A GRADIENT BOUND FOR FREE BOUNDARY GRAPHS

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Abstract. We prove an analogue for a one-phase free boundary problem of the classical gradient bound for solutions to the minimal surface equation. It follows, in particular, that every energy-minimizing free boundary that is a graph is also smooth. The method we use also leads to a new proof of the classical mimimal surface gradient bound.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$, and consider the one-phase free boundary problem ($u \geq 0$)

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = 1, & \text{on } F(u) := \partial\Omega \cap \Omega^+(u). \end{cases} \tag{1.1}$$

The set $F(u)$ is known as the free boundary. There are strong parallels between the theory of these hypersurfaces and the theory of minimal surfaces. The existence of solutions u and partial regularity (smoothness almost everywhere with respect to surface measure) of the free boundary $F(u)$ was proved by Alt and Caffarelli [AC].

We will begin by formulating our main result in the special case of energy-minimizing solutions. We call u energy-minimizing on $\bar{\Omega}$ if u minimizes the functional

$$J(v) = \int_\Omega (|\nabla v|^2 + \chi_{\{v>0\}})dx,$$

among all functions with the same boundary values as u. The first variation (Euler-Lagrange) equations for u are (1.1). Indeed, it is easy to show that u is harmonic in $\Omega^+(u)$, and it follows from deeper results of [AC, C1, C2] that u satisfies the free boundary condition $|\nabla u| = 1$ on $F(u)$ in a viscosity sense, defined in Section 2.

Define a cylinder of height $2L$, with base the ball of radius B_r in $\mathbb{R}^{n-1}$, by

$$\mathcal{C}(r, L) := B_r \times (-L, L) \subset \mathbb{R}^n; \quad (\text{and } \mathcal{C}_L := \mathcal{C}(1, L)).$$

Theorem 1.1. *If an energy-minimizing solution u on the cylinder $\mathcal{C}_L$ is monotone in the vertical direction,*

$$\partial u/\partial x_n \geq 0 \quad \text{on} \quad \mathcal{C}_L^+(u),$$

and its free boundary $F(u)$ is a fixed distance from the top and bottom of the cylinder, i. e.,

$$F(u) \subset \mathcal{C}_{L-\epsilon} \quad \text{for some } \epsilon > 0,$$

then $F(u)$ is the graph of a smooth function φ,

$$F(u) = \{(x, y) : x \in B_1; \quad y = \varphi(x)\}$$

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with

$$\sup_{x \in B_{1/2}} |\nabla \varphi(x)| \leq C$$

for a constant C *depending only on* L, ϵ, *and* n.

Let us compare our theorem with the classical gradient bound on minimal surfaces due to Bombieri, De Giorgi, and Miranda, which can be stated as follows.

Theorem 1.2. [BDM] *Let* $\phi \in C^\infty(B_1)$ *be a solution to the minimal surface equation*

$$div\left(\frac{\nabla \phi}{\sqrt{1+|\nabla \phi|^2}}\right) = 0 \quad in \quad B_1, \tag{1.2}$$

with $|\phi| \leq M$. *Then*

$$|\nabla \phi| \leq C \quad in \quad B_{1/2} \tag{1.3}$$

with C *depending on* n *and* M.

The hypothesis $\partial u/\partial x_n \geq 0$ in Theorem 1.1 implies, by the strong maximum prinicple, that $\partial u/\partial x_n > 0$. Therefore the level surfaces $\{x : u(x) = c\}$ for $c > 0$ are graphs. The hypothesis that the free boundary is a fixed distance from the top and bottom of the cylinder replaces the hypothesis in Theorem 1.2 that the oscillation of the function ϕ is bounded by M. Furthermore, the minimal surface equation (1.2) implies that the graph of ϕ is area-minimizing, so that the assumption in Theorem 1.1 that the free boundary is energy minimizing is analogous.

In the theory of minimal surfaces, it is well-known that minimal graphs are real analytic in the interior of the their domain of definition. The key first step in the proof of full regularity of the minimal graphs is to establish that the graph is Lipschitz, that is, the graph of a function with a bounded gradient. The gradient bound proved here leads, likewise, to full regularity. If the free boundary is a Lipschitz graph, then Caffarelli [C1] proved that the graph is $C^{1,\alpha}$ for some $\alpha > 0$. Higher regularity results of [KN] then yield the local analyticity of $F(u)$. So real analyticity follows if one can confirm the Lipschitz property, i. e., the gradient bound.

In [D2], an a priori gradient bound for smooth free boundary graphs is proved in the case when $n = 2, 3$. The proof given there is also motivated by the strong analogy with minimal surfaces, but is completely different. An advantage of the results here is that because they work in all dimensions, they can be expected to apply to the free boundary analogue of the Bernstein problem. The application we have in mind is to the construction (as yet unrealized) of a global solution to the free boundary problem (other than the obvious solution $u(x) = x_1^+$) whose level surfaces are graphs. This would be analogous to the counterexample to the Bernstein conjecture — a complete non-planar mimimal graph constructed in [BDG] in $\mathbb{R}^9$. In [DJ], it is shown that a certain cone in $\mathbb{R}^7$ is the free boundary analogue of the Simons cone in minimal surface theory. Based on this example, one should expect to find a free boundary whose level surfaces are non-flat graphs in $\mathbb{R}^8$.

The theorem whose proof occupies most of this paper has a more technical statement. See Section 2 for the definition of a viscosity solution and nontangentially accessible (NTA) domains.

Theorem 1.3. *Let u be a viscosity solution to (1.1) in the cylinder $\mathcal{C}_L$. Suppose that u is monotone in the vertical direction,*

$$\partial u/\partial x_n \geq 0 \quad on \quad \mathcal{C}_L^+(u),$$

and its free boundary is given as the graph of a continuous function φ, $F(u) = \{(x,y) : x \in B_1; \quad y = \varphi(x)\}$. Suppose that the oscillation of φ is bounded,

$$\max_{x \in B_1} |\varphi(x)| \leq L - 1,$$

and, finally, that there is a nontangentially accessible (NTA) domain $\mathcal{D}$ such that

$$\mathcal{C}\left(\frac{9}{10}, L - \frac{1}{2}\right) \cap \mathcal{C}_L^+(u) \subset \mathcal{D} \subset \mathcal{C}_L^+(u).$$

Then

$$\sup_{x \in B_{1/2}} |\nabla \varphi(x)| \leq C$$

for a constant C depending only on L, the NTA constants, and n.

Theorem 1.1 will follow from Theorem 1.3 using results of [D1]. Roughly speaking, [D1] shows that the hypotheses of Theorem 1.1 imply the hypotheses of Theorem 1.3. In particular, a key estimate from [D1] is that the positive phase satisfies an NTA property on any smaller cylinder. Moreover, it is also proved in [D1] that under the hypotheses of Theorem 1.1, the free boundary is the graph of a continuous function φ.

The proof of Theorem 1.3 is based on comparing $u(x)$ to its vertical translates $u(x+te_n)$. One constructs a family of supersolutions related to $u(x+te_n)$ and uses a deformation maximum principle argument to show that $u(x+te_n) \geq u(x) + ct$ for sufficiently small $t > 0$. The function $u(x)$ is comparable to the distance from x to the free boundary. The estimate shows that the change in u in the vertical direction is comparable to the change in u in the direction normal to each level surface, which is equivalent to a Lipschitz bound on the graph of the level surface.

The construction of the family of supersolutions makes use of the basic estimates on NTA domains which were the reason the notion of NTA was introduced in [JK]. The NTA property guarantees that every positive harmonic function that vanishes on the boundary vanishes at the same rate as u. The NTA property was first used in connection with regularity of free boundaries by Aguilera, Caffarelli and Spruck [ACS], who proved a partial regularity result. The NTA property also holds for the singular conic solution of Alt and Caffarelli. (This cone is not a graph, of course. Otherwise it would contradict Theorem 1.3.)

Our proof of the gradient bound for free boundaries leads to a new proof of the classical gradient bound for minimal graphs. This new proof of Theorem 1.2 is related to a much simpler proof due to N. Korevaar [K]. The hope is that this new method, while more complicated than the method in [K], will ultimately apply to classes of semilinear problems that include both free boundary problems and minimal surface problems as singular limits. An interesting aspect of our proof is that it deepens the analogy between minimal surfaces and free boundaries.

The paper is organized as follows. In Section 2 after briefly recalling some standard definitions and known results, we prove Theorem 1.3 and deduce Theorem 1.1. We present our proof of Theorem 1.2 in Section 3. In Section 4, we examine the parallels between the two proofs and especially between two key parallel ingredients, namely the boundary Harnack inequality for NTA domains and the intrinsic Harnack inequality of Bombieri and Giusti [BG].

2. Gradient bound for free boundary graphs

2.1. **Preliminaries.** We recall the definition of a viscosity solution [C1].

Definition 2.1. Let u be a nonnegative continuous function in Ω. We say that u is a viscosity solution to (1.1) in Ω if and only if the following conditions are satisfied:

(i) $\Delta u = 0$ in $\Omega^+(u)$;

(ii) If $x_0 \in F(u)$ and $F(u)$ has at x_0 a tangent ball $\mathcal{B}_\epsilon$ from either the positive or the zero side, then, for ν the unit radial direction of $\partial\mathcal{B}_\epsilon$ at x_0 into $\Omega^+(u)$,

$$u(x) = \langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \text{ as } x \to x_0.$$

Standard elliptic regularity theory implies that if $F(u)$ is a smooth surface near x_0, then u is smooth up to the free boundary near x_0 and the free boundary condition $|\nabla u| = 1$ is valid in the classical sense in such a neighborhood.

Denote by $d(x) = \text{dist}(x, F(u))$. In this section, the balls $\mathcal{B}_r = \mathcal{B}_r(0)$ and $\mathcal{B}_r(x)$ will be in $\mathbb{R}^n$ while the balls $B_r(x)$ will be in $\mathbb{R}^{n-1}$. The following result follows easily from the Hopf lemma and interior regularity of elliptic equations (see for example [CS],[D2]).

Lemma 2.2. *Let u be a viscosity solution to* (1.1) *in $\mathcal{B}_1$, $0 \in F(u)$. Then, u is Lipschitz continuous in $\mathcal{B}_{1/2}$ and there is a dimensional constant K such that*

$$\sup_{\mathcal{B}_{1/2}} |\nabla u| \leq K,$$

and

$$u(x) \leq K d(x), \quad \textit{for all } x \in \mathcal{B}_{1/2}.$$

Definition 2.3. We say that a viscosity solution u is nondegenerate in $\mathcal{B}_1$ if there is a constant $c > 0$ such that $u(x) \geq c d(x)$ for all $x \in \mathcal{B}_1^+(u)$.

We now recall the notion of nontangentially accessible (NTA) domains.

Definition 2.4. A bounded domain D in $\mathbb{R}^n$ is called NTA, when there exist constants M and $r_0 > 0$ such that:

(i) Corkscrew condition. For any $x \in \partial D$, $r < r_0$, there exists $y = y_r(x) \in D$ such that $M^{-1}r < |y - x| < r$ and $\text{dist}(y, \partial D) > M^{-1}r$;

(ii) The Lebesgue density of D^c at any of its points is bounded below uniformly by a positive constant c, i.e for all $x \in \partial D, 0 < r < r_0$,

$$\frac{|\mathcal{B}_r(x) \setminus D|}{|\mathcal{B}_r(x)|} \geq c;$$

(iii) Harnack chain condition. If $\epsilon > 0$ and x_1, x_2 belong to D, $\text{dist}(x_j, \partial D) > \epsilon$ and $|x_1 - x_2| < C_1\epsilon$, then there exists a sequence of C_2 balls of radius $c\epsilon$ such that the first ball is centered at x_1, the last at x_2, such that the centers of consecutive balls are at most $c\epsilon/2$ apart. The number of balls C_2 in the chain depends on C_1, but not on ϵ.

We recall some results about NTA domains [JK]. We start with the following boundary Harnack principle for harmonic functions.

Theorem 2.5. *(Boundary Harnack principle) Let D be an NTA domain and let V be an open set. For any compact set $K \subset V$, there exists a constant C such that for all positive harmonic functions u and v in D vanishing continuously on $\partial D \cap V$, and $x_0 \in D \cap K$,*

$$C^{-1}\frac{v(x_0)}{u(x_0)}u(x) \leq v(x) \leq C\frac{v(x_0)}{u(x_0)}u(x), \quad \textit{for all} \;\; x \in K \cap \overline{D}.$$

The boundary Harnack inequality above will be our main tool in the proof of Theorem 1.3. We will also need some further facts. First, recall that for any bounded domain $D \subset \mathbb{R}^n$ and any arbitrary $y_0 \in D$, one can define the harmonic measure ω^{y_0} of D evaluated at y_0 (for the definition see for example [JK]). We note that for any $y_1, y_2 \in D$, the measures ω^{y_1} and ω^{y_2} are mutually absolutely continuous. Hence, from now on we fix a point $y_0 \in D$ and denote $\omega = \omega^{y_0}$.

A nontangential region at $x_0 \in \partial D$ is defined as

$$\Gamma_\alpha(x_0) = \{x \in D : |x - x_0| < (1+\alpha)\mathrm{dist}(x, \partial D)\}.$$

Let u be defined on D and f on ∂D. We say that u converges to f nontangentially at $x_0 \in \partial D$ if for any α,

$$\lim_{x \to x_0} u(x) = f(x_0) \quad \text{for } x \in \Gamma_\alpha(x_0).$$

The following Fatou-type theorem was proved in [JK].

Theorem 2.6. *Let D be an NTA domain. If u is a positive harmonic function in D, then u has finite nontangential limits for ω-almost every $x_0 \in \partial D$.*

We deduce from this the following regularity result for NTA free boundaries.

Lemma 2.7. *Let u be a viscosity solution to* (1.1) *in $\mathcal{B}_1$, u non-degenerate in $\mathcal{B}_{3/4}$, and $0 \in F(u)$. Assume that there is an NTA domain D such that $D \subset \mathcal{B}_1^+(u)$ and $F(u) \cap \mathcal{B}_{3/4} \subset \partial D$. Then, $F(u) \cap \mathcal{B}_{1/2}$ is smooth almost everywhere with respect to harmonic measure ω of D.*

Proof. Since each partial derivative $\partial u/\partial x_j$ is a bounded harmonic function, Theorem 2.6 implies that for ω-almost every $x_0 \in F(u) \cap \mathcal{B}_{1/2}$, there exists $a \in \mathbb{R}^n$ such that for every $\alpha < \infty$, $\nabla u(x) \to a$ as $x \to x_0$, for $x \in \mathcal{B}_1^+(u)$, $|x - x_0| < (1+\alpha)\mathrm{dist}(x, F(u))$. We will prove that $F(u)$ is flat and hence smooth in a neighborhood of x_0. The idea of the proof is to show that for x near x_0, u is close to a linear function with gradient a. Provided that a is not the zero vector, this will show us that the level sets of u are flat and hence (by [AC, C2]) that the free boundary is smooth near x_0.

For notational simplicity assume $x_0 = 0$. Denote by u_r the rescaling of u, $u_r(x) = u(rx)/r$. We will use the notation $A_1 \approx A_2$ for positive numbers that are comparable modulo constants that depend only on the NTA constants and the ratio of $u(x)$ to the distance to the free boundary (bounded above and below by Lemma 2.2 and nondegeneracy). Consider a point $z \in \mathcal{B}_1^+(u_r)$ such that $u_r(z) \approx 1$. Note that although the point z depends on r, we require the constants in comparability of $u_r(z)$ with 1 to be independent of r as $r \to 0$. For any $x \in \mathcal{B}_1 \cap \overline{\{u_r > 0\}}$, the NTA properties imply there is a (nontangential, corkscrew) path $p(t)$ such that $p(0) = x$, $p(1) = z$, $|p'(t)| \le C$ and

$$u_r(p(t)) \approx \mathrm{dist}(p(t), F(u_r)) \approx t + u_r(p(0))$$

independent of r.

Fix $C_2 << C_1 < \infty$, $\delta > 0$, and denote,

$$T^r_\delta = \{x \in \mathcal{B}^+_{C_1}(u_r) : u_r(x) \ge \delta\},$$

$$\Gamma^r_\alpha = \{x \in \mathcal{B}^+_{1/r}(u_r) : |x| < (1+\alpha)\mathrm{dist}(x, F(u_r))\}.$$

Since $u_r(x)$ is comparable to the distance from x to $F(u_r)$, for any $x \in T^r_\delta \cap \mathcal{B}_{C_2}$, there is a constant $c > 0$ such that the path $p(t)$ from x to z belongs to $T^r_{c\delta}$. Choose α sufficiently large depending on δ and $c > 0$ and r sufficiently small depending on C_1 such that

$$T^r_{c\delta} \subset \Gamma^r_\alpha.$$

Thus there is $r_0 > 0$ (depending on C_1, δ, and α) such that that for $r < r_0$,

$$|\nabla u_r(x) - a| < \delta, \quad \text{for } x \in T^r_{c\delta}.$$

Define a linear function of x, by $L(x) = u_r(z) + a \cdot (x - z)$. For all $x \in T^r_\delta \cap \mathcal{B}_{C_2}$, since $u_r(z) - L(z) = 0$, and $p(t) \in T^r_{c\delta}$,

$$|u_r(x) - L(x)| = \left| \int_0^1 (\nabla u_r(p(t)) - a) \cdot p'(t) dt \right| \le C_3\delta.$$

In all, we have shown that for every $x \in \mathcal{B}_{C_2}$ such that $u_r(x) \ge \delta$,

$$|u_r(x) - L(x)| \le C_3\delta.$$

Next, we deduce that $|a| \approx 1$. (The upper bound $|a| \le K$ already follows from the upper bound on $|\nabla u|$.) Since $u_r(0) = 0$ and $u_r(z) \approx 1$, for some $0 < t < 1$, the point $x = tz$ satisfies $u_r(x) = \delta$. So $x \in T^r_\delta \cap \mathcal{B}_{C_2}$ and $|\delta - u_r(z) - a \cdot (x - z)| \le C_3\delta$. Hence, $|a| \ge |a \cdot (x - z)| \ge u_r(z) - \delta - C_3\delta \ge u_r(z)/2$. (All we need in what follows is that a is bounded and nonzero.)

We can now conclude that the free boundary is flat in the appropriate sense. Consider a point $x \in F(u_r) \cap \mathcal{B}_1$ and its path $p(t)$ to z. There is $t > 0$ such that $u_r(p(t)) = \delta$. Denote $y = p(t)$. Then $|y - x| \le C\delta$ and $y \in T_\delta \cap \mathcal{B}_{C_2}$. The preceding argument says $|u_r(y) - L(y)| \le C_3\delta$. Therefore,

$$|L(x)| \le |L(y)| + |L(x) - L(y)| \le |u_r(y) - L(y)| + |u_r(y)| + |a \cdot (x - y)| \le C_4\delta$$

for a larger constant C_4. Since a is bounded away from 0 in length, the bound on $L(x)$ implies that every point of $F(u_r) \cap \mathcal{B}_1$ is within a distance a constant times δ of the plane $L(x) = 0$. For sufficiently small δ, this flatness condition implies smoothness of the free boundary (see [AC, C2]).

□

2.2. **The proof of Theorem 1.3.** Throughout the proof, c_i, C_i denote constants depending on L, n, and possibly on the NTA constants. Also, a point $x \in \mathbb{R}^n$ may be denoted by (x', x_n), with $x' = (x_1, \ldots, x_{n-1})$.

We divide the proof in three steps.

Step 1: Nondegeneracy and separation of level sets.

We show first the nondegeneracy of u, namely that if $\mathcal{B}_\rho(x_0) \subset C_L^+(u)$, $\rho < 1$, then

$$u(x_0) \geq \gamma_n \rho \tag{2.1}$$

for a dimensional constant $\gamma_n > 0$.

Denote by g a strictly superharmonic function on the annulus $E = \mathcal{B}_2 \backslash \mathcal{B}_1$ such that

$$\begin{cases} g = a_n & \text{on } \partial\mathcal{B}_2, \\ g = 0 & \text{on } \partial\mathcal{B}_1, \\ |\nabla g| < 1 & \text{on } \partial\mathcal{B}_1, \end{cases}$$

with $a_n > 0$ small dimensional constant. Let $r = \rho/4$. Denote $g_r(x) = rg(x/r)$, and

$$h_t(x) = g_r(x - x_0 - te_n)$$

defined on the closed annulus $E_t = \bar{\mathcal{B}}_{2r}(x_0 + te_n) \backslash \mathcal{B}_r(x_0 + te_n)$. For t sufficiently small, $E_t \subset \{x : -L < x_n < -L+1\}$ so that $h_t(x) \geq 0 = u(x)$ for $x \in E_t$. Increasing t translates the region E_t upwards. Let t_0 be the least t for which the graph of h_t touches the graph of u, i. e., so that there is a point $z_0 \in E_t$ for which $h_t(z_0) = u(z_0) > 0$. Because h_t is a strict supersolution the point z_0 belongs to the outer boundary, $z_0 \in \partial\mathcal{B}_{2r}(x_0 + t_0 e_n)$. Furthermore, because the free boundary of u and h_t can't touch, $t_0 \leq -\rho - r < 0$. Monotonicity of u implies $u(z_0 - t_0 e_n) \geq u(z_0) = h_{t_0}(z_0) = a_n r$. Finally, since $|z_0 - t_0 e_n - x_0| = 2r = \rho/2$, Harnack's inequality comparing the value of u at $z_0 - t_0 e_n$ and x_0 implies that there is a dimensional constant $\gamma_n > 0$ such that $u(x_0) \geq \gamma_n \rho$, as required.

Next, we will show that level sets near the top of the cylinder are separated by an appropriate amount. Let $\epsilon > 0$ and denote by

$$v(x) = u(x - \epsilon e_n).$$

Since u is strictly monotone in the vertical direction, $v(x) < u(x)$ on $\mathcal{C}_L^+(u)$. We claim that

$$v(x) \leq u(x) - c_1\epsilon \quad \text{on } B_{9/10}(0) \times \{L - 1/2\} \tag{2.2}$$

for $\epsilon < \epsilon_n$ a dimensional constant, and a constant $c_1 > 0$ depending only on L and n. To prove (2.2), note first that from (2.1) it follows that $u(x) \geq b_n$ for all $x \in B_{9/10}(0) \times \{L - 1/2\}$. Write $x = (x', L - 1/2)$ and let t_n be such that $u(x', t_n) = b_n/2$, then by monotonicity $u(x', t) \geq b_n/2$ for all $t \geq t_n$. Consider the segment from (x', t_n) to $(x', L - 1/2)$. It follows from the Lipschitz bound (Lemma 2.2) that the distance from any point of the segment to the free boundary is greater than a dimensional constant. Thus by Harnack's inequality the values of $w(x) = (\partial/\partial x_n)u(x)$ on this segment are comparable with a constant depending only on n and L. Furthermore,

$$b_n - b_n/2 \leq u(x) - u(x', t_n) = \int_{t_n}^{L-1/2} w(x', t)dt.$$

Therefore, the minimum of w on this segment is bounded below by a constant $c_1 > 0$, depending only on n and L. In particular,

$$u(x) - v(x) = \int_{L-1/2-\epsilon}^{L-1/2} w(x', t)dt \geq c_1\epsilon.$$

Step 2: Construction of a family of supersolutions.

The hypothesis of Theorem 1.3 implies (by the construction of P. W. Jones [J]) that there is an NTA domain between any pair $\mathcal{C}(r_1, L-a_1)$ and $\mathcal{C}(r_2, L-a_2)$ for $r_1 < r_2 \leq 9/10$ and $a_1 > a_2 \geq 1/2$. Thus the boundary Harnack inequality, Theorem 2.5, has the following corollary.

Corollary 2.8. *Let u be as in Theorem 1.3 and let $r_1 < r_2 \leq 9/10$ and $a_1 > a_2 \geq 1/2$. Then there is a constant A depending on L, the NTA constants of $\mathcal{D}$, $r_2 - r_1 > 0$, and $a_1 - a_2 > 0$ such that if h_1 and h_2 are positive harmonic functions on $\mathcal{C}(r_2, L-a_2) \cap \mathcal{C}_L^+(u)$, vanishing on $\partial D \cap \mathcal{C}(r_2, L-a_2)$ then*

$$h_1(x)/h_2(x) \leq A h_1(y)/h_2(y)$$

for every x and y in $\mathcal{C}(r_1, L-a_1) \cap \mathcal{C}_L^+(u)$.

In this step we start our analysis on the cylinder $\mathcal{C}(9/10, L-1/2)$ which by abuse of notation we denote by $\mathcal{C}_1$. Then we restrict to smaller cylinders $\mathcal{C}_2, \mathcal{C}_3$ with base $B_{8/10}$ and $B_{7/10}$ respectively, height M with $L-1 < M < L-1/2$ and $\mathcal{C}_3 \subset\subset \mathcal{C}_2 \subset\subset \mathcal{C}_1$.

Let w be the harmonic function in $\mathcal{C}_1^+(u)$, satisfying the following boundary conditions:

$$w = 0, \quad \text{on } F(u), \tag{2.3}$$

$$v < w \leq u, \quad \text{on } \overline{\mathcal{C}_1^+(u)} \cap \partial \mathcal{C}_1, \tag{2.4}$$

$$v + \frac{c_1}{4}\epsilon < w < u - \frac{c_1}{4}\epsilon, \quad \text{on } B_{9/10} \times \{L - 1/2\}. \tag{2.5}$$

Notice that (2.5) can be achieved because of the gap (2.2) between u and v. Since v is subharmonic and u is harmonic in $\mathcal{C}_1^+(u)$, the maximum principle implies

$$v < w < u \quad \text{in } \mathcal{C}_1^+(u). \tag{2.6}$$

Moreover, $\mathcal{C}_1^+(w) = \mathcal{C}_1^+(u)$, and $F(w) = F(u) \cap \mathcal{C}_1$.

We claim next that in the smaller cylinder $\overline{\mathcal{C}}_2$,

$$|\nabla w|(x) \leq C_1, \quad x \in \overline{\mathcal{C}}_2. \tag{2.7}$$

Define $d(x) = \text{dist}(x, F(u))$. At points $x \in \overline{\mathcal{C}}_2 \cap \mathcal{C}_1^+(u)$ such that $d(x) \geq 1/10$, this follows from standard elliptic regularity and the fact that w is bounded. On the other hand, at points that are close to $F(u)$, we have that $B_{d(x)}(x) \subset \mathcal{C}_1^+(u)$ and from Lemma 2.2,

$$w(x) < u(x) \leq K d(x).$$

A standard argument using rescaling implies the bound (2.7).

Now, set $h = u - w$. Then h is a positive (see (2.6)) harmonic function on $\mathcal{C}_1^+(u)$ vanishing continuously on $F(u)$. Let H be the harmonic function in the cylinder $B_{9/10} \times (L-1, L-1/2)$, with boundary data $c_1/2$ on the top of the cylinder and vanishing on the remaining part of the boundary. Then, in view of (2.5), $h \geq \epsilon H$. Thus, $h(x_1) \geq c_1\epsilon/4$, at $x_1 = (L - 1/2 - \delta_n)e_n$ for a small dimensional constant $\delta_n > 0$. Moreover, by the Lipschizt continuity of u we get that $h(x_1) < (u-v)(x_1) \leq K\epsilon$. Using non-degeneracy and Lipschizt continuity of u we also have that $b_n \leq u(x_1) \leq 2LK$. Thus, Corollary 2.8 gives

$$c_2 \epsilon u \leq h \leq C_2 \epsilon u \ \text{ on } \overline{\mathcal{C}_2^+(u)}.$$

The upper bound on h implies,

$$w(x) \geq (1 - C_2\epsilon)u(x) \quad \text{on } \overline{\mathcal{C}_2^+(u)}, \tag{2.8}$$

while the lower bound gives

$$w(x) \leq (1 - c_2\epsilon)u(x) \quad \text{on } \overline{\mathcal{C}_2^+(u)}. \tag{2.9}$$

In particular, if $F(u)$ is smooth around a point $x_0 \in \mathcal{C}_2$ then $|\nabla u|(x_0) = 1$, which combined with (2.9) gives

$$|\nabla w|(x_0) \leq 1 - c_2\epsilon. \tag{2.10}$$

According to Lemma 2.7 we then have

$$|\nabla w| \leq 1 - c_2\epsilon \quad \omega\text{-almost everywhere on } F(u) \cap \mathcal{C}_2. \tag{2.11}$$

Next we use (2.11) to show that, by restricting on the smaller cylinder $\mathcal{C}_3$, we have

$$|\nabla w| \leq 1 - c_2\epsilon + C_3 u \quad \text{on } \mathcal{C}_3^+(u). \tag{2.12}$$

Let $\tilde{h}$ be the largest harmonic function $\tilde{h} \leq C_1$ in $\mathcal{C}_2^+(u)$ such that

$$\tilde{h} = 1 - c_2\epsilon \quad \text{on } F(u) \cap \mathcal{C}_2$$

with C_1 the constant in (2.7). Since $|\nabla w|$ is subharmonic, it satisfies (2.7)-(2.11) we get

$$|\nabla w| \leq \tilde{h}. \tag{2.13}$$

On the other hand, $\tilde{h} - (1 - c_2\epsilon)$ is a positive harmonic function on $\mathcal{C}_2^+(u)$, and it is zero on $F(u)$. Since by non-degeneracy u is bounded below by a dimensional constant on the top of $\mathcal{C}_3$, Corollary 2.8 gives

$$\tilde{h} - (1 - c_2\epsilon) \leq C_3 u \quad \text{on } \mathcal{C}_3^+(u).$$

Combining this inequality with (2.13) we obtain (2.12).

We now use (2.12) to construct a family of strict supersolutions. Define for $t \geq 0$,

$$w_t(x) = w(x) - tg(x), \quad x \in \mathcal{C}_1$$

with

$$g(x) = e^{Ax_n}\phi\left(|x'|\right)$$

where A is a positive constant to be chosen later, and $\phi \geq 0$ is a smooth bump function such that

$$\phi(r) = \begin{cases} 1, & \text{if } r < 1/2 \\ 0, & \text{if } r \geq 7/10. \end{cases}$$

Moreover, we will choose ϕ such that $\phi(r) > 0$ for $r < 7/10$

$$\phi''(r) + \frac{n-2}{r}\phi'(r) \geq 0, \quad \text{if} \quad 6/10 \leq r \leq 7/10.$$

Indeed, let $\psi(s) = e^{-2n/s}$ for $s > 0$ and $\psi(s) = 0$ for $s \leq 0$. Then for $0 \leq s \leq 1$,

$$\psi''(s) - 2n\psi'(s) = [(2n + 4n^2)/s^2 - (2n)^2/s]e^{-2n/s} \geq 0.$$

Because $(n-2)/r \leq 2n$ for $r \geq 1/2$, the function $\phi_1(r) = \psi(7/10 - r)$ satisfies the differential inequality for ϕ above in the range $r \geq 1/2$. Using a partition of unity, ϕ_1 can be modified without changing its values for $r \geq 6/10$, to obtain a function ϕ that is equal to 1 for $r \leq 1/2$. Finally, using the inequalities for ϕ,

$$\Delta g = A^2 e^{Ax_n}\phi\left(|x'|\right) + e^{Ax_n}\Delta\phi\left(|x'|\right) \geq 0$$

as long as A is a sufficiently large dimensional constant.

Thus, w_t is superharmonic on $\mathcal{C}_1^+(w_t)$. Moreover, condition (2.12) together with (2.8) imply that,

$$|\nabla w_t| \leq |\nabla w| + t|\nabla g| \leq 1 - c_2\epsilon + C_4 w + t|\nabla g|, \quad \text{on } \mathcal{C}_3^+(u).$$

In particular, on $F(w_t) \cap \mathcal{C}_3$, $t > 0$, since $w = tg$ we obtain

$$|\nabla w_t| \leq 1 - c_2\epsilon + C_4 tg + t|\nabla g|.$$

Therefore, for $0 < t \leq c_3\epsilon$, with c_3 small depending on c_2, C_4, and A, we deduce that

$$|\nabla w_t| \leq 1 - \frac{c_2}{2}\epsilon \quad \text{on } F(w_t) \cap \mathcal{C}_3. \tag{2.14}$$

Step 3: Comparison.

Observe that because g vanishes on the "sides" we have that

$$w_t = w > v \quad \text{on } (\partial B_{9/10} \times [-L, L]) \cap \overline{\mathcal{C}_1^+(w_t)}, \tag{2.15}$$

and according to (2.5) we have that

$$w_t > v \quad \text{on } B_{9/10} \times \{L - 1/2\} \quad \text{for } t \leq \frac{c_2}{4} e^{-A(L-1/2)}\epsilon = c_4\epsilon. \tag{2.16}$$

Let $E = \{t \in [0, c_4\epsilon] : v \leq w_t \text{ in } \overline{\mathcal{C}_1}\}$. We claim that $E = [0, c_4\epsilon]$. Indeed, $0 \in E$ and clearly E is closed. We need to show that E is open. Let $t_0 \in E$, then since w_t is superharmonic in its positive phase and satisfies (2.15)-(2.16) we only need to show that $w_{t_0} > v = 0$ on $F(v) \cap \mathcal{C}_1$.

In the case $t_0 = 0$, $w_0 = w > 0$ on $F(v)$ follows from the assumption that $F(u)$ is a graph in the vertical direction. In fact for all t, $w_t = w > 0$ on $F(v) \cap (\mathcal{C}_1 \setminus \mathcal{C}_3)$ because g is zero there. It remains to rule out the case, in which $t_0 > 0$, and $F(v)$ touches $F(w_{t_0})$ in $\mathcal{C}_3$, that is, where $g(x_0) \neq 0$.

Suppose by contradiction $x_0 \in F(v) \cap F(w_{t_0}) \cap \mathcal{C}_3$, $t_0 > 0$. If $\nabla w_{t_0}(x_0) \neq 0$, then by the implicit function theorem $F(w_{t_0})$ is smooth in a neighborhood of x_0 and hence there exists an exterior tangent ball $\mathcal{B}$ at x_0 for $F(v)$. Therefore, for ν the outward unit normal to $\mathcal{B}$ at x_0 we have that

$$w_{t_0}(x) \geq v(x) = (x - x_0, \nu)^+ + o(|x - x_0|)$$

as $x \to x_0$, contradicting (2.14).

On the other hand, if $\nabla w_{t_0}(x_0) = 0$, then in a small neighborhood $\mathcal{B}_r(x_0)$ we have that

$$w_{t_0}(x) \leq Cr^2.$$

However, according to the corkscrew condition, there exists a ball $\mathcal{B}_{\delta r}(y) \subset \mathcal{B}_r(x_0) \cap \mathcal{C}_1^+(v)$, for some small $\delta > 0$. By the non-degeneracy of v we then obtain

$$\sup_{\mathcal{B}_r(x_0)} v \geq cr,$$

and again we reach a contradiction.

Thus $c_4\epsilon \in E$, and

$$v \leq w_{c_4\epsilon} \quad \text{on } \overline{\mathcal{C}}_1.$$

Hence, according to the definition of g,

$$\{w \leq c_4 e^{-AL}\epsilon\} \cap \{|x'| < 1/2\} \subset \{v = 0\} \cap \{|x'| < 1/2\}.$$

Moreover, by (2.7)

$$w \leq C_1 d(x).$$

Thus,

$$\{d(x) \leq c_6\epsilon\} \cap \{|x'| < 1/2\} \subset \{v = 0\} \cap \{|x'| < 1/2\}.$$

This implies the Lipschitz continuity of $F(u) \cap \{|x'| < 1/2\}$ with bound depending only on L, n and the NTA constants. □

3. A priori gradient bound for minimal surfaces

In this section we present our proof of Theorem 1.2. Recall that B_r denotes an open $(n-1)$-dimensional ball of radius r, while $\mathcal{B}_r$, denotes an open n-dimensional ball of radius r.

Our proof is parallel to one in the free boundary setting above. One main ingredient which will allow us to apply our deformation argument will be the (weak) Harnack inequality for solutions to elliptic equations on minimal surfaces due to Bombieri and Giusti [BG]. We recall its statement in the form in which we will use it later in the proof.

Let Δ_S denotes the Laplace-Beltrami operator on the surface S.

Theorem 3.1. *Let $p < \dfrac{n-1}{n-3}$. There is a constant $C(p) < \infty$ and $\beta > 0$ depending on dimension such that if S is an area minimizing hypersurface in $\mathcal{B}_R = \mathcal{B}_R(x_0)$ and $x_0 \in S$ and v is a positive supersolution to the Laplace-Beltrami operator, $\Delta_S v \leq 0$, in $\mathcal{B}_R \cap S$, then*

$$\left(\fint_{\mathcal{B}_r \cap S} v^p dH_{n-1} \right)^{1/p} \leq C(p) \inf_{\mathcal{B}_r \cap S} v \tag{3.1}$$

for all $r \leq \beta R$.

Corollary 3.2. *Let S be an oriented surface of least area in $B_1 \times \mathbb{R} \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Assume $\mathcal{S}_{1/2} := S \cap (B_{1/2} \times \mathbb{R})$ is connected, and $S \subset B_1 \times [-M, M]$. Let v be a positive supersolution to to the Laplace-Beltrami operator, $\Delta_S v \leq 0$, in $(B_1 \times \mathbb{R}) \cap S$, such that*

$$\int_{\mathcal{S}_{1/2}} v dH_{n-1} \geq 1. \tag{3.2}$$

Then

$$v \geq c \quad \text{on} \quad \mathcal{S}_{1/2}, \tag{3.3}$$

with $c > 0$ depending only on n and M.

Proof. Let β (small) be the constant in Theorem 3.1. Decompose $\mathbb{R}^n$ into cubes of side-length $\beta/(20\sqrt{n})$. For each cube Q_i that intersects $\mathcal{S}_{1/2}$ take a ball $\widetilde{B}_i \supset Q_i$ with center x_i on $\mathcal{S}_{1/2} \cap Q_i$ and radius $\beta/20$. Clearly, the number N of balls $\widetilde{\mathcal{B}}_i$ that cover $\mathcal{S}_{1/2}$ depends only on n and M.

We say that $\widetilde{\mathcal{B}}_i \sim \widetilde{\mathcal{B}}_j$ if there exists a chain of balls $\widetilde{\mathcal{B}}_k$ connecting $\widetilde{\mathcal{B}}_i$ and $\widetilde{\mathcal{B}}_j$ such that consecutive balls intersect. This defines an equivalence relation. To each equivalence class we can associate the open set which is the union of all the elements in the class. Notice that open sets corresponding to distinct equivalence classes are disjoint. Since $\mathcal{S}_{1/2}$ is connected, we conclude that all the balls belong to the same equivalence class.

If $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ intersect then they are both contained in $B_{\beta/2}(x_1)$. Hence applying Theorem 3.1 we obtain

$$\int_{\widetilde{\mathcal{B}}_1\cap S} v dH_{n-1} \leq C_0 \fint_{B_{\beta/2}\cap S} v dH_{n-1} \leq C_1 \inf_{S_{\beta/2}(x_1)} v \leq C_2 \int_{\widetilde{\mathcal{B}}_2\cap S} v dH_{n-1}. \tag{3.4}$$

In the last inequality we used the well-known fact that

$$H_{n-1}(S \cap B_\rho(x)) \approx \rho^{n-1} \quad \text{for all} \quad x \in S. \tag{3.5}$$

It also follows from (3.2) that at least one of the balls, say $\widetilde{\mathcal{B}}_1$, satisfies

$$\int_{\mathcal{S}_{1/2}\cap\widetilde{\mathcal{B}}_1} v dH_{n-1} \geq 1/N. \tag{3.6}$$

Combining (3.4)-(3.6) with the fact that any two balls can be connected by a chain of length at most N, we obtain the desired conclusion. $\square$

Proof of Theorem 1.2. In what follows, the constants c, c_i, C, C_i depend only on n and M. Denote by S the graph of ϕ over B_1. We present the proof in three steps.

Step 1: Separation on a set of substantial measure.

Let $\epsilon > 0$ and set

$$S_\epsilon := \{(x, \phi(x) + \epsilon) : x \in B_1\}.$$

We will prove that there exists a smoothly bounded, closed set $\tilde{E} \subset B_{1/2}$ of positive measure independent of ϵ as $\epsilon \to 0$ such that

$$\text{dist}((x, \phi(x)), S_\epsilon) \geq c_0\epsilon \quad \text{for all } x \in \tilde{E} + B_\delta \tag{3.7}$$

where $\delta > 0$ depends on the (a priori) bound on the modulus of continuity of $\nabla\phi$.

Let $\eta \in C_0^\infty(B_1)$ be a smooth cut-off function such that $\eta \equiv 1$ on $B_{1/2}$. Then, since ϕ satisfies (1.2) we have that

$$\int_{B_1} \frac{\nabla\phi \cdot \nabla(\eta^2\phi)}{\sqrt{1+|\nabla\phi|^2}} dx = 0.$$

Hence,

$$\int_{B_1} \eta^2 \frac{|\nabla\phi|^2}{\sqrt{1+|\nabla\phi|^2}} dx = -2\int_{B_1} \phi\eta \frac{\nabla\phi\cdot\nabla\eta}{\sqrt{1+|\nabla\phi|^2}} \leq$$

$$2\left(\int_{B_1} \frac{\eta^2|\nabla\phi|^2}{\sqrt{1+|\nabla\phi|^2}}\right)^{1/2} \left(\int_{B_1} \frac{\phi^2|\nabla\eta|^2}{\sqrt{1+|\nabla\phi|^2}}\right)^{1/2}.$$

Thus,

$$\int_{B_1} \eta^2 \frac{|\nabla\phi|^2}{\sqrt{1+|\nabla\phi|^2}} dx \leq 4\int_{B_1} \phi^2 \frac{|\nabla\eta|^2}{\sqrt{1+|\nabla\phi|^2}} dx \leq CM^2$$

Since $\eta \equiv 1$ on $B_{1/2}$ we then get

$$\int_{B_{1/2}} |\nabla\phi| dx \leq C_0. \tag{3.8}$$

with C_0 depending on M and n only. Hence, by Chebyshev's inequality, (for $C_1 = 2C_0/|B_{1/2}|$)

$$|\{x \in B_{1/2} : |\nabla\phi| < C_1\}| \geq |B_{1/2}|/2.$$

Since ϕ is smooth, there is a closed, smoothly bounded set

$$\tilde{E} \supset \{x \in B_{1/2} : |\nabla\phi| < C_1\}$$

and $\delta > 0$ sufficiently small depending on the modulus of continuity of $\nabla\phi$ such that

$$\tilde{E} + B_\delta \subset \{x \in B_{1/2} : |\nabla\phi|^2 \leq C_1^2 + 1\}$$

This implies the desired claim (3.7), for small enough ϵ and δ, depending on the smoothness of ϕ.

In what follows we denote by $E = \{(x, \phi(x)), x \in \tilde{E}\}$. Clearly, $H_{n-1}(E) \geq |B_{1/2}|/2$.

Step 2: Construction of a family of subsolutions.

For the time being let S be any smooth surface. Denote by $H(P, S)$ the mean curvature of S at a point $P \in S$, (i.e. the trace of the second fundamental form of S at P.) Assume that S is a smooth graph over B_1, i.e. $S = \{(x, \phi(x)) : x \in B_1\}$, and let w be a C^2 non-negative function on S. Consider the surface $S_{t,\nu} := S + tw\nu$ obtained deforming S along the upward unit normal to S, that is

$$S_{t,\nu} = \{(x, \phi(x)) + tw(x, \phi(x))\nu_x, x \in B_1\},$$

with

$$\nu_x = \frac{(-\nabla\phi(x), 1)}{\sqrt{1 + |\nabla\phi(x)|^2}}.$$

Then, for t small enough, S_t is also a graph and one can compute (see for example [K])

$$H(P_t, S_{t,\nu}) = H(P, S) + t(\Delta_S w(P) + |A|_S^2 w(P)) + O(t^2), \tag{3.9}$$

$$P := (x, \phi(x)), P_t := (x, \phi(x)) + tw(x, \phi(x))\nu_x, \tag{3.10}$$

where $|A|_S$ is the norm of the second fundamental form of S. (The $O(t^2)$ term depends at most on the third derivatives of ϕ and on the second derivatives of w.)

Applying formula (3.9) to our minimal surface S we find that

$$H(P_t, S_{t,\nu}) = t(\Delta_S w(P) + |A|_S^2 w(P)) + O(t^2). \tag{3.11}$$

In order to run a continuity argument (as in the proof of Theorem 1.3), we wish to use formula (3.11) to produce a family of surfaces $S_{t,\nu}$ which are strict subsolutions to the minimal surface equation i.e. $H(\cdot, S_{t,\nu}) > 0$ at least outside $E_{t,\nu} := E + tw\nu$, with E the set from the previous step. Towards this aim we prove the following claim.

Claim. There exists a function w defined on S such that

$$\begin{aligned} &\Delta_S w + |A|_S^2 w > 0 \quad \text{on } S \setminus E, \\ &w(x, \phi(x)) = 1 \quad \text{on } \tilde{E}, \\ &w(x, \phi(x)) = 0 \quad \text{on } \partial B_1. \end{aligned}$$

Moreover

$$w(x, \phi(x)) \geq c_0 > 0 \quad \text{on } B_{1/2}, \tag{3.12}$$

with c_0 depending only on n, M and $w \in C^2(\overline{S \setminus E})$ with C^2 bounds depending on S and E.

Proof of the claim. Let w_1 be the solution to the following boundary value problem,

$$\begin{aligned} &\Delta_S w_1 = 0 \quad \text{on } S \setminus E, \\ &w_1(x, \phi(x)) = 1 \quad \text{on } \partial \tilde{E}, \\ &w_1(x, \phi(x)) = 0 \quad \text{on } \partial B_1. \end{aligned}$$

Note that the solution exists and is smooth in its domain of definition because $\tilde{E}$ is smoothly bounded. Extend $w_1 = 1$ on $\tilde{E}$. Then $\Delta_S w_1 \leq 0$ on S. Moreover, according to Step 1, we have that (using the notation of Corollary 3.2)

$$\int_{\mathcal{S}_{1/2}} w_1 dH_{n-1} \geq \int_E w_1 dH_{n-1} = H_{n-1}(E) \geq |B_{1/2}|/2.$$

Hence we can apply Corollary 3.2 to conclude that

$$w_1 \geq c \quad \text{on} \quad \mathcal{S}_{1/2} = S \cap (B_{1/2} \times \mathbb{R}). \tag{3.13}$$

Now, let w_0 be the solution to the following problem:

$$\begin{aligned} &\Delta_S w_0 = 1 \quad \text{on } S \setminus E, \\ &w_0(x, \phi(x)) = 0 \quad \text{on } \tilde{E}, \\ &w_0(x, \phi(x)) = 0 \quad \text{on } \partial B_1. \end{aligned}$$

and set $w = w_1 + \delta_1 w_0$. Clearly, $|\nabla w_0|$ is bounded (by a constant depending on S and E). Applying Hopf's lemma to w_1 on $(\partial B_1 \times \mathbb{R}) \cap S$, we obtain that, for δ_1 sufficiently small, $w > 0$ in a neighborhood of $(\partial B_1 \times \mathbb{R}) \cap S$ and hence (for a possibly smaller δ_1) $w > 0$ on S. Moreover, in view of (3.13), we can choose δ_1 so that w satisfies (3.12). Thus, w has all the required properties.

In view of the claim, according to formula (3.11), if t is sufficiently small, $0 < t \leq \epsilon_0$ then

$$H(\cdot, S_{t,\nu}) > 0, \quad \text{on } S_{t,\nu} \setminus E_{t,\nu}.$$

Step 3: Comparison.

We show that for $0 \leq t \leq c_0\epsilon \leq \epsilon_0$, the surface $S_{t,\nu}$ is below the surface S_ϵ. Indeed, this is true at $t = 0$. The first touching point cannot occur at some $x \in \partial B_1$, as our deformation leaves the ∂B_1 fixed. Moreover, for t small enough, no touching can occur on $E_{t,\nu}$ in view of (3.7) in Step 1. Finally $S_{t,\nu}$ is a strict subsolution on $S_{t,\nu} \setminus E_{t,\nu}$, hence no touching can occur there either. Since w satisfies (3.12), we can then conclude that for all sufficiently small ϵ, (recall $S_{c_0\epsilon,\nu} = S + c_0\epsilon w\nu$)

$$\text{dist}((x, \phi(x)), S_\epsilon) \geq \text{dist}((x, \phi(x)), S_{c_0\epsilon,\nu}) \geq c_1\epsilon \quad \text{on} \quad B_{1/2}$$

as desired. Note that although the size of ϵ_0 depends on the a priori bound on $\nabla\phi$, the constants $c_0 > 0$ and $c_1 > 0$ do not.

□

4. Final Remarks

The analogy between the two gradient bound proofs presented here goes farther. Not only does each proof depend crucially on a scale-invariant Harnack inequality for the second variation operator of the associated functional, but also the proofs of these two Harnack estimates follow a roughly parallel course.

The key ingredient of our proof of the gradient bound for minimal surface graphs is the Harnack inequality for the Laplace-Beltrami operator on the surface. This

Harnack inequality permits us to convert a gradient bound on average (separation on a set of substantial measure) to a gradient bound everywhere (separation everywhere). The way this Harnack inequality is proved by Bombieri and Giusti is as follows. A monotonicity formula yields (via a limiting cone argument) a measure-theoretic form of connectivity. This, in turn, implies another scale-invariant form of connectivity, an isoperimetric, or Poincaré-type, inequality. One then deduces a Harnack inequality for the Laplace-Beltrami operator on the minimal surface by a Moser-type argument.

In the free boundary case, a monotonicity formula due to Alt, Caffarelli and Friedman yields (by arguments of [ACS] and [D1]) the NTA property, a scale-invariant form of connectivity. A theorem of [JK] says that the NTA property implies a boundary Harnack inequality. The boundary Harnack inequality is used to show that separation of level surfaces of the solution function u at distances far from the free boundary implies a similar separation all the way up to the free boundary.

The parallel between these two Harnack inequalities leads to the hope that there is a Harnack estimate for the second variation operator associated to minimizers of functionals of the form
$$\int |\nabla v|^2 + F(v)$$
for wider classes of functions F.

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