LOSS OF DERIVATIVES IN THE INFINITE TYPE

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Abstract. We discuss loss of derivatives for degenerate vector fields obtained from infinite type exponentially non-degenerate hypersurfaces of $\mathbb{C}^2$.

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1. Introduction

A system of vector fields $\{L_j\}$ has subelliptic estimates when it has a gain of $\delta > 0$ derivatives in the sense that $\|\Lambda^\delta u\|^2 \underset{\sim}{<} \sum_j \|L_j u\|^2 + \|u\|^2$, $u \in C_c^\infty$. Here Λ is the standard elliptic pseudodifferential operator of order 1. A system which has finite bracket type $2m$ is a system whose commutators of order $2m-1$ span the whole tangent space. It is classical that finite type $2m$ implies δ-subelliptic estimates for $\delta = \frac{1}{2m}$. If Span$\{L, \bar{L}\}$, in $\mathbb{C} \times \mathbb{R}$, is identified to the tangential bundle $T^{1,0}M \oplus T^{0,1}M$ to a pseudoconvex hypersurface $M \subset \mathbb{C}^2$, then $\{L, \bar{L}\}$ has finite type $2m$ if and only if the contact of a complex curve γ with M is at most $2m$. If the hypersurface is "rigid", that is, graphed by $\mathsf{Re}\, w = g(z)$ for a real C^∞ function g, then with the notation $g_1 = \partial_z g$, $g_{1\bar{1}} = \partial_z \partial_{\bar{z}} g$ and $t = \mathsf{Im}\, w$, we have $L = \partial_z - i g_1(z)\partial_t$ and $[L, \bar{L}] = g_{1\bar{1}}\partial_t$. It is assumed that M is pseudoconvex, that is, $g_{1\bar{1}} \geq 0$ (this also motivates why the type is $2m$, even). In terms of g, the condition of finite type $2m$ reads

$$g_{1\bar{1}} = 0^{2(m-1)} \quad \text{but} \quad g_{1\bar{1}} \neq 0^{2m-1}. \tag{1.1}$$

In particular, if $g_{1\bar{1}} \underset{\sim}{>} |x|^{2(m-1)}$, then we have $\frac{1}{2m}$-subelliptic estimates.

A system has a superlogarithmic estimate if it has logarithmic gain of derivative with an arbitrarily large constant, that is, for any δ and for suitable c_δ

$$\|\log(\Lambda)u\|^2 \underset{\sim}{<} \delta \sum_j \|L_j u\|^2 + c_\delta \|u\|^2, \qquad u \in C_c^\infty. \tag{1.2}$$

A system which satisfies (1.2) is "precisely H^s-hypoelliptic" for any s: u is H^s exactly where the $L_j u$'s are (Kohn [7]). In particular, the system is C^∞-hypoelliptic. Let $L = \partial_z - i g_1(z)\partial_t$ for g of infinite type but exponentially non-degenerate in the sense that

$$|z|^\alpha |\log g_{1\bar{1}}| \searrow 0 \text{ as } |z| \searrow 0 \text{ for } \alpha \leq 1. \tag{1.3}$$

Under this assumption, $\{L, \bar{L}\}$ enjoys superlogarithmic estimates (cf. e.g. [12]). If we consider the perturbed system $\{\bar{L}, \bar{z}^k L\}$ (any fixed $k \geq 1$), the system has no more superlogarithmic estimates, in general; if $k > 1$, a logarithmic loss occurs (Proposition 1.4 below). However, notice that $\mathcal{L}ie\{\bar{L}, \bar{z}^k L\}$, the span of commutators of order $\leq k-1$, has superlogarithmic estimates (since it gains L). We are able to prove here that $\{\bar{L}, \bar{z}^k L\}$ has, in the terminology of Kohn [8], loss of $\frac{1}{2}$ derivative and thus, in particular, is C^∞-, but not exactly H^s-, hypoelliptic. Let ζ_0 and ζ_1 be cut-off functions in a neighborhood of 0 with $\zeta_0 \prec \zeta_1$ in the sense that $\zeta_1|_{\text{supp}\zeta_0} \equiv 1$.

Theorem 1.1. *Let $L = \partial_z - ig_1(z)\partial_t$ and assume that 0 be a point of infinite type, that is, $g_{1\bar{1}} = 0^\infty$ but not exponentially degenerate, that is,* (1.3) *be fulfilled. Then the system $\{\bar{L}, \bar{z}^k L\}$ (any k) has loss of $\frac{1}{2}$ derivatives, that is,*

$$\|\zeta_0 u\|_s^2 \lesssim \|\zeta_1 \bar{L} u\|_{s+\frac{1}{2}}^2 + \|\zeta_1 \bar{z}^k L u\|_{s+\frac{1}{2}}^2 + \|\bar{z}^k u\|_{\frac{1}{2}}^2. \tag{1.4}$$

The proof of this, and the two theorems below, follows in Section 2.

Remark 1.2. (1.4) implies local hypoellipticity. Reason is that the loss of derivative takes place only in t (whereas there is elliptic gain in z), combined with the fact that L and $\bar{L}$ have coefficients which are constant in t. Thus, if we make a partial regularization $u_\nu \to u$, $u_\nu \in C^\infty$ with respect to t, use the relation $\bar{L}u_\nu = (\bar{L}u)_\nu$ (and the same for L), and apply (1.4) to the u_ν's, we get the proof of the claim.

For $k = 1$ we have estimate for local regularity without loss

Theorem 1.3. *In the situation above, assume in addition*

$$|g_1| \lesssim g_{1\bar{1}}^{\frac{1}{2}}; \tag{1.5}$$

then

$$\|\zeta_0 u\|_s^2 \lesssim \|\zeta_1 \bar{L} u\|_s^2 + \|\zeta_1 \bar{z} L u\|_s^2 + \|\bar{z} u\|_0^2. \tag{1.6}$$

When $k > 1$, loss may occur

Proposition 1.4. *Assume that $g = e^{-\frac{1}{|z|^\alpha}}$. If*

$$\|\zeta_0 u\|_s^2 \lesssim \|(\log \Lambda)^r \zeta_1 \bar{L} u\|_s^2 + \|(\log \Lambda)^r \zeta_1 \bar{z}^k L u\|_s^2 + \|\bar{z}^k u\|_{\frac{1}{2}}^2, \tag{1.7}$$

then we must have $r \gtrsim \frac{k-(\alpha+1)}{\alpha}$.

This seems to be the first time that degenerate vector fields $\{\bar{L}, \bar{z}^k L\}$ obtained from $L = \partial_z - ig_1(z)\partial_t$ of infinite type, that satisfying $g_{1\bar{1}} = 0^\infty$, is considered. However, some additional hypothesis such as (1.3), must be required. This guarantees superlogarithmic estimates ([12]), and in turn, hypoellipticity according to Kohn [7]. Loss of derivatives for $L = \partial_z - i\bar{z}\partial_t$ was discovered by Kohn in [8]. In this case, L is the $(1,0)$ vector field tangential to the strictly pseudoconvex hypersurface $\text{Re}\, w = |z|^2$ and the loss amounts in $\frac{k-1}{2}$. The problem was further discussed by Bove, Derridj, Kohn and Tartakoff in

[1] essentially for the vector field $L=\partial_z-i\bar z|z|^{2(m-1)}\partial_t$ tangential to the hypersurface $\mathsf{Re}\,w=|z|^{2m}$ and the corresponding loss is $\frac{k-1}{2m}$. In both cases the result extends to the sum of squares $L\bar L+\bar L|z|^{2k}L$ and the loss doubles to $\frac{k-1}{m}$. For vector fields $L=\partial_z-ig_1(z)\partial_t$ tangential to general pseudoconvex hypersurfaces of finite type (with $g_{1\bar 1}$ vanishing at order $2(m-1)$), loss of $\frac{k-1}{2m}$ derivatives has been proved by the authors in [11]. Under some additional conditions, the result also extends to sums of squares (with doubled loss).

2. Technical preliminaries and Proof

Our ambient is $\mathbb{C}\times\mathbb{R}$ identified to $\mathbb{R}^3$ with coordinates $(z,\bar z,t)$ or $(\mathsf{Re}\,z,\mathsf{Im}\,z,t)$. We denote by $\xi=(\xi_z,\xi_{\bar z},\xi_t)$ the variables dual to $(z,\bar z,t)$, by Λ^s_ξ the standard symbol $(1+|\xi|^2)^{\frac{s}{2}}$, and by Λ^s the pseudodifferential operator with symbol Λ^s_ξ; this is defined by $\Lambda^s(u)=\mathcal{F}^{-1}(\Lambda^s_\xi\mathcal{F}(u))$ where $\mathcal{F}$ is the Fourier transform. We also consider the partial symbol $\Lambda^s_{\xi_t}$ and the associate pseudodifferential operator Λ^s_t. We denote by $\|u\|_s:=\|\Lambda^s u\|_0$ (resp. $\|u\|_{\mathbb{R},s}:=\|\Lambda^s_t u\|_0$) the full (resp. totally real) s-Sobolev norm. We use the notation $\underset{\sim}{>}$ and $\underset{\sim}{<}$ to denote inequalities up to multiplicative constants; we denote by $\sim$ the combination of $\underset{\sim}{>}$ and $\underset{\sim}{<}$. In $\mathbb{R}^3_\xi$, we consider a conical partition of the unity $1=\psi^++\psi^++\psi^0$ where $\psi^\pm$ have support in a neighborhood of the axes $\pm\xi_t$ and ψ^0 in a neighborhood of the plane $\xi_t=0$, and introduce a decomposition of the identity $\mathrm{id}=\Psi^++\Psi^-+\Psi^0$ by means of $\Psi^{\overset{\pm}{0}}$, the pseudodifferential operators with symbols $\psi^{\overset{\pm}{0}}$; we accordingly write $u=u^++u^-+u^0$. Since $|\xi_z|+|\xi_{\bar z}|\underset{\sim}{<}\xi_t$ over $\operatorname{supp}\psi^+$, then $\|u^+\|_{\mathbb{R},s}\sim\|u^+\|_s$.

We carry on the discussion by describing the properties of commutation of the vector fields L and $\bar L$ for $L=\partial_z-ig_1(z)\partial_t$. The crucial equality is

$$\|Lu\|^2=([L,\bar L]u,u)+\|\bar Lu\|^2,\quad u\in C^\infty_c,\tag{2.1}$$

which is readily verified by integration by parts. Note here that errors coming from derivatives of coefficients do not occur since g_1 does not depend on t. Since $\sigma(\partial_t)$, the symbol of ∂_t, is dominated by $\sigma(L)$ and $\sigma(\bar L)$ in the "elliptic region" (the support of ψ^0) and since L can be controlled by $\bar L$ with an additional $\epsilon\partial_t$ (because of (2.1)), then $\|u^0\|^2_1\underset{\sim}{<}\|\bar Lu^0\|^2_0+\|u\|^2_0$. As for u^-, recall that $[L,\bar L]=g_{1\bar 1}\partial_t$ and hence $g_{1\bar 1}\sigma(\partial_t)\le 0$ over $\mathrm{supp}\psi^-$. Thus (2.1) yields $\|Lu\|^2\underset{\sim}{<}\|\bar Lu\|^2$. It follows that, if L and $\bar L$ have superlogarithmic estimates as in our application, then

$$\|\log(\Lambda)u^-\|^2\le\delta\|\bar Lu^-\|^2+c_\delta\|u\|^2.$$

In conclusion, only estimating u^+ is relevant. We note here that, over $\mathrm{supp}\,\Psi^+$, we have $g_{1\bar{1}}\xi_t \geq 0$; thus

$$\begin{aligned}\|g_{1\bar{1}}u^+\|^2_{\frac{1}{2}} &= |([L,\bar{L}]u^+,u^+)| \\ &\leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2.\end{aligned} \tag{2.2}$$

Following Kohn [7], we introduce a microlocal modification of Λ^s, denoted by R^s; this is the pseudodifferential operator with symbol $R^s_\xi := (1+|\xi|^2)^{\frac{s\sigma(x)}{2}}$, $\sigma \in C^\infty_c$; often, what is used is in fact the partial operator in t, R^s_t with symbol $R^s_{\xi_t}$. The relevant property of R^s is

$$\|\Lambda^s\zeta_0 u\|^2 \lesssim \|R^s\zeta_0 u\|^2 + \|\zeta_0 u\|^2 \qquad \text{if } \zeta_0 \prec \sigma.$$

Thus, R^s is equivalent to Λ^s over functions supported in the region where $\sigma \equiv 1$. In addition, ζR^s better behaves with respect to commutation with L; in fact, Jacobi equality yields

$$[\zeta R^s, L] \sim \dot{\zeta}R^s + \zeta\log(\Lambda)R^s. \tag{2.3}$$

Thus, on one hand we have the disadvantage of the additional $\log(\Lambda)$ in the second term, but we gain much in the cut-off because

$$\dot{\zeta}R^s \text{ is of order 0 if } \mathrm{supp}\,\dot{\zeta} \cap \mathrm{supp}\,\sigma = \emptyset. \tag{2.4}$$

Property (2.4) is crucial in localizing regularity in presence of superlogarithmic estimates.

Proof of Theorem 1.1. As it has already been noticed, it suffices to prove (1.4) only for u^+ and for $\|\cdot\|_{\mathbb{R},\,s}$; thus we write for simplicity u and $\|\cdot\|_s$ but mean u^+ and $\|\cdot\|_{\mathbb{R},\,s}$. Moreover, we can use a cut-off $\zeta = \zeta(t)$ in t only. In fact, for a cut-off $\zeta = \zeta(z)$ we have $[L,\zeta(z)] = \dot{\zeta}$ and $\dot{\zeta} \equiv 0$ at $z = 0$. On the other hand, $z^k L \sim L$ outside $z = 0$ which yields gain of derivatives, instead of loss. In an estimate we call "good" a term in the right side (upper bound) and "absorbable" a term which comes as a fraction (small constant or sc) of a formerly encountered term. We take cut-off functions in a neighborhood of 0: $\zeta_0 \prec \sigma \prec \zeta_1 \prec \zeta'$; we have for $u \in C^\infty$

$$\begin{aligned}\|\zeta_0 u\|^2_s &= \|\zeta_0\zeta_1 u\|^2_s \\ &\leq \|R^s\zeta_1 u\|^2_0 + \|[R^s,\zeta_0]\zeta_1 u\|^2_0 + c\|u\|^2_0 \\ &\lesssim \|R^s\zeta_1 u\|^2_0 + \|u\|^2_0 \\ &\lesssim \|\zeta' R^s\zeta_1 u\|^2_0 + \|u\|^2_0,\end{aligned} \tag{2.5}$$

where the inequality in the third line follows from interpolation in Sobolev spaces and the last from $\text{supp}(1-\zeta')\cap\text{supp}\sigma=\emptyset$. We have

$$\begin{aligned}\|\zeta_0 u\|_s^2 &\underset{\text{by (2.5)}}{\lesssim} \underbrace{\|R^s\zeta_1 u\|^2}_{(a)}+\|u\|^2\\ &\underset{\text{trivial}}{\lesssim} \underbrace{\|\log(\Lambda)R^s\zeta_1 u\|^2}_{(b)}+\|u\|^2\\ &\lesssim \|\log(\Lambda)(\zeta' R^s\zeta_1)u\|^2+\|u\|^2\\ &\leq \delta\left(\|L(\zeta' R^s\zeta_1)u\|^2+\|\bar{L}(\zeta' R^s\zeta_1)u\|^2\right)+c_\delta\|u\|^2.\end{aligned}\tag{2.6}$$

Here, the inequality in the third line is analogous to the last in (2.5) in addition to the fact that $[\zeta',\log(\Lambda)]R^s=0(\Lambda^{-1})$; the inequality in the fourth line follows from superlogarithmic estimate. Using integration by parts, we estimate the first term in the last line

$$\begin{aligned}\|L(\zeta' R^s\zeta_1)u\|^2 &\lesssim \|\bar{L}(\zeta' R^s\zeta_1)u\|^2+\left|([L,\bar{L}](\zeta' R^s\zeta_1)u,(\zeta' R^s\zeta_1)u)\right|\\ &\lesssim \|\bar{L}(\zeta' R^s\zeta_1)u\|^2+lc\|[L,\bar{L}](\zeta' R^s\zeta_1)u\|^2+\underbrace{sc\|R^s\zeta_1 u\|^2}_{\text{absorbed by (a)}}.\end{aligned}\tag{2.7}$$

Observe that

$$\begin{aligned}\sigma([L,\bar{L}])=g_{1\bar{1}}\Lambda^1_\xi &\lesssim (g_{1\bar{1}}^{\frac12}\Lambda_\xi^{\frac12})|z|^k\Lambda_\xi^{\frac12}\\ &=\sigma([L,\bar{L}]^{\frac12})|z|^k\Lambda_\xi^{\frac12}.\end{aligned}\tag{2.8}$$

It follows

$$\begin{aligned}\|[L,\bar{L}](\zeta' R^s\zeta_1)u\|^2 &\leq \|[L,\bar{L}]^{\frac12}\Lambda^{\frac12}(\zeta' R^s\zeta_1)z^k u\|^2\\ &\leq \|Lz^k(\zeta' R^s\zeta_1)u\|^2_{\frac12}+\|z^k\bar{L}(\zeta' R^s\zeta_1)u\|^2_{\frac12}.\end{aligned}\tag{2.9}$$

We wish to first discard the second term in the second line of (2.9). For this, we recall Jacobi identity and get

$$\begin{aligned}[\bar{L},\zeta' R^s\zeta_1] &= [\bar{L},\zeta']R^s\zeta_1+\zeta'[\bar{L},R^s]\zeta_1+\zeta' R^s[\bar{L},\zeta_1]\\ &\sim \underbrace{\dot{\zeta}' R^s\zeta_1}_{\text{0-order by (2.4)}}+\underbrace{\zeta'\log(\Lambda)R^s\zeta_1}_{\text{by (2.3)}}+\underbrace{\zeta' R^s\dot{\zeta}_1}_{\text{0-order by (2.4)}}.\end{aligned}\tag{2.10}$$

Thus we can commutate $z^k\bar{L}$ with $\zeta' R^s\zeta_1$ in (2.9) up to an error as described in (2.10) which yiels

$$\|z^k\bar{L}(\zeta' R^s\zeta_1)u\|^2_{\frac12}\lesssim \|(\zeta' R^s\zeta_1)z^k\bar{L}u\|^2_{\frac12}+\|(\zeta'\log(\Lambda)R^s\zeta_1)z^k u\|^2_{\frac12}+\|z^k u\|^2_0.$$

On the other hand, since $[\zeta', \log(\Lambda)]R^s = 0(\Lambda^{-1})$, then

$$\begin{aligned}\|(\zeta' \log(\Lambda) R^s\zeta_1)z^k u\|^2_{\frac12} &\lesssim \|(\log(\Lambda)(\zeta' R^s\zeta_1)z^k u\|^2_{\frac12} + \|\zeta_1 z^k u\|^2_{-\frac12}\\ &\underset{\text{suplog estimate}}{\lesssim} \underbrace{\delta\left(\|L(\zeta' R^s\zeta_1)z^k u\|^2_{\frac12} + \|\bar L(\zeta' R^s\zeta_1)z^k u\|^2_{\frac12}\right)}_{\text{absorbed by } 2^{\text{nd}} \text{ line of (2.9)}} + \|\zeta_1 z^k u\|^2_{-\frac12},\end{aligned}$$

where we are using the equality $[\Lambda_t^{\frac12}, L] = 0$ as well as $[\Lambda^{\frac12}, \log(\Lambda)] = 0$. In the same way, using again (2.10), we commutate $\bar L$ with $(\zeta' R^s\zeta_1)$ in (2.6) and (2.7). What is left, is to estimate the first term in the last line of (2.9). First, from Jacobi identity we get

$$[Lz^k, \zeta' R^s\zeta_1] \sim \text{(0-order)} + z^k\zeta' \log(\Lambda)R^s\zeta_1 + \text{(0-order)},$$

so that we are eventually reduced to estimate $\|(\zeta' R^s\zeta_1)Lz^k u\|^2$. This is the most difficult operation. We have (by the trivial identity $[L, z^k] = z^{k-1}$)

$$\|(\zeta' R^s\zeta_1)Lz^k u\|^2_{\frac12} = \underbrace{\|(\zeta' R^s\zeta_1)z^k Lu\|^2_{\frac12}}_{\text{good}} + \|(\zeta' R^s\zeta_1)z^{k-1}u\|^2_{\frac12}.$$

Next,

$$\begin{aligned}\underbrace{\|(\zeta' R^s\zeta_1)z^{k-1}u\|^2_{\frac12}}_{(c)} &= (\underbrace{(\zeta' R^s\zeta_1)z^{k-1}u}_{*}, (\zeta' R^s\zeta_1)[L, z^k]u)_{\frac12}\\ &= -(*, (\zeta' R^s\zeta_1)z^k Lu)_{\frac12} + (*, (\zeta' R^s\zeta_1)Lz^k u)_{\frac12}.\end{aligned}$$

Now,

$$\begin{cases}\left|(*, (\zeta' R^s\zeta_1)z^k Lu)_{\frac12}\right| \le sc\|*\|^2_{\frac12} + \underbrace{\|(\zeta' R^s\zeta_1)z^k Lu\|^2_{\frac12}}_{\text{good}}\\ \left|(*, (\zeta' R^s\zeta_1)Lz^k u)_{\frac12}\right| \le \left|((\zeta' R^s\zeta_1)\bar L z^{k-1}u, (\zeta' R^s\zeta_1)z^k u)_{\frac12}\right|\\ \qquad + 2\left|(\underbrace{*}_{\text{absorbed by (c)}}, \underbrace{[L, (\zeta' R^s\zeta_1)]z^k u}_{(d)})_{\frac12}\right|.\end{cases}$$

We estimate (d). We notice that

$$[L, (\zeta' R^s\zeta_1)] \sim \zeta' \log(\Lambda)R^s\zeta_1 + \text{(0-order)}. \tag{2.11}$$

We also remark that

$$\begin{cases}[\Lambda^{\frac12}\zeta', \log(\Lambda)]R^s = 0(\Lambda^{-\frac12}) \quad (i)\\ [\zeta', \Lambda^{\frac12}]R^s \sim 0(\Lambda^{-\frac12}) \quad (ii)\\ [L, \Lambda^{\frac12}] = 0 \quad (iii).\end{cases} \tag{2.12}$$

Hence

$$\|(d)\|^2_{\frac12} \underset{\text{by (2.11)}}{\lesssim} \|(\zeta' \log(\Lambda) R^s \zeta_1) z^k u\|^2_{\frac12} + \|z^k u\|^2_{\frac12}$$

$$\text{(2.13)} \qquad \underset{\text{by (2.12) (i) and (ii)}}{\leq} \left\|(\log(\Lambda)\zeta' \Lambda^{\frac12} R^s \zeta_1) z^k u\right\|_0^2 + \|z^k u\|^2_{\frac12} + \|\zeta_1 z^k u\|^2_{-\frac12}$$

$$\underset{\text{by suplog estimates}}{\leq} \delta\left(\left\|L(\zeta' \Lambda^{\frac12} R^s \zeta_1 z^k u\right\|^2 + \left\|\bar{L}(\zeta' \Lambda^{\frac12} R^s \zeta_1) z^k u\right\|^2\right) + c_\delta \|z^k u\|^2_{\frac12}.$$

Now, the term with δ is absorbed by the last term in (2.9) (after we transform $\Lambda^{\frac12}$ into $\|\cdot\|_{\frac12}$ to fit into (2.9) and use the fact that $[L\zeta', \Lambda^{\frac12}] \sim \Lambda^{\frac12}$). This concludes the proof of (1.4).

□

Proof of Theorem 1.3. As above, we stay in the positive microlocal cone, the support of ψ^+, and consider only derivatives and cut-off with respect to t. From the trivial identity $[L, z] = 1$, and from $[L, \zeta_0] = \dot{\zeta}_0 g_1$, we get

$$\begin{aligned}\|\zeta_0 u\|^2 &= ([L, z]\zeta_0 u, \zeta_0 u)\\ &\lesssim \|\bar{z}\zeta_0 \bar{L} u\|^2_s + \|z\zeta_0 L u\|^2_s + \|z g_1 \zeta_1 u\|^2_s + sc\|\zeta_0 u\|^2.\end{aligned}$$

Now, the last term is absorbed. As for the term before

$$\|z g_1 \zeta_1 u\|^2_s \underset{\text{by (1.5)}}{\leq} \left\|g^{\frac12}_{1\bar{1}} \Lambda^{\frac12} \zeta_1 u\right\|^2_{s-\frac12}$$

$$\underset{\text{by (2.2)}}{\leq} \|\bar{z} L \zeta_1 u\|^2_{s-\frac12} + \|\bar{z}\zeta_1 u\|^2_{s-\frac12} \lesssim \|\zeta_1 \bar{z} L u\|^2_{s-\frac12} + \|\bar{z}\zeta_2 u\|^2_{s-\frac12} \quad \text{for } \zeta_2 \succ \zeta_1.$$

Now, $\|\bar{z}\zeta_2 u\|^2_{s-\frac12}$ is not absorbable by $\|z g_1 \zeta_1 u\|^2_s$, but can be estimated by the 0-norm using induction over j such that $\frac{j}{2} \geq s$.

□

Proof of Proposition 1.4. As ever, we stay in the positive microlocal cone and take derivatives and cut-off only in t. We prove the result for s replaced by 0 and $\frac12$ replaced by $-\epsilon$. The conclusion for general s follows from the fact that ∂_t commutes with L and $\bar{L}$. We define

$$v_\lambda = e^{-\lambda(e^{-\frac{1}{|z|^\alpha}} - it + (e^{-\frac{1}{|z|^\alpha}} - it)^2)} \qquad \lambda >> 0.$$

We denote by $-\lambda A$ the term at exponent and note that $\mathsf{Re}\,\lambda A = \lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)$. For $L = \partial_z + i g_1(z)\partial_t$, we have $\bar{L} v_\lambda = 0$ (which is the key point) and moreover

$$|\bar{z}^k L v_\lambda| \sim |z|^{k-(\alpha+1)} e^{-\lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)} e^{-\frac{1}{|z|^\alpha}}.$$

We set
$$\lambda(e^{-\frac{1}{|z|^\alpha}},t)=(\theta_1,\frac{1}{\sqrt{\lambda}}\theta_2).$$
Under this change we have, over $\operatorname{supp}\zeta_0$ and $\operatorname{supp}\zeta_1$ which implies $\theta_1 << \lambda$,
$$|z|^{k-(\alpha+1)}=\frac{1}{(\log\lambda-\log\theta_1)^{\frac{k-(\alpha+1)}{\alpha}}}.$$
Hence we interchange
$$|\bar z^kLv_\lambda| \dashrightarrow \frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}\left(\frac{\theta_1+\theta_2^2}{\left(1-\frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}}\right)e^{-(\theta_1+\theta_2^2)}.$$
Notice that $\theta_1 << \lambda$ and hence, for suitable positive c_1 and c_2, we have $c_1 < \frac{\theta_1+\theta_2^2}{\left(1-\frac{\log\theta_1}{\log\lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} < c_2$, uniformly over λ. We also interchange
$$v_\lambda \dashrightarrow e^{-(\theta_1+\theta_2^2)}.$$
Taking L^2 norms yields
$$\|\bar z^kLv_\lambda\|^2\sim\frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}\|v_\lambda\|^2.$$
So, the effect on L^2 norm of the action of $\bar z^kL$ over v_λ is comparable to $\frac{1}{(\log\lambda)^{\frac{k-(\alpha+1)}{\alpha}}}$. We describe now the effect of the pseudodifferential operator $\log(\Lambda_t)$. We claim that
$$\|\log(\Lambda_t)e^{-\lambda t^2}\|^2\sim\log\lambda\|e^{-\lambda t^2}\|^2. \tag{2.14}$$
This is a consequence of
$$\log(\Lambda_t)e^{-\lambda t^2}\sim\log\lambda e^{-\lambda t^2}+\Big(\log(\Lambda_{\tilde t})e^{-\tilde t^2}\Big)\Big|_{\tilde t=\sqrt{\lambda}t}, \tag{2.15}$$
that we go to prove now. Using the coordinate change $\tilde\theta=\sqrt{\lambda}\theta,\ \tilde\xi=\frac{\xi}{\sqrt{\lambda}}$, we get
$$\begin{aligned}&\int e^{it\xi}\log(\Lambda_\xi)\Big(\int e^{-i\xi\theta}e^{-\lambda\theta^2}d\theta\Big)d\xi\\ &\qquad=\int e^{it\sqrt{\lambda}\tilde\xi}\Big(\log(\frac{1}{\lambda}+|\tilde\xi|^2)^{\frac12}+\log(\sqrt{\lambda})\Big)\Big(\int e^{i\tilde\xi\tilde\theta-\tilde\theta^2}d\tilde\theta\Big)d\tilde\xi\\ &\qquad=\log(\sqrt{\lambda})e^{-\lambda t^2}+\Big(\log(\Lambda^\lambda_{\tilde t})e^{-\tilde t^2}\Big)\Big|_{\tilde t=\sqrt{\lambda}t},\end{aligned}$$
where $\log(\Lambda^\lambda_{\tilde t})$ is the operator with symbol $\log(\frac{1}{\lambda}+|\tilde\xi|^2)^{\frac12}$. This proves (2.15) and in turn the claim (2.14).

We compare now the effect over v_λ of $\bar{z}^k L$ with that of $\log(\Lambda_t)$. If

$$\|\zeta_0 v_\lambda\|^2 \lesssim \|\zeta_1 (\log \Lambda_t)^r \bar{z}^k L v_\lambda\|^2 + \|v_\lambda\|^2_{-\epsilon},$$

then, since the right side is estimated from above by

$$\left((\log \lambda)^r (\log \lambda)^{-\frac{k-(\alpha+1)}{\alpha}} + \lambda^{-\epsilon}\right) \|v_\lambda\|^2,$$

we must have that the logarithmic term is not infinitesimal which forces $r \geq \frac{k-(\alpha+1)}{\alpha}$. ∎

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