N)$  is torsion-free it follows that  $|H \cap N/\zeta(N)| = 1$ . Then

$$\begin{aligned} |H| &= |H : H \cap N/\zeta(N)| \cdot \underbrace{|H \cap N/\zeta(N)|}_{=1} \\ &= |H(N/\zeta(N)) : N/\zeta(N)| \leq k \quad \square \end{aligned}$$

**Lemma 4.37.** *Let  $G$  be group and assume that there exists a  $r \geq 1$  such that  $|H| \leq r$  for every finite subgroup  $H$  of  $G$ . Then every infinite virtually cyclic subgroup  $K$  of  $G$  has a infinite cyclic subgroup  $C$  with  $|K : C| \leq 2r$ .*

PROOF. Let  $N$  be the unique maximal normal finite subgroup of  $K$  such that  $K/N$  is either infinite cyclic or infinite dihedral.

If  $K/N$  is infinite cyclic, then  $K \cong N \rtimes C$  with  $C$  infinite cyclic and  $|K : C| = |N| \leq r$ . On the other hand, if  $K/N$  is infinite dihedral, then there exists  $k \in K$  such that  $kN$  generates an infinite cyclic subgroup of  $K/N$  of index 2. Then  $C := \langle k \rangle$  is an infinite cyclic subgroup of  $K$  with  $|K : C| = 2|N| \leq 2r$ .  $\square$

PROOF OF PROPOSITION 4.34. By Lemma 4.36 we know that there exists  $r \geq 0$  such that  $|H| \leq r$  for every finite subgroup  $H$  of  $G/\zeta(N)$

Let  $K$  be an extension of  $\zeta(N)$  within  $G$  such that  $K/\zeta(N)$  is virtually cyclic.

If  $K/\zeta(N)$  is finite, then

$$\begin{aligned} \underline{\text{gd}} K &\leq |K : \zeta(N)| \cdot \underline{\text{gd}} \zeta(N) && \text{(Lemma 4.28)} \\ &\leq r \cdot (\text{h}(\zeta(N)) + 2) && \text{(Proposition 4.18)} \end{aligned}$$

If  $K/\zeta(N)$  is infinite, then  $K$  has a subgroup  $C$  containing  $\zeta(N)$  as a subgroup such that  $C/\zeta(N)$  is infinite cyclic and

$$|K : C| = |K/\zeta(N) : C/\zeta(N)| \leq 2r$$

by Lemma 4.37. By assumption  $\underline{\text{gd}} C \leq k$  and thus

$$\begin{aligned} \underline{\text{gd}} K &\leq |K : C| \cdot \underline{\text{gd}} C && \text{(Lemma 4.28)} \\ &\leq 2rk \end{aligned}$$

Therefore

$$\underline{\underline{\text{gd}}} K \leq r \cdot \max(\text{h}(\zeta(N)) + 2, 2k)$$

for any  $\zeta(N) \leq K \leq G$  with  $K/\zeta(N)$  virtually cyclic. That is, the assumption 4.31 is satisfied.  $\square$

The consequence of Proposition 4.34 is that the assumption 4.31 is equivalent to the following assumption.

**4.38.** *Let  $G$  be a countable torsion-free soluble group with  $\text{h}G < \infty$ . Let  $N := \text{Fit}(G)$  and denote by  $\zeta(N)$  the centre of  $N$ . Then there exists an integer  $k \geq 0$  such that  $\underline{\underline{\text{gd}}} K \leq k$  for every  $\zeta(N) \leq K \leq G$  with  $K/\zeta(N)$  infinite cyclic.*

If  $K$  is nilpotent, then  $\underline{\underline{\text{gd}}} K \leq \text{h}(\zeta(N)) + 3$  by Proposition 4.18. In particular this bound is satisfied whenever  $K \leq N$ . If  $\zeta(N)$  is finitely generated then  $\zeta(N)$  and therefore also any infinite cyclic extension  $K$  of  $\zeta(N)$  is polycyclic. In this case  $\underline{\underline{\text{gd}}} K \leq \text{h}K + 1 = \text{h}(\zeta(N)) + 2$  by [LW07]. In the next chapter we show that under certain conditions we can ensure  $\underline{\underline{\text{gd}}} K \leq \underline{\underline{\text{gd}}} \zeta(N) + 1$ , see Proposition 5.16. However, these estimates do not cover all possibilities yet. But one may hope that one has enough constraints to be able to answer all possibilities. After all the extension  $K$  has a structure which is well understood.

Another open question is whether the assumption 4.32 does hold. The validity of this assumption is essential in the induction step which appears in the proof of Proposition 4.33. Whether or whether not the assumption holds in this form is open at the moment. However, one may relax the assumption in case one wants to restrict the attention to torsion-free soluble groups. In this case the following assumption is enough.

**4.39.** *Let  $G$  be a torsion-free soluble group with finite Hirsch length. Set  $H := G/\zeta(\text{Fit}(G))$ . Then  $\underline{\underline{\text{gd}}}(H/\Lambda(H)) < \infty$  implies  $\underline{\underline{\text{gd}}} H < \infty$ .*

In the assumption 4.32 the subgroup  $\Lambda(G)$  was allowed to be any countable soluble locally finite group. However, in the above assumption  $\Lambda(H)$  may not be anymore as arbitrary.

Finally, since an elementary amenable group  $G$  with finite Hirsch length is, modulo  $\Lambda(G)$ , a finite extension of a countable torsion-free soluble group

(Proposition 4.27) one may consider for a fixed  $n \in \mathbb{N}$  the following variation of the assumption 4.32.

**4.40.** *For any elementary amenable group  $G$  with  $|G| \leq \aleph_n$  and  $\text{h}G < \infty$  the condition  $\underline{\text{gd}}(G/\Lambda(G)) < \infty$  implies  $\underline{\text{gd}}G < \infty$ .*

Lemma 4.23 implies that the above assumption must be satisfied for all  $n \in \mathbb{N}$  if  $\underline{\text{gd}}G < \infty$  for every elementary amenable group  $G$  with  $|G| < \aleph_\omega$  and  $\text{h}G < \infty$ . Moreover we have already noted that the assumption 4.38 is a necessity, too. On the other hand, the next result says that this also suffices.

**Theorem 4.41.** *Suppose that the assumptions 4.40 is satisfied for all  $n \in \mathbb{N}$  and that the assumption 4.38 is satisfied. Then for any elementary amenable group  $G$  holds*

$$\underline{\text{gd}}G < \infty \iff |G| < \aleph_\omega \text{ and } \text{h}G < \infty.$$

PROOF. “ $\Rightarrow$ ”: This is Lemma 4.26.

“ $\Leftarrow$ ”: There exists a finite index countable torsion-free soluble subgroup  $H$  of  $G/\Lambda(G)$ . Since  $\text{h}H = \text{h}G < \infty$  it follows by Proposition 4.34 and Proposition 4.33 that  $\underline{\text{gd}}H < \infty$ . Now Lemma 4.28 and the assumption 4.40 implies that  $\underline{\text{gd}}G$  is finite, too.  $\square$

## A Geometric Interlude

### 1. A Class of Infinite Cyclic Extensions

The spectral sequence developed by Martínez-Pérez [MP02] suggests that virtually cyclic extensions are the main obstruction to understand the behaviour of Bredon dimensions under general extensions for the family of virtually cyclic subgroups. A first general answer for finite extensions has been given in [Lüc00, p. 191], see Lemma 4.28 in the previous chapter. Yet effectively nothing is known for the general case. The main objective in this chapter is to construct a model for  $\underline{EG}$  in the case that  $G$  belongs to a certain class of infinite cyclic extensions.

Infinite cyclic extensions are always split. Therefore such kind of extensions are always semidirect products. Let  $B$  be a group and let  $\varphi \in \text{Aut}(B)$ . Recall that the semidirect product

$$G := B \rtimes \mathbb{Z},$$

where  $\mathbb{Z}$  acts on  $B$  via the automorphism  $\varphi$ , is the set  $B \times \mathbb{Z}$  with the multiplication given by

$$(x, r) \cdot (y, s) = (x\varphi^r(y), r + s).$$

The identity is  $(1, 0)$  and the inverse of any element  $(x, r)$  is given by  $(x, r)^{-1} = (\varphi^{-r}(x^{-1}), -r)$ . The group  $B$  is embedded via  $x \mapsto (x, 0)$  as a normal subgroup of  $G$  and we consider  $\mathbb{Z}$  embedded as a subgroup of  $G$  via  $r \mapsto (1, r)$ .

Up to and including Section 7 of this chapter we will assume that  $G = B \rtimes \mathbb{Z}$  and that that this extension satisfy the following condition:

*The subgroup  $\mathbb{Z}$  acts via conjugation freely on the set of conjugacy classes of nontrivial elements of  $B$ .*

Under this condition we can show that there exists a suitable set of unique maximal virtually cyclic subgroups to apply a variation of Juan-Pineda and



Leary's construction (see Proposition 4.8) in order to construct a model for  $\underline{EG}$  from a model for  $\underline{EB}$ .

## 2. Technical Preparations

**Lemma 5.1.** *Assume that  $B$  is torsion-free and does not contain a subgroup isomorphic to  $\mathbb{Z}^2$ . Then  $\mathbb{Z}$  acts freely by conjugation on the set of conjugacy classes of the non-trivial elements of  $B$  if and only if  $G$  does not contain a subgroup isomorphic to  $\mathbb{Z}^2$ .*

PROOF. “ $\Leftarrow$ ”: Suppose that  $H$  is a subgroup of  $G$  which is isomorphic to  $\mathbb{Z}^2$ . It must have a non-trivial intersection with the kernel of the canonical projection  $B \rtimes \mathbb{Z} \rightarrow \mathbb{Z}$ . Therefore there exists  $(y, 0) \in B \cap H$  with  $y \neq 1$ . On the other hand,  $H$  is not contained in  $B$  and thus there exists  $(x, r) \in H \setminus B$ . Then, as  $H$  is abelian, the commutator

$$[(x, r), (y, 0)] = (x\varphi^r(y)x^{-1}y^{-1}, 0)$$

must be trivial which is the case if and only if  $x\varphi^r(y)x^{-1}y^{-1} = 1$ . This implies that  $\varphi^r(y)$  and  $y$  belong to the same conjugacy class in  $B$ . Since  $r \neq 0$  and  $y \neq 1$  this implies that  $\mathbb{Z}$  does not act freely on the set of conjugacy classes of non-trivial elements of  $B$ .

“ $\Rightarrow$ ”: Suppose that  $\mathbb{Z}$  does not act freely on the set of conjugacy classes of non-trivial elements of  $B$ . Then there exists  $1 \neq y \in B$  and  $0 \neq r \in \mathbb{Z}$  such that  $\varphi^r(y) = x^{-1}yx$  for some  $x \in B$ . This implies that the non-trivial elements  $(x, r)$  and  $(y, 0)$  commute. In general  $(x, r)$  has infinite order and since  $B$  is assumed to be torsion-free it follows that the order of  $(y, 0)$  is also infinite. Therefore  $(x, r)$  and  $(y, 0)$  generate a subgroup of  $G$  which is isomorphic to  $\mathbb{Z}^2$ .  $\square$

**Lemma 5.2.** *Let  $B$  be a non-trivial virtually cyclic group. Then  $\mathbb{Z}$  cannot act freely by conjugation on the set of conjugacy classes of the non-trivial elements of  $B$ .*

PROOF. We prove the stronger result:  $\mathbb{Z}$  cannot act freely by conjugation on the non-trivial elements of  $B$ .

Let  $N$  be the unique maximal normal finite subgroup of  $B$ . Assume towards a contradiction that  $\varphi \in \text{Aut}(B)$  such that  $\varphi^k(b) \neq b$  for all  $k \in$

$\mathbb{Z} \setminus \{0\}$  and  $b \in B \setminus \{1\}$ . Then  $X := \{\varphi^k(b) : k \in \mathbb{Z}\}$  is a infinite subset of  $B$ . If  $b \in N$  then  $X \subset N$  since  $N$  is a characteristic subgroup of  $B$ . But this is absurd since  $N$  is finite. Therefore  $N = 1$  and  $B \cong \mathbb{Z}$  or  $B \cong D_\infty$ . Then  $B$  contains a unique maximal infinite cyclic subgroup  $C$  which is characteristic in  $B$ . Then  $\varphi$  restricted to  $C$  has order at most 2. But then  $\varphi^2(b) = b$  which contradicts the assumption. Therefore  $\mathbb{Z}$  cannot act freely by conjugation on the set of non-trivial elements of  $B$ .  $\square$

The statement of the next lemma is only non-trivial if  $B$  has torsion.

**Lemma 5.3.** *Assume that  $\mathbb{Z}$  acts freely via conjugation on the set of conjugacy classes of non-trivial elements of  $B$ . If  $H$  is a virtually cyclic subgroup of  $G$  which is not a subgroup of  $B$  then  $H$  is infinite cyclic.*

PROOF. First we note that all  $x \in G \setminus B$  have infinite order. Hence the subgroup  $\tau(H)$  of  $H$  which is generated by all the elements of  $H$  which have finite order is a subgroup of  $B$ .

By assumption there exists  $(x, r) \in H$  with  $r \neq 0$ . Then the infinite cyclic subgroup of  $H$  generated by this element has trivial intersection with  $B$  and therefore it also has trivial intersection with  $\tau(H)$ . Since  $H$  is virtually cyclic this can happen only if  $\tau(H)$  is finite.

Assume towards a contradiction that there exists a non-trivial  $(y, 0) \in \tau(H)$ . Then  $(z_k, 0) := (y, 0)^{(x, r)^k}$ ,  $k \in \mathbb{N}$ , is a sequence of elements in  $\tau(H)$  such that for each  $k \in \mathbb{N}$  the element  $z_k$  is conjugate in  $B$  to  $\varphi^{-rk}(y)$ . This claim is verified by induction. The case  $k = 0$  is trivial. Thus assume that  $(z_k, 0) \in \tau(H)$  and that there exists  $u \in B$  such that  $z_k = u^{-1}\varphi^{-rk}(y)u$ . Since  $\tau(H)$  is a characteristic subgroup of  $H$  and  $(x, r) \in H$ , it follows that  $(z_{k+1}, 0) = (z_k, 0)^{(x, r)} \in \tau(H)$ . Furthermore we have that

$$(z_{k+1}, 0) = (z_k, 0)^{(x, r)} = (\varphi^{-r}(x^{-1}z_kx), 0)$$

and hence  $z_{k+1} = \varphi^{-r}(x^{-1}u^{-1}\varphi^{-rk}(y)ux) = v^{-1}\varphi^{-r(k+1)}(y)v$  with  $v := \varphi^{-r}(gx)$ . That is that  $z_{k+1}$  is conjugate in  $B$  to  $\varphi^{-r(k+1)}(y)$ . By assumption  $\mathbb{Z}$  acts freely via conjugation on the conjugacy classes of non-trivial elements of  $B$  and hence all  $z_k$  belong to different conjugacy classes. In particular they are all pairwise different. Thus  $\{(z_k, 0) : k \in \mathbb{N}\}$  forms an infinite subset of  $\tau(H)$ . But this is a contradiction as we have shown above that  $\tau(H)$  is finite!

Therefore  $\tau(H)$  must be trivial. It follows that the virtually cyclic group  $H$  does not have any torsion and thus it must be infinite cyclic.  $\square$

**Lemma 5.4.** *Assume that  $\mathbb{Z}$  acts freely via conjugation on the set of conjugacy classes of non-trivial elements of  $B$ . Then for any  $(x, r) \in G \setminus B$  and  $y \in B$  we have*

$$(x, r)^y = (x, r) \iff y = 1.$$

PROOF.  $(x, r)^y = (x, r)$  is equivalent to  $\varphi(y) = xyx^{-1}$ , which is by assumption on the action of  $\mathbb{Z}$  on  $B$  equivalent to  $y = 1$ .  $\square$

**Lemma 5.5.** *Under the assumptions of the previous lemma, if  $H$  is an infinite cyclic subgroup of  $G$  that is not a subgroup of  $B$ , and  $y \in B$ , then  $|H \cap H^y| = \infty$  if and only if  $y = 1$ .*

PROOF. The “if” statement is trivial. Therefore assume that  $y \neq 1$  and let  $(x, r)$  be a generator of  $H$ . Then  $r \neq 0$  and

$$(z, r) := (x, r)^y \neq (x, r)$$

is a generator of  $H^y$  where the inequality is due to Lemma 5.4. Suppose, for a contradiction, that  $|H \cap H^y| = \infty$ . Then there must exist  $k, l \in \mathbb{Z} \setminus \{0\}$  such that  $(x, r)^k = (z, r)^l$ . In particular this implies that  $k = l$ . But then we get

$$(z, r)^l = (z, r)^k = ((x, r)^y)^k = ((x, r)^k)^y \neq (x, r)^k,$$

where the last inequality is again due to Lemma 5.4, and so we achieve our desired contradiction. Hence we must have  $|H \cap H^y| \neq \infty$ .  $\square$

As in [LW07, p. 5] we define an equivalence relation “ $\sim$ ” on the set  $\mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}_{\text{fin}}(G)$  by

$$H \sim K \iff |H \cap K| = \infty.$$

We denote by  $[H]$  the equivalence class of the group  $H$ . If  $K$  is not finite then  $K \leq H$  implies that  $K \sim H$ . Furthermore the equivalence relation satisfies  $H \sim K$  if and only if  $H^g \sim K^g$ . Therefore the action of  $G$  by conjugation on the set  $\mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}_{\text{fin}}(G)$  gives an action of  $G$  on the set of equivalence classes. If  $[H]$  is an equivalence class, then we denote by  $G_{[H]}$  the stabiliser of  $[H]$ .

Given subgroup  $H$  of  $G$ , the *commensurator*  $\text{Comm}_G(H)$  of  $H$  in  $G$  is defined as the subgroup

$$\text{Comm}_G(H) := \{g \in G : |H : H \cap H^g| \text{ and } |H^g : H \cap H^g| \text{ are finite}\}.$$

This subgroup is also known as the *virtual normaliser*  $VN_G(H)$  of the subgroup  $H$  in  $G$ . In general it contains the normaliser  $N_G(H)$  of  $H$  in  $G$  as its subgroup. In the case that  $H$  is a virtually cyclic subgroup of  $G$  which is not finite we have

$$\text{Comm}_G(H) = \{g \in G : |H \cap H^g| = \infty\}.$$

In particular we have that  $\text{Comm}_G(H) = G_{[H]}$  in this case.

**Lemma 5.6.** *Assume that  $\mathbb{Z}$  acts freely by conjugation on the set of non-trivial conjugacy classes of non-trivial elements of  $B$ . Then the commensurator  $\text{Comm}_G(H)$  is infinite cyclic for any virtually cyclic subgroup  $H$  of  $G$  that is not a subgroup of  $B$ .*

PROOF. Any such virtually cyclic subgroup  $H$  of  $G$  is infinite cyclic by Lemma 5.3. Therefore  $G_{[H]} = \text{Comm}_G(H)$ . Suppose that  $G_{[H]}$  is not infinite cyclic. Then the canonical projection  $\pi: B \rtimes Z \rightarrow \mathbb{Z}$  cannot map  $G_{[H]}$  isomorphically onto its image. Hence there exists a non-trivial  $y \in G_{[H]} \cap \ker(\pi) = G_{[H]} \cap B$ . Since  $H$  is infinite cyclic we get  $|H \cap H^y| \neq \infty$  by Lemma 5.5 which is equivalent to  $[H] \neq [H^y]$ , and this is a contradiction to the assumption that  $y \in G_{[H]}$ . Therefore  $G_{[H]} = \text{Comm}_G(H)$  must be infinite cyclic.  $\square$

**Proposition 5.7.** *Let  $G$  be an arbitrary group and let  $\mathfrak{F}$  and  $\mathfrak{G}$  be families of subgroups of  $G$  such that*

$$\mathfrak{F}_{\text{fin}}(G) \subset \mathfrak{F} \subset \mathfrak{G} \subset \mathfrak{F}_{\text{vc}}(G).$$

*Assume that the commensurator  $\text{Comm}_G(H) \in \mathfrak{G}$  for any  $H \in \mathfrak{G} \setminus \mathfrak{F}$ , then every  $H \in \mathfrak{G} \setminus \mathfrak{F}$  is contained in a unique maximal element  $H_{\text{max}} \in \mathfrak{G}$  and  $N_G(H_{\text{max}}) = H_{\text{max}}$ .*

PROOF. Since  $H$  is an infinite virtually cyclic subgroup of  $G$  it follows that  $G_{[H]} = \text{Comm}_G(H)$  and thus  $G_{[H]} \in \mathfrak{G}$  by assumption.

Trivially we have that  $H \leq G_{[H]}$ . If  $K \in \mathfrak{G}$  with  $H \leq K$ , then  $H \sim K$  since  $H$  is not finite, and for any  $k \in K$  we get  $[H^k] = [K^k] = [K] = [H]$ .

Therefore any  $k \in K$  stabilises  $[H]$ . This implies  $K \leq G_{[H]}$  and thus  $G_{[H]}$  is maximal and unique in  $\mathfrak{G} \setminus \mathfrak{F}$ , that is  $H_{\max} = G_{[H]}$ .

Finally,  $H_{\max} \leq N_G(H_{\max}) \leq \text{Comm}_G(H_{\max}) = G_{[H_{\max}]} = H_{\max}$  and hence  $H_{\max} = N_G(H_{\max})$ .  $\square$

Together with Lemma 5.3, we get the following result:

**Corollary 5.8.** *Let  $G = B \rtimes \mathbb{Z}$  and assume that  $\mathbb{Z}$  acts freely by conjugation on the set of conjugacy classes of non-trivial elements of  $B$ . Then every  $H \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}_{\text{vc}}(B)$  is contained in a unique maximal element  $H_{\max} \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}_{\text{vc}}(B)$  and  $N_G(H_{\max}) = H_{\max}$ . Furthermore  $\mathfrak{F}_{\text{vc}}(B) \cap H = \{1\}$  for any  $H \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}_{\text{vc}}(B)$ .  $\square$*

### 3. A Generalisation of Juan-Pineda and Leary's Construction

Let  $G$  be an arbitrary group and assume that  $\mathfrak{F}$  and  $\mathfrak{G}$  are two families of subgroups of  $G$  which satisfy the conditions of Proposition 5.7. Then we have the following generalisation of Proposition 4.8.

**Proposition 5.9.** *Let  $\mathfrak{F}$  be a full family and  $\mathfrak{G}$  a semi-full family of subgroups of  $G$  with  $\mathfrak{F}_{\text{fin}}(G) \subset \mathfrak{F} \subset \mathfrak{G} \subset \mathfrak{F}_{\text{vc}}(G)$ . Assume that every  $H \in \mathfrak{G} \setminus \mathfrak{F}$  is contained in a unique maximal element  $H_{\max} \in \mathfrak{G}$  and  $N_G(H_{\max}) = H_{\max}$ . Moreover, assume that  $\mathfrak{F} \cap H \subset \mathfrak{F}_{\text{fin}}(H)$  for every  $H \in \mathfrak{G} \setminus \mathfrak{F}$ . Let  $\mathcal{C}$  be a complete set of representatives of conjugacy classes of maximal elements in  $\mathfrak{G} \setminus \mathfrak{F}$ . Denote by  $\mathcal{C}_o$  and the set of orientable elements of  $\mathcal{C}$  and denote by  $\mathcal{C}_n$  the set of non-orientable elements of  $\mathcal{C}$ . Then a model for  $E_{\mathfrak{G}}G$  can be obtained from model for  $E_{\mathfrak{F}}G$  by attaching*

- (1) orbits of 0-cells indexed by  $\mathcal{C}$ ;
- (2) orbits of 1-cells indexed by  $\mathcal{C}_o \cup \{1, 2\} \times \mathcal{C}_n$ ;
- (3) orbits of 2-cells indexed by  $\mathcal{C}$ .

*Furthermore, a model for  $B_{\mathfrak{G}}G$  can be obtained from a model for  $B_{\mathfrak{F}}G$  by attaching 2-cells indexed by  $\mathcal{C}_o$ .*

PROOF. We only need to verify that Juan-Pineda and Leary's construction works unchanged in the more general setting. We fix a model  $E$  for  $E_{\mathfrak{F}}G$ .

Let  $H \in \mathcal{C}$ . By [JPL06, p. 137] we can choose a 1-dimensional model  $E_H$  for  $\underline{E}H$  which homeomorphic to the real line and such that  $E_H/H$  is a

loop if  $H \in \mathcal{C}_o$  or a line segment if  $H \in \mathcal{C}_n$ . Denote by

$$Z_H := G \times_H E_H$$

the  $G$  space which is induced from the  $H$ -space  $E_H$  [Kaw91, pp. 52ff.]. If  $[g, x] \in Z_H$ , then  $G_{[g,x]} = G_{g[1,x]} = (G_{[1,x]})^{g^{-1}}$ . Since  $G_{[1,x]} = H_x$  and  $H_x$  is a finite subgroup of  $H$  it follows that  $G_{[g,x]} \in \mathfrak{F}_{\text{fin}}(G)$ . Therefore it follows that  $\mathfrak{F}(Z_H) \subset \mathfrak{F}_{\text{fin}}(G) \subset \mathfrak{F}$  and there exists a  $G$ -map  $f_H: Z_H \rightarrow E$  by the universal property of  $E$ .

Furthermore, we set

$$X_H := G/H,$$

which is a discrete transitive  $G$ -set. There exists a  $G$ -equivariant projection  $\pi_H: Z_H \rightarrow X_H$  which maps  $[g, x]$  to  $gH$  and an  $H$ -equivariant inclusion  $i_H: E_H \rightarrow Z_H$  given by  $i_H(x) := [1, x]$ . Denote by  $V_{gH} := \pi_H^{-1}(gH)$ . Clearly  $i_H(E_H) \subset V_H$ . On the other hand, if  $[h, x] \in V_H$ , then  $hx \in E_H$  such that  $i_H(hx) = [1, hx] = [h, x]$ . That is  $i_H(E_H) \subset V_H$  and we have the equality  $i_H(E_H) = V_H$ . Since  $G$  is discrete, it follows that  $i_H$  is an open map and in particular it maps  $E_H$  homeomorphically onto  $V_H$ . Let  $R_H$  be a complete system of representatives of the left cosets  $G/H$ . Since  $G/H$  is discrete we have that  $Z_H$  is the disjoint union

$$Z_H = \coprod_{g \in R_H} V_{gH} = \coprod_{g \in R_H} gV_H$$

of contractible subspaces  $gV_H$ ,  $g \in R_H$ , which are permuted by the action of  $G$ .

**Claim 1.** Let  $K$  be a finite subgroup of  $G$ . Then the projection  $\pi_H$  induces a homotopy equivalence

$$\pi_H: (Z_H)^K \rightarrow (X_H)^K.$$

Let  $[g, x] \in (Z_H)^K$ . Without any loss of generality we may assume that  $g \in R_H$ . Now  $[g, x] \in (Z_H)^K$  implies

$$\begin{aligned} & \forall k \in K: k[g, x] = [g, x] \\ \iff & \forall k \in K: [g^{-1}kg, x] = [1, x] \\ \iff & \forall k \in K: \exists h \in H: g^{-1}kgh^{-1} = 1 \text{ and } hx = x \\ \iff & \forall k \in K: \exists h \in H_x: g^{-1}kg = h \\ \iff & K^g \leq H_x \end{aligned}$$

$\Rightarrow x \in \{y \in E_H : K^g \leq H_y\} = (E_H)^{K^g}$  and  $K$  is finite.

Thus  $[1, x] \in (V_H)^{K^g} = \{[1, x] : x \in (E_H)^{K^g}\}$  and therefore  $[g, x] = g[1, x] \in g((V_H)^{K^g})$ . On the other hand, let  $[1, x] \in (V_H)^{K^g}$ . Then

$$k[g, x] = g(g^{-1}kg)[1, x] = g[1, x] = [g, x]$$

for all  $k \in K$ . Therefore  $[g, x] \in (Z_H)^K$  (and  $K$  is finite as before).

Altogether this shows

$$(Z_H)^K = \coprod_{g \in R_H} g((V_H)^{K^g})$$

is the disjoint union of the subspaces  $g((V_H)^{K^g})$ . Since  $K$  is assumed to be finite we have that

- (1)  $(V_H)^{K^g}$  is contractible if  $K^g \leq H$ ;
- (2)  $(V_H)^{K^g} = \emptyset$  otherwise.

On the other hand  $gH \in (G/H)^K$  if and only if  $K^g \leq H$ . It follows that  $\pi_H$  induces map

$$\pi_H: (Z_H)^K \rightarrow (X_H)^K$$

which maps the contractible components  $g((V_H)^{K^g})$  of  $Z_H$  in an one-to-one way onto the discrete space  $(X_H)^K = (G/H)^K$ . It follows that  $\pi_H$  induces a homotopy equivalence  $(Z_H)^K \rightarrow (X_H)^K$  and the claim follows.

We set

$$Z := \coprod_{H \in \mathcal{C}} Z_H$$

and the  $G$ -maps  $f_H$  and  $\pi_H$  give rise to  $G$ -maps  $f: Z \rightarrow E$  and  $\pi: Z \rightarrow X$  with  $X := \coprod_{H \in \mathcal{C}} X_H$ . Note that  $\pi$  induces a homotopy equivalence  $Z^K \rightarrow X^K$  for every finite subgroup  $K$  of  $G$ .

**Claim 2.** The  $G$ -map

$$(f, \pi): Z \rightarrow E \times X$$

given by  $[g, x] \mapsto (f([g, x]), \pi([g, x]))$  is a  $G$ -homotopy equivalence.

Let  $K$  be a subgroup of  $G$  such that  $(E \times X)^K = E^K \times X^K \neq \emptyset$ . Then  $E^K \neq \emptyset$  and  $X^K \neq \emptyset$ . The condition  $E^K \neq \emptyset$  implies that  $K \in \mathfrak{F}$  and  $X^K \neq \emptyset$  implies that there exists a  $H \in \mathcal{C}$  and a  $g \in G$  such that  $K^g \leq H$ . Thus  $K^g = K^g \cap H \in \mathfrak{F} \cap H \leq \mathfrak{F}_{\text{fin}}(H)$ . Therefore  $K$  is a finite subgroup of  $G$ . Since  $\mathfrak{F}_{\text{fin}}(G) \subset \mathfrak{F}$  it follows that  $E^K$  is contractible by the universal property

of  $E$ . Moreover, since  $K$  is finite, it follows that  $\pi$  induces a homotopy equivalence  $Z^K \rightarrow X^K$ .

It follows that  $(f, \pi)$  induces a homotopy equivalence  $Z^K \rightarrow E^K \times X^K = (E \times X)^K$ . To see this, denote by  $\theta$  the homotopy inverse of  $\pi$  restricted to  $X^K$ , and denote by  $p$  the projection  $E^K \times X^K \rightarrow X^K$ . Let

$$\tilde{f}: E^K \times X^K \rightarrow Z^K$$

be the composite map  $\tilde{f} := \theta \circ p$ . Then  $\tilde{f} \circ (f, \pi) = \theta \circ p \circ (f, \pi) = \theta \circ \pi \simeq \text{id}$ . On the other hand we have that  $\pi \circ \theta = \text{id}$  since  $X$  is discrete. Therefore

$$(f, \pi) \circ \tilde{f} = (f \circ \theta \circ p, \pi \circ \theta \circ p) = (f \circ \theta \circ p, p)$$

maps  $(x, gH) \mapsto ((f \circ \theta)(gH), gH)$  for all  $(x, gH) \in E^K \times X^K$ . Since  $E^K$  is contractible it follows that  $(f, \pi) \circ \tilde{f} \simeq \text{id}$ . Altogether this shows that  $(f, \pi)$  has a homotopy inverse and therefore it is a homotopy equivalence.

Since  $(f, \pi)$  is a  $G$ -map and  $K$  has been an arbitrary subgroup of  $G$  such that  $(E \times X)^K \neq \emptyset$  we can apply the Equivariant Whitehead Theorem [Lüc89, p. 36] and we get that  $(f, \pi)$  is a  $G$ -homotopy equivalence and this proves Claim 2.

As in [JPL06, p. 140] we can attach  $Z \times [0, 1]$  to the disjoint union of  $E$  and  $X$ , identifying  $(z, 0)$  with  $f(z) \in E$  and  $(z, 1)$  with  $\pi(z) \in X$ . Denote this space by  $\tilde{E}$ . Since  $Z \simeq_G E \times X$  it follows that  $\tilde{E}$  is  $G$ -homotopy equivalent to the join  $E * X$  of  $E$  and  $X$ . Note that the join inherits a natural  $G$ -CW-complex structure from  $E$  and  $X$ .

**Claim 3.** The join  $E * X$  is a model for  $E_{\mathfrak{G}}G$ .

If  $K \in \mathfrak{F}(E * X)$ , then at least one of the following cases does hold:

- (1)  $K \in \mathfrak{F}(E) \subset \mathfrak{F} \subset \mathfrak{G}$ ;
- (2)  $K \in \mathfrak{F}(X) \subset \mathfrak{G}$ ;
- (3)  $K = K_1 \cap K_2$  with  $K_1 \in \mathfrak{F} \subset \mathfrak{G}$  and  $K_2 \in \mathfrak{G}$ .

Since  $\mathfrak{G}$  is semi-full it follows that also in the last case  $K \in \mathfrak{G}$  holds. Altogether  $\mathfrak{F}(E * X) \subset \mathfrak{G}$ .

If  $K \in \mathfrak{F}$  then  $E^K$  is contractible and if  $K \in \mathfrak{G} \setminus \mathfrak{F}$ , then  $X^K$  consists of a single point and is therefore contractible. It follows that  $(E * X)^K$  is contractible in both cases, that is for every  $K \in \mathfrak{G}$ .



Altogether  $E * X$  is a model for  $E_{\mathfrak{G}}G$  by Proposition 2.4 and this proves the claim.

Since  $E * X \simeq_G \tilde{E}$  it follows that  $\tilde{E}$  is a model for  $E_{\mathfrak{G}}G$ , too. It follows that  $\tilde{E}$  is obtained from  $E$  by attaching orbits of 0-, 1- and 2-cells as described in the proposition.

The remaining claim about the construction of a model for  $B_{\mathfrak{G}}G$  from a model for  $B_{\mathfrak{F}}G$  follows from the argument which proved Corollary 10 in [JPL06, p. 141]. This concludes the proof of Proposition 5.9.  $\square$

Note that in the case  $\mathfrak{G} = \mathfrak{F}_{\text{vc}}(G)$  and  $\mathfrak{F} = \mathfrak{F}_{\text{fin}}(G)$  we recover the original statement of Proposition 4.8. However we apply it to the case that  $G = B \rtimes \mathbb{Z}$ ,  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(B)$  and  $\mathfrak{G} = \mathfrak{F}_{\text{vc}}(G)$ . If  $\mathbb{Z}$  acts freely by conjugation on the set of conjugacy classes of non-trivial elements of  $B$ , then Corollary 5.8 tells us that we can use Proposition 5.9 in order to construct a model for  $\underline{E}G$  from a model for  $E_{\mathfrak{F}_{\text{vc}}(B)}G$ . However, in order to obtain this way a nice model for  $\underline{E}G$  we need to have a nice model for  $E_{\mathfrak{F}_{\text{vc}}(B)}G$  to start with. In the next section we will give a general construction for such a model if a nice model for  $\underline{E}B$  is given.

#### 4. Constructing a Model for $E_{\mathfrak{F}}G$ from a Model for $E_{\mathfrak{F}}B$

We carry out the construction in a setting that is more general than in the previous section. Let  $G := B \rtimes \mathbb{Z}$  be an arbitrary infinite cyclic extension, where  $\mathbb{Z}$  acts on  $B$  via an automorphism  $\varphi \in \text{Aut}(B)$ . Let  $\mathfrak{F}$  be a family of subgroups of  $B$ . We assume that  $\mathfrak{F}$  is invariant under the automorphism  $\varphi$ , that is  $\varphi^k(H) \in \mathfrak{F}$  for every  $H \in \mathfrak{F}$  and  $k \in \mathbb{Z}$ . This implies that  $H \in \mathfrak{F}$  if and only if  $\varphi(H) \in \mathfrak{F}$  for any subgroup  $H$  of  $B$ . Furthermore this implies that  $\mathfrak{F}$  is not just a family of subgroups of  $B$  but also a family of subgroups of  $G$ .

We begin our construction with the assumption that we are given a model  $X$  for  $E_{\mathfrak{F}}B$ . For each  $k \in \mathbb{Z}$  let  $X_k$  be a copy of  $X$  seen as a set. We define a  $B$ -action

$$\Phi_k: B \times X_k \rightarrow X_k$$

on  $X_k$  by  $\Phi_k(g, x) := \varphi^{-k}(g)x$ .

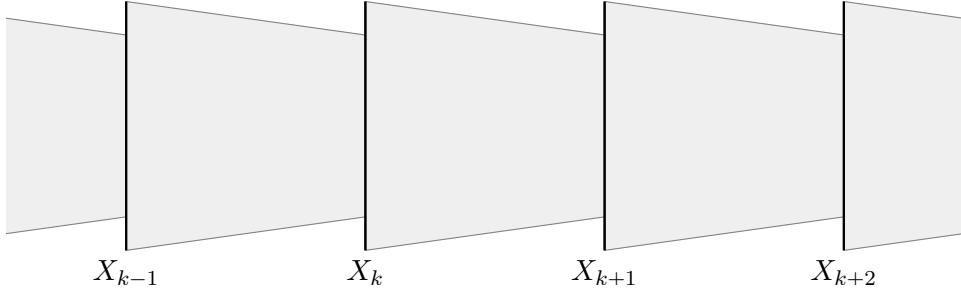


FIGURE 5. A schematic picture of the  $B$ -CW-complex  $Y$ .

Since  $X_0$  and  $X_1$  are models for  $\underline{E}B$  there exists a  $B$ -map  $f: X_0 \rightarrow X_1$ . In other words  $f$  is a continuous map  $f: X \rightarrow X$  which satisfies  $f(gx) = \varphi^{-1}(g)f(x)$  for every  $x \in X$  and  $g \in B$ . By the equivariant Cellular Approximation Theorem [Lüc89, p. 32] we may assume without loss of generality that  $f$  is cellular. Denote by  $X_\infty$  the disjoint union of  $B$ -spaces

$$X_\infty := \coprod_{k \in \mathbb{Z}} (X_k \times [0, 1])$$

and let  $Y$  be the quotient space

$$Y := X_\infty / \sim$$

under the equivalence relation generated by  $(x, 1) \sim (f(x), 0)$  whenever  $x \in X_k$  and  $f(x) \in X_{k+1}$  for some  $k \in \mathbb{Z}$ . Since  $f$  is a cellular  $B$ -map it follows that  $Y$  is a  $B$ -CW-complex. Essentially, it is a mapping telescope which extends to infinity in both directions, see Figure 5. Note that if  $X$  is an  $n$ -dimensional  $B$ -CW-complex, then  $Y$  is  $(n+1)$  dimensional  $B$ -CW-complex.

**Lemma 5.10.** *The  $B$ -CW-complex  $Y$  is a model for  $E_{\mathfrak{F}}B$ .*

PROOF. Let  $H$  be a subgroup of  $B$  such that  $H \notin \mathfrak{F}$  and let  $x \in X_k$  for some  $k \in \mathbb{Z}$ . Since  $\mathfrak{F}$  is assumed to be invariant under the automorphism  $\varphi$  we have  $\varphi^{-k}(H) \notin \mathfrak{F}$ . Therefore there exists a  $h \in H$  such that  $\varphi^{-k}(h)x \neq x$ . But then

$$\Phi_k(h, x) = \varphi^{-k}(h)x \neq x,$$

which implies that  $x \notin X_k^H$ . It follows that  $X_k^H = \emptyset$  and therefore also  $Y_k^H = \emptyset$  for all  $k \in \mathbb{Z}$ . Hence  $Y^H = \emptyset$ .

On the other hand, consider the case that  $H \in \mathfrak{F}$ . Since the family  $\mathfrak{F}$  is assumed to be invariant under the automorphism  $\varphi$  it follows that

$\varphi^k(H) \in \mathfrak{F}$  for every  $k \in \mathbb{Z}$ . Then  $X_k^H = X^{\varphi^k(H)}$  is contractible for every  $k \in \mathbb{Z}$ . Choose for every  $k \in \mathbb{Z}$  an element  $x_k \in X_k^H$  and a contracting homotopy  $h_k: X_k^H \simeq \{x_k\}$ . This induces a contracting homotopy  $h: Y^H \simeq L$  where  $L$  is a subspace of  $Y^H$  which is homeomorphic to the real line  $\mathbb{R}$ . Since  $L$  is contractible it follows that  $Y^H$  is contractible.  $\square$

For every  $(x, t) \in X_k \times [0, 1]$  and  $(g, r) \in G$  set

$$\Psi(g, r) := (\Phi_{k+r}(g, x), t) \in X_{k+r} \times [0, 1].$$

Straight forward calculation shows that this induces a well defined action

$$\Psi: G \times Y \rightarrow Y$$

of  $G$  on  $Y$ , which extends the already existing  $B$ -action on  $Y$ . If  $(g, r) \in G \setminus B$ , then  $r \neq 0$  and therefore clearly  $\Psi((g, r), x) \neq x$  for any  $x \in Y$ . Then together with Lemma 5.10 this implies that  $Y$  is an  $(n + 1)$ -dimensional model for  $E_{\mathfrak{F}}G$ . Altogether we have then shown the following result.

**Proposition 5.11.** *Let  $G = B \rtimes \mathbb{Z}$  be an arbitrary infinite cyclic extension where  $\mathbb{Z}$  acts on  $B$  via an automorphism  $\varphi \in \text{Aut}(B)$ . Let  $\mathfrak{F}$  be a family of subgroups of  $B$  which is invariant under the automorphism  $\varphi$ . If there exists an  $n$ -dimensional model for  $E_{\mathfrak{F}}B$  then there exists an  $(n + 1)$ -dimensional model for  $E_{\mathfrak{F}}G$ .  $\square$*

## 5. Examples

Strictly descending HNN-extensions are a natural source for candidates for infinite cyclic extensions  $G = B \rtimes \mathbb{Z}$  where  $\mathbb{Z}$  acts freely by conjugation on the set of conjugacy classes of the non-trivial elements of  $B$ .

The general setup is the following. Let  $B_0$  be a group and let  $\varphi: B_0 \rightarrow B_0$  a monomorphism. Recall that the descending HNN-extension determined by this data is the group  $G$  given by the presentation

$$G := \langle B_0, t \mid t^{-1}xt = \varphi(x) \text{ for all } x \in B_0 \rangle$$

and this group is usually denoted by  $B_0 *_{\varphi}$  in the literature. The group  $B_0$  is called the base group of the HNN-extension. The HNN-extension is called *strictly descending* if the monomorphism  $\varphi$  is not an isomorphism. We consider  $B_0$  as a subgroup of  $G$  in the obvious way.

Conjugation by  $t \in G$  defines an automorphism of  $G$  which agrees on  $B_0$  with  $\varphi$  which we will therefore denote by the same symbol. In other words, the monomorphism  $\varphi: B_0 \rightarrow B_0$  extends to the whole group  $G$  if we set

$$\varphi: G \rightarrow G, x \mapsto \varphi(x) := t^{-1}xt.$$

For each  $k \in \mathbb{Z}$  we set  $B_k := \varphi^k(B_0)$ . In this way we obtain a descending sequence

$$\dots \supset B_{-2} \supset B_{-1} \supset B_0 \supset B_1 \supset B_2 \supset \dots$$

of subgroups of  $G$ . This sequence of subgroups is strictly descending if and only if the HNN-extension is strictly descending. We denote the directed union of all these  $B_k$  by  $B$ . The automorphism  $\varphi$  restricts to an automorphism of  $B$  which is therefore a normal subgroup of  $G$ . It is standard fact that we can write  $G$  as the semidirect product  $G = B \rtimes \mathbb{Z}$  where  $\mathbb{Z}$  acts on  $B$  via the automorphism  $\varphi$  restricted to  $B$ .

**Lemma 5.12.** *Assume that for every non-trivial  $x \in B_0$  there exists a  $k \in \mathbb{N}$  such that  $x \notin \varphi^k(B_0)$ . Given  $x \in B_0$ , denote by  $[x]$  the set of all elements in  $B_0$  which are conjugate in  $B_0$  to  $x$ . Assume that for each  $x \in B$  we are given a finite subset  $[x]' \subset [x]$  such that  $\varphi([x]') \subset [\varphi(x)]'$  for every  $x \in B_0$ . Then  $\mathbb{Z}$  acts freely on the set of conjugacy classes of non-trivial elements of  $B$ .*

PROOF. We suppose that  $\mathbb{Z}$  does not act freely on the set of conjugacy classes of non-trivial elements of  $B$ . Then there exists  $x \in B$  and  $n \geq 1$  such that  $\varphi^n(x)$  is conjugate in  $B$  to  $x$ . Without any loss of generality we may assume that  $x \in B_0$  (otherwise replace  $x$  by  $\varphi^k(x)$  for a suitable  $k \in \mathbb{N}$ ). Furthermore, without any loss of generality we may assume that  $x \in [x]'$ . Finally we may assume without any loss of generality that  $\varphi^n(x)$  is actually conjugate in  $B_0$  to  $x$  (otherwise, again, replace  $x$  by  $\varphi^k(x)$  for a suitable  $k \in \mathbb{N}$ ).

Now  $\varphi^{rn}(x) \in [x]'$  for any  $r \geq 1$ . Since  $[x]'$  is finite this implies that  $\varphi^{rn}(x) = \varphi^{sn}(x)$  for some  $s > r$ . Therefore  $\varphi^{(s-r)n}(x) = x$  and since  $(s-r)n > 0$  it follows that  $x \in \varphi^k(B_0)$  for any  $k \in \mathbb{N}$ . However, this is a contradiction on the hypothesis that  $\mathbb{Z}$  does not act freely on the set of conjugacy classes of non-trivial elements of  $B$ . Therefore the opposite must be true.  $\square$

Note that the requirement that for every non-trivial element  $x \in B_0$  there exists a  $k \in \mathbb{N}$  such that  $x \notin \varphi^k(B_0)$  implies that the descending HNN-extension  $G = B_0 *_\varphi$  is actually strictly descending. Furthermore, we can conclude from it that the intersection of the groups  $B_k$ ,  $k \in \mathbb{Z}$ , is trivial.

**Example 5.13.** Let  $B_0$  be an abelian group and  $\varphi: B_0 \rightarrow B_0$  a monomorphism such that for every non-trivial  $x \in B_0$  there exists a  $k \in \mathbb{N}$  such that  $x \notin \varphi^k(B_0)$ . Since  $B_0$  is abelian, each conjugacy class  $[x]$  of elements in  $B_0$  contains precisely one element. Therefore Lemma 5.12 states that  $\mathbb{Z}$  acts freely by conjugation on the set of non-trivial elements of  $B$ . In particular we can use Proposition 5.9 to obtain a model for  $\underline{EG}$  from a model for  $E_{\mathfrak{F}_{\text{vc}}(B)}G$ .

Let  $B_0$  be a free group. An element  $x \in B_0$  is called *cyclically reduced* if it cannot be written as  $x = u^{-1}yu$  for some non-trivial  $u, y \in B_0$ . It follows from [MKS76, pp. 33ff.] that every element  $x \in B_0$  is conjugate to a cyclically reduced element  $x'$  and that there are only finitely many cyclically reduced elements in  $B_0$  which are conjugate to  $x$ . Therefore

$$[x]' := \{x' \in [x] : x' \text{ is cyclically reduced}\}$$

is a finite subset of  $[x]$  for every  $x \in B_0$ . Then the following two assumptions on the monomorphism  $\varphi: B_0 \rightarrow B_0$  are necessary in order to apply Lemma 5.12:

- (1) For every non-trivial  $x \in B_0$  there exists a  $k \in \mathbb{N}$  such that  $x \notin \varphi^k(B_0)$ ;
- (2) If  $x$  is a cyclically reduced element in  $B_0$ , then so  $\varphi(x)$ .

**Example 5.14.** Let  $X$  be an arbitrary non-empty set and let  $B_0 := F(X)$  be the free group on the basis  $X$ . Let  $\{\alpha_x\}_{x \in X}$  be a collection of integers such that  $|\alpha_x| \geq 2$  for every  $x \in X$ . Consider the endomorphism  $\varphi: B_0 \rightarrow B_0$  that maps any basis element  $x$  to  $x^{\alpha_x}$ . Then  $\varphi$  is a monomorphism which satisfies the assumptions (1) and (2) above. Lemma 5.12 tells us then that we can use Proposition 5.9 to construct a model for  $\underline{EG}$  from a model for  $E_{\mathfrak{F}_{\text{vc}}(B)}G$ .

**Example 5.15.** Another example of a strictly descending HNN-extension (in disguise) is the restricted wreath product  $A \wr \mathbb{Z}$  of an arbitrary group  $A$  by  $\mathbb{Z}$  which is defined as follows. Let  $A_k$  be a copy of  $A$  for each  $k \in \mathbb{Z}$ . Let

$B$  be the coproduct of all these  $A_k$  and let  $\mathbb{Z}$  act on  $B$  via  $\varphi$  which maps  $A_k$  identically onto  $A_{k+1}$  for all  $k \in \mathbb{Z}$ . Then

$$A \wr \mathbb{Z} := B \rtimes \mathbb{Z}.$$

Since each  $A_k$  is normal in  $B$  the above definition of  $\varphi$  forces the action of  $\mathbb{Z}$  on the set of conjugacy classes of non-trivial elements of  $B$  to be free. Therefore we can apply Proposition 5.9 in this case, too.

## 6. Dimensions

Given a family  $\mathfrak{F}$  of subgroups of  $G$ , a model for  $E_{\mathfrak{F}}G$  is only defined uniquely up to  $G$ -homotopy. Consider a model for  $E_{\mathfrak{F}}G$ . One particular invariant of the group  $G$  is called the *geometric dimension* of  $G$  with respect to the family  $\mathfrak{F}$ , and this is defined as being the least possible dimension of a model for  $E_{\mathfrak{F}}G$ . It is denoted by  $\text{gd}_{\mathfrak{F}}G$  and may be infinite. In the case that  $\mathfrak{F} = \{1\}$  we recover the classical geometric dimension of the group  $G$ . In the case that  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$  we denote the geometric dimension by  $\underline{\text{gd}}G$ .

**Proposition 5.16.** *Let  $G = B \rtimes \mathbb{Z}$  and assume that  $\mathbb{Z}$  acts freely via conjugation on the conjugacy classes of non-trivial elements of  $B$ . Then  $B$  is not virtually cyclic and*

$$\underline{\text{gd}}B \leq \underline{\text{gd}}G \leq \underline{\text{gd}}B + 1.$$

PROOF. Since (in general) a model for  $\underline{E}G$  is always a model for  $\underline{E}B$  via restriction, we have that the second inequality is the only non-trivial one. If  $X$  is an  $n$ -dimensional model for  $\underline{E}B$ , then the telescope construction in Section 4 gives an  $(n+1)$ -dimensional model for  $E_{\mathfrak{F}_{\text{vc}}(B)}G$ .

By Lemma 5.2 the group  $B$  cannot be virtually cyclic. Therefore  $n+1 \geq 2$  and attaching cells of dimension at most 2 does not increase the dimension of the resulting space. Hence Proposition 5.9 yields an  $(n+1)$ -dimensional model for  $\underline{E}G$  and this concludes the proof.  $\square$

**Corollary 5.17.** *Let  $G = B_0 *_\varphi$  be a descending HNN-extension as in Section 5. If  $G = B \rtimes \mathbb{Z}$  satisfies the conditions of the previous proposition then*

$$\underline{\text{gd}}B_0 \leq \underline{\text{gd}}G \leq \underline{\text{gd}}B_0 + 2.$$

PROOF. As exploited previously, since  $B_0$  is a subgroup of  $G$ , the second inequality is the only non-trivial part of the statement. The group  $B$  is the countable direct union of the conjugates of  $B_0$  in  $G$ . Therefore an  $n$ -dimensional model for  $\underline{E}B_0$  gives rise to an  $(n + 1)$ -dimensional model for  $\underline{E}B$  by a construction of Lück and Weiermann [LW07, pp. 11ff.]. Now the claim follows from the previous proposition.  $\square$

**Example 5.18.** Let  $G = B_0 *_{\varphi}$  be a descending HNN-extension with  $B_0$  a free group. If  $B_0$  has rank 1, then  $G$  is a soluble Baumslag–Solitar group and this case is treated below in Theorem 5.20. Thus we may assume that  $B_0$  has rank at least 2. Free groups are torsion-free and act freely on a tree which is therefore a 1-dimensional model for  $\underline{E}B_0$ . Free groups are word hyperbolic and therefore Proposition 9 in [JPL06] states the existence of a 2-dimensional model for  $\underline{E}B_0$ . On the other hand by Remark 16 in [JPL06] there cannot exist a model for  $\underline{E}B_0$  less than 2. Therefore  $\underline{\text{gd}} B_0 = 2$ . Now the direct union  $B$  of all conjugates of  $B_0$  in  $G$  is locally free and therefore does not contain a subgroup isomorphic to  $\mathbb{Z}^2$ . Then Lemma 5.1 states that we can apply Corollary 5.17 if and only if  $G$  does not contain a subgroup isomorphic to  $\mathbb{Z}^2$ . Therefore we get in this case the estimation  $2 \leq \underline{\text{gd}} G \leq 4$ .

**Example 5.19.** Consider the restricted wreath product  $G = A \wr \mathbb{Z}$  where  $A$  is a countable locally finite group. Then

$$B := \prod_{k \in \mathbb{Z}} A$$

is also a countable locally finite group. Since  $B$  is not finite it follows that  $\underline{\text{gd}} B = 1$  by Lemma 4.28. We have seen that  $G$  does satisfy the requirements of Proposition 5.16. Therefore we get the estimate  $1 \leq \underline{\text{gd}} G \leq 2$ . We will see in the next chapter with Corollary 6.3, that  $\underline{\text{gd}} G = 1$  implies that  $G$  is locally virtually cyclic. However  $G$  is not locally virtually cyclic and therefore we  $\underline{\text{gd}} G \neq 1$ . Thus we have altogether

$$\underline{\text{gd}} G = 2.$$

Note that the smallest concrete example of a group of this type is the Lamplighter group  $L = \mathbb{Z}_2 \wr \mathbb{Z}$  where  $\mathbb{Z}_2$  is the cyclic group of the integers modulo 2.

## 7. Soluble Baumslag–Solitar Groups

We conclude this chapter with a complete answer to the geometric dimension of the soluble Baumslag–Solitar groups with respect to the family of virtually cyclic subgroups. These groups belong to a class of two-generator and one-relator groups introduced by Baumslag and Solitar in [BS62]. Their class contains all the groups

$$BS(m, n) = \langle x, t \mid t^{-1}x^mt = x^n \rangle.$$

where  $m$  and  $n$  are non-zero integers. The soluble Baumslag–Solitar groups are the groups of the form  $BS(1, m)$ ,  $m \neq 0$  and these groups can also be written as

$$BS(1, m) = \mathbb{Z}[1/m] \rtimes \mathbb{Z},$$

where  $\mathbb{Z}[1/m]$  is the subgroup of the rational numbers  $\mathbb{Q}$  generated by all powers of  $1/m$  and where  $\mathbb{Z}$  acts on  $\mathbb{Z}[1/m]$  by multiplication with  $m$ . The group  $BS(1, 1)$  is  $\mathbb{Z}^2$  and  $BS(1, -1)$  is the Klein bottle group  $\mathbb{Z} \times \mathbb{Z}$ . If  $|m| \geq 2$ , then  $BS(1, m)$  belongs to the case described in Example 5.13, as well as to the case described in Example 5.14.

**Theorem 5.20.** *Let  $G = \mathbb{Z}[1/m] \rtimes \mathbb{Z}$  be a soluble Baumslag–Solitar group. Then*

$$\underline{\text{hd}} G = \underline{\text{cd}} G = \underline{\text{gd}} G = \begin{cases} 3 & \text{if } |m| = 1, \\ 2 & \text{if } |m| \geq 2. \end{cases}$$

PROOF. The case  $|m| = 1$  has been answered in the previous chapter. Thus we assume that  $|m| \geq 2$ . In this case  $G$  is the fundamental group of a graph  $(G, Y)$  of groups in the sense of [Ser80] where  $Y$  is a loop and where the vertex groups are all infinite cyclic. Let  $X$  be the Bass–Serre tree associated with this graph of groups. Then  $T$  is not only a model for  $\underline{\underline{E}}\mathbb{Z}[1/m]$  but also a model for  $E_{\mathfrak{F}_{\text{vc}}(\mathbb{Z}[1/m])}G$ . We can apply Proposition 5.9 and obtain a model  $X$  for  $\underline{\underline{E}}G$  by attaching cells of dimension less or equal to 2 to  $T$ . Therefore we get  $\underline{\text{gd}} G \leq 2$ .

In order to see that  $\underline{\text{hd}} G \geq 2$  we calculate  $H_2(\underline{\underline{B}}G)$ . Note that  $Y = X/G$  is a model for  $B_{\mathfrak{F}_{\text{vc}}(B)}G$  and  $Y$  consists of one 0-cell and one 1-cell. The second part of Proposition 5.9 states that we can obtain a model for  $\underline{\underline{B}}G$  by attaching 2-cells to  $Y$  indexed by the conjugacy classes of maximal virtually cyclic



subgroups of  $G$  that are not contained in  $\mathbb{Z}[1/m]$ . But there are infinitely many of them. Therefore  $H_2(\underline{BG}) \neq 0$  which implies that  $\underline{\text{hd}} G \geq 2$ .

Altogether we get  $2 \leq \underline{\text{hd}} G \leq \underline{\text{cd}} G \leq \underline{\text{gd}} G \leq 2$  and thus equality holds.  $\square$

## 8. Relative Hyperbolic Groups and Free Products

In the literature a common strategy to construct a model for  $\underline{EG}$  is to begin with a known model for  $\underline{EG}$  and attach cells in order to obtain the a model for  $\underline{EG}$ . One key idea in the construction of models for  $\underline{EG}$  in this chapter has been to begin with a model for  $E_{\mathfrak{F}}G$  with  $\mathfrak{F}_{\text{fin}}(G) \subset \mathfrak{F} \subset \mathfrak{F}_{\text{vc}}(G)$  where  $\mathfrak{F}$  is a family of subgroups of  $G$  which in general is larger than  $\mathfrak{F}_{\text{fin}}(G)$ . In what follows we give another example for a fruitful application of this idea.

Let  $G$  be a group and  $H_\lambda$ ,  $\lambda \in \Lambda$ , a collection of subgroups of  $G$ . Assume that  $G$  is relatively hyperbolic with respect to the subgroups  $H_\lambda$  in the sense of [Osi06]. The subgroups  $H_\lambda$  are called the peripheral subgroups of  $G$ . Consider the set

$$\mathfrak{F} := \{H^g : H \in \mathfrak{F}_{\text{vc}}(H_\lambda), \lambda \in \Lambda, g \in G\} \cup \mathfrak{F}_{\text{fin}}(G), \quad (5.1)$$

that is,  $\mathfrak{F}$  consists of all virtually cyclic subgroups which are subconjugate to one of the peripheral subgroups of  $G$  together with all finite subgroups of  $G$ . This is clearly a full family of subgroups of  $G$ .

Lafont and Ortiz have shown in [LO07, p. 532f.] using results of Osin that if  $G$  is relative hyperbolic in the sense of Bowditch [Bow99] that the following is true:

- (1)  $\mathfrak{F} \cap H \subset \mathfrak{F}_{\text{fin}}(H)$  for every  $H \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}$ ;
- (2) every  $H \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}$  is contained in a unique maximal  $H_{\text{max}} \in \mathfrak{F}_{\text{vc}}(G)$ ;
- (3)  $N_G(H_{\text{max}}) = H_{\text{max}}$  for every  $H \in \mathfrak{F}_{\text{vc}}(G) \setminus \mathfrak{F}$ .

The definition of relative hyperbolicity in [Osi06] extends the definition of relative hyperbolicity in [Bow99]. Furthermore, the proof in [LO07] of the above result is also correct for relative hyperbolic groups in the sense of [Osi06]. Therefore we can apply Proposition 5.9 in the current setting. That is, one can obtain a model for  $\underline{EG}$  by attaching orbits of at most 2-dimensional cells to any model for  $E_{\mathfrak{F}}G$ .

Lafont and Ortiz have constructed in [LO07] a model for  $\underline{EG}$  for relative hyperbolic groups in the sense [Bow99] by forming the join  $X * Y$  where  $X$  is a model for  $\underline{EG}$  and  $Y$  is the disjoint union of models for  $\underline{EH}_\lambda$ ,  $\lambda \in \Lambda$ , and a set of discrete points. Their construction is also valid for relatively hyperbolic groups in the sense of [Osi06] and from the join construction one obtains

$$\begin{aligned} \dim(X * Y) &= \dim(X) + \dim(Y) + 1 \\ &= \underline{\text{gd}} G + \sup\{\underline{\text{gd}} H_\lambda : \lambda \in \Lambda\} + 1 \end{aligned}$$

This is the lowest dimension one can achieve with Lafont and Ortiz's construction. There is no example known where  $\underline{\text{gd}} G > \underline{\text{gd}} G + 1$  and this has raised the question whether the bound  $\underline{\text{gd}} G \leq \underline{\text{gd}} G + 1$  for every group  $G$ , see [LW07, p. 3]. Thus, if the peripheral subgroups contain groups which are not virtually cyclic, then the dimension of  $X * Y$  is strictly larger than  $\underline{\text{gd}} G + 1$  and suggests that in this case  $X * Y$  is not a model of minimal dimension.

However, if a nice model is known for  $E_{\mathfrak{F}}G$ , where  $\mathfrak{F}$  is as in (5.1), then Proposition 5.9 can give a model of minimal dimension for  $\underline{EG}$ . We conclude with an example where we can construct a nice model for  $E_{\mathfrak{F}}G$  such that we obtain a model for  $\underline{EG}$  of minimal dimension.

Let  $G$  be a free product

$$G := H_1 * \cdots * H_n$$

of finitely many groups  $H_i$ . It follows straight from the definition in [Osi06] that  $G$  is relatively hyperbolic with respect to the factors  $H_i$ ,  $i = 1, \dots, n$ . For simplicity we assume in the following that  $n = 2$  and in order to avoid triviality we assume that  $G$  is not virtually cyclic.

Since  $G$  is not virtually cyclic it follows that  $G$  has free subgroup of rank 2. Thus  $\underline{\text{gd}} G \geq 2$  and since also  $G$  contains  $H_1$  and  $H_2$  as subgroups we get altogether

$$\underline{\text{gd}} G \geq \max(\underline{\text{gd}} H_1, \underline{\text{gd}} H_2, 2).$$

Similarly to Example 4.10 in [Lüc05, p. 290] we construct a  $G$ -CW-complex  $X$  which is obtained from the Bass–Serre tree  $T$  associated with the free product  $H_1 * H_2$  by replacing the vertices  $v$  of  $T$  equivariantly by

models for  $\underline{E}H_1$  and  $\underline{E}H_2$ . More precisely, for  $i = 1, 2$  choose once and for all  $x_i \in X_i$  where  $X_i$  is a model for  $\underline{E}H_i$  and define  $G$ -equivariant maps

$$\begin{aligned} F_i: G &\rightarrow G \times_{H_i} X_i, \\ g &\mapsto [g, x_i], \end{aligned}$$

where  $G \times_{H_i} X_i$  denotes the  $G$ -space induced from the  $H_i$ -space  $X_i$ . We obtain  $X$  as a  $G$ -equivariant cellular pushout

$$\begin{array}{ccc} G \times \{0, 1\} & \xrightarrow{F_1 \amalg F_2} & (G \times_{H_1} X_1) \amalg (G \times_{H_2} X_2) \\ \downarrow & & \downarrow \\ G \times [0, 1] & \longrightarrow & X \end{array}$$

It follows that  $X$  is a model for  $E_{\mathfrak{F}}G$  where

$$\mathfrak{F} := \{H^g : H \in \mathfrak{F}_{\text{vc}}(H_1) \cup \mathfrak{F}_{\text{vc}}(H_2) \text{ and } g \in G\},$$

that is  $\mathfrak{F}$  is the family of all virtually cyclic subgroups of  $G$  which are subconjugate to one of the on of the factors  $H_1$  or  $H_2$ . Since any finite subgroup of  $G$  is conjugate to one of the factors  $H_1$  or  $H_2$  we have that  $\mathfrak{F}$  includes all finite subgroups of  $G$  [Ser80, p. 36]. Thus this family agrees with the family defined in (5.1). By construction we have

$$\dim X = \max(\underline{\text{gd}} H_1, \underline{\text{gd}} H_2, 1).$$

Now we can apply Proposition 5.9 to obtain a model  $Z$  for  $\underline{E}G$  by attaching to  $X$  orbits of cells in dimension 2 and less. Thus

$$\underline{\text{gd}} G \leq \dim Z = \max(\underline{\text{gd}} H_1, \underline{\text{gd}} H_2, 2).$$

**Theorem 5.21.** *Let  $G := H_1 * \cdots * H_n$  be a free product of finitely many groups. If  $G$  is not virtually cyclic then*

$$\underline{\text{gd}} G = \max(\underline{\text{gd}} H_1, \dots, \underline{\text{gd}} H_n, 2)$$

PROOF. This statement follows either by adapting the above construction to general values of  $n$ . Alternatively one can prove it by induction on  $n$  and using the fact  $H_1 * \cdots * H_n \cong (H_1 * \cdots * H_{n-1}) * H_n$ .  $\square$

## Groups with Low Bredon Dimension for the Family $\mathfrak{F}_{\text{vc}}$

### 1. Groups $G$ with $\underline{\text{gd}}G = 0$

The classification of groups with  $\underline{\text{gd}}G = 0$  is a straight forward consequence of Proposition 3.19 and Proposition 3.20 applied to the family  $\mathfrak{F} = \mathfrak{F}_{\text{vc}}(G)$ .

**Proposition 6.1.** *Let  $G$  be a group. Then  $\underline{\text{gd}}G = 0$  if and only if  $\underline{\text{cd}}G = 0$  if and only if  $G$  is virtually cyclic.  $\square$*

### 2. Groups $G$ with $\underline{\text{gd}}G = 1$

**Proposition 6.2.** *Let  $G$  be a group with  $\underline{\text{gd}}G = 1$ . Then  $G$  is not finitely generated.*

PROOF. By assumption  $G$  has a tree  $T$  as a model for  $\underline{EG}$ . Assume towards a contradiction that  $G$  is finitely generated. For every cyclic subgroup  $\langle g \rangle$  of  $G$  the fixed point set  $T^{\langle g \rangle} \neq \emptyset$  since  $T$  is a model for  $\underline{EG}$ . Hence every element of  $G$  has fixed points and Corollary 3 to Proposition 25 in [Ser80, pp. 64f.] implies that  $T^G \neq \emptyset$ . This can only happen if  $G$  is virtually cyclic. Then Proposition 6.1 implies that  $\underline{\text{gd}}G = 0$ , which is a contradiction to the assumption that  $\underline{\text{gd}}G = 1$ . Therefore  $G$  cannot be finitely generated.  $\square$

**Corollary 6.3.** *A group  $G$  with  $\underline{\text{gd}}G = 1$  is locally virtually cyclic.*

PROOF. If  $H$  is a finitely generated subgroup of  $G$  then  $\underline{\text{gd}}H \leq \underline{\text{gd}}G = 1$ . Then Proposition 6.2 implies  $\underline{\text{gd}}H \neq 1$  and therefore we must have  $\underline{\text{gd}}H = 0$ . Hence  $H$  is virtually cyclic by Proposition 6.1.  $\square$

**Corollary 6.4.** *If  $G$  is a group with  $\underline{\text{gd}}G = 1$ , then  $\underline{\text{cd}}G = 1$  and  $\underline{\text{hd}}G = 0$ .*

PROOF. This is true for every locally virtually cyclic group by Corollary 3.48.  $\square$

Using a result of Lück and Weiermann, we can now prove the following classification of countable groups  $G$  with  $\underline{\underline{\text{gd}}}G = 1$ .

**Proposition 6.5.** *Let  $G$  be a countable group. Then  $\underline{\underline{\text{gd}}}G = 1$  if and only if  $G$  is locally virtually cyclic but not virtually cyclic.*

PROOF. The “only if” part is covered by Corollary 6.3.

Conversely, assume that  $G$  is locally virtually cyclic. Since  $G$  is countable, it has only countably many virtually cyclic subgroups and the claim follows from Lemma 4.2 and Theorem 4.3 in [LW07, pp. 12ff.] together with Proposition 6.1.  $\square$

A natural question which arises is the following: does  $\underline{\underline{\text{cd}}}G = 1$  imply  $\underline{\underline{\text{gd}}}G = 1$ ? If not, under which conditions on the group  $G$  does this implication hold?

**Theorem 6.6.** *Let  $G$  be a countable, torsion-free, soluble group. Then*

$$\underline{\underline{\text{cd}}}G = 1 \iff \underline{\underline{\text{gd}}}G = 1.$$

PROOF. “ $\Leftarrow$ ”: This is Corollary 6.4.

“ $\Rightarrow$ ”: Theorem 3.37 implies that  $\underline{\underline{\text{cd}}}G \leq \underline{\underline{\text{cd}}}G + 1 = 2$ . Since  $G$  is assumed to be torsion free, we have that  $\text{cd}G = \underline{\underline{\text{cd}}}G$  and thus  $\text{cd}G \leq 2$ .

Now  $\text{cd}G = 0$  if and only if  $G$  is trivial, and in this case  $\underline{\underline{\text{cd}}}G = 0$  which is a contradiction. Furthermore  $\text{cd}G = 1$  if and only if  $G$  is a free group. Since free groups of rank greater or equal to two are not soluble,  $G$  must necessarily be cyclic and in this case we obtain the contradiction  $\underline{\underline{\text{cd}}}G = 0$ . Thus we must have that  $\text{cd}G = 2$ .

By the classification of soluble groups of cohomological dimension 2 due to Gildenhuys [Gil79] lists the following possibilities for  $G$ :

- (1)  $G \cong BS(1, m)$  for some integer  $m \neq 0$ ;
- (2)  $G$  is isomorphic to a non-cyclic subgroup of  $\mathbb{Q}$ .

In the first case we have  $\underline{\underline{\text{cd}}}G \geq 2$  by Theorem 5.20. However, this contradicts the assumption  $\underline{\underline{\text{cd}}}G = 1$ . Thus  $G$  must be isomorphic to a non-cyclic subgroup of  $\mathbb{Q}$ . In this case  $G$  is locally virtually cyclic but not virtually cyclic. Thus  $\underline{\underline{\text{gd}}}G = 1$  by Proposition 6.5.  $\square$

### 3. Groups $G$ with $\underline{\underline{\text{gd}}}G = 2$ or $\underline{\underline{\text{gd}}}G = 3$

There is not much known about which groups  $G$  have  $\underline{\underline{\text{gd}}}G = 2$ , and even less about which groups  $G$  have  $\underline{\underline{\text{gd}}}G = 3$ . We conclude with a summary of the results obtained in this thesis for groups which belong to this class of groups:

- (1) Let  $G$  be a Gromov-hyperbolic group with  $\underline{\underline{\text{gd}}}G \leq 2$ . If  $G$  is not virtually cyclic, then

$$\underline{\underline{\text{hd}}}G = \underline{\underline{\text{cd}}}G = \underline{\underline{\text{gd}}}G = 2$$

by Proposition 4.9. In particular this includes the cases where  $G$  is a free group of rank at least 2 (Corollary 4.10) and where  $G$  is the fundamental group of a finite graph of finite groups (Proposition 4.12).

- (2) If  $G \cong \mathbb{Z}[1/m] \rtimes \mathbb{Z}$  is a soluble Baumslag–Solitar group,  $|m| \neq 1$ , then we have by Theorem 5.20

$$\underline{\underline{\text{hd}}}G = \underline{\underline{\text{cd}}}G = \underline{\underline{\text{gd}}}G = 2.$$

- (3) For any virtually polycyclic group  $G$  with  $\text{vcd}G = 2$  we have

$$\underline{\underline{\text{hd}}}G = \underline{\underline{\text{cd}}}G = \underline{\underline{\text{gd}}}G = 3$$

by Proposition 4.7. In particular this holds for  $\mathbb{Z}^2$  and  $\mathbb{Z} \rtimes \mathbb{Z}$ .

In particular the above cases are not counter examples for the Eilenberg–Ganea Conjecture for Bredon cohomology with respect to the family of virtually cyclic subgroups (cf. Section 4 in Chapter 3).

Furthermore, if  $G$  is the restricted wreath product  $A \wr \mathbb{Z}$  where  $A$  is a non-trivial, countable, locally finite group, then we have seen in Example 5.19 in the previous chapter that

$$\underline{\underline{\text{gd}}}G = 2.$$

In particular this is true for the Lamplighter group  $L = \mathbb{Z}_2 \wr \mathbb{Z}$ .

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