Subgroup Distortion in Wreath Products of Cyclic Groups

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October 5, 2010

Abstract

We study the effects of subgroup distortion in the wreath products A wr $\mathbb{Z}$, where A is finitely generated abelian. We show that every finitely generated subgroup of A wr $\mathbb{Z}$ has distortion function bounded above by some polynomial. Moreover, for A infinite, and for any polynomial, there is a 2-generated subgroup of A wr $\mathbb{Z}$ having distortion function equivalent to the given polynomial.

1 Introduction

The notion of subgroup distortion was first formulated by Gromov in [Gr]. For a group G with finite generating set T and a subgroup H of G finitely generated by S, the distortion function of H in G is

$$\Delta_H^G(l) = \max\{|w|_S : w \in H, |w|_T \leq l\},$$

where $|w|_S$ represents the word length with respect to the given generating set S, and similarly for $|w|_T$. This function measures the difference in the word metrics on G and on H.

As usual, we only study distortion up to a natural equivalence relation. For functions f and g on $\mathbb{N}$, we say that $f \preceq g$ if there exists an integer $C > 0$ such that $f(l) \leq Cg(Cl + C) + Cl$ for all $l \geq 0$. We say two functions are equivalent, written $f \approx g$, if $f \preceq g$ and $g \preceq f$. When considered up to this equivalence, the distortion function becomes independent of the choice of finite generating sets. A subgroup H of G is said to be undistorted if $\Delta_H^G(l) \approx l$. If a subgroup H is not undistorted, then it is said to be distorted, and its distortion refers to the equivalence class of $\Delta_H^G(l)$.

Here we study the effects of distortion in various subgroups of the wreath products $\mathbb{Z}^k$ wr $\mathbb{Z}$, for $0 < k \in \mathbb{Z}$, and more generally, in A wr $\mathbb{Z}$ where A is finitely generated abelian. The group $\mathbb{Z}$ wr $\mathbb{Z}$ is the simplest example of a finitely generated (though not finitely presented) group containing a free abelian group of infinite rank. In [GS] the group $\mathbb{Z}$ wr $\mathbb{Z}$ is studied in connection with diagram groups and in particular with Thompson's group. In the same paper, it is shown that in $H_d = (\cdots(\mathbb{Z} \text{ wr } \mathbb{Z}) \text{ wr } \mathbb{Z})\cdots \text{ wr } \mathbb{Z})$, where the group $\mathbb{Z}$ appears d times, there is a subgroup $K \leq H_d \times H_d$ having distortion function $\Delta_K^{H_d \times H_d}(l) \succeq l^d$. In contrast to the study of these iterated wreath products, here we obtain polynomial distortion of arbitrary degree in the group $\mathbb{Z}$ wr $\mathbb{Z}$ itself. In [C] the distortion of $\mathbb{Z}$ wr $\mathbb{Z}$ in Baumslag's metabelian group is shown to be at least exponential, and an undistorted embedding of $\mathbb{Z}$ wr $\mathbb{Z}$ in Thompson's group is constructed.

In this note, rather than embedding the group $\mathbb{Z}$ wr $\mathbb{Z}$ into larger groups, or studying multiple wreath products, we will study distorted and undistorted subgroups in the wreath products A wr $\mathbb{Z}$ with A finitely generated abelian. The main results are as follows.

Theorem 1.1. *Let A be a finitely generated abelian group.*

1. *For any finitely generated subgroup $H \leq A$ wr $\mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in A wr $\mathbb{Z}$ is*
$$\Delta_H^{A \text{ wr } \mathbb{Z}}(l) \preceq l^m.$$
2. *If A is finite, then $m = 1$; that is, all subgroups are undistorted.*
3. *If A is infinite, then for every $m \in \mathbb{N}$, there is a 2-generated subnormal subgroup H of A wr $\mathbb{Z}$ having distortion function*
$$\Delta_H^{A \text{ wr } \mathbb{Z}}(l) \approx l^m.$$

Theorem 1.1 will be proved in Section 10.

Corollary 1.2. *It follows from the proof of Theorem 1.1 that the 2-generated subgroups of $\mathbb{Z}$ wr $\mathbb{Z}$ having distortion function equivalent to l^m can be given explicitly. If we let the standard generating set for $\mathbb{Z}$ wr $\mathbb{Z}$ be $\{a, b\}$, then the distorted subgroup is given by $H = \langle b, [\cdots[a, b], b], \cdots, b]\rangle$, where the commutator is $(m-1)$-fold. Because the subgroup $\langle [a, b], b\rangle$ of $\mathbb{Z}$ wr $\mathbb{Z}$ is normal, it follows by induction that the distorted subgroup H is subnormal.*

Remark 1.3. *There are distorted embeddings from the group $\mathbb{Z}$ wr $\mathbb{Z}$ into itself as a normal subgroup. For example, the map defined on generators by $b \mapsto b, a \mapsto [a, b]$ extends to an embedding, and the image is a quadratically distorted subgroup by Corollary 1.2. By Lemma 9.3, $\mathbb{Z}$ wr $\mathbb{Z}$ is the smallest example of a metabelian group embeddable to itself as a normal subgroup with distortion.*

Corollary 1.4. *There is a distorted embedding of $\mathbb{Z}$ wr $\mathbb{Z}$ into Thompson's group F.*

Under the embedding of Remark 1.3, $\mathbb{Z}$ wr $\mathbb{Z}$ embeds into itself as a distorted subgroup. It is proved in [GS] that $\mathbb{Z}$ wr $\mathbb{Z}$ embeds to F. Therefore, Corollary 1.4 is true.

Question 1.5. *Is there a finitely generated subgroup $H \leq \mathbb{Z}$ wr $\mathbb{Z}$ whose distortion is not equivalent to a polynomial?*

The following will be explained in Section 3.

Corollary 1.6. *For every $m \in \mathbb{N}$, there is a 2-generated subgroup H of the free n-generated metabelian group $K_{n,2}$ having distortion function*

$$\Delta_H^{K_{n,2}}(l) \succeq l^m.$$

2 Background and Preliminaries

2.1 Subgroup Distortion

Here we provide some examples of distortion as well as some basic facts to be used later on.

Example 2.1.

1. Consider the three-dimensional Heisenberg group $\mathcal{H}^3 = \langle a, b, c | c = [a, b], [a, c] = [b, c] = 1\rangle$. It has cyclic subgroup $\langle c\rangle_\infty$ with quadratic distortion, which follows from the equation $c^{l^2} = [a^l, b^l]$.

2. The Baumslag-Solitar Group $BS(1, 2) = \langle a, b | bab^{-1} = a^2\rangle$ has cyclic subgroup $\langle a\rangle_\infty$ with at least exponential distortion, because $a^{2^l} = b^l ab^{-l}$.

However, there are no similar mechanisms distorting subgroups in $\mathbb{Z}$ wr $\mathbb{Z}$. Therefore, a natural conjecture would be that free metabelian groups in general or the group $\mathbb{Z}$ wr $\mathbb{Z}$ in particular do not contain distorted subgroups. This conjecture was brought to the attention of the authors by Denis Osin. The result of Theorem 1.1 shows that the conjecture is not true.

The following facts are well-known and easily verified. When we discuss distortion functions, it is assumed that the groups under consideration are finitely generated.

Lemma 2.2.

1. If $H \leq G$ and $[G : H] < \infty$ then $\Delta_H^G(l) \approx l$.

2. If $H \leq K \leq G$ then $\Delta_H^K(l) \preceq \Delta_H^G(l)$.

3. If H is a retraction of G then $\Delta_H^G(l) \approx l$.

4. If G is a finitely generated abelian group, and $H \leq G$, then $\Delta_H^G(l) \approx l$.

2.2 Wreath Products

Let A be a finitely generated abelian group. The group A wr G, where G is any group, is defined as follows.

Definition 2.3. *The group A wr G is the semidirect product given by the action of G on $\bigoplus_{g\in G} A(g)$: for $f \in \bigoplus_{g\in G} A(g), g_1 \in G$ we have that $(f \circ g_1)(g_2) = f(g_1g_2)$ for any $g_2 \in G$.*

The following is a presentation for the group A wr G.

Lemma 2.4. *The group wr G has presentation given by generators and defining relations*

$$\langle y_1, \ldots, y_s, x_1, \ldots, x_t | [y_i, g^{-1} y_j g], R, S : g \in G, 1 \leq i, j \leq s\rangle,$$

where $G = \langle x_1, \ldots, x_t | R\rangle$, and $A = \langle y_1, \ldots, y_s | S\rangle$.

A proof may be found in [DS].

Remark 2.5. *The group $\mathbb{Z}$ wr $\mathbb{Z}$ is isomorphic to the subgroup G of 2×2 real matrices generated by two elements*

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$$

where ζ is any transcendental number.

2.3 Connections with Free Solvable Groups

In [M], Magnus shows that if $F = F_k$ is an absolutely free group of rank k with normal subgroup N, then the group $F/[N, N]$ embeds into $\mathbb{Z}^k$ wr $F/N = \mathbb{Z}^k$ wr G. For more information in an easy to read exposition, refer to [RS].

Remark 2.6. *The monomorphism $\alpha : F/[N, N] \to \mathbb{Z}^k$ wr G is called the Magnus embedding.*

Lemma 2.7. *Consider the group $\mathbb{Z}$ wr $\mathbb{Z} = W\lambda\langle b\rangle$. Let $1 \neq w \in W, x \notin W$. Then $gp\langle w, x\rangle \cong \mathbb{Z}$ wr $\mathbb{Z}$.*

This follows because the mapping $\phi : \mathbb{Z} \text{ wr } \mathbb{Z} \to \text{gp}\langle w, x\rangle : a \mapsto w, b \mapsto x$ preserves all defining relations, and it is easy to see that the kernel is trivial.

We let $K_{k,l}$ denote the k-generated class l free solvable group.

Lemma 2.8. *If $k, l \geq 2$, then the group $K_{k,l}$ contains a subgroup isomorphic to $\mathbb{Z}$ wr $\mathbb{Z}$.*

Proof. It suffices to show that the free metabelian group of rank 2, $K_{2,2}$, contains a subgroup isomorphic to $\mathbb{Z}$ wr $\mathbb{Z}$. This follows because for any $H \leq K_{2,2}$ we may use the Nielsen-Schrier Theorem to identify $H \leq F_k^{(l-2)}/F_k^{(l)} \leq F_k/F_k^{(l)} \cong K_{k,l}$.

Let $K_{2,2}$ have free generators x, y. Because $\mathbb{Z}$ wr $\mathbb{Z}$ is metabelian, we have a homomorphism

$$\phi : K_{2,2} \to \mathbb{Z} \text{ wr } \mathbb{Z} : x \mapsto a, y \mapsto b,$$

where a, b are the usual generators of $\mathbb{Z}$ wr $\mathbb{Z}$. Let H be the subgroup of $K_{2,2}$ generated by $[x, y]$ and y. Then by Lemmas 2.4, 2.7 and Dyck's Theorem we have that $H \cong \mathbb{Z}$ wr $\mathbb{Z}$. $\square$

It should be noted that by results of [S], the group $\mathbb{Z}$ wr $\mathbb{Z}^2$ can not be embedded into any free metabelian or free solvable groups.

Subgroup distortion has connections with the membership problem. It was observed in [Gr] and proved in [F] that for a finitely generated subgroup H of a finitely generated group G with solvable word problem, the membership problem is solvable in H if and only if the distortion function $\Delta_H^G(l)$ is bounded by a recursive function.

By Theorem 2 of [U], the membership problem for free solvable groups of length greater than two is undecidable. Therefore, because of the connections between subgroup distortion and the membership problem just mentioned, we restrict our primary attention to the case of free metabelian groups.

Lemma 2.8 motivates us to study distortion in $\mathbb{Z}$ wr $\mathbb{Z}$ in order to better understand distortion in free metabelian groups. Distortion in free metabelian groups is similar to distortion in wreath products of free abelian groups, by Lemma 2.8 and the Magnus embedding. In particular, if $k \geq 2$ then

$$\mathbb{Z} \text{ wr } \mathbb{Z} \leq K_{k,2} \leq \mathbb{Z}^k \text{ wr } \mathbb{Z}^k.$$

Thus by Lemma 2.2, given $H \leq \mathbb{Z}$ wr $\mathbb{Z}$ we have

$$\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \preceq \Delta_H^{K_{k,2}}(l).$$

This explains Corollary 1.6. On the other hand, given $L \leq K_{k,2}$ then we have

$$\Delta_L^{K_{k,2}}(l) \preceq \Delta_L^{\mathbb{Z}^k \text{ wr } \mathbb{Z}^k}(l).$$

Based on this discussion, we ask the following. An answer would be helpful in order to more fully understand subgroup distortion in free metabelian groups.

Question 2.9.
What effects of subgroup distortion are possible in $\mathbb{Z}^k$ wr $\mathbb{Z}^k$ for $k > 1$?

3 Canonical Forms and Word Metric

Here we aim to further understand the form of elements in $\mathbb{Z}^k$ wr $\mathbb{Z}$ as well as the word metric in these groups.

Using the presentation of $\mathbb{Z}^k$ wr $\mathbb{Z}$ of Lemma 2.4, and the definition of $\mathbb{Z}^k$ wr $\mathbb{Z}$ we may write any element in a canonical form. The form described in Remark 3.1 below is useful for understanding the structure of group elements.

We will use the notation that $(x)_i$ equals the conjugate $b^{-i}xb^i$ for $i \in \mathbb{Z}$ and $x \in \mathbb{Z}^k$, where $\mathbb{Z}^k$ wr $\mathbb{Z} = \mathbb{Z}^k$ wr $\langle b \rangle$.

Remark 3.1. *Arbitrary element in $\mathbb{Z}^k$ wr $\mathbb{Z} = gp\langle a_1, \dots, a_k, b\rangle$ is of the form*

$$b^t w = b^t \prod_{i=-\infty}^{\infty}{}^{\circ} (a_1)_i^{m_{i,1}} (a_2)_i^{m_{i,2}} \cdots (a_k)_i^{m_{i,k}},$$

where the product is finite, indicated by the $\circ$ symbol.

The form is unique.

The normal form described in Remark 3.2 for elements of A wr $\mathbb{Z}$, where A is a finitely generated abelian group, is necessary to obtain a general formula for computing the word length.

Remark 3.2. *Arbitrary element of A wr $\mathbb{Z}$ may be written in a normal form, following [CT], as*

$$b^t (u_1)_{\iota_1} \cdots (u_N)_{\iota_N} (v_1)_{-\epsilon_1} \cdots (v_M)_{-\epsilon_M}$$

where $0 \leq \iota_1 < \cdots < \iota_N, 0 < \epsilon_1 < \cdots < \epsilon_M$, and $u_1, \ldots, u_N, v_1, \ldots, v_M$ are minimal length representatives of elements in $A - \{1\}$.

The following formula for the word length in A wr $\mathbb{Z}$ is given in [CT].

Lemma 3.3. *Given an element in A wr $\mathbb{Z}$ having normal form as in Remark 3.2, its length is given by the formula*

$$\sum_{i=1}^{N} |u_i|_A + \sum_{i=1}^{M} |v_i|_A + \min\{2\epsilon_M + \iota_N + |t - \iota_N|, 2\iota_N + \epsilon_M + |t + \epsilon_M|\}.$$

Because the subgroup W of $G = \mathbb{Z}$ wr $\mathbb{Z} = W \lambda \mathbb{Z}$ is abelian, we also use additive notation to represent elements of W.

We will use the following notation in the case of $\mathbb{Z}$ wr $\mathbb{Z}$.

Remark 3.4. *In the case of $\mathbb{Z}$ wr $\mathbb{Z} = \langle a \rangle$ wr $\langle b \rangle$, we use module language to write any element as*

$$w = af(x) \text{ where } f(x) = \sum_{i=-\infty}^{\infty} {}^{\circ} m_i x^i$$

is a Laurent polynomial.

4 Structure of Some Subgroups of $\mathbb{Z}$ wr $\mathbb{Z}$

Lemma 4.1. *Let G be a group having normal subgroup W and cyclic $G/W = \langle bW \rangle$. Then any finitely generated subgroup H of G may be generated by elements of the form $b^t w_1, w_2, \ldots, w_s$ where $w_i \in W$.*

The proof is elementary and follows from the assumption that G/W is cyclic.

Remark 4.2. *It follows that any finitely generated subgroup in $\mathbb{Z}^k$ wr $\mathbb{Z}$ or $\mathbb{Z}_n^k$ wr $\mathbb{Z}$ can be generated by elements $b^t w_1, w_2, \ldots, w_s$ where $w_i \in W$.*

Definition 4.3. *For $k > 0$ fixed and any $t > 0$, the group L_t is the subgroup of $\mathbb{Z}^k$ wr $\mathbb{Z}$ generated by the subgroup W and by the element b^t.*

The following discussion helps further the understanding of the structure of $\mathbb{Z}^k$ wr $\mathbb{Z}$ and its subgroups, and will be used again in later Sections.

Lemma 4.4. *The group* $L_t \cong \mathbb{Z}^{tk}$ *wr* $\mathbb{Z}$.

Proof. By [DS] we have that a presentation of $\mathbb{Z}^{tk}$ wr $\mathbb{Z}$ is given by

$$\langle b', a'_1, \ldots, a'_{tk} | [a'_i, a'_j], [a'_i, b'^{-l} a'_j b'^l], l > 0, 1 \le i, j \le tk \rangle.$$

Observe that L_t may be generated by the elements $b^t, a_1, b^{-1}a_1 b, \ldots, b^{1-t}a_1 b^{t-1}$, $\ldots, a_k, b^{-1}a_k b, \ldots, b^{1-t}a_k b^{t-1}$. The map $\phi : \mathbb{Z}^{tk} \text{ wr } \mathbb{Z} \to L_t : b' \mapsto b^t, a'_1 \mapsto b^{-1}a_1 b, a'_2 \mapsto b^{-2}a_1 b, \ldots, a'_t \mapsto b^{1-t}a_1 b^{t-1}, a'_{t+1} \mapsto a_2, \ldots, a'_{kt} \mapsto b^{1-t}a_k b^{t-1}$ is easily checked to be an isomorphism.

□

Lemma 4.5. *For any* $w \in W$ *there is an isomorphism* $L_t \to L_t$ *identical on* W *such that* $b^t w \to b^t$*, provided* $t \neq 0$.

This follows because the actions of b^t and $b^t w$ on W coincide.

Lemma 4.6. *Let* H *be a finitely generated subgroup of* $\mathbb{Z}^k$ *wr* $\mathbb{Z}$ *not contained in* W*. Then* H *is a subgroup of* L_t *for some* t*. Under the isomorphism* $L_t \to \mathbb{Z}^{tk}$ *wr* $\mathbb{Z}$ *of Lemma 4.4, the subgroup* H *has generators of the form* $b, x_1, \ldots, x_s$ *where* $\mathbb{Z}^{tk} \text{ wr } \mathbb{Z} = W \lambda \langle b \rangle_\infty, W = \prod_{\mathbb{Z}} \mathbb{Z}^{tk}$*, and* $x_1, \ldots, x_s \in W$*. Moreover, the distortion of* H *in* $\mathbb{Z}^k$ *wr* $\mathbb{Z}$ *is equivalent to the distortion of* $gp\langle b, x_1, \ldots, x_s \rangle$ *in* $\mathbb{Z}^{tk}$ *wr* $\mathbb{Z}$.

Proof. By Lemma 4.1 the generators of H may be chosen to have the form $b^t w_0, w_1, \ldots, w_s$ where $w_i \in W$. Therefore, for this value of t we have that H is a subgroup of L_t. Using the isomorphisms of Lemmas 4.4 and 4.5 we have that H is a subgroup of $\mathbb{Z}^{kt}$ wr $\mathbb{Z}$ generated by the image of $b^t w_0, w_1, \ldots, w_s$ under the two isomorphisms: elements $b, x_1, \ldots, x_s$. Finally, because $[\mathbb{Z}^k \text{ wr } \mathbb{Z} : L_t] < \infty$ we have by Lemma 2.2 that the distortion of H in $\mathbb{Z}^k$ wr $\mathbb{Z}$ is equivalent to the distortion of its image in $\mathbb{Z}^{tk}$ wr $\mathbb{Z}$. □

5 Undistorted Abelian Subgroups

Lemma 5.1. *Any subgroup of the form* $H = gp\langle b^k w \rangle$*,* $k \neq 0$*, where* $w \in W$ *is undistorted in* $\mathbb{Z}$ *wr* $\mathbb{Z}$.

Proof. Consider the subgroup $L_k = \text{gp}\langle W, b^k w \rangle$ of $\mathbb{Z}$ wr $\mathbb{Z}$. We have that the index $[\mathbb{Z} \text{ wr } \mathbb{Z} : L_k] < \infty$. Moreover, from $L_k = W \cdot \langle b^k w \rangle$, $W \trianglelefteq L_k$, and $W \cap \langle b^k w \rangle = \{1\}$ it follows that H is a retraction of L_k. Therefore, by Lemma 2.2, H is undistorted in $\mathbb{Z}$ wr $\mathbb{Z}$. □

Lemma 5.2. *Any finitely generated subgroup of* $\mathbb{Z}$ *wr* $\mathbb{Z}$ *contained in the free abelian subgroup* W *of* $\mathbb{Z}$ *wr* $\mathbb{Z}$ *is undistorted.*

Proof. Let $H \leq \mathbb{Z} \text{ wr } \mathbb{Z}$, $H \subset W$. Let the generating set for $\mathbb{Z} \text{ wr } \mathbb{Z}$ be the usual one, $\{a, b\}$, and let H have generating set $\{h_1, \ldots, h_s\}$, where without loss of generality s is the minimum possible number of generators. Then for some $n \geq s$ we have that $H \leq K = \text{gp}\langle a_{i_1}, \ldots, a_{i_n}\rangle$, where $a_j = b^{-j}ab^j$ in terms of generators of $\mathbb{Z} \text{ wr } \mathbb{Z}$. This follows by letting $\{a_{i_1}, \ldots, a_{i_n}\}$ be the collection of distinct constituents occuring in the canonical form of the generators of H. Then we have that the free abelian group H, considered as a subgroup of the free abelian group K is undistorted by Lemma 2.2. It suffices to show that K is undistorted in $\mathbb{Z} \text{ wr } \mathbb{Z}$. Let $h \in K$, so there is an expression $h = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n}$ for some $\alpha_i \in \mathbb{Z}$. Then by Lemma 3.3 we have that

$$|h|_{\mathbb{Z} \text{ wr } \mathbb{Z}} \geq \sum_{j=1}^{n} |\alpha_j| = |h|_K.$$

Thus the subgroup is undistorted.

□

Here we are able to prove that all finitely generated abelian subgroups of $\mathbb{Z} \text{ wr } \mathbb{Z}$ are undistorted. It should be remarked that the authors are aware that an independent proof of this fact is available in [GS]. In that paper it is shown that $\mathbb{Z}^k \text{ wr } \mathbb{Z}$ is a subgroup of the Thompson group F, and that every finitely generated abelian subgroup of F is undistorted. However, our proof is elementary and so we include it.

Proof. By Lemmas 4.4 and 2.2, it suffices to consider the case where $k = 1$. Let H be a finitely generated subgroup of $\mathbb{Z} \text{ wr } \mathbb{Z}$. Then by Remark 4.2, H can be generated by elements $b^k w_1, w_2, \ldots, w_s$ where $w_i \in W$. If $k = 0$ then $H \subset W$ so by Lemma 5.2 it is undistorted. If $s = 1$ then by Lemma 5.1, H is undistorted. Thus is remains to observe that if $s > 1$ and $k \neq 0$ that such an H is nonabelian because $b^k w_1$ and w_2 do not commute.

□

6 Lower Bounds on Distortion in $\mathbb{Z}$ wr $\mathbb{Z}$.

Lemma 6.1. *Let $m \in \mathbb{N}$. For any $l \in \mathbb{N}$ let there be polynomials $f_l(x) \in \mathbb{Z}[x]$ such that the sum of modules of coefficients of $f_l(x)$ is equivalent to l^m, while the sum of modules of coefficients of $g_l(x) = (1-x)^{m-1} f_l(x)$ is at most linear in l. Then the subgroup H of $\mathbb{Z}$ wr $\mathbb{Z}$ generated by $h = a(1-x)^{m-1} \in W$ and b has distortion $\Delta_H^{\mathbb{Z} \, wr \, \mathbb{Z}}(l) \succeq l^m$.*

Proof. We let $H = \text{gp}\langle h, b\rangle$. By Lemma 2.7 we have that $H \cong \mathbb{Z} \text{ wr } \mathbb{Z}$ under the obvious isomorphism $b \mapsto b, h \mapsto a$. We fix the notation that $h_i = b^{-i}hb^i$. Suppose we have $f_l(x)$ and $g_l(x)$ as in the statement of Lemma 6.1. Then if $f_l(x) = \sum_{i=0}^{l} a_i x^i$, we have that in the module language, $hf_l(x) = h^{a_0} h_1^{a_1} \cdots h_l^{a_l}$, so by Lemma 3.3 and by hypothesis, we have that

$$|hf_l(x)|_H = \sum_{i=0}^{l} |a_i| + 2l \approx l^m.$$

On the other hand, in $\mathbb{Z}$ wr $\mathbb{Z} = \langle a, b\rangle$ we have that

$$hf_l(x) = af_l(x)(1-x)^{m-1} = ag_l(x).$$

Let $g_l(x) = \sum_{i=0}^{l+m-1} b_i x^i$. Then

$$hf_l(x) = a\sum_{i=0}^{l+m-1} b_i x^i = a_0^{b_0} a_1^{b_1} \cdots a_{l+m-1}^{b_{l+m-1}}.$$

Therefore,

$$|hf_l(x)|_{\mathbb{Z} \text{ wr } \mathbb{Z}} = \sum_{i=0}^{l+m-1} |b_i| + 2(l+m-1) \approx l.$$

Therefore, H is distorted in $\mathbb{Z}$ wr $\mathbb{Z}$ of order at least l^m. □

We will use the following formula involving binomial coefficients during the proof of Theorem 1.1.

Lemma 6.2. *For any $m, N \in \mathbb{N}$ we have that*

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (N+1-i)^{m-1} = (m-1)!.$$

Proof. This is the derivative of formula 1.14 in [Go]. □

We are now prepared to prove the following fact.

Proposition 6.3. *For any $m \in \mathbb{N}$, there exist polynomials $f_l(x)$ as in Lemma 6.1. That is to say, the 2-generated subgroup $\langle b, a(1-x)^{m-1}\rangle = H \leq \mathbb{Z}$ wr $\mathbb{Z}$ has distortion $\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \succeq l^m$.*

Proof. We will construct concrete polynomials as described in Lemma 6.1. Let m be fixed. We will define

$$f_{ml-1}(x) = a_0 + a_1 x + \cdots + a_{ml-1}x^{ml-1}.$$

The coefficient a_{sl+t}, for $s = 0, 1, \ldots, m-1, t = 0, 1, \ldots, l-1$ is

$$a_{sl+t} = \sum_{i=0}^{s} d_i((s-i)l + t + 1)^{m-1},$$

where $d_0 = 1$ and $d_1, \ldots, d_s$ are constants to be determined later. Observe that it suffices to find $d_1, \ldots, d_s \in \mathbb{Q}$ with bounded denominators not depending on

l. For in this case, we can multiply $f_l(x)$ and $g_l(x)$ by an integer (independent of l) and obtain polynomials with integer coefficients and satisfying Lemma 6.1.

Then we have that

$$\sum_{i=0}^{ml-1} |a_i| \geq \sum_{i=0}^{l-1} |a_i| = \sum_{i=1}^{l} i^{m-1} \approx l^m.$$

The equality $\sum_{i=0}^{l-1} |a_i| = \sum_{i=1}^{l} i^{m-1}$ follows by the definition of

$$a_{0l+t} = \sum_{i=0}^{0} d_i((0-i)l+t+1)^{m-1} = (t+1)^{m-1}.$$

It remains to prove that each of the coefficients $|c_j|, j = 0, \ldots, (m-1)+(ml-1)$, of $(1-x)^{m-1} f_{ml-1}(x) = \sum_{i=0}^{(m-1)+(ml-1)} c_j x^j$ is bounded by a constant, if we choose $d_1, \ldots, d_s$ properly. We fix the notation that $(1-x)^{m-1} = \sum_{i=0}^{m-1} b_i x^i$ so that

$$b_i = (-1)^i \binom{m-1}{i}.$$

For $j \in [sl, (s+1)l-1]$ and $s \in \{0, \ldots, m-1\}$ then

$$a_j = \sum_{k=0}^{s} d_k((s-k)l + j - sl + 1)^{m-1}.$$

Let $j \in [sl+m-1, (s+1)l-1]$ and $s \in \{0, \ldots, m-1\}$. Then we have that

$$c_j = \sum_{i=0}^{m-1} b_i a_{j-i} = \sum_{i=0}^{m-1} b_i (\sum_{k=0}^{s} d_k((s-k)l + j - i - sl + 1)^{m-1}).$$

Let $\gamma_{i,j,k,l} = (s-k)l + j - i - sl + 1 = -kl + j - i + 1$ be a temporary shorthand. Then we have that

$$c_j = \sum_{i=0}^{m-1} b_i (\sum_{k=0}^{s} d_k(\gamma_{i,j,k,l})^{m-1} = \sum_{k=0}^{s} d_k (\sum_{i=0}^{m-1} b_i(\gamma_{i,j,k,l})^{m-1}).$$

By Lemma 6.2 with $N = j - kl$, which is a constant within each summand where k and j are fixed, we have

Consider now an index $j \in \mathbb{N}$ inside an interval of the form $[sl, sl+m-2]$ for $0 \leq s \leq m-1$. We may write $j = sl + u$ for $0 \leq u \leq m-2$. Then by definition we have that

$$a_{j-i} = \begin{cases} \displaystyle\sum_{k=0}^{s} d_k((s-k)l + j - i - sl + 1)^{m-1} & \text{if } 0 \leq i \leq j - sl, \\ \displaystyle\sum_{k=0}^{s-1} d_k((s-1-k)l + j - i - (s-1)l + 1)^{m-1} & \text{if } j - sl + 1 \leq i \leq m-1. \end{cases}$$

Therefore, for such j we compute that

$$c_j = \sum_{i=0}^{j-sl} b_i a_{j-i} + \sum_{i=j-sl+1}^{m-1} b_i a_{j-i}$$

$$= \sum_{i=0}^{j-sl} b_i (\sum_{k=0}^{s} d_k(-kl + j - i + 1)^{m-1}) + \sum_{i=j-sl+1}^{m-1} b_i (\sum_{k=0}^{s-1} d_k(-kl + j - i + 1)^{m-1})$$

$$= \sum_{i=0}^{j-sl} b_i (\sum_{k=0}^{s-1} d_k(-kl + j - i + 1)^{m-1}) + \sum_{i=0}^{j-sl} d_s b_i(-sl + j - i + 1)^{m-1}$$

$$+ \sum_{i=j-sl+1}^{m-1} b_i (\sum_{k=0}^{s-1} d_k(-kl + j - i + 1)^{m-1})$$

$$= \sum_{i=0}^{m-1} b_i \sum_{k=0}^{s-1} d_k(-kl + j - i + 1)^{m-1} + d_s \sum_{i=0}^{u} b_i(u - i + 1)^{m-1}$$

$$= (m-1)! \sum_{k=0}^{s-1} d_k + d_s \sum_{i=0}^{u} b_i(u - i + 1)^{m-1}.$$

The last equation follows from Lemma 6.2. The final formula obtained is a constant independent of l.

Therefore, we have shown that for all $j \in \{0, 1, \ldots, ml - 1\}$ that c_j is a constant independent of l. It remains to prove that $d_1, \ldots, d_{m-1}$ may be chosen in such a way that the remaining coefficients

$$c_{ml+m-2} = (-1)^{m-1} a_{ml-1}, \ldots, c_{ml} = b_1 a_{ml-1} + b_2 a_{ml-2} + \cdots + b_{m-1} a_{ml-m+1}$$

are bounded by a constant. By the triangle inequality, it suffices to bound the quantities $|a_{ml-1}|, \ldots, |a_{ml-m+1}|$ by a constant independent of l. This takes into account the fact that $\{a_{ml}, \ldots, a_{ml-m+1}\}$ and $\{a_{l-1}, \ldots, a_0\}$ are disjoint since without loss of generality, $m \geq 2$. For each $j \in \{1, \ldots, m-1\}$ we have by definition that

$$a_{ml-j} = a_{(m-1)l+(l-j)} = \sum_{i=0}^{m-1} d_i((m - 1 - i)l + l - j + 1)^{m-1}$$

$$\sum_{i=0}^{m-1} d_i (\sum_{k=0}^{m-1} \binom{m-1}{k} ((m-i)l)^k (1-j)^{m-1-k})$$

$$= l^{m-1} (\sum_{k=1}^{m} d_{m-k}(1-j)^0 k^{m-1}) + l^{m-2} (\sum_{k=1}^{m} d_{m-k} \binom{m-1}{m-2} (1-j)^1 k^{m-2}) + \cdots$$

$$+ l (\sum_{k=1}^{m} d_{m-k} \binom{m-1}{1} (1-j)^{m-2} k^1) + C$$

where C is the constant term of this polynomial and hence irrelevant. It suffices to show that for each $p \in \{1, \dots, m-1\}$ that

$$0 = \binom{m-1}{p}(1-j)^{m-1-p}(m^p + d_1(m-1)^p + \cdots + d_{m-1}(1)^p).$$

We will select $d_1, \dots, d_{m-1}$ so that

$$d_1(m-1)^p + \cdots + d_{m-1}(1)^p = -m^p$$

for each $p = 1, \dots, m-1$. The matrix of the linear system is a non-singular Vandermonde matrix, whose determinent $\det(A) = \pm 1 \cdot \prod_{i=2}^{m-2} i^2(m-1) \prod_{1 \le i < j \le m-2}(j-i)$ does not depend on l. By Cramer's rule, the required $d_1, \dots, d_{m-1}$ exist.

□

7 Auxilliary Computations

7.1 Some Linear Algebra

In order to obtain upper bounds on distortion in $\mathbb{Z}$ wr $\mathbb{Z}$ we require some facts from linear algebra. Fix an integer $k \ge 1$ and let $n > 0$ be arbitrary.

Lemma 7.1. *Let $Y_1, \dots, Y_n, C_1, \dots, C_n$ be $k \times 1$ column vectors. Suppose that the modulus of each coordinate of each C_i is bounded by a constant b. Suppose that the modulus of each coordinate of Y_1 and Y_n is bounded by bc_1 for some constant $c_1 \ge 1$. Suppose further we have the relationship*

$$Y_i = AY_{i-1} + C_i, i = 2, \dots, n$$

where A is a $k \times k$ matrix. Then the modulus of each coordinate of arbitrary $Y_i, 2 \le i \le n-1$ is bounded by cc_1bn^k where c depends on A only. All matrix entries are assumed to be complex.

Proof. There exists a Jordan decomposition, $A = S^{-1}A'S$, where S depends on A only and

$$A' = \begin{pmatrix} J_1 & 0 \dots & 0 \\ 0 & J_2 \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 \dots & J_l \end{pmatrix}.$$

where each block is of the form

$$J_i = \begin{pmatrix} \lambda_i & 0 & 0 \dots & 0 \\ 1 & \lambda_i & 0 \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 \dots & 1 & \lambda_i \end{pmatrix}$$

for some λ_i. Let $S = (s_{i,j})_{1\le i,j\le k}$ and let $s = \max|s_{i,j}|$. Then for $C_i' = SC_i$ and $Y_i' = SY_i$ we have that

$$Y_i' = A'Y_{i-1}' + C_i'. \tag{1}$$

By hypothesis, the coordinates of C_i' are bounded by $b' = ksb$ for each i, and the coordinates of Y_1' and Y_n' are bounded by $ksbc_1 = b'c_1$. As we will explain, our problem can be reduced to the similar problem for Y_i' in (1). Suppose that the modules of coordinates of every Y_i' are bounded by $dc_1b'n^k$ where d depends on A only. Then, letting $S^{-1} = (\tilde{s_{i,j}})_{1\le i,j\le k}$ and $\tilde{s} = \max|\tilde{s_{i,j}}|$ we have by definition of Y_i' that arbitrary element of Y_i has modulus bounded above by $k\tilde{s}dc_1b'n^k = cc_1bn^k$ where $c = s\tilde{s}dk^2$ only depends on A', as required.

If there is more than one Jordan block present in A', the problem is decomposed into at most k subproblems, each with only one Jordan block of size smaller than k. By induction it suffices to prove Lemma 7.1 in the case where A' has only one Jordan block. Let λ be the eigenvalue of A'. We will consider two cases.

- First suppose that $|\lambda| > 1$.

We introduce notation: let $Y_i' = [y_{i,1}, \cdots, y_{i,k}]^T$ and $C_i' = [c_{i,1}, \cdots, c_{i,k}]^T$. Consider the constant $\frac{b'}{|\lambda|-1}$. If for each $1 \le i \le n-1$ we have that $|y_{i,1}| \le \frac{b'}{|\lambda|-1}$, then all $|y_{i,1}|$ are already bounded. Otherwise we have by formula (1) that

$$y_{i,1} = \lambda y_{i-1,1} + c_{i,1}$$

and $|y_{i-1,1}| > \frac{b'}{|\lambda|-1}$ for some $i-1 > 0$. This implies that

$$|\lambda||y_{i-1,1}| = |y_{i,1} - c_{i,1}| \le |y_{i,1}| + |c_{i,1}| < |y_{i,1}| + b' < |y_{i,1}| + |y_{i-1,1}|(|\lambda| - 1),$$

which in turn implies that $|y_{i-1,1}| < |y_{i,1}|$. We similarly obtain that

$$|y_{i,1}| < |y_{i+1,1}| < \cdots < |y_{n,1}| \le b'c_1.$$

Let $\alpha_1 = \max\{\frac{b'}{|\lambda|-1}, b'c_1\}$. Then for any i we have that $|y_{i,1}| < \alpha_1$.

By induction we have that $|y_{i,k}| < \alpha_k$ for all i. Therefore, all $|y_{i,j}|$ are bounded by $\max\{\alpha_1, \ldots, \alpha_k\}$. Observe that this constant is of the form $(\mu c_1 + \nu)b'$ where μ, ν depend on A only. Therefore, we may select the constant $d \ge \mu + \nu \ge \frac{\mu c_1 + \nu}{c_1}$ so that the modulus of any coordinate in Y_i' is bounded above by $dc_1b' \le dc_1b'n^k$.

- Now suppose that $|\lambda| \le 1$.

From Formula (1) we derive:

$$\begin{aligned} Y_i' &= A'(A'Y_{i-2}' + C_{i-1}') + C_i' = (A')^2Y_{i-2}' + A'C_{i-1}' + C_i' = \cdots \\ &= (A')^{i-1}Y_1' + (A')^{i-2}C_2' + \cdots + A'C_{i-1}' + C_i'. \end{aligned} \tag{2}$$

We have already obtained that the modules of coordinates of Y_1' are bounded by $b'c_1$ and the modules of coordinates of $C_2', \ldots, C_n'$, are bounded by b', so it suffices to bound the elements of $(A')^r$ from above, where $r \leq n-1$. The following formula for $(A')^r$ is well-known because A' is assumed to be a Jordan block; it may also be checked easily using induction. We have that

$$(A')^r = \begin{pmatrix} \lambda^r & & 0 & 0 \ldots & 0 \\ r\lambda^{r-1} & & \lambda^r & 0 \ldots & 0 \\ \frac{r(r-1)}{2!}\lambda^{r-2} & & r\lambda^{r-1} & \lambda^r \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \frac{r!}{(r-(k-1))!(k-1)!}\lambda^{r-(k-1)} & \ldots & \frac{r(r-1)}{2!}\lambda^{r-2} & r\lambda^{r-1} & \lambda^r \end{pmatrix},$$

with the understanding that if $r < k-1$, any terms of the form $\binom{r}{j}\lambda^{r-j}$ where $r < j$ are 0. Arbitrary nonzero element of the matrix $(A')^r$ is of the form $\binom{r}{j}\lambda^{r-j}$ for some $j \leq k-1$.

Because $|\lambda| \leq 1$ we have that

$$\begin{aligned} |\binom{r}{j}\lambda^{r-j}| \leq \binom{r}{j} &= \frac{r(r-1)\cdots(r-(j-1))}{j!} \\ &\leq r(r-1)\cdots(r-(j-1)) \leq r^j \leq n^{k-1}. \end{aligned} \tag{3}$$

Using the triangle inequality and Equation (7.1) we will obtain an upper bound on the entries of arbitrary Y_i', $2 \leq i \leq n-1$. By Equation (7.1), we have that arbitrary summand is of the form

$$(A')^{i-1}Y_1', C_i', \text{ or } (A')^{i-j}C_j'$$

for $2 \leq j \leq i-1$. The modulus of entries in $(A')^{i-1}Y_1'$ is bounded by $kn^{k-1}b'c_1$; in C_i' is bounded by b'; and in $(A')^{i-j}C_j'$ is bounded by $kn^{k-1}b'$. Since there are at most n summands in Equation (7.1), we have that every entry of Y_i' is bounded above by $kb'c_1n^k$, and k depends upon A. □

We will use Lemma 7.1 to prove the following.

Lemma 7.2. *Let the $(n+k)\times n$ matrix M have j^{th} column, for $j = 1, \ldots, n$, given by $[0, \ldots, 0, d_0, d_1, \ldots, d_k, 0, \ldots, 0]^T$, where $d_0, d_k \neq 0$ and d_0 first appears as the j^{th} entry in this j^{th} column. Suppose that $X = [x_1, x_2, \ldots, x_n]^T$ is a solution to the system of equations $MX = B$, where $B = [b_1, b_2, \ldots, b_{n+k}]^T$. Then it is possible to bound the modules of all coordinates $x_1, \ldots, x_n$ of the vector X such that $|x_i| \leq cbn^k$ where $b = \max_j\{|b_j|\}$ for every $1 \leq j \leq n+k$ and the constant c depends upon $d_0, \ldots, d_k$ only.*

Prior to proving Lemma 7.2 we prove an easier special case.

Lemma 7.3. *It is possible to bound the coordinates $x_1, \ldots, x_k$ of the vector X from Lemma 7.2 from above by $b\tilde{\gamma}$ where $b = \max\{|b_j|\}_{j=1,\ldots,n+k}$ and $\tilde{\gamma} = \tilde{\gamma}(d_0, \ldots, d_{k-1})$.*

Proof. By Cramer's Rule, we have the explicit formula that

$$|x_i| = \left|\frac{\det(L_i)}{\det(L)}\right|$$

where L is the $k \times k$ upper left submatrix of M corresponding to the first k equations, and L_i is obtained by replacing column i in L with $[b_1, \ldots, b_k]^T$. Because $\det(L) = d_0^k$, it suffices to show that the desired bounds exist for $\det(L_i)$; that is, we must show that there exists a constant $\tilde{\gamma}$ depending on $d_0, \ldots, d_{k-1}$ only such that $|\det(L_i)| \leq b\tilde{\gamma}$ for $i = 1, \ldots, k$. By expanding along the i^{th} column in L_i, we find that

$$\det(L_i) = \pm b_1 f_1(d_0, \ldots, d_{k-1}) \pm b_2 f_2(d_0, \ldots, d_{k-1}) \pm \cdots \pm b_k f_k(d_0, \ldots, d_{k-1}),$$

where for each $i = 1, \ldots, k$, f_i is a function of $d_0, \ldots, d_{k-1}$ only obtained as the determinant of a submatrix containing none of $b_1, \ldots, b_k$. The proof is complete by the triangle inequality. □

Note that the $|x_j|$ for $j = n-k+1, \ldots, n$ are similarly bounded by $b\overline{\gamma}$ for the same b and some $\overline{\gamma} = \overline{\gamma}(d_0, \ldots, d_{k-1})$ as in Lemma 7.3. It is clear according to the proof of Lemma 7.3 that we may assume that $|x_i| \leq b\gamma$ for the same $\gamma = \gamma(d_0, \ldots, d_{k-1})$ for all $i = 1, \ldots, k, n-k+1, \ldots, n$.

We proceed with the Proof of Lemma 7.2.

Proof. It suffices to obtain upper bounds for $|x_i|$ when $n - k \geq i \geq k+1$.

For such indices, we have that

$$d_k x_{i-k} + d_{k-1} x_{i+1-k} + \cdots + d_0 x_i = b_i.$$

In other words,

$$x_i = \xi_i + a_1 x_{i-k} + a_2 x_{i+1-k} + \cdots + a_k x_{i-1},$$

where $\xi_i = \frac{b_i}{d_0}$ and $a_j = -\frac{d_{k-j+1}}{d_0}$. Let $X_i = [x_{i-k+1}, \ldots, x_i]^T$ and let $\Xi_i = [0, \ldots, 0, \xi_i]^T$. Then for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \ldots & 0 & 1 \\ a_1 & a_2 \ldots & a_{k-1} & a_k \end{pmatrix}$$

we have the recursive relationship

$$X_i = AX_{i-1} + \Xi_i$$

for $i = k, \ldots, n$. Observe that the matrix A depends on $d_0, \ldots, d_k$ only.

We see by Lemma 7.3 that Lemma 7.1 applies to our situation. Therefore, the modules of coordinates of arbitrary X_i, $k+1 \leq i \leq n-k$ are bounded by $c'b\gamma(n-k+1)^k \leq cb\gamma n^k$, where $c = c'\gamma$ depends only on $d_0, \ldots, d_k$. □

7.2 Estimating Word Length

In order to prove Theorem 1.1 we will establish a looser way of computing lengths in $\mathbb{Z}^r$ wr $\mathbb{Z}$, $r \geq 1$ than the formula introduced in Lemma 3.3.

Lemma 7.4. *Let $\mathbb{Z}^r$ wr $\mathbb{Z}$ have standard generating set $\{a_1, \dots, a_r, b\}$. Let a subgroup H of $\mathbb{Z}^r$ wr $\mathbb{Z}$ have generators of the form $b, w_1, \dots, w_k$ where each $w_i \neq 1$ is in the normal closure of a_i for $i = 1, \dots, k$. Then H is isomorphic to $\mathbb{Z}^k$ wr $\mathbb{Z}$.*

This follows from what has been established already. Each w_i generates a free cyclic $\mathbb{Z}[\langle b \rangle]$ submodule. By hypothesis, all w_i's are in different direct summands, so they generate a free $\mathbb{Z}[\langle b \rangle]$ module of rank k.

We will only consider subgroups of $\mathbb{Z}^r$ wr $\mathbb{Z}$ that are of a special form, such as in the statement of Lemma 7.4. Such a subgroup H has generators $b, w_1, \dots, w_k$ where $w_i \in W$, and further, for each $i = 1, \dots, k$ we have that

$$w_i = \sum_{j=0}^{t_i} d_{i,j} (a_i)_j. \tag{4}$$

This follows without loss of generality by conjugating by a power of b. Then for any element $g \in H$, we may write

$$g = b^n \sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} z_q (w_i)_q \tag{5}$$

for some $s_i, z_q \in \mathbb{Z}, p_i \geq 0$. In the generators of $\mathbb{Z}^r$ wr $\mathbb{Z}$ we may also write this element as

$$b^n \sum_{i=1}^{k} \sum_{j=s_i}^{s_i+p_i+t_i} y_{i,j} (a_i)_j, \tag{6}$$

for some $y_{i,j} \in \mathbb{Z}$. For this element, consider the norms

$$e(g) = \sum_{i=1}^{k} \sum_{j=s_i}^{s_i+p_i+t_i} |y_{i,j}| \text{ and } e_H(g) = \sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} |z_q|.$$

Letting $\iota = \max_i\{t_i + s_i + p_i, 0\}, \varepsilon = \min_i\{s_i, 0\}, \iota_H = \max_i\{s_i + p_i, 0\}$ we define $u_H(g) = \iota_H - \varepsilon$ and $u(g) = \iota - \varepsilon$.

Consider the function

$$h(l) = \max\{e_H(g) : g \in H \cap W, e(g) \leq l \text{ and } u(g) \leq l\}.$$

The following Lemma shows that we may reduce computations of word length to computations with coefficients of polynomials.

Lemma 7.5. *Let $H \leq \mathbb{Z}^r$ wr $\mathbb{Z}$ be of the form $H = gp\langle b, w_1, \dots, w_k \rangle$ where for each $i = 1, \dots, k$ we have that $w_i = \sum_{j=0}^{t_i} d_{i,j} (a_i)_j$, and $r \geq k$. Then we have that*

$$\Delta_H^{\mathbb{Z}^r \text{ wr } \mathbb{Z}}(l) \approx h(l).$$

Proof. Recall that by Lemma 3.3 as well as Lemma 7.4, we have the following formulas. For $g \in H$ with the notation established above, we have that: $|g|_H = e_H(g) + \min\{-2\varepsilon + \iota_H + |n - \iota_H|, 2\iota_H - \varepsilon + |n - \varepsilon|\}$ and $|g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}} = e(g) + \min\{-2\varepsilon + \iota + |n - \iota|, 2\iota - \varepsilon + |n - \varepsilon|\}$.

The following inequality follows from the definitions:

$$\max\{e(g), u(g), |n|\} \leq |g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}}. \tag{7}$$

Similarly, we have that

$$|g|_H \leq e_H(g) + 2u_H(g) + |n| \text{ and } |g|_{\mathbb{Z}^r \text{ wr } \mathbb{Z}} \leq e(g) + 2u(g) + |n|. \tag{8}$$

Observe that for $g \in H \cap W$ we have that

$$|g|_H \geq \max\{e_H(g), u_H(g)\}. \tag{9}$$

Observe that

$$\max\{u_H(g) : g \in H, u(g) \leq l\} \leq l. \tag{10}$$

Thus,

$$\Delta_H^{\mathbb{Z}^r \text{ wr } \mathbb{Z}}(l) \leq \max\{e_H(g) : g \in H, e(g) \leq l, u(g) \leq l\} + \max\{2u_H(g) : g \in H, u(g) \leq l\}$$
$$+ \max\{|n| : g \in H, |n| \leq l\} \leq h(l) + 3l.$$

The first inequality follows from Equation (7), the second from Equation (8).

On the other hand, we have that

$$\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \geq \max\{e_H(g) : g \in H \cap W, e(g) \leq l/4, u(g) \leq l/4\}$$
$$- \max\{u_H(g) : g \in H \cap W, e(g) \leq l/4, u(g) \leq l/4\} \geq h(l/4) - l/4.$$

The first inequality follows from Equation (8), the second from Equation (9), and the third from Equation (10).

Thus $\Delta_H^{\mathbb{Z}^r \text{ wr } \mathbb{Z}}(l)$ and $h(l)$ are equivalent. ∎

7.3 Some Modules

In order to later obtain upper bounds on distortion of some subgroups of $\mathbb{Z}^r$ wr $\mathbb{Z}$ we will need the following auxiliary remarks about module theory. As usual, $\mathbb{Z}^r$ wr $\mathbb{Z}$ has standard generating set $\{a_1, \ldots, a_r, b\}$.

Let $H \leq \mathbb{Z}^r$ wr $\mathbb{Z}$ be generated by b, as well as any elements $w_1, \ldots, w_k \in W$. Let V be the normal closure of $w_1, \ldots, w_k$ in $\mathbb{Z}^r$ wr $\mathbb{Z}$.

The following is proved in [FS].

Lemma 7.6. *The ring $\mathbb{Q}[\langle b \rangle]$ is a principal ideal ring.*

Let $\overline{V} = V \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\overline{W} = W \otimes_{\mathbb{Z}} \mathbb{Q}$. Observe that the groups $\overline{W}$ and $\overline{V}$ are free modules over $\mathbb{Q}[\langle b \rangle]$ of respective ranks r and $l \leq k$.

Lemma 7.7. *The free $\mathbb{Q}[\langle b\rangle]$-modules $\overline{V}$ and $\overline{W}$ have bases $e'_1,\dots,e'_l$ and $f'_1,\dots,f'_r$ respectively such that*

$$e'_i = u'_i f'_i, i = 1,\dots,l$$

for some $u'_i \in \mathbb{Q}[\langle b\rangle]$.

Proof. The statement of Lemma 7.7 is a result from module theory. It follows because by Lemma 7.6 $\overline{W}$ is a free module over a prinicipal ideal ring with submodule $\overline{V}$. See for instance, [B]. □

Remark 7.8. *It follows that there exist $0 < m, n \in \mathbb{Z}$ with $(me'_i) = u_i(nf'_i)$ where $e_i = me'_i \in V, f_i = nf'_i \in W, u_i \in \mathbb{Z}[\langle b\rangle]$. Moreover, the modules generated by $\{e_1,\dots,e_l\}$ and $\{f_1,\dots,f_r\}$ are free.*

Remark 7.9. *There is a bijective correspondence between the set of finitely generated $\mathbb{Z}[\langle b\rangle]$ submodules of $\mathbb{Z}[\langle b\rangle]^r$ and the set of subgroups $K \cap W$ of $\mathbb{Z}^r$ wr $\mathbb{Z}$ such that the finite set of generators of K is of the form $b, w_1,\dots,w_k$, $w_i \in W$.*

Remark 7.10. *Let V_1 and W_1 be generated as submodules over $\mathbb{Z}[\langle b\rangle]$ by the elements from Remark 7.8: $e_1,\dots,e_l$ and $f_1,\dots,f_r$ respectively. Let H_1 and G_1 be subgroups of $\mathbb{Z}^r$ wr $\mathbb{Z}$ generated by $\{b, V_1\}$ and $\{b, W_1\}$ respectively. It follows by Remark 7.8 that that $G_1 \cong \mathbb{Z}^r$ wr $\mathbb{Z}$ and $H_1 \cong \mathbb{Z}^l$ wr $\mathbb{Z}$.*

Remark 7.11. *Observe that under the correspondence of Remark 7.9 that each generator of the group H_1 is in the normal closure of only one generator of G_1. That is, for each i, $e_i = u_i f_i$ for $u_i \in \mathbb{Z}[\langle b\rangle]$ means that there exist expressions $e_i = \sum_{p=1}^{t_i} n_{i,p}(f_i)_{j_{i,p}}$.*

Lemma 7.12. *There exists $0 < n', m' \in \mathbb{N}$ so that $n'W \subset W_1 \subset W$, and $m'V \subset V_1 \subset V$.*

Proof. By Remark 7.9 we have that V is a finitely generated $\mathbb{Z}[\langle b\rangle]$ module with generators $w_1,\dots,w_k$. For each w_i, we have that the element $w_i \otimes 1 \in \overline{V}$. Therefore, by Lemma 7.7, there are $\lambda_{i,j} \in \mathbb{Q}[\langle b\rangle]$ so that $w_i = \sum_{j=1}^{l} \lambda_{i,j} e'_j$. First observe that $mw_i = \sum_{j=1}^{l} \lambda_{i,j} e_j$, because $e_i = me'_i \in V$.

Next, there exists $M_i \in \mathbb{N}$ so that $M_i m w_i = \sum_{j=1}^{l} \mu_{i,j} e_j \in V_1$ where $\mu_{i,j} \in \mathbb{Z}[\langle b\rangle]$. Let $m' = M_1 \dots M_k m$. Then for any $v \in V$, we have that $v = \sum_{i=1}^{k} v_i w_i$ where $v_i \in \mathbb{Z}[\langle b\rangle]$, and therefore, $m'v \in V_1$ as required. A similar argument works for obtaining n'. □

Lemma 7.13. *Let $\mathbb{Z}^r$ wr $\mathbb{Z} = G = W\lambda\langle b\rangle$ and let $K = \langle\langle w_1,\dots,w_k\rangle\rangle^G \leq G$ be the normal closure of elements $w_i \in W$. Suppose that there exists $n \in \mathbb{N}$ and a finitely generated subgroup $K' \leq K$ so that $nK \leq K'$. Then*

$$\Delta^G_{\langle b,K'\rangle}(l) \approx \Delta^G_{\langle b,K\rangle}(l).$$

Proof. We will use the notation that $K_1 = \mathrm{gp}\langle K, b\rangle, K_1' = \mathrm{gp}\langle K', b\rangle, K_1'' = \mathrm{gp}\langle nK, b\rangle$. Observe that the mapping $\phi : G \to G : b \to b, w \to nw$ for $w \in W$ is an injective homomorphism which restricts to an isomorphism $K_1 \to K_1''$. An easy computation which uses Lemma 3.3 and the definition of ϕ shows that for any $g \in K_1$, we have that

$$|g|_G \le |\phi(g)|_G \le n|g|_G \tag{11}$$

where the lengths are computed in G with respect to the usual generating set $\{a_1, \ldots, a_r, b\}$.

Observe that under the map ϕ we have that

$$\text{for } h \in K_1, |h|_{K_1} = |\phi(h)|_{K_1''}, \tag{12}$$

where the lengths in K_1'' are computed with respect to the images under ϕ of a fixed generating set of K_1.

By their definitions, we have the embeddings

$$K_1'' \le K_1' \le K_1 \overset{\phi}{\hookrightarrow} K_1''. \tag{13}$$

By Equation (13) there exists $k' > 0$ depending only on the chosen generating sets of K_1 and K_1' so that

$$\text{for any } h \in K_1', |h|_{K_1} \le k'|h|_{K_1'}. \tag{14}$$

It also follows by by Equation (13) that there exists a constant $k > 0$ depending only on the chosen generating sets of K_1'' and K_1' so that

$$\text{for any } h \in K_1'', |h|_{K_1'} \le k|h|_{K_1''}. \tag{15}$$

First we show that $\Delta^G_{K_1''}(l) \preceq \Delta^G_{K_1}(l)$.

Let $g \in K_1''$ be such that $|g|_G \le l$ and $|g|_{K_1''} = \Delta^G_{K_1''}(l)$. Then there exists $g' \in K_1$ such that $\phi(g') = g$. Therefore, it follows that $\Delta^G_{K_1''}(l) = |g|_{K_1''} = |\phi(g')|_{K_1''} = |g'|_{K_1} \le \Delta^G_{K_1}(l)$. The first and second equalities follow by definition, the third by Equation (12), and the inequality is true because by Equation (11) we have that $|g'|_G \le |\phi(g)|_G = |g|_G \le l$.

We claim that $\Delta^G_{K_1}(l) \preceq \Delta^G_{K_1'}(l)$.

Let $g \in K_1$ be such that $|g|_{K_1} = \Delta^G_{K_1}(l)$. Then $|g|_{K_1} \le |\phi(g)|_{K_1} \le k'|\phi(g)|_{K_1'} \le k'\Delta^G_{K_1'}(nl)$, which follows from Equations (14), (11) and by definition.

On the other hand, we will show that $\Delta^G_{K_1'}(l) \preceq \Delta^G_{K_1''}(l)$. Let $g \in K_1'$ be such that $|g|_{K_1'} = \Delta^G_{K_1'}(l)$. Then $|g|_{K_1'} \le |\phi(g)|_{K_1'} \le k|\phi(g)|_{K_1''} \le k\Delta^G_{K_1''}(nl)$, which follows from Equations (15), (11) and by definition.

Therefore, we have that $\Delta^G_{K_1}(l) \preceq \Delta^G_{K_1'}(l) \preceq \Delta^G_{K_1''}(l) \preceq \Delta^G_{K_1}(l)$. □

Remark 7.14. *Recall that the groups G_1 and H_1 were defined in Lemma 7.10. It follows from Lemmas 7.12 and 7.13 that the distortion functions*

$$\Delta^G_{G_1}(l) \approx \Delta^G_G(l) \approx l \text{ and } \Delta^G_{H_1}(l) \approx \Delta^G_H(l).$$

8 Upper Bounds on Distortion in $\mathbb{Z}^r$ wr $\mathbb{Z}$

Our goal in this section is to obtain upper bounds on distortion of certain subgroups of $\mathbb{Z}^r$ wr $\mathbb{Z}$ having a special form. We use that $\{a_1, \ldots, a_r, b\}$ is the standard generating set of $\mathbb{Z}^r$ wr $\mathbb{Z}$.

Lemma 8.1. *Let a subgroup H of $\mathbb{Z}^r$ wr $\mathbb{Z}$ have generators of the form $b, w_1, \ldots, w_k$ where each w_i is in the normal closure of a_i for $i = 1, \ldots, k$. Then the distortion of H in $\mathbb{Z}^r$ wr $\mathbb{Z}$ is at most polynomial.*

Proof. After conjugating by a power of b, we may assume without loss of generality that for each $i = 1, \ldots, k$ we have expressions as in Equation (4), where $d_{i,0}, d_{i,t_i} \neq 0$. Let us have an element $g \in H$. Then we may write g as in Equation (5) where for each $i = 1, \ldots, k$ we have that $s_i \in \mathbb{Z}$ and $p_i \geq 0$. In the generators of $\mathbb{Z}^r$ wr $\mathbb{Z}$ this expression becomes that of Equation (6) for some $y_{i,j} \in \mathbb{Z}$. By Lemma 7.5, and using the notation introduced there, it suffices to show that the function $h(l)$ is bounded above by a polynomial. That is, we may suppose that $n = 0, e(g) \leq l$ and $u(g) \leq l$. We need to show that $\sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} |z_q|$ is bounded from above by a polynomial in l.

For each $i = 1, \ldots, k$ the expressions (5) and (6) yield a linear system of equations $M_i Z_i = Y_i$, where $Z_i = [z_{s_i}, \ldots, z_{s_i+p_i}]^T, Y_i = [y_{i,s_i}, \ldots, y_{i,t_i+s_i+p_i}]^T$ and

$$M_i = \begin{pmatrix} d_{i,0} & 0 & 0 & \ldots & 0 \\ d_{i,1} & d_{i,0} & 0 & \ldots & 0 \\ d_{i,2} & d_{i,1} & d_{i,0} & \ldots & 0 \\ \vdots & \ldots & \ddots & \ddots & \vdots \\ d_{i,t_i} & d_{i,t_i-1} & \ldots & d_{i,1} \ldots & 0 \\ 0 & d_{i,t_i} & \ldots & d_{i,2} \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & d_{i,t_i} & d_{i,t_i-1} \\ 0 & \ldots & 0 & 0 & d_{i,t_i} \end{pmatrix}$$

is an $(p_i + t_i + 1) \times (p_i + 1)$ matrix.

By Lemma 7.2 we have that for each $i = 1, \ldots, k$ and for each $q = s_i, \ldots, s_i + p_i$ that $|z_q| \leq c y_i (p_i + 1)^{t_i}$ where $c = c(d_{i,j}), y_i = \max\{|y_{i,j}|\}_{j=s_i,\ldots,s_i+t_i+p_i}$. Moreover, we know that for each i, $y_i \leq \sum_{j=s_i}^{t_i+s_i+p_i} |y_{i,j}| \leq e(g) \leq l$. It is not hard to check as well that $p_i \leq u(g) \leq l$ for each i. Therefore, letting $t = \max_i\{t_i\}$ we have that

$$\sum_{i=1}^{k} \sum_{q=s_i}^{s_i+p_i} |z_q| \leq \sum_{i=1}^{k} (p_i + 1) c y_i (p_i + 1)^{t_i} \leq kc(l+1)^{t+2}.$$

This completes the proof, because k, c and t depend only on the choice of generating set of H.

∎

Remark 8.2. *As the proof of Proposition 7.4 shows, for a subgroup H in $\mathbb{Z}$ wr $\mathbb{Z}$ generated by elements b, w only, for $w = \sum_{j=0}^{t} d_j a_j$ where $d_0, d_t \neq 0 \in W$, the polynomial upper bound has degree equal to the number $t+2$ appearing in the canonical form of w.*

Applying Remark 8.2 to Proposition 6.3, we see that the subgroup $H = \langle b, a(1-x)^{m-1} \rangle$ has distortion function at most l^{m+1}, up to equivalence. In fact, we can prove the following stronger result.

Lemma 8.3. *Let $H = \langle b, w = a(1-x)^{m-1} \rangle \leq \mathbb{Z}$ wr $\mathbb{Z}$. Then the distortion of H in $\mathbb{Z}$ wr $\mathbb{Z}$ is at most l^m, up to equivalence.*

Proof. We apply Lemma 7.5. It suffices to show that the function $h(l) \preceq l^m$. Let $h \in H \cap W$. We have an expression $h = wf(x)$ where f is a Laurent polynomial

$$f(x) = \sum_{q=s}^{s+p} z_s x^s$$

for some $s \in \mathbb{Z}, p \geq 0$. Then in the generators of $\mathbb{Z}$ wr $\mathbb{Z}$ we have that

$$h = a(1-x)^{m-1} f(x) = ag(x), \text{ where } g(x) = \sum_{j=s}^{m-1+s+p} y_j x^j.$$

As in the proof of Lemma 8.1, we may suppose that $\sum_j |y_j| \leq l$ and $p \leq l$, and it suffices to prove that for each q, $|z_q| \leq (\sum_j |y_j|) l^{m-2}$. We have that in the ring of formal power series with integral coefficients,

$$f(x) = \frac{g(x)}{(1-x)^{m-1}} = g(x) \big[\sum_{i=0}^{\infty} x^i \big]^{m-1}.$$

It follows by an easy induction argument that if $\big[\sum_{i=0}^{\infty} x^i\big]^{m-1} = \sum_{i=0}^{\infty} c_i x^i$ then for $i = 0, \ldots, l$ we have that $|c_i| \leq (l+1)^{m-2}$. Now consider arbitrary coefficient z_{s+j} for $0 \leq j \leq p$. Then we have the formula $z_{s+j} = \sum_{i=s}^{s+j} y_i c_{s+j-i}$ and so

$$|z_{s+j}| \leq \sum_{i=s}^{s+j} |y_i| |c_{s+j-i}| \leq (l+1)^{m-2} \sum_{i=s}^{s+j} |y_i|$$

because $p \leq l$. Therefore the required bounds exist. $\square$

9 Distortion in $\mathbb{Z}^k$ wr $\mathbb{Z}$

We are now able to formulate and prove the following, which constitutes a large component of the proof of Theorem 1.1.

Lemma 9.1. *Let $0 < k \in \mathbb{Z}$ be fixed.*

(1) For every $m \in \mathbb{N}$, there is a 2-generated subgroup H of $\mathbb{Z}^k$ wr $\mathbb{Z}$ having distortion function

$$\Delta_H^{\mathbb{Z}^k \text{ wr } \mathbb{Z}}(l) \approx l^m.$$

(2) For any finitely generated subgroup $H \leq \mathbb{Z}^k$ wr $\mathbb{Z}$ there exists $m \in \mathbb{N}$ such that the distortion of H in $\mathbb{Z}^k$ wr $\mathbb{Z}$ is

$$\Delta_H^{\mathbb{Z}^k \text{ wr } \mathbb{Z}}(l) \preceq l^m.$$

First, we prove Part (1) of Lemma 9.1 for the group $\mathbb{Z}$ wr $\mathbb{Z}$. In this case where $k = 1$, part (1) of Lemma 9.1 follows in part from Proposition 6.3, which provides the polynomial lower bound of degree m on distortion. All that remains to be shown is that for the particular subgroup H constructed there, that $\Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l) \preceq l^m$, which follows from Lemma 8.3. Now, for $k > 1$ fixed, we may consider the subgroup H_1 obtained by intersecting H with $L_k = \text{gp}\langle b^k, W\rangle \cong \mathbb{Z}^k$ wr $\mathbb{Z}$. Then we have that H_1 is subnormal in L_k and that the distortion $\Delta_{H_1}^{\mathbb{Z}^k \text{ wr } \mathbb{Z}}(l) \approx \Delta_H^{\mathbb{Z} \text{ wr } \mathbb{Z}}(l)$, since the indices $[\mathbb{Z} \text{ wr } \mathbb{Z} : L_k]$ and $[H : H_1]$ are finite.

Remark 9.2. *If we adopt the notation that the commutator $[a, b] = a^{-1}b^{-1}ab$, then we see that in $\mathbb{Z}$ wr $\mathbb{Z}$, the element of W corresponding to the polynomial $a(1-x)^{m-1}$ is $[\cdots[a, b], b], \cdots, b]^{-1}$ where the commutator is $(m-1)$-fold. This explains Corollary 1.2.*

To prove part (2) of Lemma 9.1, we will set up some notation. Let H be any finitely generated subgroup of $\mathbb{Z}^k$ wr $\mathbb{Z}$ not contained in W. By Lemma 4.6, we may identify H with a subgroup of $G = \mathbb{Z}^r \text{ wr } \mathbb{Z} = W\lambda\langle b\rangle$ for some $r \geq 1$ such that the generators of H are of the form $b, w_1, \ldots, w_s$, where $w_i \in W$. The distortion of H in $\mathbb{Z}^r$ wr $\mathbb{Z}$ under this identification is equivalent to its distortion in $\mathbb{Z}^k$ wr $\mathbb{Z}$. By the results of Section 7.3, we obtain subgroups

$$H_1 \leq G_1 \leq G$$

as in Remark 7.10.

In particular, by Remark 7.14, the embedding $G_1 \leq G$ is undistorted, which together with Lemma 2.2 implies that $\Delta_{H_1}^{G_1}(l) \approx \Delta_{H_1}^{G}(l)$. By Remark 7.14, we also have that $\Delta_{H_1}^{G}(l) \approx \Delta_H^G(l)$. It follows that

$$\Delta_H^G(l) \approx \Delta_{H_1}^{G_1}(l).$$

By Remarks 7.10, 7.11 and Lemma 8.1 we have that $\Delta_{H_1}^{G_1}(l)$ is at most polynomial. That is to say, the subgroup H has at most polynomial distortion in $\mathbb{Z}^r$ wr $\mathbb{Z}$, and so part (2) of Lemma 9.1 is proved.

We now return to one of the motivating ideas of this paper, and complete the explanation of Remark 1.3.

Lemma 9.3. *The group $\mathbb{Z}$ wr $\mathbb{Z}$ is the smallest metabelian group which embeds to itself as a normal distorted subgroup in the following sense. For any metabelian group G, if there is an embedding $\phi : G \to G$ such that $\phi(G) \trianglelefteq G$ and $\phi(G)$ is a distorted subgroup in G, then there exists some subgroup H of G for which $H \cong \mathbb{Z}$ wr $\mathbb{Z}$.*

Proof. By Lemma 2.2, we have that the group $G/\phi(G)$ is infinite, else $\phi(G)$ would be undistorted. Being a finitely generated solvable group, G must have a subnormal factor isomorphic to $\mathbb{Z}$. Because $\phi(G) \cong G$, one may repeat this argument to obtain a subnormal series in G with arbitrarily many infinite cyclic factors. Therefore, the derived subgroup G' has infinite (rational) rank.

Since the group $B = G/G'$ is finitely presented, the action of B by conjugation makes G' a finitely generated left B module. Hence, $G' = \langle B \circ C\rangle$ for some finitely generated $C \leq G'$. Because it is a finitely generated abelian group, $B = \langle b_k\rangle \cdots \langle b_1\rangle$ is a product of cyclic groups. Therefore for some i we have a subgroup $A = \langle\langle b_{i-1}\rangle \cdots \langle b_1\rangle \circ C\rangle$ of finite rank in G' but $\langle\langle b_i\rangle \circ A\rangle$ has infinite rank. Then A has an element such that the $\langle b_i\rangle$ submodule generated by a has infinite rank, and so it is a free $\langle b_i\rangle$ module. It follows that a and b, where $b_i = bG'$, generate a subgroup of the form $\mathbb{Z}$ wr $\mathbb{Z}$. ☐

10 The Case of A wr $\mathbb{Z}$

In this section, we will prove Theorem 1.1 Part (2). First we recall some basic similarities and differences between the groups $\mathbb{Z}_n^k$ wr $\mathbb{Z}$ and $\mathbb{Z}^k$ wr $\mathbb{Z}$. Let $G = \mathbb{Z}_n^k$ wr $\mathbb{Z}$, for $n \geq 2, k \geq 1$. It follows from [DS] that there is a presentation $\mathbb{Z}_n^k$ wr $\mathbb{Z} = \langle a_1, \ldots, a_k, b | a_i^n, [a_i, a_j], [a_i, b^{-x} a_j b^x], x \geq 0, 1 \leq i, j \leq k\rangle$. Moreover, by [C] we have the length formula as in Lemma 3.3.

Remark 10.1. *By Lemma 4.1 and an analogue of Lemma 4.6 we have that for any finitely generated nonabelian subgroup H of G, it suffices to consider generators of the form $b, w_1, \ldots, w_s$ where $w_i \in W = \bigoplus_{\mathbb{Z}} \mathbb{Z}_n$ is a free module of rank k over the group ring $R = \mathbb{Z}_n[\langle b\rangle]$.*

Although the notion of equivalence has only been defined for functions from $\mathbb{N}$ to $\mathbb{N}$, we would like to define a notion of equivalence for functions on a finitely generated group. We say that two functions $f, g : G \to \mathbb{N}$ are equivalent if there exists $C > 0$ such that for any $h \in G$ we have

$$\frac{1}{C} f(h) - C \leq g(h) \leq C f(h) + C.$$

If there is a function $f : G \to \mathbb{N}$ such that $f \approx |\cdot|_G$, then for any subgroup H of G, $\Delta_H^G(l) \approx \max\{|h|_H : h \in H, f(h) \leq l\}$.

Lemma 10.2. *For any $g \in G$, the following function $f : G \to \mathbb{N}$ is equivalent to the length in G. We have that*

$$f(g) = |t| + \epsilon_M + \iota_N \approx |g|_G.$$

Proof. First let $g \in G$ have normal form as in the statement of Lemma 3.2. Then by Lemma 3.3 it follows that

$$|g|_G \leq (N+M)(n-1)+2(\iota_N+\varepsilon_M)+|t| \leq (\iota_N+1+\varepsilon_M)(n-1)+2(\iota_N+\varepsilon_M)+|t|$$

$$\leq (n+1)(\iota_N+\epsilon_M)+|t|+(n-1) \leq Cf(g)+C,$$

where $C = n+1$. The computations follow from the definitions, as well as the fact that $\varepsilon_M \geq M, \iota_N \geq N-1$ and the length in $\mathbb{Z}_n^k$ of each u_i, v_j is bounded from above by $n-1$. On the other hand, observe that $|g|_G \geq \max\{|t|, \iota_N+\varepsilon_M\}$. Therefore, $2|g|_G \geq f(g)$, so the two functions are equivalent. $\square$

We are now able to prove the following special case of Theorem 1.1 Part (2).

Lemma 10.3. *If p is a prime, then any finitely generated subgroup of $G = \mathbb{Z}_p^k$ wr $\mathbb{Z}$ is undistorted.*

Proof. If p is a prime, then $\mathbb{Z}_p$ is a field. This implies that the ring $R = \mathbb{Z}_p[\langle b\rangle]$ is a principal ideal ring, by [FS]. Consider a finitely generated subgroup H of G. Let $V = H \cap W$. Then V is a free R-module, being a finitely generated submodule of the free module W over the PIR R. Just as in Lemma 7.7, we have that V and W have bases $e_1, \ldots, e_l$ and $f_1, \ldots, f_k$ respectively, for $l \leq k$ such that

$$e_i = g_i f_i, i = 1, \ldots, l \tag{16}$$

for some $g_i \in R$. Thus we can choose the generators for G and H to be $\{b, f_1, \ldots, f_k\}$ and $\{b, e_1, \ldots, e_l\}$, respectively.

Without loss of generality, the g_i are regular (not Laurent) polynomials. Observe that $H \cong \mathbb{Z}_p^l$ wr $\mathbb{Z}$. Let $h \in H$ have normal form in the generators of H given by

$$h = b^t(u_1)_{\iota_1} \cdots (u_N)_{\iota_N}(v_1)_{-\varepsilon_1} \cdots (v_M)_{-\varepsilon_M}.$$

Then by Lemma 10.2, $|h|_H \leq (p+1)(|t|+\iota_N+\varepsilon_M)+(p+1)$. We wish to compare this to the length of h in G, and by Lemma 10.2, it suffices to compare $|h|_H$ to $f(h)$. By Equation (18), we may obtain an expression for the normal form of h in the generators of G. For instance, $(u_N)_{\iota_N} = (e_1^{\alpha_1} \cdots e_l^{\alpha_l})_{\iota_N}$, where at least one of $\alpha_1, \ldots, \alpha_l$ is nonzero modulo p. Introducing the notation that $g_i = \sum_{j=0}^{k_i} \beta_j x^j$ we have that $(u_N)_{\iota_N} = ((f_1)_0^{\beta_1\alpha_1} \cdots (f_1)_{k_1}^{\beta_k\alpha_1} \cdots (f_l)_0^{\beta_1\alpha_l} \cdots (f_l)_{k_l}^{\beta_k\alpha_l})_{\iota_N}$, where at least one term $(f_i)_0^{\beta_1\alpha_i} \cdots (f_i)_{k_i}^{\beta_k\alpha_i}$ is nontrivial. Therefore, the largest subscript occurring in the normal form in the generators of G is of the form $\iota_N + j$, where $j \in \bigcup_{i=1}^{l}\{0, \ldots, k_i\}$. Using similar considerations on $(v_M)_{-\varepsilon_M}$, we see that the smallest negative subscript is of the form $\varepsilon_M - q$ for $q \in \bigcup_{i=1}^{l}\{0, \ldots, k_i\}$. Letting $k = \max_i\{k_i\}$, which is a constant that depends only on the choice of finite generating sets of G and H, we have that

$f(h) = |t| + \iota_N + \varepsilon_M + j - q \geq |t| + \iota_N + \varepsilon_M - k \geq (\frac{1}{p+1})[|h|_H - (p+1)] - k$. This implies that $|h|_H \leq cf(h) + c$ for $c = (p+1)(k+1)$, which implies that H is undistorted. $\square$

We are now prepared to prove Theorem 1.1 Part (1).

Proof. Let A be a finitely generated abelian group and consider $G = A \text{ wr } \mathbb{Z} = A \text{ wr } \langle b \rangle$. There exists a series

$$A = A_0 > A_1 > \cdots > A_m \cong \mathbb{Z}^k$$

for $k \geq 0$ where A_{i-1}/A_i has prime order for $i = 1, \dots, m$. We claim that for any finitely generated subgroup $H \leq G$, there exists $n \geq 0$ such that $\Delta_H^G(l) \preceq l^n$.

We induct on m. If $m = 0$, then $A \cong \mathbb{Z}^k$ and the claim holds by Theorem 1.1. Now let $m > 0$. Observe that A_1 is a finitely generated abelian group with a series $A_1 > \cdots > A_m \cong \mathbb{Z}^k$ of length $m - 1$. Therefore, by induction, any finitely generated subgroup in $A_1 \text{ wr } \mathbb{Z}$ has distortion at most equivalent to a polynomial.

By Lemma 10.3, all finitely generated subgroups of $G_1 = (A/A_1) \text{ wr } \mathbb{Z}$ are undistorted. By induction, all finitely generated subgroups of $G_2 = A_1 \text{ wr } \mathbb{Z}$ have at most polynomial distortion. Denote the natural homomorphism by $\phi : G \to G_1$. Let

$$U = \bigoplus_{\langle b \rangle} A_1 = \ker(\phi).$$

Observe that $U \cdot \langle b \rangle \cong G_2$. The product is semidirect because U is a normal subgroup which meets $\langle b \rangle$ trivially, and it is isomorphic to the wreath product by definition: the action of b on the module $\bigoplus_{\langle b \rangle} A_1$ is the same. Let H be a finitely generated subgroup in G. Suppose that H is not contained in W. It follows in this case by Remark 10.1 that we may assume that b is contained among the generators.

Let $R = \mathbb{Z}[\langle b \rangle]$. Observe that R is a Noetherian ring. This follows from basic algebra because $\mathbb{Z}$ is a commutative Noetherian ring, so $\mathbb{Z}[[x]] \cong R$ is as well. Therefore, W is a finitely generated module over the Noetherian ring R, hence is Noetherian itself. Thus, the R-submodule $H \cap U$ is finitely generated. Let $\{w_1', \dots, w_r'\}$ generate $H \cap U$ as a R-module. Let $\{b, w_1, \dots, w_s\}$ be a set of generators of H modulo U; that is, the images of these elements generate the subgroup $H_1 = HU/U \cong H/H \cap U$ of G_1. Then the set $\{b, w_1, \dots, w_s, w_1', \dots, w_r'\}$ generates H. Furthermore, the collection $\{b, w_1', \dots, w_r'\}$ generates the subgroup $H_2 = \langle b \rangle \cdot (H \cap U)$ of G_2.

Let $g \in H$ have $|g|_G \leq l$. Then the image $g_1 = \phi(g)$ in G_1 belongs to H_1, because $g \in H$, and has length $|g|_{G_1} \leq l$ by Lemma 10.2 and definition of ϕ and G_1. It follows by Lemma 10.3 that H_1 is undistorted in G_1. Therefore, there exists a linear function $f : \mathbb{N} \to \mathbb{N}$ (which does not depend on the choice of g) such that $|g_1|_{H_1} \leq f(l)$. That is to say, there exists a product P of at most $f(l)$

of the chosen generators $\{b, w_1, \dots, w_s\}$ of H_1 such that $P = g_1^{-1}$ in H_1. Taking preimages, we obtain that $gP \in U$.

Because H is a subgroup of G, there exists a constant c depending only on the choice of finite generating set of H such that for any $h \in H$ we have that

$$|h|_G \leq c|h|_H. \tag{17}$$

It follows by Equation (17) that

$$|gP|_G \leq |g|_G + |P|_G \leq |g|_G + c|P|_H \leq l + cf(l). \tag{18}$$

Observe that $gP \in H_2$. This follows because $gP \in U$ by construction, and $g \in H$ by choice. Further, $P \in H$ because it is a product of some of the generators of H. Since $H_2 = \langle b \rangle \cdot (H \cap U)$ we see that $gP \in H_2$. Using the fact that G and G_2 are wreath products together with the length formula in Lemma 3.3, we have that for any $h \in G_2$,

$$|h|_{G_2} \leq |h|_G. \tag{19}$$

By induction, the finitely generated subgroup H_2 of G_2 has at most polynomial distortion. Therefore, there exists a function $F : \mathbb{N} \to \mathbb{N}$ such that $F(l) \approx l^n$ for some $n \geq 1$ and such that for any $h \in H_2$,

$$|h|_{H_2} \leq F(|h|_{G_2}). \tag{20}$$

Since $gP \in H_2$, we have that

$$|gP|_{H_2} \leq F(|gP|_{G_2}) \leq F(|gP|_G) \leq F(l + cf(l)).$$

The first inequality follows from Equation (20), the second from Equation (19), and the last from Equation (18).

Because $H_2 \leq H$ there is a constant k such that for any $h \in H_2, |h|_H \leq k|h|_{H_2}$.

Combining all previous estimates, we compute that

$$|g|_H \leq |gP|_H + |P|_H \leq k|gP|_{H_2} + f(l) \leq kF(l + cf(l)) + f(l).$$

The right-hand side is bounded by a polynomial function since f is linear, and F is polynomial.

If the subgroup H had been abelian, it follows by induction that it is undistorted, because the finitely generated group $H \cap U$ is also abelian, and so its distortion in G_2 is linear. $\square$

Remark 10.4. *It follows by the same induction argument above that all finitely generated subgroups in A wr $\mathbb{Z}$ where A is finite abelian are undistorted. For in this case, $k = 0$ and so $F(l)$ is linear. Therefore, Theorem 1.1 Part* (2) *is also proved.*

Now we complete the proof of Theorem 1.1, Part (3). Let A be a finitely generated abelian group of rank k. Consider the 2-generated subgroup $H \leq \mathbb{Z}^k$ wr $\mathbb{Z}$ constructed in Lemma 9.1 Part (2). By the above induction argument, we have that the distortion of the subgroup H in A wr $\mathbb{Z}$ is at most equivalent to its distortion in $\mathbb{Z}^k$ wr $\mathbb{Z}$. The required lower bound on distortion follows from the fact that $\mathbb{Z}^k$ wr $\mathbb{Z}$ is a subgroup of A wr $\mathbb{Z}$.

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