RATIONAL EQUIVARIANT RIGIDITY

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Abstract. We prove that for a finite or profinite group G, the homotopy information of rational G-spectra is entirely determined by the triangulated structure of their homotopy category.

Introduction

Quillen equivalences between stable model categories give rise to triangulated equivalences of their homotopy categories. The converse is not necessarily true - there are numerous examples of model categories that have equivalent homotopy categories but completely different homotopical behaviour. One example, see [SS02, 2.6], is module spectra over the Morava K-theories $K(n)$ and differential graded $K(n)_*$-modules. Their homotopy categories are equivalent triangulated categories, yet they cannot be Quillen equivalent as they have different mapping spaces.

As this converse statement is a very strong formality property, it is of great interest to find stable model categories $\mathcal{C}$ whose homotopical information is entirely determined by the triangulated structure of the homotopy category $\mathrm{Ho}(\mathcal{C})$. Such homotopy categories are called *rigid*.

The first major result was found by Stefan Schwede who proved the rigidity of the stable homotopy category $\mathrm{Ho}(Sp)$: any stable model category whose homotopy category is triangulated equivalent to $\mathrm{Ho}(Sp)$ is Quillen equivalent to the model category of spectra [Sch07a]. To investigate further into the internal structure of the stable homotopy category, Bousfield localisations of the stable homotopy category have subsequently been considered. The second author showed in [Roi07] that the K-local stable homotopy category at the prime 2 is rigid. Astonishingly, this is not true for odd primes, as a counterexample by Jens Franke shows in [Fra96], see also [Roi08].

The main focus of the proofs of the above results is the respective sphere spectra and their endomorphisms. For both the stable homotopy category and its p-local K-theory localisation, the sphere is a *compact generator*, meaning that it "generates" the entire homotopy category under exact triangles and coproducts. Studying the endomorphisms of a generator is essentially Morita theory. The idea is that all homotopical information of a model category can be deduced from a certain endomorphism ring object of its compact generators [SS03]. In the case of a category

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with one compact generator, this endomorphism ring object is a symmetric ring spectrum.

The cases mentioned above all monogenic, that is, they have a single compact generator (the sphere spectrum). In this paper we are working with homotopy categories that have multiple generators. The examples we study are model categories of rational G-equivariant spectra $G\,\mathrm{Sp}_{\mathbb{Q}}$ for either a finite group or a profinite group G. Recall that a profinite group is an inverse limit of an inverse system of finite groups, with the p-adic numbers being the canonical example. Any finite group is, of course, profinite, but whenever we talk of a profinite group we assume that the group is infinite. In the case of a finite group the category $G\,\mathrm{Sp}_{\mathbb{Q}}$ has been extensively studied by Greenlees, May and the first author. For a profinite group the non-rationalised category is introduced and examined in [Fau08]. Equivariant stable homotopy theory is of great general interest because of the prevalence of group actions in mathematics, so it makes good sense to use this as our example of a non-monogenic example of rigidity. The goal of this paper is to prove the following for G finite or profinite.

Theorem (**Rational G-equivariant Rigidity Theorem**). *Let $\mathcal{C}$ be a proper, cofibrantly generated, simplicial, stable model category and let*

$$\Phi : \mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

be an equivalence of triangulated categories. Then $\mathcal{C}$ and $G\,\mathrm{Sp}_{\mathbb{Q}}$ are Quillen equivalent.

The homotopy category $\mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}})$ is not monogenic, a set of compact generators is, in the finite case, given by $\mathcal{G}_{top} = \{\Sigma^\infty G/H_+\}$, the suspension spectra of the homogeneous spaces G/H for H a subgroup of G. In the profinite case the generators are $\mathcal{G}_{top} = \{\Sigma^\infty G/H_+\}$ as H runs over the open subgroups of G. Hence, instead of studying an endomorphism ring object, we consider a "ring spectrum with several objects", a small spectral category. Via the above equivalence Φ, $\mathcal{G}_{top}$ also provides a set of compact generators $\mathcal{X} = \Phi(\mathcal{E}_{top})$ for the homotopy category of $\mathcal{C}$, from which we can form its endomorphism category $\mathcal{E}(\mathcal{X})$. Generally, a triangulated equivalence on homotopy category would not be sufficient to extract enough information from $\mathcal{E}(\mathcal{X})$. However, in our case computations by Greenlees and May (for finite groups) and the first author (for profinite groups) allow us to construct a Quillen equivalence.

This theorem is particularly notable as it provides an example of rigidity in the case of multiple generators rather than just a monogenic homotopy category. We note that in the finite case, the coproduct of all the generators is also a compact generator, so technically speaking, $G\,\mathrm{Sp}_{\mathbb{Q}}$ can be thought of as monogenic in the finite case. In the profinite setting (where we have assumed the group to be infinite) there are countably many generators and no finite subset will suffice, furthermore the coproduct of this infinite collection of generators will not be compact, so in the profinite case $G\,\mathrm{Sp}_{\mathbb{Q}}$ cannot be monogenic.

Organisation. In Section 1 we establish some notations and conventions before discussing the notions of generators and compactness.

Section 2 provides a summary of Schwede's and Shipley's Morita theory result which relates model categories to categories of modules over an endomorphism category of generators. More precisely, assume that $\mathcal{C}$ is a model category which satisfies some further minor technical assumptions and which has a set of generators $\mathcal{X}$. Then one can define the *endomorphism category* $\mathcal{E}(\mathcal{X})$ and the model category of modules over it. Schwede and Shipley then give a sequence of Quillen equivalences

$$\mathcal{C} \simeq_Q \text{mod-}\mathcal{E}(\mathcal{X}).$$

In Section 3 we recover some definitions and properties about rational G-spectra and describe its endomorphism category $\mathcal{E}(\mathcal{G}_{top})$.

These three sections provide enough information to produce a Quillen equivalence between the given model category $\mathcal{C}$ and $G\,\mathrm{Sp}_{\mathbb{Q}}$, which we describe in in Section 4. In detail, we use the Morita theory of Section 2 to obtain a Quillen equivalence

$$\text{mod-}\mathcal{E}(\mathcal{G}_{top}) \rightleftarrows G\,\mathrm{Sp}_{\mathbb{Q}}$$

and a zig-zag of Quillen equivalences $\mathcal{C} \simeq_Q \text{mod-}\mathcal{E}(\mathcal{X})$. We then use the computations of Section 3 to produce a series of Quillen equivalences relating mod-$\mathcal{E}(\mathcal{G}_{top})$ and mod-$\mathcal{E}(\mathcal{X})$.

1. Stable model categories and generators

We assume that the reader is familiar with the basics of Quillen model categories. We provide only a brief summary of the main notions in order to establish notation and other conventions.

A model category $\mathcal{C}$ is a category with three distinguished classes of morphisms denoted *weak equivalences* $\xrightarrow{\sim}$, *fibrations* $\twoheadrightarrow$ and *cofibrations* $\rightarrowtail$ satisfying some strong but rather natural axioms. A good reference is [DS95]. The main purpose of a model structure is enabling us to define a reasonable notion of homotopy between morphisms. One can then form the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ using homotopy classes of morphisms.

For a pointed model category $\mathcal{C}$ one can define a *suspension functor* and *loop functor* as follows. Let X be an object in $\mathcal{C}$, without loss of generality let X be fibrant and cofibrant. We then choose a factorisation

$$X \rightarrowtail C \xrightarrow{\sim} *$$

of the map $X \longrightarrow *$. The suspension ΣX is now defined as the pushout of the diagram

$$* \longleftarrow X \rightarrowtail C\ .$$

The loop functor Ω is defined dually. Suspension and loop functors form an adjunction

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \Omega.$$

Note that when we write down an adjunction of functors the top arrow always denotes the left adjoint.

In the case of pointed topological spaces this recovers the usual suspension and loop functors. For the derived category of an abelian category the suspension functor is the shift functor of degree +1 and the loop functor the shift functor of degree -1. So in the latter case suspension and loop functors are inverse equivalences of categories, which is not the case for topological spaces.

Definition 1.1. *A pointed model category $\mathcal{C}$ is called* stable *if*

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \Omega$$

are inverse equivalences of categories.

One reason why stable model categories are of interest is because their homotopy categories are triangulated, which gives us a wealth of additional structure to make use of. Examples of stable model categories are chain complexes of modules over a ring R with either the projective or injective model structure [Hov99, 2.3] or symmetric spectra Sp in the sense of [HSS00].

We also need to consider functors respecting the model structures on categories.

Definition 1.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint functor pair. Then (F, G) is called a* Quillen adjunction *if F preserves cofibrations and acyclic cofibrations, or equivalently, G preserves fibrations and acyclic fibrations.*

Note that a Quillen functor pair induces an adjunction

$$LF : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : RG$$

[Hov99, 1.3.10]. The functors LF and RG are called the *derived functors* of the Quillen functors F and G. If the adjunction (LF, RG) provides an equivalence of categories, then (F, G) is called *Quillen equivalence*. But Quillen equivalences do not only induce equivalences of homotopy categories, they also give rise to equivalences on all 'higher homotopy constructions' on $\mathcal{C}$ and $\mathcal{D}$. To summarise, Quillen equivalent model categories have the same homotopy theory.

In the case of $\mathcal{C}$ and $\mathcal{D}$ being stable model categories, the derived functors LF and RG are *exact* functors. This means that they respect the triangulated structures. For a triangulated category $\mathcal{T}$, we denote the morphisms in $\mathcal{T}$ by $[-,-]_*^{\mathcal{T}}$. In the case of $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$, we abbreviate this to $[-,-]_*^{\mathcal{C}}$.

Definition 1.3. *Let $\mathcal{T}$ be a triangulated category. A set $\mathcal{X} \subseteq \mathcal{T}$ is called* a set of generators *if it detects isomorphisms, i.e. a morphism $f : A \longrightarrow B$ in $\mathcal{T}$ is an isomorphism if and only if*

$$f_* : [X, A]_*^{\mathcal{T}} \longrightarrow [X, B]_*^{\mathcal{T}}$$

is an isomorphism for all $X \in \mathcal{X}$.

Definition 1.4. *If $\mathcal{T}$ is a triangulated category that has infinite coproducts, then an object $Y \in \mathcal{T}$ is called* compact *(or* small*) if $[Y, -]_*^{\mathcal{T}}$ commutes with coproducts.*

The importance of those definitions can be seen in the following: if $\mathcal{T}$ has a set of compact generators $\mathcal{X}$, then any triangulated subcategory of $\mathcal{T}$ that contains $\mathcal{X}$ that is closed under coproducts must already be $\mathcal{T}$ itself, [Kel94, 4.2]. Note that if $\Phi : \mathcal{T} \longrightarrow \mathcal{T}'$ is an equivalence of triangulated categories and $\mathcal{X} \subseteq \mathcal{T}$ a set of generators, it is immediate that $\Phi(\mathcal{X})$ is a set of generators in $\mathcal{T}'$.

Examples of compact generators include the following.

- The sphere spectrum S^0 is a compact generator for the stable homotopy category $\mathrm{Ho}(Sp)$.
- Consider a smashing Bousfield localisation with respect to a homology theory E_*. Then the E-local sphere $L_E S^0$ is a compact generator of the E-local stable homotopy category $\mathrm{Ho}(L_E Sp)$. However, if the localisation is not smashing, then $L_E S^0$ is a generator but not compact [HPS97, 3.5.2].
- Let R be a commutative ring. Then the free R-module of rank one is a compact generator of the derived category $\mathcal{D}(R\text{-mod})$.

For a more detailed list of examples, [SS03] is an excellent source, which also gives examples of non-monogenic triangulated categories such as G-spectra. The category of G-spectra will be discussed in detail in Section 3.

2. Morita theory for stable model categories

We are going to summarize some results and techniques of [SS03] in this section. Schwede and Shipley show that, given a few minor technical assumptions, any stable model category is Quillen equivalent to a category of modules over an endomorphism ring object. In the case of a model category with a single generator X, this endomorphism object is a symmetric ring spectrum. However, in the case of a stable model category with a set of several generators such as rational G-spectra, the endomorphism object is a small category enriched over ring spectra, or a "ring spectrum with several objects".

We are going to assume that our stable model category $\mathcal{C}$ is a simplicial model category [Hov99, 4.3] which is also proper and cofibrantly generated. As mentioned, being proper and cofibrantly generated are only minor technical assumptions which hold in most examples of reasonable model categories and being simplicial is less of a restriction than it may seem, via [RSS01] or [Dug06]. In the latter reference, Dugger shows that any stable model category which is also "presentable" is Quillen equivalent to a spectral model category as defined below. The conditions of being cofibrantly generated, proper and simplicial will also be the assumptions of our main theorem in Section 4 later.

A *spectral category* is a category $\mathcal{O}$ enriched, tensored and cotensored over symmetric spectra, see e.g. [SS03, 3.3.1] or [Hov99, 4.1.6]. A *spectral model category* is a model category which is also a spectral category and further, the spectral structure is compatible with the model structure via the axiom (SP). The axiom (SP) is analogous to the axiom (SM7) that makes a simplicial category into a simplicial model category, see [SS03, 3.5.1] or [Hov99, 4.2.18]. A *module over the spectral category $\mathcal{O}$* is a spectral functor

$$M : \mathcal{O}^{op} \longrightarrow Sp.$$

A spectral functor consists of a symmetric spectrum $M(X)$ for each $X \in \mathcal{O}$ together with maps

$$M(X) \wedge \mathcal{O}(Y, X) \longrightarrow M(Y)$$

satisfying associativity and unit conditions, see [SS03, 3.3.1]. The category mod-$\mathcal{O}$ of modules over the spectral category $\mathcal{O}$ can be given a model structure such that the weak equivalences are element-wise stable equivalences of symmetric spectra and fibrations are element-wise stable fibrations [SS03, Theorem A.1.1].

To define the endomorphism category of a cofibrantly generated, simplicial, proper stable model category $\mathcal{C}$, we first have to replace $\mathcal{C}$ by a spectral category. In [SS03, 3.6] Schwede and Shipley describe the category $Sp(\mathcal{C})$ of symmetric spectra over $\mathcal{C}$, i.e. symmetric spectra with values in $\mathcal{C}$ rather than pointed simplicial sets. Theorem 3.8.2 of [SS03] states how $\mathcal{C}$ can be replaced by $Sp(\mathcal{C})$.

Theorem 2.1 (Schwede-Shipley)**.** *The category $Sp(\mathcal{C})$ can be given a model structure, the stable model structure, which makes $Sp(\mathcal{C})$ into a spectral model category that is Quillen equivalent to $\mathcal{C}$ via the adjunction*

$$\Sigma^\infty : \mathcal{C} \rightleftarrows Sp(\mathcal{C}) : Ev_0.$$

Now let $\mathcal{D}$ be a spectral model category with a set of compact generators $\mathcal{X}$. We define the *endomorphism category* $\mathcal{E}(\mathcal{X})$ as having objects $X \in \mathcal{X}$ and morphisms

$$\mathcal{E}(\mathcal{X})(X_1, X_2) = \mathrm{Hom}_{\mathcal{D}}(X_1, X_2).$$

Here $\mathrm{Hom}_{\mathcal{D}}(-,-)$ denotes the homomorphism spectrum. This is an object in the category of symmetric spectra and comes as a part of the spectral enrichment of $\mathcal{D}$. The category $\mathcal{E}(\mathcal{X})$ is obviously a small spectral category. In the case of $\mathcal{X} = \{X\}$, $\mathcal{E}(\mathcal{X})(X, X)$ is a symmetric ring spectrum, the *endomorphism ring spectrum of* X.

Without loss of generality we assume our generators to be both fibrant and cofibrant. One can now define an adjunction

$$- \wedge_{\mathcal{E}(\mathcal{X})} \mathcal{X} : \text{mod-}\mathcal{E}(\mathcal{X}) \rightleftarrows \mathcal{D} : \mathrm{Hom}(\mathcal{X}, -)$$

where the right adjoint is given by $\mathrm{Hom}(\mathcal{X}, Y)(X) = \mathrm{Hom}_{\mathcal{D}}(X, Y)$, [SS03, 3.9.1]. The left adjoint is given via

$$M \wedge_{\mathcal{E}(\mathcal{X})} \mathcal{X} = \mathrm{coeq}\Big(\bigvee_{X_1, X_2 \in \mathcal{X}} M(X_2) \wedge \mathcal{E}(\mathcal{X})(X_1, X_2) \wedge X_1 \rightrightarrows \bigvee_{X \in \mathcal{X}} M(X) \wedge X \Big).$$

One map in the coequaliser is induced by the module structure, the other one comes from the evaluation map

$$\mathcal{E}(\mathcal{X})(X_1, X_2) \wedge X_1 \longrightarrow X_2.$$

Schwede and Shipley then continue to prove [SS03, Theorem 3.9.3] which says that in the case of all generators being compact, the above adjunction forms a Quillen equivalence. Combining this with Theorem 2.1 one arrives at the following.

Theorem 2.2 (Schwede-Shipley)**.** *Let $\mathcal{C}$ be a cofibrantly generated, simplicial proper stable model category with a set of compact generators $\mathcal{X}$. Then there is a zig-zag of simplicial Quillen equivalences*

$$\mathcal{C} \simeq_Q \textit{mod-}\mathcal{E}(\mathcal{X}).$$

3. Rational G-spectra and their endomorphism category

For a finite group G, there are several Quillen equivalent constructions of G-spectra. We prefer to use equivariant orthogonal spectra from [MM02] as they can be generalised to the profinite case [Fau08]. Recall that when we talk of a profinite group, we have assumed that group to be infinite.

Briefly, a G-spectrum X consists of a collection of G-spaces $X(U)$, one for each finite dimensional real representation U of G, with G-equivariant suspension maps

$$S^V \wedge X(U) \to X(U \oplus V).$$

Here, S^V is the one-point-compactification of the vector space V. A map of G-spectra is then a collection of G-maps

$$f(U) : X(U) \to Y(U)$$

commuting with the suspension structure maps. An orthogonal G-spectrum has more structure still, but the underlying idea is the same. We denote the category of orthogonal G-spectra by $G\,\mathrm{Sp}$.

There are several model structures on $G\,\mathrm{Sp}$, which vary according to what subgroups of G are of interest. We are concerned with the case of all subgroups for a finite group and the open subgroups for a profinite groups. These cases are the ones of most interest to topologists as they contain the most information about the G-behaviour of spaces and spectra. From now on we only talk of open subgroups, as all subgroups of a finite group are open. Following [Bar09] these model categories can be rationalised, to form $L_{\mathbb{Q}}G\,\mathrm{Sp}$ which we denote as $G\,\mathrm{Sp}_{\mathbb{Q}}$.

Theorem 3.1. *There is a model structure on $G\,\mathrm{Sp}_{\mathbb{Q}}$ such that the weak equivalences are those maps f such that $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all open subgroups H of G. This model category is proper, cofibrantly generated, monoidal and spectral.*

The homotopy category $\mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}})$ has a finite set of compact generators in the case of a finite groups and a countable collection in the case of a profinite group.

Lemma 3.2. *The fibrant replacements of the spectra $\Sigma^\infty G/H_+$ for H an open subgroup form a set of compact generators denoted $\mathcal{G}_{top}$ for $\mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}})$.*

For a proof of this, see e.g. [Fau08, 4.6].

The spectra $\Sigma^\infty G/H_+$ themselves are usually chosen to form $\mathcal{G}_{top}$, but for technical reasons we would like the generators to be fibrant and cofibrant. They are cofibrant to begin with and taking the fibrant replacement obviously does not change their property as generators. We denote these replacements by $\{\Sigma^\infty_f G/H_+\}$.

We now take a closer look at the endomorphism category $\mathcal{E}_{top} := \mathcal{E}(\mathcal{G}_{top})$ of $G\,\mathrm{Sp}_{\mathbb{Q}}$. We remember from Theorem 2.2 that $G\,\mathrm{Sp}_{\mathbb{Q}}$ is Quillen equivalent to the category of modules mod-$\mathcal{E}_{top}$ over a spectral category $\mathcal{E}_{top}$. The category $\mathcal{E}_{top}$ has objects $\mathcal{G}_{top} = \{\Sigma^\infty_f G/H_+\}$. For $\sigma_1, \sigma_2 \in \mathcal{G}_{top}$, the morphisms are defined as

$$\mathcal{E}_{top}(\sigma_1, \sigma_2) = \mathrm{Hom}_{G\,\mathrm{Sp}_{\mathbb{Q}}}(\sigma_1, \sigma_2).$$

We now introduce another bit of notation. Let $\sigma_1, \sigma_2 \in \mathcal{G}_{top}$ be two generators. Let $\underline{A}(\sigma_1, \sigma_2)$ denote $\pi_0(\mathcal{E}_{top}(\sigma_1, \sigma_2))$. Using the composition

$$\mathcal{E}_{top}(\sigma_2, \sigma_3) \wedge \mathcal{E}_{top}(\sigma_1, \sigma_2) \longrightarrow \mathcal{E}_{top}(\sigma_1, \sigma_3)$$

we see that $\underline{A}$ forms a *ringoid* or *ring with several objects*, i.e. a category with objects $\mathcal{G}_{top}$ and morphisms $\underline{A}(\sigma_1, \sigma_2)$ together with composition maps

$$\underline{A}(\sigma_2, \sigma_3) \otimes \underline{A}(\sigma_1, \sigma_2) \longrightarrow \underline{A}(\sigma_1, \sigma_3)$$

satisfying associativity and unital conditions.

Applying the Eilenberg-MacLane functor H (see e.g. [HSS00] or [Sch08]) then yields another spectral category $H\underline{A}$.

Let us return to $\mathcal{E}_{top}$, in [GM95, Appendix A], Greenlees and May computed the groups

$$[\Sigma^\infty G/H_+, \Sigma^\infty G/K_+]_*^{G\,\mathrm{Sp}} \otimes \mathbb{Q}$$

for subgroups H and K of a finite group G. Using the Segal-tom Dieck splitting result [Fau08, 7.10] one can similarly compute these groups in the case of a profinite group.

Theorem 3.3. *In degrees away from zero, $[\Sigma^\infty G/H_+, \Sigma^\infty G/K_+]_*^{G\,\mathrm{Sp}}$ is torsion. Hence, the homotopy groups of the spectrum $\mathcal{E}_{top}(\sigma_1, \sigma_2)$ are concentrated in degree zero.*

It is not too surprising that a symmetric spectrum with homotopy groups concentrated in degree zero is weakly equivalent to an Eilenberg-MacLane spectrum. For a statement like this, see [Sch07b, Theorem 4.22]. However, we are also after a statement about the category of modules over this spectral category. Schwede and Shipley prove the following in [SS03, Theorem A.1.1 and Proposition B.2.1]

Theorem 3.4 (Schwede-Shipley)**.** *Let $\underline{R}$ be a spectral category whose morphism spectra are fibrant in Sp and have homotopy groups concentrated in degree zero. Then the module categories mod-$\underline{R}$ and mod-$H\underline{\pi_0 R}$ are related by a chain of Quillen equivalences.*

We can apply this theorem to our case, we have chosen our generators $\mathcal{G}_{top}$ to be fibrant and cofibrant. Hence $\mathcal{E}_{top}(\sigma_1, \sigma_2) = \mathrm{Hom}_G\,\mathrm{Sp}_\mathbb{Q}(\sigma_1, \sigma_2)$ is fibrant as for cofibrant σ_1, $\mathrm{Hom}_G\,\mathrm{Sp}_\mathbb{Q}(\sigma_1, -)$ is a right Quillen functor by definition.

Corollary 3.5. *The categories mod-$\mathcal{E}_{top}$ and mod-$H\underline{A}$ are Quillen equivalent.*

Combining this with Theorem 2.2 yields the following corollary which, in the finite case, is [SS03, 5.1.2] A more detailed version for finite groups that considers the monoidal structure appears in [Bar09].

Corollary 3.6. *There is a chain of Quillen equivalences between rational G-spectra $G\,\mathrm{Sp}_\mathbb{Q}$ and mod-$H\underline{A}$.*

4. The Quillen equivalence

We are finally going to put together the results from the previous section to obtain our main theorem.

Theorem 4.1. *Let G be either a finite group or a profinite group. Let $\mathcal{C}$ be a cofibrantly generated, proper, simplicial, stable model category together with an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

Then $G\,\mathrm{Sp}_{\mathbb{Q}}$ and $\mathcal{C}$ are Quillen equivalent.

Proof. We showed in Corollary 3.6 at the end of Section 3 how $G\,\mathrm{Sp}_{\mathbb{Q}}$ is Quillen equivalent to the category of modules over the Eilenberg-MacLane "ring spectrum of several objects" mod-$H\underline{A}$.

Let $\mathcal{X}$ consist of fibrant and cofibrant replacements of $\Phi(\sigma)$ where $\sigma \in \mathcal{G}_{top}$ runs over the generators of $\mathrm{Ho}(G\,\mathrm{Sp}_{\mathbb{Q}})$. The set $\mathcal{X}$ is then a set of generators for $\mathrm{Ho}(\mathcal{C})$. By Theorem 2.2 we have Quillen equivalences

$$G\,\mathrm{Sp}_{\mathbb{Q}} \simeq_Q \text{mod-}\mathcal{E}_{top} \quad \text{and} \quad \mathcal{C} \simeq_Q \text{mod-}\mathcal{E}(\mathcal{X}).$$

We are now going to compare mod-$\mathcal{E}_{top}$ to mod-$\mathcal{E}(\mathcal{X})$ by relating them both to mod-$H\underline{A}$. Let X_1 be a cofibrant and fibrant replacement for $\Phi(\sigma_1)$ where $\sigma_1 \in \mathcal{G}_{top}$ and define X_2 analogously. Remember that $\mathcal{E}(\mathcal{X})(X_1, X_2)$ was defined as the homomorphism spectrum of the fibrant replacement of the suspension spectrum of X_1 and X_2 in $Sp(\mathcal{C})$ [SS03, Definition 3.7.5], so

$$\mathcal{E}(\mathcal{X})(X_1, X_2) = \mathrm{Hom}_{Sp(\mathcal{C})}(\Sigma^\infty_f X_1, \Sigma^\infty_f X_2).$$

By adjunction and using the Quillen equivalence $\Sigma^\infty : \mathcal{C} \rightleftarrows Sp(\mathcal{C}) : Ev_0$ we obtain

$$[S^0, \mathcal{E}(\mathcal{X})(X_1, X_2)]^{Sp}_* \cong [\Sigma^\infty X_1, \Sigma^\infty X_2]^{Sp(\mathcal{C})}_* \cong [X_1, X_2]^{\mathcal{C}}_*.$$

Via the equivalence Φ and again by adjunction this equals

$$[\sigma_1, \sigma_2]^{G\,\mathrm{Sp}_{\mathbb{Q}}}_* \cong [S^0, \mathcal{E}_{top}(\sigma_1, \sigma_2)]^{Sp}_*.$$

Thus, $\mathcal{E}_{top}(\sigma_1, \sigma_2)$ and $\mathcal{E}(\mathcal{X})(X_1, X_2)$ have the same homotopy groups. By Lemma 3.3, these are concentrated in degree zero where they equal $\underline{A}(\sigma_1, \sigma_2)$. As the generators $X_i \in \mathcal{X}$ have been chosen to be fibrant and cofibrant, the spectra $\mathcal{E}(\mathcal{X})(X_1, X_2)$ are all fibrant in Sp. Hence Theorem 3.4 applies, giving us a chain of Quillen equivalences between mod-$\mathcal{E}(\mathcal{X})$ and mod-$H\underline{A}$. Thus we have arrived at a collection of Quillen equivalences

$$G\,\mathrm{Sp}_{\mathbb{Q}} \leftrightarrows \text{mod-}\mathcal{E}_{top} \simeq_Q \text{mod-}H\underline{A} \simeq_Q \text{mod-}\mathcal{E}(\mathcal{X}) \rightleftarrows \mathcal{C}$$

which concludes our proof of the G-equivariant Rigidity Theorem. □

Hence we have presented a nontrivial example of rigidity that is not monogenic. It is a subject of further research whether rigidity also holds for G-spectra $G\,\mathrm{Sp}$ before rationalisation.

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