Decomposition Theorem for Perverse sheaves on Artin stacks

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Abstract

We generalize the decomposition theorem for perverse sheaves to Artin stacks with affine automorphism groups over finite fields and over the complex numbers. During the proof, we also give the generic base change theorem for f_* on stacks, and some results on comparison between derived categories of different topoi.

1 A counter-example: BE.

Let E be a complex elliptic curve, and let f : pt=Spec $\mathbb{C} \to BE$ be the natural projection; this is a representable proper map. There is a natural non-zero morphism $\underline{\mathbb{C}}_{BE} \to Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}$, adjoint to the isomorphism $f^*\underline{\mathbb{C}}_{BE} \simeq \underline{\mathbb{C}}_{\mathrm{pt}}$, but there is no non-zero morphism in the other direction, because

$$Hom(Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{BE}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, f^!\underline{\mathbb{C}}_{BE}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}[2]) = 0.$$

Here the Hom's are taken in the derived categories. Similarly, the non-zero natural map $Rf_*\underline{\mathbb{C}}_{\mathrm{pt}} \to R^2f_*\underline{\mathbb{C}}_{\mathrm{pt}}[-2] = \underline{\mathbb{C}}_{BE}[-2]$ lies in

$$Hom(Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{BE}[-2]) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, f^!\underline{\mathbb{C}}_{BE}[-2]) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}, \underline{\mathbb{C}}_{\mathrm{pt}}) = \mathbb{C},$$

but the Hom set in the other direction is zero:

$$Hom(\underline{\mathbb{C}}_{BE}[-2], Rf_*\underline{\mathbb{C}}_{\mathrm{pt}}) = Hom(f^*\underline{\mathbb{C}}_{BE}[-2], \underline{\mathbb{C}}_{\mathrm{pt}}) = Hom(\underline{\mathbb{C}}_{\mathrm{pt}}[-2], \underline{\mathbb{C}}_{\mathrm{pt}}) = 0.$$

Therefore, $Rf_*\mathbb{C}$ is not semi-simple of geometric origin (since it is not a direct sum of the ${}^p\mathscr{H}^i(f_*\underline{\mathbb{C}})[-i]$'s). The same argument applies to finite fields, with $\underline{\mathbb{C}}$ replaced by $\overline{\mathbb{Q}}_\ell$.

Remark 1.1. This example was first given by Drinfeld, who asked for the reason of the failure of the usual argument for schemes. Later, it was communicated by J. Bernstein to Y. Varshavsky, who asked M. Olsson in an email correspondence. Olsson kindly shared this email with me, and explained to me that the reason is the failure of the upper bound of weights in [4] for stacks.

In the following we explain why the usual proof (as in [3]) fails for f. The proof in [3] of the decomposition theorem over $\mathbb{C}$ relies on the decomposition theorems over finite fields ([3], 5.3.8, 5.4.5), so it suffices to explain why the proof of ([3], 5.4.5) fails for f, for an elliptic curve $E/\mathbb{F}_q$.

Let $K_0 = Rf_*\overline{\mathbb{Q}}_\ell$. The perverse t-structure agrees with the trivial t-structure on Spec $\mathbb{F}_q$, and by definition ([12], 4), we have ${}^p\mathscr{H}^iK_0 = \mathscr{H}^{i+1}(K_0)[-1]$ on BE, and so

$$\bigoplus_i ({}^p\mathscr{H}^iK)[-i] = \bigoplus_i (\mathscr{H}^iK)[-i].$$

Each $R^if_*\overline{\mathbb{Q}}_\ell[-i]$ is pure of weight 0. In the proof of ([3], 5.4.5), the exact triangles

$$\tau_{<i}K_0 \longrightarrow \tau_{\leq i}K_0 \longrightarrow (\mathscr{H}^iK_0)[-i] \longrightarrow$$

split geometrically, because $Ext^1((\mathscr{H}^iK)[-i],\tau_{<i}K)$ has weights >0. We will see that for $f:\operatorname{Spec}\mathbb{F}_q\to BE$, this group is pure of weight 0, and in fact has 1 as a Frobenius eigenvalue. For simplicity, we denote $H^i(\mathscr{X},\overline{\mathbb{Q}}_\ell)$ by $H^i(\mathscr{X})$.

Let $\pi:BE\to\operatorname{Spec}\mathbb{F}_q$ be the structural map; then $\pi\circ f=\mathrm{id}$. Since E is connected, the sheaves $R^if_*\overline{\mathbb{Q}}_\ell$ are inverse images of some sheaves on $\operatorname{Spec}\mathbb{F}_q$, namely $f^*R^if_*\overline{\mathbb{Q}}_\ell$. By smooth base change, they are isomorphic to $\pi^*H^i(E)$ as $\mathrm{Gal}(\mathbb{F}_q)$-modules. In particular, $R^0f_*\overline{\mathbb{Q}}_\ell=\overline{\mathbb{Q}}_\ell$, $R^1f_*\overline{\mathbb{Q}}_\ell\cong\pi^*H^1(E)$ and $R^2f_*\overline{\mathbb{Q}}_\ell=\overline{\mathbb{Q}}_\ell(-1)$. Then the exact triangle above becomes

$$i=2:\qquad \tau_{\le 1}K_0\longrightarrow K_0\longrightarrow\overline{\mathbb{Q}}_\ell(-1)[-2]\longrightarrow$$

$$i=1:\qquad \overline{\mathbb{Q}}_\ell\longrightarrow\tau_{\le 1}K_0\longrightarrow\pi^*H^1(E)[-1]\longrightarrow.$$

Apply $Ext^*(\overline{\mathbb{Q}}_\ell(-1)[-2],-)$ to the second triangle. One can compute $H^*(BE)$ by a theorem of Borel (see ([17], 7)): $H^{2i-1}(BE)=0$, and $H^{2i}(BE)=\mathrm{Sym}^iH^1(E)$. Let α and β be the eigenvalues of the Frobenius F on $H^1(E)$. We have

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2],\overline{\mathbb{Q}}_\ell)=Ext^3(\overline{\mathbb{Q}}_\ell,\overline{\mathbb{Q}}_\ell(1))=H^3(BE)(1)=0,$$

and

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2],\pi^*H^1(E)[-1])=H^2(BE)\otimes H^1(E)(1)=H^1(E)\otimes H^1(E)(1)=End(H^1(E)),$$

which is 4-dimensional with eigenvalues $\alpha/\beta,\beta/\alpha,1,1$, and

$$Ext^2(\overline{\mathbb{Q}}_\ell(-1)[-2],\overline{\mathbb{Q}}_\ell)=H^4(BE)(1),$$

which is 3-dimensional with eigenvalues $\alpha/\beta,\beta/\alpha,1$. This implies that the kernel

$$Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2],\tau_{\le 1}K)=\mathrm{Ker}\Big(Ext^1(\overline{\mathbb{Q}}_\ell(-1)[-2],\pi^*H^1(E)[-1])\to Ext^2(\overline{\mathbb{Q}}_\ell(-1)[-2],\overline{\mathbb{Q}}_\ell)\Big)$$

is non-zero, pure of weight 0, and has 1 as a Frobenius eigenvalue. So the first exact triangle above does not necessarily (in fact does not, as the argument in the beginning shows) split geometrically. Also

$$Ext^1(\pi^*H^1(E)[-1],\overline{\mathbb{Q}}_\ell)=Ext^2(\overline{\mathbb{Q}}_\ell,\pi^*H^1(E)^\vee)=H^1(E)\otimes H^1(E)^\vee=End(H^1(E))$$

is 4-dimensional and has eigenvalues $\alpha/\beta,\beta/\alpha,1,1$, hence the proof for the geometric splitting of the second exact triangle fails too.

In [12], Laszlo and Olsson generalized the theory of perverse sheaves to Artin stacks locally of finite type over some field. In [17], we proved that for Artin stacks of finite type over a finite field, with affine automorphism groups (defined below), Deligne's upper bound of weights for the compactly supported cohomology groups still applies. In this paper, we will show that for such stacks, similar argument as in [3] gives the decomposition theorem.

Organization. In §2 we complete the proof of the structure theorem for ι-mixed sheaves on stacks, as claimed in ([17], 2.7). In §3, we generalize the decomposition theorem for perverse sheaves on stacks over finite fields, using weight theory. In §4 we prove the generic base change for f_* and $R\mathscr{H}om$, and in §5 we use this result to prove a comparison between bounded derived categories with prescribed stratification over the complex numbers and over an algebraic closure of a finite field, as well as a comparison between the lisse-étale topos and the lisse-analytic topos of a $\mathbb{C}$-stack, and then finish the proof of the decomposition theorem over $\mathbb{C}$.

Notations and Conventions 1.2. We fix an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_q$ with q elements. Let F or F_q be the q-geometric Frobenius, namely the q-th root automorphism on $\mathbb{F}$. Let ℓ be a prime number, $\ell \neq p$, and fix an isomorphism of fields $\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \mathbb{C}$. For simplicity, let $|\alpha|$ denote the complex absolute value $|\iota\alpha|$, for $\alpha \in \overline{\mathbb{Q}}_\ell$.

By an Artin stack, or an algebraic stack, we mean an algebraic stack in the sense of M. Artin ([9], 4.1) *of finite type* over the base.

Objects over $\mathbb{F}_q$ will be denoted with an index $_0$. For instance, if K_0 is a $\overline{\mathbb{Q}}_\ell$-sheaf complex on an Artin stack $\mathscr{X}_0$ over $\mathbb{F}_q$, then K denotes its inverse image on $\mathscr{X} := \mathscr{X}_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

For a field k, let $\mathrm{Gal}(k)$ denote its absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. By a variety over k we mean a separated reduced k-scheme of finite type.

For a map $f : X \to Y$ and a sheaf complex K on Y, we sometimes write $H^n(X, K)$ for $H^n(X, f^*K)$. We will write $H^n(\mathscr{X})$ for $H^n(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, and $h^n(\mathscr{X}, \mathscr{F})$ for $\dim H^n(\mathscr{X}, \mathscr{F})$, and ditto for $H^n_c(\mathscr{X})$ and $h^n_c(\mathscr{X}, \mathscr{F})$.

We will denote $Rf_*, Rf_!, Lf^*$ and $Rf^!$ by $f_*, f_!, f^*$ and $f^!$ respectively. We will only consider the middle perversity.

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2 The prototype: the structure theorem of mixed sheaves on stacks.

We generalize the structure theorem of ι-mixed sheaves ([4], 3.4.1) to stacks. This result is independent from other results in this paper, but it is the prototype, in some sense I think, of the corresponding results (e.g. weight filtrations and the decomposition theorem) for perverse sheaves.

Theorem 2.1. *(stack version of ([4], 3.4.1)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack.*

(i) Every ι-mixed sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ has a unique decomposition $\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$, called the decomposition according to the weights mod $\mathbb{Z}$, *such that the punctual ι-weights of $\mathscr{F}_0(b)$ are all in the coset b. This decomposition, in which almost all the $\mathscr{F}_0(b)$'s are zero, is functorial in $\mathscr{F}_0$.*

(ii) Every ι-mixed lisse sheaf $\mathscr{F}_0$ with integer punctual ι-weights on $\mathscr{X}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the weight filtration, *such that Gr^W_i is punctually ι-pure of weight i. Every morphism between such sheaves on $\mathscr{X}_0$ is strictly compatible with their weight filtrations.*

(iii) If $\mathscr{X}_0$ is a normal algebraic stack, and $\mathscr{F}_0$ is a lisse and punctually ι-pure sheaf on $\mathscr{X}_0$, then $\mathscr{F}$ on $\mathscr{X}$ is semi-simple.

Proof. (i) and (ii) are proved in ([17], 2.6.1), where (iii) is claimed to hold without giving a detailed proof. Here we complete the proof of (iii).

First of all, note that we may replace $\mathscr{X}_0$ and $\mathscr{F}_0$ by $\mathscr{X}_0 \otimes \mathbb{F}_{q^v}$ and $\mathscr{F}_0 \otimes \mathbb{F}_{q^v}$, for any finite base change $\mathbb{F}_{q^v}/\mathbb{F}_q$.

From the proof of ([12], 8.3), we see that if $\mathscr{U} \subset \mathscr{X}$ is an open substack, and $\mathscr{G}_{\mathscr{U}}$ is a subsheaf of $\mathscr{F}|_{\mathscr{U}}$, then it extends to a unique subsheaf $\mathscr{G} \subset \mathscr{F}$. Therefore, we may shrink $\mathscr{X}$ to a dense open substack $\mathscr{U}$, and replace $\mathscr{X}_0$ by some model of $\mathscr{U}$ over a finite extension $\mathbb{F}_{q^v}$. We can assume $\mathscr{X}_0$ is smooth and geometrically connected.

Following the proof ([4], 3.4.5), it suffices to show ([4], 3.4.3) for stacks. We claim that, if $\mathscr{F}_0$ is lisse and punctually ι-pure of weight w, then $H^1(\mathscr{X},\mathscr{F})$ is ι-mixed of weights $\geq 1+w$. The conclusion follows from this claim.

Let $D=\dim\mathscr{X}_0$. By Poincaré duality, it suffices to show that, for every lisse sheaf $\mathscr{F}_0$, punctually ι-pure of weight w, $H_c^{2D-1}(\mathscr{X},\mathscr{F})$ is ι-mixed of weights $\leq 2D-1+w$. To show this, we may shrink $\mathscr{X}_0$ to open substacks, and hence we may assume that the inertia $\mathscr{I}_0\to\mathscr{X}_0$ is flat. As in the proof of ([17], 1.4), we have the spectral sequence

$$H_c^r(X,R^k\pi_!\mathscr{F})\Longrightarrow H_c^{r+k}(\mathscr{X},\mathscr{F}),$$

so let $r+k=2D-1$. Note that k can only be of the form $-2i-2d$, for $i\geq 0$, where $d=\mathrm{rel.dim}(\mathscr{I}_0/\mathscr{X}_0)$. So we have $r=2\dim X_0+2i-1$, and in order for $H_c^r(X,-)$ to be non-zero, $i=0$. Then

$$H_c^{2D-1}(\mathscr{X},\mathscr{F})=H_c^{2\dim X-1}(X,R^{-2d}\pi_!\mathscr{F}).$$

It suffices to show that $H_c^{-2d}(BG,\mathscr{F})$ has weights $\leq w-2d$, where G_0 is an algebraic group of dimension d, and $\mathscr{F}_0$ is a lisse punctually ι-pure sheaf on BG_0 of weight w. In fact, $R^{-2d}\pi_!\mathscr{F}$ is punctually ι-pure of weight $w-2d$. We reduce to the case where G_0 is connected, and the claim is clear. $\square$

3 Decomposition theorem for stacks over $\mathbb{F}_q$.

For an algebraic stack $\mathscr{X}_0/\mathbb{F}_q$, let $D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ be the full subcategory of ι-mixed sheaf complexes in $D_c(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ ([17], 2.3iii). It is stable under the perverse truncations ${}^p\tau_{\leq 0}$ and ${}^p\tau_{\geq 0}$. This can be checked smooth locally, and hence follows from ([17], 2.10) and ([3], 5.1.6). The core of $D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ with respect to this induced perverse t-structure is called the category of ι-mixed perverse sheaves on $\mathscr{X}_0$, denoted $\mathrm{Perv}_m(\mathscr{X}_0)$. This is a Serre subcategory of $\mathrm{Perv}(\mathscr{X}_0)$.

In fact, Lafforgue proved the conjecture of Deligne that, all (Weil) sheaves are ι-mixed, for any ι. Using this result, $D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)=D_c(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ and $\mathrm{Perv}_m(\mathscr{X}_0)=\mathrm{Perv}(\mathscr{X}_0)$. But to emphasize the condition of ι-mixedness, we still write "D_m" and "Perv_m" in this paper.

Definition 3.1. *Let $K_0\in D_m(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$.*

(i) We say that K_0 has ι-weights $\leq w$ *if for each $i\in\mathbb{Z}$, the punctual ι-weights of $\mathscr{H}^iK_0$ are $\leq i+w$, and we denote by $D_{\leq w}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ the subcategory of such complexes. We say that K_0* has ι-weights $\geq w$ *if its Verdier dual DK_0 has ι-weights $\leq -w$, and denote by $D_{\geq w}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ the subcategory of such complexes.*

(ii) For a coset $b\in\mathbb{R}/\mathbb{Z}$, we say that K_0 has ι-weights in b *if the punctual ι-weights of $\mathscr{H}^iK_0$ are in b, for all $i\in\mathbb{Z}$.*

Lemma 3.2. *Let $P:\mathscr{X}_0'\to\mathscr{X}_0$ be a representable surjection of $\mathbb{F}_q$-algebraic stacks, and $K_0\in D_c(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$. Then K_0 is ι-mixed of weights $\leq w$ (resp. $\geq w$) if and only if P^*K_0 (resp. $P^!K_0$) is so.*

Proof. It suffices to consider only the case where K_0 has weights $\leq w$, since the other statement is dual to this one. The "only if" part is obvious. The "if" part for ι-mixedness follows from ([17], 2.8), and the "if" part for the weights follows from the surjectivity of P. $\square$

In particular, this applies to the case where P is a presentation.

We say that an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ has *affine automorphism groups* if for every integer $v\geq 1$ and every $x\in\mathscr{X}_0(\mathbb{F}_{q^v})$, the automorphism group scheme Aut_x over $k(x)$ is affine. In the following, some results require the stack to have affine automorphism groups. We will first give results that apply to all stacks, and then give those that require this condition.

The following lemma is the perverse sheaf version of (2.1i).

Lemma 3.3. *Every ι-mixed perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ has a unique decomposition $\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}}\mathscr{F}_0(b)$ into perverse subsheaves, called the* decomposition according to the weights mod $\mathbb{Z}$, *such that for each coset b, the ι-weights of $\mathscr{F}_0(b)$ belong to b. This decomposition, in which almost all the $\mathscr{F}_0(b)$'s are zero, is functorial in $\mathscr{F}_0$.*

Proof. By descent theory ([12], 7.1) we reduce to the case where $\mathscr{X}_0 = X_0$ is a scheme. One can further replace X_0 by the disjoint union of finitely many open affines, and assume X_0 is separated. We want to reduce to the case where X_0 is proper.

Let $j : X_0 \hookrightarrow Y_0$ be a Nagata compactification, i.e. an open dense immersion into a proper scheme Y_0, and assume we have the existence and uniqueness of the decomposition of any ι-mixed perverse sheaf on Y_0 according to the weights mod $\mathbb{Z}$, and the decomposition is functorial. Let

$$j_{!*}\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} G_0(b)$$

be the decomposition for $j_{!*}\mathscr{F}_0$. Applying j^* we get a decomposition

$$\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} j^*G_0(b).$$

Note that j^* takes a perverse sheaf to a perverse sheaf. This shows the existence. For uniqueness, let $\mathscr{F}_0 = \bigoplus_b \mathscr{F}_0(b)$ be another such decomposition. Then we have

$$j_{!*}\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} j_{!*}\mathscr{F}_0(b).$$

Following the proof in [3] we see that $j_{!*}\mathscr{F}_0(b)$ is ι-mixed of weights in b (by twisting, we may assume $\mathscr{F}_0(b)$ is ι-mixed of integer weights, then follow the proof in ([3], 5.3.1) to show $j_{!*}$ preserves ι-mixedness with integer weights, and finally twist back). By uniqueness of the decomposition for $j_{!*}\mathscr{F}_0$ we have $j_{!*}\mathscr{F}_0(b) = G_0(b)$, and so $\mathscr{F}_0(b) = j^*G_0(b)$. For functoriality, given a morphism $\mathscr{F}_0 \to \mathscr{G}_0$ between ι-mixed perverse sheaves on X_0, we get a morphism $j_{!*}\mathscr{F}_0 \to j_{!*}\mathscr{G}_0$ of ι-mixed sheaves on Y_0, which respects their decompositions by assumption, and then apply j^*.

So we may assume that $X_0/\mathbb{F}_q$ is proper. Let a be the structural map of $X_0/\mathbb{F}_q$. Let K_0 and L_0 in $D_c^b(X_0,\overline{\mathbb{Q}}_\ell)$ be ι-pure complexes of ι-weights w and w', respectively, and assume $w - w' \notin \mathbb{Z}$. Then we claim that $Ext^1(K_0, L_0) = 0$. From the exact sequence ([3], 5.1.2.5)

$$0 \longrightarrow Ext^{i-1}(K,L)_F \longrightarrow Ext^i(K_0,L_0) \longrightarrow Ext^i(K,L)^F \longrightarrow 0$$

we see it suffices to show that 1 cannot be a Frobenius eigenvalue on $Ext^i(K,L)$, for every i. Note that $R\mathscr{H}om(K_0,L_0) = D(K_0 \otimes^L DL_0)$ is ι-pure of weight $w' - w$, by the spectral sequence

$$\mathscr{H}^i(K_0 \otimes^L \mathscr{H}^j DL_0) \Longrightarrow \mathscr{H}^{i+j}(K_0 \otimes^L DL_0)$$

and the similar one for the first factor K_0. Consider the spectral sequence

$$R^i a_* R^j \mathscr{H}om(K_0,L_0) \Longrightarrow R^{i+j}(a_*\mathscr{H}om)(K_0,L_0).$$

Since $a_* = a_!$, by ([4], 3.3.10) we see that the ι-weights of $Ext^i(K,L)$ cannot be integers. Therefore $Ext^1(K_0,L_0) = 0$.

For every $b \in \mathbb{R}/\mathbb{Z}$, we apply ([3], 5.3.6) to $\mathrm{Perv}_{\iota\text{-mix}}(X_0)$, taking S^+ (resp. S^-) to be the set of isomorphism classes of simple ι-mixed perverse sheaves (and hence ι-pure (3.5)) of weight not in b (resp. in b). Then for every ι-mixed perverse sheaf $\mathscr{F}_0$, we get a unique subobject $\mathscr{F}_0(b)$ with ι-weights in b, such that $\mathscr{F}_0/\mathscr{F}_0(b)$ has ι-weights not in b, and $\mathscr{F}_0(b)$ is functorial in $\mathscr{F}_0$. As we see from the argument above, this extension splits: $\mathscr{F}_0 = \mathscr{F}_0(b) \oplus \mathscr{F}_0/\mathscr{F}_0(b)$, so by induction we get the decomposition, which is unique and functorial. $\square$

Lemma 3.4. *(stack version of ([3], 5.3.2)) Let $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ be an immersion of algebraic stacks. Then for any real number w, the intermediate extension $j_{!*}$ ([12], 6) respects* $\mathrm{Perv}_{\geq w}$ *and* $\mathrm{Perv}_{\leq w}$. *In particular, if $\mathscr{F}_0$ is an ι-pure perverse sheaf on $\mathscr{U}_0$, then $j_{!*}\mathscr{F}_0$ is ι-pure of the same weight.*

Proof. For a closed immersion i, we see that i_* respects $D_{\geq w}$ and $D_{\leq w}$, so we may assume that j is an open immersion. We only need to consider the case for $\mathrm{Perv}_{\leq w}$, since the case for $\mathrm{Perv}_{\geq w}$ follows from $j_{!*}D = Dj_{!*}$.

Let $P : X_0 \to \mathscr{X}_0$ be a presentation, and let the following diagram be 2-Cartesian:

$$\begin{array}{ccc} U_0 & \xrightarrow{j'} & X_0 \\ {\scriptstyle P'}\downarrow & & \downarrow{\scriptstyle P} \\ \mathscr{U}_0 & \xrightarrow[j]{} & \mathscr{X}_0. \end{array}$$

For $\mathscr{F}_0 \in \mathrm{Perv}_{\leq w}(\mathscr{U}_0)$, by (3.2) it suffices to show that $P^*j_{!*}\mathscr{F}_0 \in D_{\leq w}(X_0, \overline{\mathbb{Q}}_\ell)$. Let d be the relative dimension of P. By ([12], 6.2) we have

$$P^*j_{!*}\mathscr{F}_0 = (P^*(j_{!*}\mathscr{F}_0)[d])[-d] = j'_{!*}(P'^*\mathscr{F}_0[d])[-d].$$

Since $P'^*\mathscr{F}_0 \in D_{\leq w}$, $P'^*\mathscr{F}_0[d] \in D_{\leq w+d}$, and by ([3], 5.3.2), $j'_{!*}(P'^*\mathscr{F}_0[d]) \in \mathrm{Perv}_{\leq w+d}$, and by definition $P^*j_{!*}\mathscr{F}_0 \in D_{\leq w}$. $\square$

Corollary 3.5. *(stack version of ([3], 5.3.4)) Every ι-mixed simple perverse sheaf $\mathscr{F}_0$ on an algebraic stack $\mathscr{X}_0$ is ι-pure.*

Proof. By ([12], 8.2ii), there exists a d-dimensional irreducible substack $j : \mathscr{V}_0 \hookrightarrow \mathscr{X}_0$ such that $\mathscr{V}_{\mathrm{red}}$ is smooth, and a simple ι-mixed (hence ι-pure) lisse sheaf L_0 on $\mathscr{V}_0$ such that $\mathscr{F}_0 \cong j_{!*}L_0[d]$. The result follows from (3.4). $\square$

The stack version of ([3], 5.3.5) is given in ([12], 9.2), and the following is a variant for ι-mixed perverse sheaves with integer weights (3.1ii), which is the perverse sheaf version of (2.1ii).

Theorem 3.6. *Let $\mathscr{F}_0$ be an ι-mixed perverse sheaf on $\mathscr{X}_0$ with integer weights. Then there exists a unique finite increasing filtration W of $\mathscr{F}_0$ by perverse subsheaves, called the* weight filtration, *such that $\mathrm{Gr}_i^W\mathscr{F}_0$ is ι-pure of weight i, for each i. Every morphism between such perverse sheaves on $\mathscr{X}_0$ is strictly compatible with their weight filtrations.*

Proof. As in ([12], 9.2), we may assume $\mathscr{X}_0 = X_0$ is a scheme. The proof in ([3], 5.3.5) still applies. Namely, by (3.9ii), if $\mathscr{F}_0$ and $\mathscr{G}_0$ are ι-pure simple perverse sheaves on X_0, of ι-weights f and g respectively, and $f > g$, then $Ext^1(\mathscr{G}_0, \mathscr{F}_0) = 0$. Then take S^+ (resp. S^-) to be the set of isomorphism classes of ι-pure simple perverse sheaves on X_0 of ι-weights $> i$ (resp. $\leq i$) for each integer i, and apply ([3], 5.3.6). $\square$

Theorem 3.7. *(stack version of ([3], 5.4.1, 5.4.4)) Let $K_0 \in D^b_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Then K_0 has ι-weights $\le w$ (resp. $\ge w$) if and only if ${}^p\mathscr{H}^i K_0$ has ι-weights $\le w+i$ (resp. $\ge w+i$), for each $i \in \mathbb{Z}$. In particular, K_0 is ι-pure of weight w if and only if each ${}^p\mathscr{H}^i K_0$ is ι-pure of weight $w+i$.*

Proof. The case of " $\ge$ " follows from the case of " $\le$ " and ${}^p\mathscr{H}^i \circ D = D \circ {}^p\mathscr{H}^{-i}$. So we only need to show the case of " $\ge$ ".

Let $P : X_0 \to \mathscr{X}_0$ be a presentation of relative dimension d. Then K_0 has ι-weights $\le w$ if and only if (3.2) P^*K_0 has ι-weights $\le w$, if and only if ([3], 5.4.1) each ${}^p\mathscr{H}^i(P^*K_0)$ has ι-weights $\le w+i$. We have ${}^p\mathscr{H}^i(P^*K_0) = {}^p\mathscr{H}^i(P^*(K_0[-d])[d]) = P^{*p}\mathscr{H}^i(K_0[-d])[d] = P^*({}^p\mathscr{H}^{i-d}K_0)[d]$, so $P^*({}^p\mathscr{H}^{i-d}K_0)$, and hence ${}^p\mathscr{H}^{i-d}K_0$, has ι-weights $\le w+i-d$. □

In the following results, except (3.8i, ii, iv, v), we will need the assumption of affine automorphism groups.

Proposition 3.8. *(stack version of ([3], 5.1.14)) (i) The Verdier dual D interchanges $D_{\le w}$ and $D_{\ge -w}$.*

(ii) For every morphism f of $\mathbb{F}_q$-algebraic stacks, f^* respects $D_{\le w}$ and $f^!$ respects $D_{\ge w}$.

(iii) For every morphism $f : \mathscr{X}_0 \to \mathscr{Y}_0$, where $\mathscr{X}_0$ is an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, $f_!$ respects $D^{-,\text{stra}}_{\le w}$ and f_* respects $D^{+,\text{stra}}_{\ge w}$.

(iv) $\otimes^L$ takes $D^-_{\le w} \times D^-_{\le w'}$ into $D^-_{\le w+w'}$.

(v) $R\mathscr{H}om$ takes $D^-_{\le w} \times D^+_{\ge w'}$ into $D^+_{\ge w'-w}$.

Proof. (i), (ii) and (iv) are clear, and (v) follows from (iv). For (iii), if $\mathscr{X}_0$ has affine automorphism groups, so are all fibers $f^{-1}(y)$, for $y \in \mathscr{Y}_0(\mathbb{F}_{q^v})$, and the claim for $f_!$ follows from the spectral sequence

$$H^i_c(f^{-1}(\overline{y}), \mathscr{H}^j K) \Longrightarrow H^{i+j}_c(f^{-1}(\overline{y}), K)$$

and ([17], 1.4), and the claim for f_* follows from ([17], 2.10, 3.8) and the claim for $f_!$. □

Corollary 3.9. *(stack version of ([3], 5.1.15)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, and let $a : \mathscr{X}_0 \to$ Spec $\mathbb{F}_q$ be the structural map. Let K_0 (resp. L_0) be in $D^-_{\le w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (resp. $D^+_{>w}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$) for some real number w. Then*

(i) $a_*R\mathscr{H}om(K_0, L_0)$ is in $D^+_{>0}(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$.

(ii) $Ext^i(K_0, L_0) = 0$ for $i > 0$.

If $L_0 \in D^+_{\ge w}$, then $a_*R\mathscr{H}om(K_0, L_0)$ is in $D^+_{\ge 0}$, and we have

(iii) $Ext^i(K, L)^F = 0$ for $i > 0$. In particular, for $i > 0$, the morphism $Ext^i(K_0, L_0) \to Ext^i(K, L)$ is zero.

The proof is the same as ([3], 5.1.15), using the above stability result for stacks with affine automorphism groups.

The following is a perverse sheaf version of (2.1iii).

Theorem 3.10. *(stack version of ([3], 5.3.8)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups. Then every ι-pure perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ is geometrically semi-simple (i.e. $\mathscr{F}$ is semi-simple), hence $\mathscr{F}$ is a direct sum of perverse sheaves of the form $j_{!*}L[d_{\mathscr{U}}]$, for inclusions $j : \mathscr{U} \hookrightarrow \mathscr{X}$ of $d_{\mathscr{U}}$-dimensional irreducible substacks that are essentially smooth, and for simple ι-pure lisse sheaves L on $\mathscr{U}$.*

Proof. Let $\mathscr{F}'$ be the sum in $\mathscr{F}$ of simple perverse subsheaves; it is a direct sum, and is the largest semi-simple perverse subsheaf of $\mathscr{F}$. Then $\mathscr{F}'$ is stable under Frobenius, hence descends to a perverse subsheaf $\mathscr{F}_0' \subset \mathscr{F}_0$ (([3], 5.1.2) holds for stacks also). Let $\mathscr{F}_0'' = \mathscr{F}_0/\mathscr{F}_0'$. By (3.9iii), the extension

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

splits, because $\mathscr{F}_0'$ and $\mathscr{F}_0''$ have the same weight ([12], 9.3). Then $\mathscr{F}''$ must be zero, since otherwise it contains a simple perverse subsheaf, and this contradicts the maximality of $\mathscr{F}'$. Therefore $\mathscr{F} = \mathscr{F}'$ is semi-simple. The other claim follows from ([12], 8.2ii). □

Theorem 3.11. *(stack version of ([3], 5.4.5)) Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack with affine automorphism groups, and let $K_0 \in D_m^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be ι-pure. Then K on $\mathscr{X}$ is isomorphic to the direct sum of the shifted perverse cohomology sheaves $({}^p\mathscr{H}^i K)[-i]$.*

Proof. By (3.7), both ${}^p\tau_{<i}K_0$ and $({}^p\mathscr{H}^i K_0)[-i]$ are ι-pure of the same weight as that of K_0. Therefore, by (3.9iii), the exact triangle

$${}^p\tau_{<i}K_0 \longrightarrow {}^p\tau_{\leq i}K_0 \longrightarrow ({}^p\mathscr{H}^i K_0)[-i] \longrightarrow$$

is geometrically split, i.e. we have

$${}^p\tau_{\leq i}K \simeq {}^p\tau_{<i}K \oplus ({}^p\mathscr{H}^i K)[-i],$$

and the result follows by induction. □

4 Generic base change.

In the next section we will prove the decomposition theorem for stacks over the complex numbers $\mathbb{C}$. A technical step in the proof (as in [3]) will be to compare the derived categories of the fiber over $\mathbb{C}$ and the fiber over $\mathbb{F}$ of some stack over a DVR with mixed characteristics. For doing that, we prove a stack version of the generic base change theorem ([5], Th. finitude) in this section.

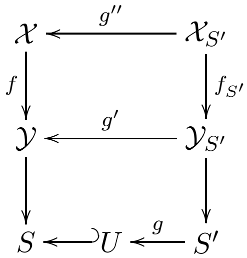
4.1. Let S be a scheme satisfying the following condition denoted (LO): it is a noetherian affine excellent finite-dimensional scheme in which ℓ is invertible, and all S-schemes of finite type have finite ℓ-cohomological dimension. The theory of derived categories and the six operations in [10, 11] then applies to algebraic stacks over S locally of finite type. Let $(\Lambda, \mathfrak{m})$ be a complete DVR of mixed characteristic, with finite residue field Λ_0 of characteristic ℓ and uniformizer λ. Let $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$. As mentioned in §1, we will only consider algebraic stacks that are *of finite type* over the base. Let $\mathscr{A} = \mathscr{A}(\mathcal{X}) := \mathrm{Mod}(\mathcal{X}^{\mathbb{N}}_{\text{lis-ét}}, \Lambda_\bullet)$.

We refer to ([17], §3) for the definition and basic properties of stratifiable complexes in detail; we only give a quick review of the definition here.

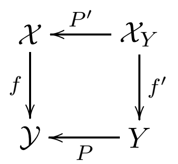
For a pair $(\mathscr{S}, \mathcal{L})$, where $\mathscr{S}$ is a stratification of the stack $\mathcal{X}$, and $\mathcal{L}$ assigns to every stratum $\mathcal{U} \in \mathscr{S}$ a finite set $\mathcal{L}(\mathcal{U})$ of isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{U}$, we define $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ to be the full subcategory of $\mathscr{D}_c(\mathscr{A})$ consisting of the complexes of projective systems $K = (K_n)_n$ such that, for all $i, n \in \mathbb{Z}$ and for every $\mathcal{U} \in \mathscr{S}$, the restriction $\mathscr{H}^i(K_n)|_{\mathcal{U}}$ is lcc with Jordan-Hölder components contained in $\mathcal{L}(\mathcal{U})$. Define $D_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda)$ to be its essential image under the localization $\mathscr{D}_c(\mathscr{A}) \to D_c(\mathcal{X}, \Lambda)$; in other words, it is the quotient of $\mathscr{D}_{\mathscr{S},\mathcal{L}}(\mathscr{A})$ by the thick subcategory of AR-null complexes. It is a triangulated category.

4.2. For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of S-algebraic stacks and $K \in D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $D_c^+(\mathcal{X}, \Lambda)$), we say that *the formation of f_*K commutes with generic base change,* if there exists an open dense subset $U \subset S$ such that for any morphism $g : S' \to U \subset S$

with S' satisfying (LO), the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ is an isomorphism. The base change morphism is defined to be the one corresponding by adjunction (g'^*, g'_*) to $f_* K \to g'_* f_{S'*} g''^* K \simeq f_* g''_* g''^* K$, obtained by applying f_* to the adjunction map $K \to g''_* g''^* K$.



Lemma 4.3. *(i) Let $P : Y \to \mathcal{Y}$ be a presentation, and let the following diagram be 2-Cartesian:*

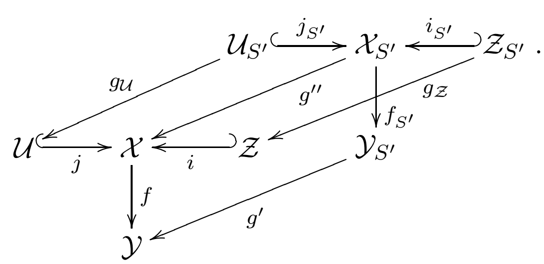


Then for $K \in D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $K \in D_c^+(\mathcal{X}, \Lambda)$), the formation of $f_* K$ commutes with generic base change if and only if the formation of $f'_* P'^* K$ commutes with generic base change.

(ii) Let $K' \to K \to K'' \to K'[1]$ be an exact triangle in $D_c^+(\mathcal{X}, \Lambda_n)$ (resp. $D_c^+(\mathcal{X}, \Lambda)$), and let $f : \mathcal{X} \to \mathcal{Y}$ be an S-morphism. If the formations of $f_* K'$ and $f_* K''$ commute with generic base change, then so is the formation of $f_* K$.

(iii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a schematic morphism, and let $K \in D^+_{\{\mathcal{X}\},\mathcal{L}}(\mathcal{X}, \Lambda)$ for some finite set $\mathcal{L}$ of isomorphism classes of simple Λ_0-modules on $\mathcal{X}$. Then the formation of $f_* K$ commutes with generic base change.

(iv) Let $K \in D_c^+(\mathcal{X}, \Lambda)$, and let $j : \mathcal{U} \to \mathcal{X}$ be an open immersion with complement $i : \mathcal{Z} \to \mathcal{X}$. For $g : S' \to S$, consider the following diagram obtained by base change:



Suppose the base change morphisms

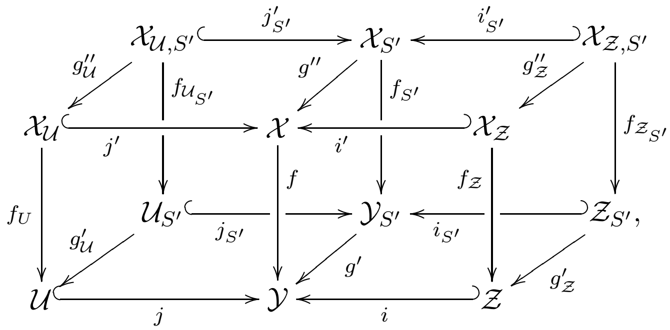
$$g'^*(fj)_* j^* K \longrightarrow (f' j_{S'})_* g_{\mathcal{U}}^* j^* K,$$
$$g'^*(fi)_* i^! K \longrightarrow (f' i_{S'})_* g_{\mathcal{Z}}^* i^! K \qquad \text{and}$$
$$g''^* j_* j^* K \longrightarrow j_{S'*} g_{\mathcal{U}}^* j^* K$$

are isomorphisms, then the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ is also an isomorphism.

(v) Let $f : \mathcal{X} \to \mathcal{Y}$ be a schematic morphism of S-Artin stacks, and let $K \in D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda)$. Then the formation of $f_* K$ commutes with generic base change on S.

(vi) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-Artin stacks, and let $j : \mathcal{U} \to \mathcal{Y}$ be an open immersion with complement $i : \mathcal{Z} \to \mathcal{Y}$. Let $K \in D_c^+(\mathcal{X}, \Lambda)$ (or $D_c^+(\mathcal{X}, \Lambda_n)$). For a map

$g : S' \to S$, consider the following diagram, in which the squares are 2-Cartesian:

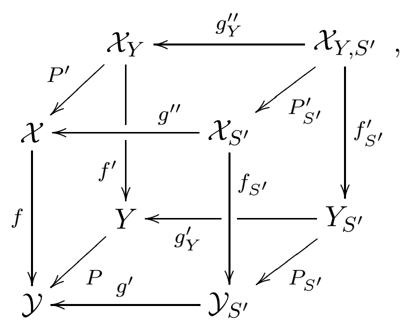


and assume that the base change morphisms

$$g'^*_{\mathcal{U}} f_{\mathcal{U}*} j'^* K \to f_{\mathcal{U}_{S'}*} g''^*_{\mathcal{U}} j'^* K \quad \textit{and} \quad g'^*_{\mathcal{Z}} f_{\mathcal{Z}*} i'^! K \to f_{\mathcal{Z}_{S'}*} g''^*_{\mathcal{Z}} i'^! K$$

are isomorphisms. Then after shrinking S, the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ is an isomorphism.

Proof. (i) Given a map $g : S' \to S$, consider the following diagram



where all squares are 2-Cartesian. For the base change morphism $g'^* f_* K \to f_{S'*} g''^* K$ to be an isomorphism on $\mathcal{Y}_{S'}$, it suffices for it to be an isomorphism locally on $Y_{S'}$. In the following commutative diagram

$$\begin{array}{ccc}
P^*_{S'} g'^* f_* K & \xrightarrow[(0)]{} & P^*_{S'} f_{S'*} g''^* K \\
(1) \Vert & & \downarrow (2) \\
g'^*_Y P^* f_* K & & f'_{S'*} P'^*_{S'} g''^* K \\
(3) \downarrow & & \Vert (4) \\
g'^*_Y f'_* P'^* K & \xrightarrow{(5)} & f'_{S'*} g''^*_Y P'^* K,
\end{array}$$

(1) and (4) are canonical isomorphisms given by "$P^* g^* \simeq g^* P^*$", (2) and (3) are canonical isomorphisms given by "$P^* f_* = f_* P^*$", which follows from the definition of f_* on the lisse-étale site. Therefore, (0) is an isomorphism if and only if (5) is an isomorphism.

(ii) This follows easily from the axioms of a triangulated category (or 5-lemma):

$$\begin{array}{ccccccc}
g'^* f_* K' & \longrightarrow & g'^* f_* K & \longrightarrow & g'^* f_* K'' & \longrightarrow & \\
\downarrow \wr & & \downarrow & & \downarrow \wr & & \\
f_{S'*} g''^* K' & \longrightarrow & f_{S'*} g''^* K & \longrightarrow & f_{S'*} g''^* K'' & \longrightarrow & .
\end{array}$$

(iii) By (i) we may assume that $f : X \to Y$ is a morphism of S-schemes. Note that the property of being trivialized by a pair of the form $(\{\mathcal{X}\}, \mathcal{L})$ is preserved when passing to a presentation. By definition f_*K is the class of the system $(f_*\widehat{K}_n)_n$, so it suffices to show that there exists a nonempty open subscheme of S over which the formation of $f_*\widehat{K}_n$ commutes with base change, for every n. By the spectral sequence

$$R^p f_* \mathscr{H}^q(\widehat{K}_n) \Longrightarrow R^{p+q} f_* \widehat{K}_n$$

and (ii), it suffices to show the existence of a nonempty open subset of S, over which the formations of f_*L commute with generic base change, for all $L \in \mathcal{L}$. This follows from ([5], Th. finitude).

(iv) Consider the commutative diagram

$$\begin{array}{ccccc} g'^*f_*i_*i^!K & \xrightarrow[(2)]{} & f_{S'*}g''^*i_*i^!K & \xrightarrow[(3)]{} & f_{S'*}i_{S'*}g_{\mathcal{Z}}^*i^!K \\ {\scriptstyle(1)}\Vert & & & & \Vert{\scriptstyle(4)} \\ g'^*(fi)_*i^!K & & \xrightarrow{(5)} & & (f_{S'}i_{S'})_*g_{\mathcal{Z}}^*i^!K. \end{array}$$

(1) and (4) are canonical isomorphisms, (5) is an isomorphism by assumption, and (3) is the base change morphism for i_*, which is an isomorphism by ([11], 12.5.3), since $i_* = i_!$ (note that i_* has finite cohomological dimension, so it is defined on complexes unbounded in both directions). Therefore, (2) is an isomorphism. Similarly, consider the commutative diagram

$$\begin{array}{ccccc} g'^*f_*j_*j^*K & \xrightarrow[(2)]{} & f_{S'*}g''^*j_*j^*K & \xrightarrow[(3)]{} & f_{S'*}j_{S'*}g_{\mathcal{U}}^*j^*K \\ {\scriptstyle(1)}\Vert & & & & \Vert{\scriptstyle(4)} \\ g'^*(fj)_*j^*K & & \xrightarrow{(5)} & & (f_{S'}j_{S'})_*g_{\mathcal{U}}^*j^*K. \end{array}$$

(1) and (4) are canonical isomorphisms, and (3) and (5) are isomorphisms by assumption, so (2) is an isomorphism. Then applying (ii) to the exact triangle $i_*i^!K \to K \to j_*j^*K \to$, we are done.

(v) By (i), we may assume that $f : X \to Y$ is a morphism of S-schemes. Assume K is trivialized by $(\mathscr{S}, \mathcal{L})$, and let $j : U \to X$ be the immersion of an open stratum in $\mathscr{S}$ with complement $i : Z \to X$. Then $j^*K \in D^+_{\{U\},\mathcal{L}(U)}(U, \Lambda)$, so by (iii), the formation of $j_*(K|_U)$ commutes with generic base change. This is the third base change isomorphism in the assumption of (iv). By noetherian induction and (iv), we replace X by U and assume that $\mathscr{S} = \{X\}$. The result follows from (iii).

(vi) In the commutative diagrams

$$\begin{array}{ccccc} g'^*j_*f_{\mathcal{U}*}j'^*K & & \xrightarrow{(1)} & & j_{S'*}g_{\mathcal{U}}'^*f_{\mathcal{U}*}j'^*K \\ {\scriptstyle(2)}\downarrow & & & & \downarrow{\scriptstyle(3)} \\ g'^*(fj')_*j'^*K & \xrightarrow[(4)]{} & (f_{S'}j'_{S'})_*g_{\mathcal{U}}''^*j'^*K & \xrightarrow[(5)]{} & j_{S'*}f_{\mathcal{U}_{S'}*}g_{\mathcal{U}}''^*j'^*K \end{array}$$

and

$$\begin{array}{ccccc} g'^*i_*f_{\mathcal{Z}*}i'^!K & & \xrightarrow{(6)} & & i_{S'*}g_{\mathcal{Z}}'^*f_{\mathcal{Z}*}i'^!K \\ {\scriptstyle(7)}\downarrow & & & & \downarrow{\scriptstyle(8)} \\ g'^*(fi')_*i'^!K & \xrightarrow[(9)]{} & (f_{S'}i'_{S'})_*g_{\mathcal{Z}}''^*i'^!K & \xrightarrow[(10)]{} & i_{S'*}f_{\mathcal{Z}_{S'}*}g_{\mathcal{Z}}''^*i'^!K, \end{array}$$

(2), (5), (7) and (10) are canonical isomorphisms, (3) and (8) are isomorphisms by assumption, (6) is an isomorphism by proper base change, and (1) is an isomorphism after shrinking

S by (v). Therefore, (4) and (9) are isomorphisms. Also by (iii), the base change morphism $g''^*j'_*j'^*K \to j'_{S'*}g''^*_{\mathcal{U}}j'^*K$ becomes an isomorphism after shrinking S. Hence by (iv), the base change morphism $g'^*f_*K \to f_{S'*}g''^*K$ is an isomorphism after shrinking S. $\square$

4.4. For $K \in D_c^-(\mathcal{X}, \Lambda)$ and $L \in D_c^+(\mathcal{X}, \Lambda)$, and for a morphism $g : \mathcal{Y} \to \mathcal{X}$, the base change morphism $g^*R\mathscr{H}om_{\mathcal{X}}(K, L) \to R\mathscr{H}om_{\mathcal{Y}}(g^*K, g^*L)$ is defined as follows. By adjunction (g^*, g_*), it corresponds to the morphism

$$R\mathscr{H}om_{\mathcal{X}}(K, L) \to g_*R\mathscr{H}om_{\mathcal{Y}}(g^*K, g^*L) \simeq R\mathscr{H}om_{\mathcal{X}}(K, g_*g^*L)$$

obtained by applying $R\mathscr{H}om_{\mathcal{X}}(K, -)$ to the adjunction morphism $L \to g_*g^*L$.

The following is the main result of this section.

Theorem 4.5. *(i) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. For every $K \in D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda)$ (resp. $D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda_n)$), the formation of f_*K commutes with generic base change on S.*

(ii) For every $K, L \in D_c^b(\mathcal{X}, \Lambda)$ (resp. $D_c^b(\mathcal{X}, \Lambda_n)$), the formation of $R\mathscr{H}om_{\mathcal{X}}(K, L)$ commutes with generic base change on S.

Proof. (i) We can always replace a stack by its maximal reduced closed substack, so we will assume all stacks in the proof are reduced.

Suppose K is $(\mathscr{S}, \mathcal{L})$-stratifiable for some pair $(\mathscr{S}, \mathcal{L})$. By (4.3i,iii,iv), we can replace $\mathcal{Y}$ by a presentation and replace $\mathcal{X}$ by an open stratum in $\mathscr{S}$, to assume that $\mathcal{Y} = Y$ is a scheme, that $\mathscr{S} = \{\mathcal{X}\}$, that the relative inertia $\mathcal{I}_f$ is flat and has components over $\mathcal{X}$ ([2], 5.1.14), and let

$$\mathcal{X} \xrightarrow{\pi} X \xrightarrow{b} Y$$

be the rigidification with respect to $\mathcal{I}_f$. Replacing $\mathcal{X}$ by the inverse image of an open dense subscheme of the S-algebraic space X, we may assume X is a scheme. Let $\mathscr{F} = \pi_*K$, which is stratifiable ([17], 3.8). By (4.3v), the formation of $b_*\mathscr{F}$ commutes with generic base change. To finish the proof, we shall show that the formation of π_*K commutes with generic base change. As in the proof of (4.3iii), it suffices to show that there exists an open dense subscheme of S, over which the formations of π_*L commute with any base change $g : S' \to U$, for all $L \in \mathcal{L}$.

By ([2], 5.1.5), π is smooth, so étale locally it has a section. By (4.3i) we may assume that $\pi : BG \to X$ is a neutral gerbe, associated to a flat group space G/X. By (4.3vi) we can use dévissage and shrink X to an open subscheme. Using the same technique as the proof of ([17], 3.8), we can reduce to the case where G/X is smooth. For the reader's convenience, we briefly recall this reduction. Shrinking X, we may assume X is an integral scheme with function field $k(X)$, and G/X is a group scheme. There exists a finite field extension $L/k(X)$ such that G_{red} is smooth over Spec L. Factor $L/k(X)$ as a separable extension $L'/k(X)$ and a purely inseparable extension L/L'. Purely inseparable morphisms are universal homeomorphisms. By taking the normalization of X in these field extensions, we get a finite generically étale surjection $X' \to X$, such that G_{red} is generically smooth over X'. Shrinking X and X' we may assume $X' \to X$ is an étale surjection, and replacing X by X' (4.3i) we may assume G_{red} is generically smooth over X, and shrinking X we may assume G_{red} is smooth over X. Replacing G by G_{red} (since the morphism $BG_{\mathrm{red}} \to BG$ is representable and radicial) one can assume G/X is smooth.

Now $P : X \to BG$ is a presentation. Consider the associated smooth hypercover, and let $f_i : G^i \to X$ be the structural maps. We have the spectral sequence ([11], 10.0.9)

$$R^jf_{i*}f_i^*P^*L \Longrightarrow R^{i+j}\pi_*L.$$

As in the proof of ([17], 3.8), we can regard the map f_i as a product $\prod_i f_1$ and apply Künneth formula (shrinking X we can assume X is affine, so X satisfies the condition (LO), and we can apply ([11], 11.0.14))

$$f_{i*}f_i^*P^*L = f_{1*}f_1^*P^*L \otimes^L f_{1*}\Lambda_0 \otimes^L \cdots \otimes^L f_{1*}\Lambda_0.$$

Shrink S so that the formations of $f_{1*}f_1^*P^*L$ and $f_{1*}\Lambda_0$ commute with any base change on S. From the base change morphism of the spectral sequences

$$\begin{array}{ccc}
g^*R^jf_{i*}f_i^*P^*L & \Longrightarrow & g^*R^{i+j}\pi_*L \\
\| & & \\
\mathscr{H}^jg^*(f_{1*}f_1^*P^*L\otimes^L_{\Lambda_0} f_{1*}\Lambda_0\otimes^L_{\Lambda_0}\cdots\otimes^L_{\Lambda_0} f_{1*}\Lambda_0) & & \\
\| & & \\
\mathscr{H}^j\big((g^*f_{1*}f_1^*P^*L)\otimes^L_{\Lambda_0}(g^*f_{1*}\Lambda_0)\otimes^L_{\Lambda_0}\cdots\otimes^L_{\Lambda_0}(g^*f_{1*}\Lambda_0)\big) & & \\
\downarrow\wr & & \downarrow(1) \\
\mathscr{H}^j\big((f_{1*}g^*f_1^*P^*L)\otimes^L_{\Lambda_0}(f_{1*}g^*\Lambda_0)\otimes^L_{\Lambda_0}\cdots\otimes^L_{\Lambda_0}(f_{1*}g^*\Lambda_0)\big) & & \\
\| & & \\
\mathscr{H}^j\big((f_{1*}f_1^*P^*g^*L)\otimes^L_{\Lambda_0}(f_{1*}\Lambda_0)\otimes^L_{\Lambda_0}\cdots\otimes^L_{\Lambda_0}(f_{1*}\Lambda_0)\big) & & \\
\| & & \\
R^jf_{i*}f_i^*P^*g^*L & \Longrightarrow & R^{i+j}\pi_*g^*L
\end{array}$$

we see that the base change morphism (1) is an isomorphism.

(ii) For K and $L\in D_c^b(\mathcal{X},\Lambda)$, the complex $R\mathscr{H}om(K,L)$ is defined to be the image in $D_c(\mathcal{X},\Lambda)$ of the projective system $R\mathscr{H}om_{\Lambda_\bullet}(\widehat{K},\widehat{L})$, so we only need to prove the case where K and L are in $D_c^b(\mathcal{X},\Lambda_n)$.

Note that for an algebraic stack $\mathcal{X}$, $R\mathscr{H}om_{\mathcal{X}}$ takes $D_c^{b,\mathrm{op}}\times D_c^b$ into D_c^b. To see this, take a presentation $P:X\to\mathcal{X}$ of relative dimension d, for some locally constant function d on X. For bounded complexes K and L on $\mathcal{X}$, to show $R\mathscr{H}om_{\mathcal{X}}(K,L)$ is bounded, it suffices to show that $P^*R\mathscr{H}om_{\mathcal{X}}(K,L)$ is bounded. We have

$$\begin{aligned}
P^*R\mathscr{H}om_{\mathcal{X}}(K,L) &= P^!R\mathscr{H}om_{\mathcal{X}}(K,L)\langle -d\rangle = R\mathscr{H}om_X(P^*K,P^!L)\langle -d\rangle \\
&= R\mathscr{H}om_X(P^*K,P^*L),
\end{aligned}$$

which is bounded on X.

Let $g:S'\to S$ be any morphism, and consider the 2-Cartesian diagrams

$$\begin{array}{ccccc}
X_{S'} & \xrightarrow{P'} & \mathcal{X}_{S'} & \longrightarrow & S' \\
{\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow[P]{} & \mathcal{X} & \longrightarrow & S.
\end{array}$$

For the base change morphism

$$g'^*R\mathscr{H}om_{\mathcal{X}}(K,L)\to R\mathscr{H}om_{\mathcal{X}_{S'}}(g'^*K,g'^*L)$$

to be an isomorphism, we can check it locally on $X_{S'}$. Consider the commutative diagram

$$\begin{array}{ccc}
P'^*g'^*R\mathscr{H}om_{\mathcal{X}}(K,L) & \xrightarrow{(1)} & P'^*R\mathscr{H}om_{\mathcal{X}_{S'}}(g'^*K,g'^*L) \\
{\scriptstyle(2)}\downarrow & & \downarrow{\scriptstyle(3)} \\
g''^*P^*R\mathscr{H}om_{\mathcal{X}}(K,L) & & R\mathscr{H}om_{X_{S'}}(P'^*g'^*K,P'^*g'^*L) \\
{\scriptstyle(4)}\downarrow & & \downarrow{\scriptstyle(5)} \\
g''^*R\mathscr{H}om_X(P^*K,P^*L) & \xrightarrow{(6)} & R\mathscr{H}om_{X_{S'}}(g''^*P^*K,g''^*P^*L),
\end{array}$$

where (2) and (5) are canonical isomorphisms, (3) and (4) are proved to be isomorphisms above, and (6) is an isomorphism after shrinking S ([5], Th. finitude, 2.10). Therefore (1) is an isomorphism after shrinking S. ☐

Remark 4.5.1. This result generalizes ([16], 9.10ii), in that the open subscheme in S can be chosen to be independent of the index i as in $R^i f_* F$.

5 Complex analytic stacks.

In this section, we give some fundamental results on constructible sheaves and derived categories on the lisse-analytic topos of the analytification of a complex algebraic stack, and prove a comparison between the lisse-étale topos and the lisse-analytic topos of the stack.

5.1 Lisse-analytic topos.

For the definition of analytic stacks, we follow [14, 18]. Strictly speaking, Toen only discussed analytic Deligne-Mumford stacks in [18], and Noohi only discussed topological stacks in [14] (and mentioned analytic stacks briefly). I believe that they could have done the theory of analytic stacks in their papers. For completeness, we give a definition as follows.

Definition 5.1.1. *Let* **Ana-Sp** *be the site of complex analytic spaces with the analytic topology. A stack $\mathfrak{X}$ over this site is called an* analytic stack*, if the following hold:*

(i) the diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representible (by analytic spaces) and quasi-compact,

(ii) there exists an analytic smooth surjection $P : X \to \mathfrak{X}$, where X is an analytic space.

5.1.2. Similar to the lisse-étale topos of an algebraic stack, one can define the *lisse-analytic topos* $\mathfrak{X}_{\text{lis-an}}$ of an analytic stack $\mathfrak{X}$ to be the topos associated to the *lisse-analytic site* Lis-an$(\mathfrak{X})$ defined as follows:

• Objects: pairs $(U, u : U \to \mathfrak{X})$, where U is an complex analytic space and u is a smooth morphism (or an analytic *submersion*, in the topological terminology);

• Morphisms: a morphism $(U, u \in \mathfrak{X}(U)) \to (V, v \in \mathfrak{X}(V))$ is given by a pair (f, α), where $f : U \to V$ is a morphism of analytic spaces and $\alpha : vf \cong u$ is a 2-isomorphism in $\mathfrak{X}(U)$;

• Open coverings: $\{(j_i, \alpha_i) : (U_i, u_i \in \mathfrak{X}(U_i)) \to (U, u \in \mathfrak{X}(U))\}_{i \in I}$ is an open covering if the maps j_i's are open immersions of analytic subspaces and their images cover U.

To give a sheaf $F \in \mathfrak{X}_{\text{lis-an}}$ is equivalent to giving the data

• for every $(U, u) \in$ Lis-an$(\mathfrak{X})$, a sheaf F_u in the analytic topos U_{an} of U, and

• for every morphism $(f, \alpha) : (U, u) \to (V, v)$, a morphism $f^* : f^{-1}F_v \to F_u$.

The sheaf F is *Cartesian* if f^* is an isomorphism, for every (f, α). By abuse of notation, we will also denote "F_u" by "F_U", if there is no confusion about the reference to u.

This topos is equivalent to the "lisse-étale" topos $\mathfrak{X}_{\text{lis-ét}}$ associated to the site Lis-ét$(\mathfrak{X})$ with the same underlying category as that of Lis-an$(\mathfrak{X})$, but the open coverings are surjective families of local isomorphisms. This is because the two topologies are cofinal: for a local isomorphism $V \to U$ of analytic spaces, there exists an open covering $\{V_i \subset V\}_i$ of V by analytic subspaces, such that for each i, the composition $V_i \subset V \to U$ is isomorphic to the natural map from a disjoint union of open analytic subspaces of U to U.

5.2 Locally constant sheaves and constructible sheaves.

For a sheaf on the analytic site of an analytic space, we say that the sheaf is *locally constant constructible*, abbreviated as *lcc*, if it is locally constant with respect to the analytic topology, and has finite stalks.

Let $\mathfrak{X}$ be an analytic stack. For a Cartesian sheaf $F \in \mathfrak{X}_{\text{lis-an}}$, we say that F is *locally constant* (resp. *lcc*) if the conditions in the following (5.2.1) hold. This lemma is an analytic version of ([16], 9.1).

Lemma 5.2.1. *Let $F \in \mathfrak{X}_{\text{lis-an}}$ be a Cartesian sheaf. Then the following are equivalent.*

(i) For every $(U, u) \in \text{Lis-an}(\mathfrak{X})$, the sheaf F_U is locally constant (resp. lcc).

(ii) There exists an analytic presentation $P : X \to \mathfrak{X}$ such that F_X is locally constant (resp. lcc).

Proof. It suffices to show that (ii)⇒(i), which is similar to that of ([16], 9.1). There exists an open covering $U = \cup U_i$, such that over each U_i, the smooth surjection $X \times_{P,\mathfrak{X},u} U \to U$ has a section s_i :

$$\begin{array}{ccccc} & & X \times_{\mathfrak{X}} U & \longrightarrow & X \\ & {\scriptstyle s_i} \nearrow & \downarrow & & \downarrow {\scriptstyle P} \\ U_i & \hookrightarrow & U & \xrightarrow{u} & \mathfrak{X}. \end{array}$$

Therefore $F_{U_i} \simeq s_i^* F_{X \times_{\mathfrak{X}} U}$, which is locally constant (resp. lcc). □

5.2.2. Let $\mathcal{X}$ be a complex algebraic stack. Following ([14], 20), one can define its associated analytic stack $\mathcal{X}^{\text{an}}$ as follows. If $X_1 \rightrightarrows X_0 \to \mathcal{X}$ is a smooth groupoid presentation, then $\mathcal{X}^{\text{an}}$ is defined to be the analytic stack given by the presentation $X_1^{\text{an}} \rightrightarrows X_0^{\text{an}}$, and it can be proved that this is independent of the choice of the presentation, up to an isomorphism that is unique up to 2-isomorphism. Similarly, for a morphism $f : \mathcal{X} \to \mathcal{Y}$ of complex algebraic stacks, one can choose their presentations so that f lifts to a morphism of groupoids, hence induces a morphism of their analytifications, denoted $f^{\text{an}} : \mathcal{X}^{\text{an}} \to \mathcal{Y}^{\text{an}}$. The analytification functor preserves 2-Cartesian products.

5.2.3. Let $\mathfrak{X} = \mathcal{X}^{\text{an}}$ for a complex algebraic stack $\mathcal{X}$, and let $P : X \to \mathcal{X}$ be a presentation. For a Cartesian sheaf $F \in \mathfrak{X}_{\text{lis-an}}$, we say that F is *constructible*, if for every $(U, u) \in \text{Lis-ét}(\mathcal{X})$, the sheaf $F_{U^{\text{an}}}$ is constructible, i.e. lcc on each stratum in an *algebraic* stratification of the analytic space U^{an}.

One could also define a notion of *analytic constructibility*, using analytic stratifications rather than algebraic ones, but this notion will not give us a comparison between the constructible derived categories of the lisse-étale topos and of the lisse-analytic topos.

Lemma 5.2.4. *Let $F \in \mathfrak{X}_{\text{lis-an}}$ be a Cartesian sheaf. Then the following are equivalent.*

(i) F is constructible.

(ii) $F_{X^{\text{an}}}$ is constructible on X^{an} (in the algebraic sense above).

(iii) There exists an algebraic stratification $\mathscr{S}^{\text{an}}$ on $\mathfrak{X}$, such that for each stratum U^{an}, the sheaf $F_{U^{\text{an}}}$ is lcc.

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii). Let $\mathscr{S}_X$ be a stratification of the scheme X, such that for each $U \in \mathscr{S}_X$, the sheaf $F_{U^{\text{an}}}$ is lcc. Let U be an open stratum, and let V be the image of U under the map P; then V is an open substack of $\mathcal{X}$, and $P_U : U \to V$ is a presentation. Let $V' \to V^{\text{an}}$ be an analytic presentation. There exists an analytic open covering $V' = \cup V'_i$, over which P_U^{an} has a section:

$$\begin{array}{ccccccc} & & & & U^{\text{an}} & \hookrightarrow & X^{\text{an}} \\ & & {\scriptstyle s_i} \nearrow & & \downarrow {\scriptstyle P_U^{an}} & & \downarrow {\scriptstyle P^{an}} \\ V'_i & \hookrightarrow & V' & \longrightarrow & V^{\text{an}} & \hookrightarrow & \mathfrak{X}, \end{array}$$

so $F_{V'_i} \simeq s_i^{-1} F_{U^{\text{an}}}$ is lcc, therefore $F_{V'}$ (and hence $F_{V^{\text{an}}}$, by (5.2.1)) is lcc. Note that $X - P^{-1}(V) \to \mathcal{X} - V$ gives an algebraic presentation of $(\mathcal{X} - V)^{\text{an}} = \mathfrak{X} - V^{\text{an}}$, and

$$(F|_{\mathfrak{X}-V^{\text{an}}})_{(X-P^{-1}(V))^{\text{an}}} \simeq F_{X^{\text{an}}}|_{(X-P^{-1}(V))^{\text{an}}}$$

is still constructible, so by noetherian induction we are done.

(iii)⇒(i). Let $(U, u) \in \text{Lis-ét}(\mathcal{X})$. Then $u^{\text{an},*}\mathscr{S}^{\text{an}} = (u^*\mathscr{S})^{\text{an}}$ is an algebraic stratification of U^{an}, and it is clear that $F_{U^{\text{an}}}$ is lcc on each stratum of this stratification. $\square$

5.2.5. A *constructible Λ_n-module* on $\mathfrak{X}_{\text{lis-an}}$ is a Λ_n-sheaf, which is constructible as a sheaf of sets. They form a full subcategory of $\text{Mod}(\mathfrak{X}, \Lambda_n)$ that is closed under kernels, cokernels and extensions (i.e. it is a Serre subcategory). It suffices to show that Cartesian sheaves form a Serre subcategory, because lcc Λ_n-modules form a Serre subcategory, and one can use (5.2.4iii).

Let $(f, \alpha) : (U, u) \to (V, v)$ be a morphism in $\text{Lis-an}(\mathfrak{X})$. Note that the functor $F \mapsto f^*F : \text{Mod}(V_{\text{an}}, \Lambda_n) \to \text{Mod}(U_{\text{an}}, \Lambda_n)$ is exact, because $f^*F = \Lambda_{n,U} \otimes_{f^{-1}\Lambda_{n,V}} f^{-1}F = f^{-1}F$. Let $a : F \to G$ be a morphism of Cartesian sheaves. Then $\text{Ker}(f^*a_V : f^*F_V \to f^*G_V) = f^*\text{Ker}(a_V)$, and it is clear that the induced morphism $f^*\text{Ker}(a_V) \to \text{Ker}(a_U)$ is an isomorphism:

$$\begin{array}{ccccc} f^*\text{Ker}(a_V) & \longrightarrow & f^*F_V & \xrightarrow{f^*a_V} & f^*G_V \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ \text{Ker}(a_U) & \longrightarrow & F_U & \xrightarrow{a_U} & G_U. \end{array}$$

The proof for cokernels and extensions (using 5-lemma) is similar. One can also mimic the proof in ([16], 3.8, 3.9) to prove a similar statement for analytic stacks, in the more general situation where the coefficient ring is a *flat sheaf*. In this paper, we will only need the case of a constant coefficient ring.

5.3 Derived categories.

5.3.1. Again assume $\mathfrak{X} = \mathcal{X}^{\text{an}}$. Let $D(\mathfrak{X}_{\text{lis-an}}, \Lambda_n)$ be the ordinary derived category of Λ_n-modules on $\mathfrak{X}$. By (5.2.5), we have the triangulated subcategory $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda_n)$ of complexes with constructible cohomology sheaves. We follow [11] and define the *derived category $D_c(\mathfrak{X}_{lis\text{-}an}, \Lambda)$ of constructible Λ-adic sheaves* (by abuse of language, as usual) as follows. A complex of projective systems K in the ordinary derived category $\mathscr{D}(\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}, \Lambda_\bullet)$ of the simplicial topos $\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}$ ringed by $\Lambda_\bullet = (\Lambda_n)_n$, is called a *$\lambda$-complex* if for every i and n, the sheaf $\mathscr{H}^i(K_n)$ is constructible and the cohomology system $\mathscr{H}^i(K)$ is AR-adic. A *λ-module* is a λ-complex concentrated in degree 0. Then we define $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ to be the quotient of the full subcategory $\mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}_{\text{lis-an}}, \Lambda_\bullet)$ of λ-complexes by the full subcategory of AR-null complexes (i.e. those with AR-null cohomology systems).

This quotient has a natural t-structure, and we define the category $\Lambda\text{-Sh}_c(\mathfrak{X})$ of *constructible Λ-adic sheaves* on $\mathfrak{X}_{\text{lis-an}}$ to be its core, namely the quotient of the AR-adic projective systems with constructible components by the thick full subcategory of AR-null systems. By ([8], p.234), this is equivalent to the category of adic systems, i.e. those projective systems $F = (F_n)_n$, such that for each n, F_n is a constructible Λ_n-module on $\mathfrak{X}_{\text{lis-an}}$, and the induced morphism $F_n \otimes_{\Lambda_n} \Lambda_{n-1} \to F_{n-1}$ is an isomorphism.

Using localization and 2-colimit, one can also define the categories $D_c(\mathfrak{X}_{\text{lis-an}}, E_\lambda)$ and $D_c(\mathfrak{X}_{\text{lis-an}}, \overline{\mathbb{Q}}_\ell)$, and their cores, the categories of constructible E_λ or $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathfrak{X}_{\text{lis-an}}$.

5.3.2. Let $\text{Mod}(\mathfrak{X}, \mathbb{C})$ be the category of sheaves of $\mathbb{C}$-vector spaces on $\mathfrak{X}_{\text{lis-an}}$, with $\mathbb{C}$-linear morphisms, and define the category $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$ of *constructible* $\mathbb{C}_{\mathfrak{X}}$*-modules* to be the full subcategory of $\text{Mod}(\mathfrak{X}, \mathbb{C})$ consisting of those sheaves M, such that there exists an *algebraic stratification* $\mathscr{S}$ of $\mathfrak{X}$, over each stratum of which M is locally constant, and stalks of M are finite dimensional $\mathbb{C}$-vector spaces. Note that, in order for $M|_U$ to be constant, we may have to refine $\mathscr{S}$ to an analytic stratification that is not necessarily algebraic. Then we define $D_c(\mathfrak{X}_{\text{lis-an}}, \mathbb{C})$ to be the full subcategory of the ordinary derived category of $\mathbb{C}_{\mathfrak{X}}$-modules, consisting of those sheaf complexes with constructible cohomology sheaves. The core of the natural t-structure on $D_c(\mathfrak{X}_{\text{lis-an}}, \mathbb{C})$ is $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$.

Similarly, one can also define the category $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$ of *constructible* $\Lambda_{\mathfrak{X}}$*-modules*, i.e. $\Lambda_{\mathfrak{X}}$-sheaves for which there exists an algebraic stratification of $\mathfrak{X}$, such that over each stratum the sheaf is locally constant, and stalks are finitely generated Λ-modules. Then we define $\mathscr{D}_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ (and also with E_λ- and $\overline{\mathbb{Q}}_\ell$-coefficients) to be the full subcategory of the ordinary derived category (denoted $\mathscr{D}(\mathfrak{X}_{\text{lis-an}}, \Lambda)$) of $\Lambda_{\mathfrak{X}}$-modules, consisting of those with constructible cohomology. In (5.5.4), we will show that the two derived categories $D_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ and $\mathscr{D}_c(\mathfrak{X}_{\text{lis-an}}, \Lambda)$ are equivalent.

For simplicity, for any coefficient Ω, we will usually drop "lis-an" in $D_c(\mathfrak{X}_{\text{lis-an}}, \Omega)$, if there is no confusion. Also we will drop "lis-ét" in $D_c(\mathcal{X}_{\text{lis-ét}}, \Omega)$.

In the following lemma, we show that the category $\Lambda\text{-Sh}_c(\mathfrak{X})$ admits a similar description as $\text{Mod}_c(\mathfrak{X}, \mathbb{C})$.

Lemma 5.3.3. *There is a natural equivalence between* $\Lambda\text{-Sh}_c(\mathfrak{X})$ *and* $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$.

Proof. Firstly, we define the functor $\phi : \Lambda\text{-Sh}_c(\mathfrak{X}) \to \mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$. Let $F = (F_n)_n$ be an adic sheaf on $\mathfrak{X}_{\text{lis-an}}$, and define $\phi(F)$ to be $\varprojlim_n (F_n)_n$. For a morphism $b : F \to G$ of adic sheaves, define $\phi(b)$ to be the induced morphism on their inverse limits.

Then we show it is well-defined. Let $P : X \to \mathcal{X}$ be a presentation. Then $F_{X^{\text{an}}} := (F_{n,X^{\text{an}}})_n$ is an adic sheaf on X^{an}. By the comparison ([3], 6.1.2, (A'')), $F_{X^{\text{an}}}$ is algebraic, i.e. it comes from a constructible Λ-adic sheaf G on $X_{\text{ét}}$. Since X is noetherian, G is lisse over the strata of a stratification $\mathscr{S}_X$ of X. Let $U \in \mathscr{S}_X$ be an open stratum, and let V be its image under P. Then $V \subset \mathcal{X}$ is an open substack and $P_U : U \to V$ is a presentation. We have $(G_U)^{\text{an}} = \phi(F)_{U^{\text{an}}} \cong P_U^{\text{an},*}(\phi(F)_{V^{\text{an}}})$, and it is the Λ-local system on $U(\mathbb{C})$ obtained by restricting the continuous representation ρ_{G_U} of $\pi_1^{\text{ét}}(U, \overline{u})$ corresponding to the lisse sheaf G_U to $\pi_1^{\text{top}}(U^{\text{an}}, \overline{u})$:

$$\pi_1^{\text{top}}(U^{\text{an}}, \overline{u}) \longrightarrow \pi_1^{\text{ét}}(U, \overline{u}) \xrightarrow{\rho_{G_U}} \text{GL}(G_{U,\overline{u}}).$$

The sheaf $\phi(F)_{U^{\text{an}}}$ is locally constant because U^{an} is covered by contractible analytic open subspaces.

As in the proof of (5.2.4), one can take an analytic presentation $V' \to V^{\text{an}}$, and cover the analytic space V' by analytic open subspaces V'_i, such that P_U^{an} has a section s_i over each V'_i, and so $\phi(F)_{V'_i}$ is locally constant with stalks finitely generated Λ-modules, and the same is true for $\phi(F)_{V^{\text{an}}}$. Finally apply noetherian induction to the complement $\mathfrak{X} - V^{\text{an}}$ to finish the proof that ϕ is well-defined.

Then we define a functor $\psi : \mathfrak{Mod}_c(\Lambda_{\mathfrak{X}}) \to \Lambda\text{-Sh}_c(\mathfrak{X})$. Given a $\Lambda_{\mathfrak{X}}$-module M, let $M_n = M \otimes_\Lambda \Lambda_n$, and define $\psi(M)$ to be the adic system $(M_n)_n$. For a morphism $c : M \to N$ of constructible $\Lambda_{\mathfrak{X}}$-modules, define $\psi(c)_n$ to be $c \otimes \Lambda_n$.

We need to show $\psi(M)$ gives a constructible Λ-adic sheaf on $\mathfrak{X}_{\text{lis-an}}$. It is clearly adic. To show each M_n is constructible, by (5.2.4), it suffices to show that there exists an algebraic stratification of $\mathfrak{X}$, such that over each stratum M_n is lcc. This follows from the definition of $\mathfrak{Mod}_c(\Lambda_{\mathfrak{X}})$.

Finally, note that ϕ and ψ are quasi-inverse to each other. □

5.4 Comparison between the derived categories of lisse-étale and lisse-analytic topoi.

Given an algebraic stack $\mathcal{X}/\mathbb{C}$, let $\mathfrak{X} = \mathcal{X}^{\text{an}}$, and let $P : X \to \mathcal{X}$ be a presentation, with analytification $P^{\text{an}} : X^{\text{an}} \to \mathfrak{X}$. Let $\epsilon : X_\bullet \to \mathcal{X}$ be the associated strictly simplicial smooth hypercover, and let $\epsilon^{\text{an}} : X_\bullet^{\text{an}} \to \mathfrak{X}$ be the analytification. They induce morphisms of topoi, denoted by the same symbol. Consider the following morphisms of topoi:

$$\begin{array}{ccccc} \mathfrak{X}_{\text{lis-an}} & \xleftarrow{\gamma^{\text{an}}} & \mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\text{an}}} & \xrightarrow{\delta_\bullet^{\text{an}}} & X_{\bullet,\text{an}}^{\text{an}} \\ & \multicolumn{3}{c}{\curvearrowleft \ \epsilon^{\text{an}}} & \downarrow \xi_\bullet \\ \mathcal{X}_{\text{lis-ét}} & \xleftarrow{\gamma} & \mathcal{X}_{\text{lis-ét}}|_{X_\bullet} & \xrightarrow{\delta_\bullet} & X_{\bullet,\text{ét}}. \\ & \multicolumn{3}{c}{\curvearrowleft \ \epsilon} & \end{array}$$

Following ([11], 10.0.6), we define the derived category $D_c(X_\bullet^{\text{an}}, \Lambda)$ as follows. A sheaf $F \in \text{Mod}(X_\bullet^{\text{an},\mathbb{N}}, \Lambda_\bullet)$ is *AR-adic* if it is Cartesian (in the $\bullet$-direction) and $F|_{X_{n,\text{an}}^{\text{an},\mathbb{N}}}$ is AR-adic for every n. A complex $C \in \mathscr{D}(X_\bullet^{\text{an},\mathbb{N}}, \Lambda_\bullet)$ is a *λ-complex* (resp. an *AR-null complex*) if the cohomology sheaf $\mathscr{H}^i(C)$ is AR-adic and $\mathscr{H}^i(C_m)|_{X_n^{\text{an}}}$ is constructible, for every i, m, n (resp. $C|_{X_n^{\text{an}}}$ is AR-null, for every n). Finally we define $D_c(X_\bullet^{\text{an}}, \Lambda)$ to be the quotient of the full subcategory $\mathscr{D}_c(X_\bullet^{\text{an},\mathbb{N}}, \Lambda_\bullet) \subset \mathscr{D}(X_\bullet^{\text{an},\mathbb{N}}, \Lambda_\bullet)$ consisting of all λ-complexes by the full subcategory of AR-null complexes.

Using the diagram above, we will show that $R\epsilon_* \circ R\xi_{\bullet,*} \circ \epsilon^{\text{an},*}$ gives an equivalence between $D_c(\mathfrak{X}, \Lambda)$ and $D_c(\mathcal{X}, \Lambda)$, and it is compatible with pushforwards. It is proved in ([11], 10.0.8) that, $(\epsilon^*, R\epsilon_*)$ induce an equivalence between the triangulated categories $D_c(\mathcal{X}, \Lambda)$ and $D_c(X_\bullet, \Lambda)$. We mimic the proof to give a proof of the analytic analogue.

Proposition 5.4.1. *(i) The functors $(\epsilon^{\text{an},*}, R\epsilon_*^{\text{an}})$ induce an equivalence between the triangulated categories $D_c(\mathfrak{X}, \Lambda)$ and $D_c(X_\bullet^{\text{an}}, \Lambda)$.*

(ii) Let X be a $\mathbb{C}$-scheme, and let $\xi : X^{\text{an}} \to X_{\text{ét}}$ be the natural morphism of topoi. Then $R\xi_*$ is defined on the unbounded derived category, and the functors $(\xi^*, R\xi_*)$ induce an equivalence between $D_c(X^{\text{an}}, \Lambda)$ and $D_c(X, \Lambda)$.

(iii) Let $f : X \to Y$ be a morphism of $\mathbb{C}$-schemes, and let ξ_X, ξ_Y be as in (ii). Then for every $F \in D_c^+(X, \Lambda)$, the natural morphism

$$\xi_Y^* f_* F \to f_*^{\text{an}}(\xi_X^* F)$$

is an isomorphism.

Proof. (i) Firstly, note that $\delta_{\bullet,*}^{\text{an}} : \text{Ab}(\mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\text{an}}}) \to \text{Ab}(X_{\bullet,\text{an}}^{\text{an}})$ is exact, since the topologies are the same. So in fact, $R\delta_{\bullet,*}^{\text{an}} = \delta_{\bullet,*}^{\text{an}}$. The functor $\delta_{n,*}^{\text{an}}$ is the restriction functor, and $\delta_n^{\text{an},*}$

takes a sheaf $F \in X_{n,\mathrm{an}}^{\mathrm{an}}$ to the sheaf $\delta_n^{\mathrm{an},*}F$ that assigns to the object

$$\begin{array}{ccc} U & \xrightarrow{u} & X_n^{\mathrm{an}} \\ & \searrow & \downarrow \\ & & \mathfrak{X} \end{array}$$

the sheaf u^*F on U_{an}. It is clear that $(\delta_n^{\mathrm{an},*}, \delta_{n,*}^{\mathrm{an}})$ induce an equivalence between the category $\mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X_n^{\mathrm{an}}}, \Lambda_m)$ of Cartesian sheaves on the localized topos $\mathfrak{X}|_{X_n^{\mathrm{an}}}$ and $\mathrm{Mod}(X_n^{\mathrm{an}}, \Lambda_m)$. For $K \in D(X_n^{\mathrm{an}}, \Lambda_m)$, we see that the adjunction morphism

$$K \to \delta_{n,*}^{\mathrm{an}} \delta_n^{\mathrm{an},*} K$$

is an isomorphism by applying $\mathscr{H}^i$:

$$\mathscr{H}^i K \to \mathscr{H}^i(\delta_{n,*}^{\mathrm{an}} \delta_n^{\mathrm{an},*} K) \simeq \delta_{n,*}^{\mathrm{an}} \delta_n^{\mathrm{an},*} \mathscr{H}^i(K),$$

noting that $\delta_{n,*}^{\mathrm{an}}$ is exact. Similarly, if $K \in D(\mathfrak{X}|_{X_n^{\mathrm{an}}}, \Lambda_m)$ has Cartesian cohomology sheaves, the coadjunction morphism

$$\delta_n^{\mathrm{an},*} \delta_{n,*}^{\mathrm{an}} K \to K$$

is an isomorphism. Hence $(\delta_\bullet^{\mathrm{an},*}, \delta_{\bullet,*}^{\mathrm{an}})$ induce an equivalence

$$D_{\mathrm{cart}}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \leftrightarrow D(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

We will show later that constructible sheaves form a Serre subcategory in $\mathrm{Mod}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$, and then it is also clear that $(\delta_\bullet^{\mathrm{an},*}, \delta_{\bullet,*}^{\mathrm{an}})$ gives an equivalence

$$D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \leftrightarrow D_c(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

To show $\gamma^{\mathrm{an},*}$ induces an equivalence on the torsion level Λ_m, we will apply ([10], 2.2.3). For the morphism $\gamma^{\mathrm{an}} : (\mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \to (\mathfrak{X}_{\text{lis-an}}, \Lambda_m)$, all the transition morphisms of topoi in the strictly simplicial ringed topos $(\mathfrak{X}_{\text{lis-an}}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$ as well as γ^{an} are flat. Let $\mathscr{C}$ be the category of constructible Λ_m-modules on $\mathfrak{X}_{\text{lis-an}}$, which is a Serre subcategory (5.2.5). We need to verify the assumption ([10], 2.2.1), which has two parts.

• ([10], 2.1.2) for the ringed site $(\text{Lis-an}(\mathfrak{X})|_{X_n^{\mathrm{an}}}, \Lambda_m)$ with $\mathscr{C}$ = constructible Λ_m-modules. This means, for every object U in this site, we need to show that there exist an analytic open covering $U = \cup U_i$ and an integer n_0, such that for every constructible Λ_m-module F on this site and $n \geq n_0$, we have $H^n(U_i, F) = 0$. This follows from ([6], 3.1.5, 3.4.1).

• $\gamma^{\mathrm{an},*} : \mathscr{C} \to \mathscr{C}_\bullet$ is an equivalence with quasi-inverse $R\gamma_*^{\mathrm{an}}$. Here $\mathscr{C}_\bullet$ is the essential image of $\mathscr{C}$ under $\gamma^{\mathrm{an},*} : \mathrm{Mod}(\mathfrak{X}, \Lambda_m) \to \mathrm{Mod}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$, called the *category of constructible sheaves* in the target. Recall that, an object in $\mathrm{Mod}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$ is given by a family of objects $F_i \in \mathrm{Mod}(\mathfrak{X}|_{X_i^{\mathrm{an}}}, \Lambda_m)$ indexed by i, together with transition morphisms $a^*F_j \to F_i$ for each $a : i \to j$ in the strictly simplicial set $\Delta^{+,\mathrm{op}}$. Consider the commutative diagram

$$\begin{array}{ccc} X_i^{\mathrm{an}} & \xrightarrow{a} & X_j^{\mathrm{an}} \\ & \searrow & \downarrow \\ & & \mathfrak{X}. \end{array}$$

For $F \in \mathrm{Mod}(\mathfrak{X}, \Lambda_m)$, its image $\gamma^{\mathrm{an},*}F$ is given by $F_i = F_{X_i^{\mathrm{an}}} \in \mathrm{Mod}(X_i^{\mathrm{an}}, \Lambda_m) \simeq \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X_i^{\mathrm{an}}}, \Lambda_m)$, and the transition morphisms $a^*F_j \to F_i$ are part of the data in the definition of F. One can prove the analytic version of ([16], 4.4, 4.5) stated as follows.

Let $\mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$ be the category of pairs (F, α), where $F \in \mathrm{Mod}(X^{\mathrm{an}}, \Lambda_m)$, and $\alpha : p_1^* F \to p_2^* F$ is an isomorphism on the analytic topos $X^{\mathrm{an}}_{1,\mathrm{an}}$ (where p_1 and p_2 are the natural projections $X_1^{\mathrm{an}} \rightrightarrows X_0^{\mathrm{an}} = X^{\mathrm{an}}$), such that $p_{13}^*(\alpha) = p_{23}^*(\alpha) \circ p_{12}^*(\alpha) : \bar{p}_1^* F \to \bar{p}_3^* F$ on X_2^{an}. Here $\bar{p}_i : X_2 \to X_0$ are the natural projections. There is a natural functor $A : \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}, \Lambda_m) \to \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$, sending M to (F, α), where $F = M_{X^{\mathrm{an}}}$ and α is the composite

$$p_1^* F \xrightarrow{p_1^*} M_{X_1^{\mathrm{an}}} \xrightarrow{(p_2^*)^{-1}} p_2^* F.$$

There is also a natural functor $B : \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an},+}, \Lambda_m) \to \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m)$ sending $F = (F_i)_i$ to (F_0, α), where α is the composite

$$p_1^* F_0 \xrightarrow{\mathrm{can}} F_1 \xrightarrow{\mathrm{can}^{-1}} p_2^* F_0,$$

and the cocycle condition is verified as in ([16], 4.5.4).

Lemma 5.4.2. *The natural functors in the diagram*

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an}}, \Lambda_m) & \xrightarrow{\mathrm{res}} & \mathrm{Mod}_{\mathrm{cart}}(X_\bullet^{\mathrm{an},+}, \Lambda_m) \\ {\scriptstyle \epsilon^{\mathrm{an},*}}\uparrow & & \downarrow{\scriptstyle B} \\ \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}, \Lambda_m) & \xrightarrow{A} & \mathrm{Des}(X^{\mathrm{an}}/\mathfrak{X}, \Lambda_m) \end{array}$$

are all equivalences, and the diagram is commutative up to natural isomorphism.

The proof in ([16], 4.4, 4.5) carries verbatim to analytic stacks, so we do not write down the proof again. This finishes the verification of ([10], 2.2.1). In particular, $\mathscr{C}_\bullet = \mathrm{Mod}_{\mathrm{cart}}(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m)$ is a Serre subcategory ([10], 2.2.2), so as we mentioned before, $(\delta_\bullet^{\mathrm{an},*}, \delta_{\bullet,*}^{\mathrm{an}})$ give an equivalence

$$D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \leftrightarrow D_c(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

By ([10], 2.2.3), the functors $(\gamma^{\mathrm{an},*}, R\gamma_*^{\mathrm{an}})$ induce an equivalence

$$D_c(\mathfrak{X}, \Lambda_m) \leftrightarrow D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m).$$

It is clear that the composition of equivalences

$$D_c(\mathfrak{X}, \Lambda_m) \xrightarrow{\gamma^{\mathrm{an},*}} D_c(\mathfrak{X}|_{X_\bullet^{\mathrm{an}}}, \Lambda_m) \xrightarrow{\delta_{\bullet,*}^{\mathrm{an}}} D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$$

is just $\epsilon^{\mathrm{an},*}$ (they are both restrictions). Since $\delta_{\bullet,*}^{\mathrm{an}}$ is the quasi-inverse of $\delta_\bullet^{\mathrm{an},*}$, it is both a left adjoint and a right adjoint of $\delta_\bullet^{\mathrm{an},*}$. This implies that $R\epsilon_*^{\mathrm{an}} = R\gamma_*^{\mathrm{an}} \circ \delta_\bullet^{\mathrm{an},*}$ and it is a quasi-inverse of the equivalence

$$\epsilon^{\mathrm{an},*} : D_c(\mathfrak{X}, \Lambda_m) \to D_c(X_\bullet^{\mathrm{an}}, \Lambda_m).$$

Note that for $M \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$ (resp. $\mathscr{D}_c(X_\bullet^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet)$), each level M_m is in $D_c(\mathfrak{X}, \Lambda_m)$ (resp. $D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$), and the property of M being AR-adic (resp. AR-null) is intrinsic ([8], V, 3.2.3). So the notions of AR-adic (resp. AR-null) on the two sides correspond under this equivalence. Therefore, we get equivalences

$$(\epsilon^{\mathrm{an},*}, R\epsilon_*^{\mathrm{an}}) : \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet) \leftrightarrow \mathscr{D}_c(X_\bullet^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet)$$

and

$$(\epsilon^{\mathrm{an},*}, R\epsilon_*^{\mathrm{an}}) : D_c(\mathfrak{X}, \Lambda) \leftrightarrow D_c(X_\bullet^{\mathrm{an}}, \Lambda).$$

(ii) This is a generalization of ([3], 6.1.2 (B'')), which says that $\xi^* : D_c^b(X,\Lambda) \to D_c^b(X^{\mathrm{an}},\Lambda)$ is an equivalence. We prove it on the torsion level first.

For a Λ_n-module G on X^{an}, the sheaf $R^i\xi_*G$ on $X_{\text{ét}}$ is the sheafification of the presheaf

$$(U \to X) \mapsto H^i(U^{\mathrm{an}}, G).$$

By ([6], 3.1.5, 3.4.1), $R^i\xi_*G = 0$ for all sheaves G and all $i > 1 + 2\dim_{\mathbb{C}} X$, so $R\xi_*$ has finite cohomological dimension, hence it extends to a functor

$$R\xi_* : D(X^{\mathrm{an}}, \Lambda_n) \to D(X, \Lambda_n).$$

It takes the full subcategory $D_c(X^{\mathrm{an}}, \Lambda_n)$ into $D_c(X, \Lambda_n)$, since for any i there exist integers a and b such that $R^i\xi_*G = R^i\xi_*\tau_{[a,b]}G$.

Given $F \in D_c(X, \Lambda_n)$, we want to show that the adjunction morphism $F \to R\xi_*\xi^*F$ is an isomorphism. Recall that ξ^* is the analytification functor, which is exact. For each $i \in \mathbb{N}$, we want to show the morphism

$$\mathscr{H}^i F \to R^i\xi_*\xi^*F$$

is an isomorphism. Consider the spectral sequence

$$R^p\xi_*\xi^*\mathscr{H}^q F \Longrightarrow R^{p+q}\xi_*\xi^*F,$$

where $R^p\xi_*\xi^*\mathscr{H}^q F$ is the sheafification of the functor

$$(U \to X) \mapsto H^p(U^{\mathrm{an}}, \xi^*\mathscr{H}^q F) = H^p(U^{\mathrm{an}}, (\mathscr{H}^q F)^{\mathrm{an}}).$$

By the comparison theorem of Artin ([1], XVI, 4.1), we have $H^p(U^{\mathrm{an}}, (\mathscr{H}^q F)^{\mathrm{an}}) = H^p(U, \mathscr{H}^q F)$, and this presheaf sheafifies to zero if $p > 0$ ([13], 10.4). When $p = 0$, the sheafification is obviously $\mathscr{H}^q F$. Therefore, the spectral sequence degenerates to isomorphisms

$$\mathscr{H}^i F = \xi_*\xi^*\mathscr{H}^i F \simeq R^i\xi_*\xi^*F,$$

and the adjunction morphism is an isomorphism.

Given $G \in D_c(X^{\mathrm{an}}, \Lambda_n)$, we want to show that the coadjunction morphism $\xi^*R\xi_*G \to G$ is an isomorphism. Consider the spectral sequence

$$\xi^*R^p\xi_*\mathscr{H}^q G \Longrightarrow \xi^*R^{p+q}\xi_*G,$$

where $\xi^*R^p\xi_*\mathscr{H}^q G$ is the analytification of the sheafification of the presheaf on Ét(X)

$$(U \to X) \mapsto H^p(U^{\mathrm{an}}, \mathscr{H}^q G).$$

By the comparison ([3], 6.1.2 (A')), the constructible Λ_n-sheaf $\mathscr{H}^q G$ is algebraic, therefore by Artin's comparison theorem ([1], XVI, 4.1) again, $\xi^*R^p\xi_*\mathscr{H}^q G = 0$ for $p > 0$, and the spectral sequence degenerates to isomorphisms

$$\mathscr{H}^i G = \xi^*\xi_*\mathscr{H}^i G \simeq \xi^*R^i\xi_*G.$$

This proves that we have an equivalence

$$(\xi^*, R\xi_*) : D_c(X^{\mathrm{an}}, \Lambda_n) \leftrightarrow D_c(X, \Lambda_n)$$

for each n. As in the proof of (i), the notions of being AR-adic (resp. AR-null) for complexes in $\mathscr{D}(X^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet)$ and in $\mathscr{D}(X^{\mathbb{N}}, \Lambda_\bullet)$ are the same, therefore, we have equivalences

$$(\xi^*, R\xi_*) : \mathscr{D}_c(X^{\mathrm{an},\mathbb{N}}, \Lambda_\bullet) \leftrightarrow \mathscr{D}_c(X^{\mathbb{N}}, \Lambda_\bullet)$$

and

$$(\xi^*, R\xi_*) : D_c(X^{\mathrm{an}}, \Lambda) \leftrightarrow D_c(X, \Lambda).$$

(iii) Applying $\mathscr{H}^i$ on both sides, we should show that

$$\xi_Y^* R^i f_* F \to R^i f_*^{\mathrm{an}}(\xi_X^* F)$$

is an isomorphism. Replacing F by various levels $\widehat{F}_n$ of its normalization, we reduce to the case where $F \in D_c^+(X, \Lambda_n)$. We know that f_* and f_*^{an} have finite cohomological dimension (for instance by generic base change), so one can replace F by $\tau_{[a,b]}F$ and reduce to the case where F is bounded. Taking truncations again and using 5-lemma, we reduce to the case where F is a constructible Λ_n-sheaf, and this follows from Artin's comparison ([1], XVI, 4.1). ☐

5.4.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of $\mathbb{C}$-algebraic stacks. Choose a commutative diagram

$$\begin{array}{ccc} X_\bullet & \xrightarrow{\widetilde{f}} & Y_\bullet \\ {\scriptstyle \epsilon_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \epsilon_{\mathcal{Y}}} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}. \end{array}$$

Then by construction, the diagram

$$\begin{array}{ccc} D_c^+(\mathfrak{X}, \Lambda) & \xrightarrow{\epsilon_{\mathcal{X}}^{\mathrm{an},*}} & D_c^+(X_\bullet^{\mathrm{an}}, \Lambda) \\ {\scriptstyle f_*^{\mathrm{an}}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*^{\mathrm{an}}} \\ D_c^+(\mathfrak{Y}, \Lambda) & \xrightarrow{\epsilon_{\mathcal{Y}}^{\mathrm{an},*}} & D_c^+(Y_\bullet^{\mathrm{an}}, \Lambda) \end{array}$$

commutes. On the algebraic side, the equivalence ϵ^* is also compatible with taking cohomology (cf. [11], p.202). As a summary, we have the following commutative diagram

$$\begin{array}{ccccccc} D_c^+(\mathfrak{X}, \Lambda) & \xrightarrow{\epsilon_{\mathcal{X}}^{\mathrm{an},*}} & D_c^+(X_\bullet^{\mathrm{an}}, \Lambda) & \xleftarrow{\xi_{X_\bullet}^*} & D_c^+(X_\bullet, \Lambda) & \xleftarrow{\epsilon_{\mathcal{X}}^*} & D_c^+(\mathcal{X}, \Lambda) \\ \downarrow{\scriptstyle f_*^{\mathrm{an}}} & & \downarrow{\scriptstyle \widetilde{f}_*^{\mathrm{an}}} & & \downarrow{\scriptstyle \widetilde{f}_*} & & \downarrow{\scriptstyle f_*} \\ D_c^+(\mathfrak{Y}, \Lambda) & \xrightarrow[\epsilon_{\mathcal{Y}}^{\mathrm{an},*}]{} & D_c^+(Y_\bullet^{\mathrm{an}}, \Lambda) & \xleftarrow[\xi_{Y_\bullet}^*]{} & D_c^+(Y_\bullet, \Lambda) & \xleftarrow[\epsilon_{\mathcal{Y}}^*]{} & D_c^+(\mathcal{Y}, \Lambda), \end{array}$$

where the horizontal arrows are all equivalences of triangulated categories.

5.5 Comparison between the two derived categories on the lisse-analytic topos.

In (5.3.1) and (5.3.2), we defined two derived categories, denoted $D_c(\mathfrak{X}, \Lambda)$ and $\mathscr{D}_c(\mathfrak{X}, \Lambda)$ respectively. Before proving they are equivalent, we give some preparation on the analytic analogues of some concepts and results in [11].

5.5.1. As in [7], let $\pi : \mathfrak{X}^{\mathbb{N}} \to \mathfrak{X}$ be the morphism of topoi, with $\pi_* = \varprojlim$. We have derived functors $R\pi_*$ and $L\pi^*$ between $\mathscr{D}(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$ and $\mathscr{D}(\mathfrak{X}, \Lambda)$. Denote $\mathrm{Mod}(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$ by $\mathscr{A}(\mathfrak{X})$ or just $\mathscr{A}$.

Lemma 5.5.2. *Let M be an AR-null complex in $\mathscr{D}(\mathscr{A})$. Then $R\pi_* M = 0$.*

Proof. Each of $\mathscr{H}^i(M)$ and $\tau_{>i}M$ is AR-null, so by ([7], 1.1) we have $R\pi_*\mathscr{H}^i(M) \cong R\pi_*\tau_{>i}M = 0$. By ([6], 3.1.5, 3.4.1), the assumption ([10], 2.1.7) for the ringed topoi $(\mathfrak{X}_{\text{lis-an}}, \Lambda_n)$ with $\mathscr{C}_n =$ all Λ_n-sheaves is satisfied, so by ([10], 2.1.10) we have $R\pi_* M = 0$. ☐

Therefore the functor $R\pi_* : \mathscr{D}_c(\mathscr{A}) \to \mathscr{D}_c(\mathfrak{X}, \Lambda)$ factors through the quotient category $D_c(\mathfrak{X}, \Lambda)$:

$$\mathscr{D}_c(\mathscr{A}) \xrightarrow{Q} D_c(\mathfrak{X}, \Lambda) \xrightarrow{\overline{R\pi}_*} \mathscr{D}_c(\mathfrak{X}, \Lambda) \xrightarrow{L\pi^*} \mathscr{D}(\mathscr{A}).$$

One can also define the *normalization functor* to be $K \mapsto \widehat{K} := L\pi^* \overline{R\pi}_* K$. For $M \in \mathscr{D}_c(\mathscr{A})$, we will also write $\widehat{M}$ for $\widehat{Q(M)}$, if there is no confusion. A complex M is *normalized* if the natural map $\widehat{M} \to M$ is an isomorphism. The analytic versions of ([11], 2.2.1, 3.0.11, 3.0.10) hold, as we state in the following.

Proposition 5.5.3. *(i) For $U \to \mathfrak{X}$ in* Lis-an$(\mathfrak{X})$ *and $M \in \mathscr{D}(\mathscr{A}(\mathfrak{X}))$, we have $R\pi_*(M_U) = (R\pi_* M)_U$ in $\mathscr{D}(U_{\mathrm{an}}, \Lambda)$.*

(ii) For $U \to \mathfrak{X}$ in Lis-an$(\mathfrak{X})$ *and $M \in \mathscr{D}(\mathfrak{X}, \Lambda)$, we have $L\pi^*(M_U) = (L\pi^* M)_U$ in $\mathscr{D}(\mathscr{A}(U_{\mathrm{an}}))$.*

(iii) For $M \in \mathscr{D}(\mathscr{A}(\mathfrak{X}))$, it is normalized if and only if the natural map

$$M_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \to M_{n-1}$$

is an isomorphism for each n.

They can be proved in the same way as in [11], and we do not repeat the proof here.

Proposition 5.5.4. *(i) The functors $(Q \circ L\pi^*, \overline{R\pi}_*)$ induce an equivalence $D_c(\mathfrak{X}, \Lambda) \leftrightarrow \mathscr{D}_c(\mathfrak{X}, \Lambda)$.*

(ii) Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of complex algebraic stacks, and let $f^{\mathrm{an}} : \mathfrak{X} \to \mathfrak{Y}$ be its analytification. Then the following diagram commutes:

$$\begin{array}{ccc} D_c^+(\mathfrak{X}, \Lambda) & \xrightarrow{\overline{R\pi}_{\mathcal{X},*}} & \mathscr{D}_c^+(\mathfrak{X}, \Lambda) \\ \downarrow f_*^{\mathrm{an}} & & \downarrow f_*^{\mathrm{an}} \\ D_c^+(\mathfrak{Y}, \Lambda) & \xrightarrow{\overline{R\pi}_{\mathcal{Y},*}} & \mathscr{D}_c^+(\mathfrak{Y}, \Lambda). \end{array}$$

Proof. (i) We will show that the adjunction and coadjunction maps are isomorphisms. For coadjunction maps, this is an analogue of ([11], 3.0.14).

Lemma 5.5.5. *Let $M \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$. Then $\widehat{M}$ is constructible and the coadjunction map $\widehat{M} \to M$ has an AR-null cone. In particular, $\widehat{M} \in \mathscr{D}_c(\mathfrak{X}^{\mathbb{N}}, \Lambda_\bullet)$.*

Proof. It can be proved in the same way as ([11], 3.0.14). We go over the proof briefly.

Let $P : X \to \mathfrak{X}$ be an algebraic presentation, i.e. the analytification of a presentation of the algebraic stack $\mathcal{X}$. By (5.5.3), the restriction of the natural map $\widehat{M} \to M$ to X gives the natural map $\widehat{N} \to N$ in $\mathscr{D}(\mathscr{A}(X))$, where $N = M|_X$. It suffices to show the cone of $\widehat{N} \to N$ is AR-null, and $\widehat{M}$ is Cartesian.

1. By ([6], 3.1.5, 3.4.1) and ([11], 2.1.i), the cohomological dimension of $R\pi_*$ on X_{an} is finite. Since $L\pi^*$ also has finite cohomological dimension, the same is true for the normalization functor, namely there exists an integer d, such that for every a and $N \in \mathscr{D}^{\geq a}(X^{\mathbb{N}})$ (resp. $\mathscr{D}^{\leq a}(X^{\mathbb{N}})$), we have $\widehat{N} \in \mathscr{D}^{\geq a-d}(X^{\mathbb{N}})$ (resp. $\mathscr{D}^{\leq a+d}(X^{\mathbb{N}})$).

2. One reduces to the case where N is a λ-module. This is because

$$\mathscr{H}^i(\widehat{N}) = \mathscr{H}^i(\widehat{\tau_{\geq i-d}\tau_{\leq i+d}N})$$

and hence one can assume N is bounded, and then a λ-module.

3. One reduces to the case where N is an adic system. There exists an adic system K with an AR-isomorphism $K \to N$, whose normalization $\widehat{K} \to \widehat{N}$ is an isomorphism (5.5.2).

4. By comparison ([3], 6.1.2 (A'')), the adic system N on X is *algebraic*, so by ([5], Rapport sur la formule des traces, 2.8) there exists an n_0 such that $N/\mathrm{Ker}(\lambda^{n_0})$ is torsion-free. Hence one reduces to two cases: N is torsion-free, or $\lambda^{n_0}N = 0$.

5. Assume N is torsion-free and adic. Then the component N_n is flat over Λ_n, for each n, and the natural map

$$N_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \simeq N_n \otimes_{\Lambda_n} \Lambda_{n-1} \xrightarrow{\sim} N_{n-1}$$

is an isomorphism, i.e. N is normalized. Then the cone of $\widehat{N} \to N$ is zero.

6. Assume $\lambda^{n_0}N = 0$. One reduces to the case where $n_0 = 1$ by considering the λ-filtration. This means the map

$$(N_n)_n \to (N_n/\lambda N_n)_n = (N_0)_n$$

is an AR-isomorphism, so N and $\pi^*N_0 = (N_0)_n$ have the same normalization. Note that $R\pi_*(N_0)_n = N_0$ ([11], 2.2.3), hence $\widehat{\pi^*N_0} = L\pi^*N_0$. By (5.2.4), N_0 is lcc on each stratum of an algebraic stratification of X, and one can check if $L\pi^*N_0 \to \pi^*N_0$ is an isomorphism on each stratum. By ([3], 6.1.2 (A')), N_0 is algebraic, and one can replace each stratum by an étale cover on which N_0 is constant. Finally by additivity we reduce to the case $N_0 = \Lambda_0$, which is proved by computing $L\pi^*\Lambda_0$ via the 2-term flat Λ-resolution of Λ_0 (cf. ([11], 3.0.10)).

The proof for $\widehat{M}$ being Cartesian is also the same as ([11], 3.0.14) (note that the analytic version of ([11], 3.0.13) holds). Let us not to repeat it here.

In particular, $\widehat{M} \in \mathscr{D}_c(\mathscr{A}(\mathfrak{X}))$, since the cone (which is AR-null) is AR-adic, and λ-complexes form a triangulated subcategory. ☐

We prove that the adjunction map is an isomorphism in the following lemma. This will be the crucial step; it only holds in the analytic category.

Lemma 5.5.6. *Let $M \in \mathscr{D}_c(\mathfrak{X}, \Lambda)$. Then the adjunction map $M \to R\pi_*L\pi^*M$ is an isomorphism.*

Proof. For simplicity, let us denote $R\pi_*L\pi^*M$ by $\check{M}$. Note that if $M' \to M \to M'' \to M'[1]$ is an exact triangle, and the adjunction maps for M' and M'' are isomorphisms, then the same holds for M, since $\check{M}' \to \check{M} \to \check{M}'' \to \check{M}'[1]$ is also an exact triangle.

1. That the map $M \to \check{M}$ is an isomorphism is a local property, since it is equivalent to the vanishing of all the cohomology sheaves of the cone, which can be checked locally. So we may replace $\mathfrak{X}$ by the algebraic presentation X.

2. On the analytic topos X_{an}, the functor $R\pi_*$ has finite cohomological dimension ([11], 2.1.i). Then as explained in (5.5.5), since the functor $M \mapsto \check{M}$ has finite cohomological dimension, one reduces to the case where M is a constructible Λ_X-module. This case follows from ([3], 6.1.2 (B'')), but for the reader's convenience, we continue to finish the proof.

By (5.3.3), M is the limit of some adic sheaf $F \in \Lambda\text{-}\mathrm{Sh}_c(X_{\mathrm{an}})$, and by comparison ([3], 6.1.2 (A'')) we see that F is algebraic. Therefore by ([5], Rapport sur la formule des traces, 2.8), we reduce to two cases: M is torsion-free (i.e. stalks are free Λ-modules of finite type), or M is killed by λ. The second case follows from ([11], 2.2.3).

3. Assume M is a torsion-free constructible sheaf. We want to use noetherian induction to reduce to the case where M is locally constant. Let $j : U \hookrightarrow X$ be the open immersion of a Zariski open subspace over which M is locally constant (by definition; see (5.3.3)), and let $i : Z \hookrightarrow X$ be the complement. Consider the exact triangle

$$i_*F \longrightarrow M \longrightarrow Rj_*M_U \longrightarrow,$$

where $F = Ri^! M \in \mathscr{D}_c(Z_{\mathrm{an}}, \Lambda)$. It suffices to show that the adjunction maps for $i_* F$ and $Rj_* M_U$ are isomorphisms.

We have the following commutative diagram of topoi

$$\begin{array}{ccc} Z_{\mathrm{an}}^{\mathbb{N}} & \xrightarrow{i^{\mathbb{N}}} & X_{\mathrm{an}}^{\mathbb{N}} \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Z_{\mathrm{an}} & \xrightarrow{i} & X_{\mathrm{an}}, \end{array}$$

so $R\pi_{X,*} \circ i_*^{\mathbb{N}} \simeq i_* \circ R\pi_{Z,*}$. Also $L\pi_X^* \circ i_* \simeq i_*^{\mathbb{N}} \circ L\pi_Z^*$, since i_* is just extension by zero, and $i_*(F \otimes_\Lambda^L \Lambda_n) \simeq i_* F \otimes_\Lambda^L \Lambda_n$. Therefore, the adjunction map for $i_* F$ on X is obtained by applying i_* to the adjunction map for F on Z :

$$i_* F \to R\pi_{X,*} L\pi_X^* i_* F \simeq i_* R\pi_{Z,*} L\pi_Z^* F,$$

which is an isomorphism by noetherian hypothesis.

We have the commutative diagram of topoi

$$\begin{array}{ccc} U_{\mathrm{an}}^{\mathbb{N}} & \xrightarrow{j^{\mathbb{N}}} & X_{\mathrm{an}}^{\mathbb{N}} \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ U_{\mathrm{an}} & \xrightarrow{j} & X_{\mathrm{an}}, \end{array}$$

so $R\pi_{X,*} \circ Rj_*^{\mathbb{N}} \simeq Rj_* \circ R\pi_{U,*}$. Also $Rj_*^{\mathbb{N}} \circ L\pi_U^* \simeq L\pi_X^* \circ Rj_*$. For each n we have a natural morphism $\Lambda_n \otimes_\Lambda^L Rj_* F \to Rj_*(F \otimes_\Lambda^L \Lambda_n)$. Let $P^\bullet$ be the flat Λ-resolution $0 \to \Lambda \xrightarrow{\lambda^{n+1}} \Lambda \to \Lambda_n$ of Λ_n, and let $F \to I^\bullet$ be an injective resolution of the sheaf F. Then $I^\bullet \otimes_\Lambda P^\bullet$ is also a complex of injectives, and it is clear that $j_*(I^\bullet \otimes P^\bullet) = j_*(I^\bullet) \otimes P^\bullet$. Therefore, the adjunction map for $Rj_* M_U$ on X is obtained by applying Rj_* to the adjunction map for M_U on U. Hence we reduce to the case where M is a locally constant sheaf on X.

4. Since the question is local for the analytic topology, we may cover X by analytic open subspaces over which M is constant, and hence reduce to the case where M is constant, defined by a free module Λ^r of finite rank. By additivity we may assume $M = \Lambda$. Then $L\pi^* \Lambda = (\Lambda_n)_n$, and $\pi_*(\Lambda_n)_n = \varprojlim (\Lambda_n)_n = \Lambda$. To finish the proof, we shall show $R^i \pi_*(\Lambda_n)_n = 0$ for $i \neq 0$.

Recall that $R^i \pi_*(\Lambda_n)_n$ is the sheafification of the presheaf on X_{an}

$$U \mapsto H^i(\pi^* U, (\Lambda_n)_n).$$

Consider the exact sequence ([11], 2.1.i)

$$0 \longrightarrow R^1 \varprojlim H^{i-1}(U, \Lambda_n) \longrightarrow H^i(\pi^* U, \Lambda_\bullet) \longrightarrow \varprojlim H^i(U, \Lambda_n) \longrightarrow 0.$$

Since X is locally contractible, and $R^1 \varprojlim H^0(U, \Lambda_n) = R^1 \varprojlim \Lambda_\bullet = 0$ for U connected, we see that the sheafification $R^i \pi_* \Lambda_\bullet$ is zero for $i \neq 0$. This proves that the adjunction morphism $M \to \check{M}$ is an isomorphism. □

Therefore, $(Q \circ L\pi^*, \overline{R\pi_*})$ induce an equivalence between $D_c(\mathfrak{X}, \Lambda)$ and $\mathscr{D}_c(\mathfrak{X}, \Lambda)$.

(ii) If $X_\bullet \to \mathfrak{X}$ is a strictly simplicial algebraic smooth hypercover, we have $D_c(\mathfrak{X}, \Lambda) \simeq D_c(X_\bullet, \Lambda)$ by (5.4.1i). Similarly, $\mathscr{D}_c(\mathfrak{X}, \Lambda)$ is naturally equivalent to $\mathscr{D}_c(X_\bullet, \Lambda)$. This can be proved in the same way as we prove "$D_c(\mathfrak{X}, \Lambda_m) \simeq D_c(X_\bullet^{\mathrm{an}}, \Lambda_m)$" in (5.4.1).

So we may assume that $\mathfrak{X} = X$ is the analytic space associated to an algebraic scheme. By definition of $\overline{R\pi_*}$, it suffices to show the following diagram commutes

$$\begin{array}{ccc} \mathscr{D}_c^+(\mathscr{A}(X)) & \xrightarrow{R\pi_{X,*}} & \mathscr{D}_c^+(X, \Lambda) \\ \downarrow f_*^{\mathbb{N}} & & \downarrow f_* \\ \mathscr{D}_c^+(\mathscr{A}(Y)) & \xrightarrow{R\pi_{Y,*}} & \mathscr{D}_c^+(Y, \Lambda), \end{array}$$

and this follows from the commutativity of the diagram of topoi

$$\begin{array}{ccc} X_{\mathrm{an}}^{\mathbb{N}} & \xrightarrow{\pi_X} & X_{\mathrm{an}} \\ \downarrow f^{\mathbb{N}} & & \downarrow f \\ Y_{\mathrm{an}}^{\mathbb{N}} & \xrightarrow{\pi_Y} & Y_{\mathrm{an}}. \end{array}$$

Note that the corresponding diagram for $f : \mathfrak{X} \to \mathfrak{Y}$ does not even make sense, since if f is not smooth, it does not necessarily induce a morphism of their lisse-analytic topoi. □

Similarly, $\overline{R\pi_{\mathcal{X},*}}$ induces a fully faithful functor $D_c(\mathfrak{X}, \Lambda) \to \mathscr{D}_c(\mathfrak{X}, \Lambda)$, which is compatible with f_*.

6 Over $\mathbb{C}$.

Let $(\Lambda, \mathfrak{m})$ be a complete DVR as before, with residue characteristic $\ell \neq 2$. Let $\mathcal{X}$ be an algebraic stack over Spec $\mathbb{C}$. We first prove a comparison theorem between the lisse-étale topoi over $\mathbb{C}$ and over $\mathbb{F}$, and then use this together with (5.4.3) to deduce the decomposition theorem for $\mathbb{C}$-algebraic stacks with affine diagonal.

6.1 Comparison between the lisse-étale topoi over $\mathbb{C}$ and over $\mathbb{F}$.

Let $(\mathscr{S}, \mathcal{L})$ be a pair on $\mathcal{X}$ defined on the level of Λ. By refining we may assume all strata in $\mathscr{S}$ are essentially smooth (i.e. their maximal reduced substack is smooth) and connected. Let $A \subset \mathbb{C}$ be a subring of finite type over $\mathbb{Z}$, large enough so that there exists a triple $(\mathcal{X}_S, \mathscr{S}_S, \mathcal{L}_S)$ over $S :=$ Spec A giving rise to $(\mathcal{X}, \mathscr{S}, \mathcal{L})$ by base change, and $1/\ell \in A$. Then S satisfies the condition (LO); the hypothesis on ℓ-cohomological dimension follows from ([1], X, 6.2). We may shrink S to assume that strata in $\mathscr{S}_S$ are smooth over S with geometrically connected fibers, which is possible because one can take a presentation $P : X_S \to \mathscr{X}_S$ and shrink S so that the strata in $P^*\mathscr{S}_S$ are smooth over S with geometrically connected fibers. Let $a : \mathcal{X}_S \to S$ be the structural map.

Let $A \subset V \subset \mathbb{C}$, where V is a strictly henselian DVR whose residue field is an algebraic closure of a finite residue field of A. Let $(\mathcal{X}_V, \mathscr{S}_V, \mathcal{L}_V)$ be the triple obtained by base change to V, and let $(\mathcal{X}_s, \mathscr{S}_s, \mathcal{L}_s)$ be its special fiber. Then we have morphisms

$$\mathcal{X} \xrightarrow{u} \mathcal{X}_V \xleftarrow{i} \mathcal{X}_s.$$

Proposition 6.1.1. *(stack version of ([3], 6.1.9)) For S small enough, the functors*

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda_n) \xleftarrow{u_n^*} D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda_n) \xrightarrow{i_n^*} D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \Lambda_n)$$

and

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda) \xleftarrow{u^*} D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda) \xrightarrow{i^*} D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \Lambda)$$

are equivalences of triangulated categories.

Proof. Clearly, they are all triangulated functors.

By (4.5), we can shrink $S = \text{Spec } A$ so that for any F and G of the form $j_! L$, where $j : \mathcal{U}_S \to \mathcal{X}_S$ in $\mathscr{S}_S$ and $L \in \mathcal{L}_S(\mathcal{U}_S)$, the formations of $R\mathscr{H}om_{\mathcal{X}_S}(F, G)$ commute with base change on S, and the complexes $a_* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G)$ on S are lcc and of formation compatible with base change, i.e. the cohomology sheaves are lcc, and for any $g : S' \to S$, the base change morphism for a_* :

$$g^* a_* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*} g'^* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G)$$

is an isomorphism. Then using the same argument as in [3], the claim for u_n^* and i_n^* follows. For the reader's convenience, we explain the proof in [3] in more detail.

Note that the spectra of $V, \mathbb{C}$ and s have no non-trivial étale surjections mapping to them, so their small étale topoi are the same as Sets. In particular, Ra_{V*} (resp. $Ra_{\mathbb{C}*}$ and Ra_{s*}) is just $R\Gamma$. Let us show the full faithfulness of u_n^* and i_n^* first. For $K, L \in D^b_{\mathscr{S}_V, \mathcal{L}_V}(\mathcal{X}_V, \Lambda_n)$, let $K_{\mathbb{C}}$ and $L_{\mathbb{C}}$ (resp. K_s and L_s) be their images under u_n^* (resp. i_n^*). Then the full faithfulness follows from the more general claim that, the maps

$$Ext^i_{\mathcal{X}}(K_{\mathbb{C}}, L_{\mathbb{C}}) \xleftarrow{u_n^*} Ext^i_{\mathcal{X}_V}(K, L) \xrightarrow{i_n^*} Ext^i_{\mathcal{X}_s}(K_s, L_s)$$

are bijective for all i.

Since $Hom_{D_c(\mathcal{X}, \Lambda_n)}(K, -)$ and $Hom_{D_c(\mathcal{X}, \Lambda_n)}(-, L)$ are cohomological functors, by 5-lemma we may assume that $K = F$ and $L = G$ are Λ_n-sheaves. Let $j : \mathcal{U}_S \to \mathcal{X}_S$ be the immersion of an open stratum in $\mathscr{S}_S$, with complement $i : \mathcal{Z}_S \to \mathcal{X}_S$. Using the short exact sequence

$$0 \longrightarrow j_{V!} j_V^* F \longrightarrow F \longrightarrow i_{V*} i_V^* F \longrightarrow 0$$

and noetherian induction on the support of F and G, we may assume that they take the form $j_{V!} L$, where j is the immersion of some stratum in $\mathscr{S}_S$, and L is a sheaf in $\mathcal{L}_V$. The spectral sequence

$$R^p a_{\square,*} \mathscr{E}xt^q_{\mathcal{X}_\square}(F_\square, G_\square) \Longrightarrow Ext^{p+q}_{\mathcal{X}_\square}(F_\square, G_\square)$$

is natural in the base $\square$, which can be $V, \mathbb{C}$ or s. The assumption on S made before implies that the composite base change morphism

$$g^* a_* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*} g'^* \mathscr{E}xt^q_{\mathcal{X}_S}(F, G) \to a_{S'*} \mathscr{E}xt^q_{\mathcal{X}_{S'}}(g'^* F, g'^* G)$$

is an isomorphism, for all $g : S' \to S$. Therefore, the maps

$$Ext^i_{\mathcal{X}}(F_{\mathbb{C}}, G_{\mathbb{C}}) \xleftarrow{u_n^*} Ext^i_{\mathcal{X}_V}(F, G) \xrightarrow{i_n^*} Ext^i_{\mathcal{X}_s}(F_s, G_s)$$

are bijective for all i. The claim (hence the full faithfulness of u_n^* and i_n^*) follows.

This claim also implies their essential surjectivity. To see this, let us give a lemma first.

Lemma 6.1.2. *Let $F : \mathscr{C} \to \mathscr{D}$ be a triangulated functor between triangulated categories. Let $A, B \in \text{Obj } \mathscr{C}$, and let $F(A) \xrightarrow{v} F(B) \to C \to F(A)[1]$ be an exact triangle in $\mathscr{D}$. If the map*

$$F : Hom_{\mathscr{C}}(A, B) \to Hom_{\mathscr{D}}(F(A), F(B))$$

is surjective, then C is in the essential image of F.

Proof. Let $u : A \to B$ be a morphism such that $F(u) = v$. Let C' be the mapping cone of u, i.e. let the triangle $A \xrightarrow{u} B \to C' \to A[1]$ be exact. Then its image

$$F(A) \xrightarrow{v} F(B) \longrightarrow F(C') \longrightarrow F(A)[1]$$

is also an exact triangle. This implies that $C \simeq F(C')$. ☐

Now we can show the essential surjectivity of u_n^* and i_n^*. For $K \in D^b_{\mathscr{S}_{\mathbb{C}},\mathcal{L}_{\mathbb{C}}}(\mathcal{X}, \Lambda_n)$, to show that K lies in the essential image of u_n^*, using the truncation exact triangles and (6.1.2), we reduce to the case where K is a sheaf. Using noetherian induction on the support of K, we reduce to the case where $K = j_!L$, where $j : \mathcal{U} \to \mathcal{X}$ is the immersion of a stratum in $\mathscr{S}$, and $L \in \mathcal{L}(\mathcal{U})$. This is the image of the corresponding $j_{V!}L_V$, since they are all defined over S. Similarly, i_n^* is also essentially surjective.

Next, we prove that u^* and i^* are equivalences.

We claim that for $K, L \in D^b_c(\mathcal{X}_V, \Lambda)$, if the morphisms

$$u_n^* : Hom_{D_c(\mathcal{X}_V,\Lambda_n)}(\widehat{K}_n, \widehat{L}_n) \to Hom_{D_c(\mathcal{X},\Lambda_n)}(\widehat{K}_{n,\mathbb{C}}, \widehat{L}_{n,\mathbb{C}})$$

and

$$i_n^* : Hom_{D_c(\mathcal{X}_V,\Lambda_n)}(\widehat{K}_n, \widehat{L}_n) \to Hom_{D_c(\mathcal{X}_s,\Lambda_n)}(\widehat{K}_{n,s}, \widehat{L}_{n,s})$$

are bijective for all n, then the morphisms

$$u^* : Hom_{D_c(\mathcal{X}_V,\Lambda)}(K, L) \to Hom_{D_c(\mathcal{X},\Lambda)}(K_{\mathbb{C}}, L_{\mathbb{C}})$$

and

$$i^* : Hom_{D_c(\mathcal{X}_V,\Lambda)}(K, L) \to Hom_{D_c(\mathcal{X}_s,\Lambda)}(K_s, L_s)$$

are bijective. Let $\square$ be one of the bases $V, \mathbb{C}$ or s. Since K and L are bounded, we see from the spectral sequence

$$R^p a_{\square,*} \mathscr{E}xt^q_{\mathcal{X}_\square}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square}) \Longrightarrow Ext^{p+q}_{\mathcal{X}_\square}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square})$$

and the finiteness of $R\mathscr{H}om$ and $Ra_{\square,*}$ ([10], 4.2.2, 4.1) that, the groups $Ext^{-1}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square})$ are finite for all n, hence they form a projective system satisfying the condition (ML). By ([11], 3.1.3), we have an isomorphism

$$Hom_{D_c(\mathcal{X}_\square,\Lambda)}(K_\square, L_\square) \xrightarrow{\sim} \varprojlim_n Hom_{D_c(\mathcal{X}_\square,\Lambda_n)}(\widehat{K}_{n,\square}, \widehat{L}_{n,\square}),$$

natural in the base $\square$, and the claim follows.

Since when restricted to $D^b_{\mathscr{S}_\square,\mathcal{L}_\square}$, the functors u_n^* and i_n^* are fully faithful for all n, we deduce that u^* and i^* are also fully faithful.

Finally we prove the essential surjectivity of u^* and i^*. In the following 2-commutative diagram

$$\begin{array}{ccc} \mathscr{D}^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathscr{A}(\mathcal{X}_V)) & \xrightarrow{Q_V} & D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda) \\ \downarrow{\scriptstyle u'^*} & & \downarrow{\scriptstyle u^*} \\ \mathscr{D}^b_{\mathscr{S},\mathcal{L}}(\mathscr{A}(\mathcal{X})) & \xrightarrow{Q} & D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda), \end{array}$$

the localization functors Q and Q_V are essentially surjective. Given $K \in D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \Lambda)$, to show that K lies in the essential image of u^*, it suffices to show that $\widehat{K} \in \mathscr{D}^b_{\mathscr{S},\mathcal{L}}(\mathscr{A}(\mathcal{X}))$ lies in the essential image of u'^*.

Let $M = \widehat{K} = (M_n)_n$; it is a normalized complex ([11], 3.0.8). Let $\rho_n : M_n \to M_{n-1}$ be the transition maps. Since u_n^* is an equivalence, there exists (uniquely up to isomorphism) an $M_{n,V} \in D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V, \Lambda_n)$ such that $u_n^*(M_{n,V}) \simeq M_n$, for each n. Also there exists $\rho_{n,V} \in Hom_{D^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathcal{X}_V,\Lambda_n)}(M_{n,V}, M_{n-1,V})$, which is mapped to ρ_n via u_n^*. We see that the induced morphism $\overline{\rho}_{n,V} : M_{n,V} \otimes^L_{\Lambda_n} \Lambda_{n-1} \to M_{n-1,V}$ is an isomorphism, since its image under the equivalence u_{n-1}^* is an isomorphism ([11], 3.0.10).

For $M_V = (M_{n,V})_n$ to be an object of $\mathscr{D}^b_{\mathscr{S}_V,\mathcal{L}_V}(\mathscr{A}(\mathcal{X}_V))$, we need to show it is an object of $\mathscr{D}_c(\mathscr{A}(\mathcal{X}_V))$, i.e. the cohomology systems $\mathscr{H}^i(M_V)$ are AR-adic ([11], 3.0.6).

Let $N^i = (N^i_n)_n$ be the universal image of the projective system $\mathscr{H}^i(M)$. Recall ([8], V, 3.2.3) that, since $\mathscr{H}^i(M)$ is AR-adic, it satisfies the condition (MLAR) (so that it makes sense to talk about its system of the universal images), and there exists an integer $k \geq 0$ such that $l_k(N^i) := (N^i_{n+k}/\lambda^{n+1}N^i_{n+k})_n$ is an adic system. Let $r \geq 0$ be an integer such that N^i_n is the image of $\mathscr{H}^i(M_{n+r}) \to \mathscr{H}^i(M_n)$, for each n. Then for every $s \geq r$, we have

$$\frac{\mathrm{Im}(\mathscr{H}^i(M_{n+r,V}) \to \mathscr{H}^i(M_{n,V}))}{\mathrm{Im}(\mathscr{H}^i(M_{n+s,V}) \to \mathscr{H}^i(M_{n,V}))} = 0,$$

since its image under the equivalence u^*_n is zero. This shows that $\mathscr{H}^i(M_V)$ also satisfies the condition (MLAR), with universal images $N^i_{n,V} = \mathrm{Im}(\mathscr{H}^i(M_{n+r,V}) \to \mathscr{H}^i(M_{n,V}))$. Also, the projective system $l_k(N^i_V)$ is adic, since the image under u^*_n of the transition map

$$\left(N^i_{n+k+1,V}/\lambda^{n+2}N^i_{n+k+1,V}\right) \otimes_{\Lambda_{n+1}} \Lambda_n \to N^i_{n+k,V}/\lambda^{n+1}N^i_{n+k,V}$$

is an isomorphism. By ([8], V, 3.2.3) again, the system $\mathscr{H}^i(M_V)$ is AR-adic. This finishes the proof that u^* (and similarly, i^*) is essentially surjective. $\square$

6.2 The proof.

Let $P : X \to \mathcal{X}$ be a presentation and let $\mathfrak{X} = \mathcal{X}^{\mathrm{an}}$ be the associated analytic stack.

6.2.1. Following [12], one can define Ω-perverse sheaves (for $\Omega = \mathbb{C}, E_\lambda$ or $\overline{\mathbb{Q}}_\ell$) on $\mathfrak{X}$ as follows. Let $p = p_{1/2}$ be the middle perversity on X^{an}. Let $d : \pi_0(X) \to \mathbb{N}$ be the dimension of the smooth map P. Define ${}^pD^{\leq 0}_c(\mathfrak{X}, \Omega)$ (resp. ${}^pD^{\geq 0}_c(\mathfrak{X}, \Omega)$) to be the full subcategory of objects $K \in D_c(\mathfrak{X}, \Omega)$ such that $P^{\mathrm{an},*}K[d]$ is in ${}^pD^{\leq 0}_c(X^{\mathrm{an}}, \Omega)$ (resp. ${}^pD^{\geq 0}_c(X^{\mathrm{an}}, \Omega)$). As in ([12], 4.1, 4.2), one can show that these subcategories do not depend on the choice of the presentation P, and they define a t-structure, called the (middle) perverse t-structure on $\mathfrak{X}$.

6.2.2. Following ([3], 6.2.4), one can define the sheaf complexes *of geometric origin* as follows. Let $\mathscr{F}$ be a Ω-perverse sheaf on $\mathfrak{X}$ (resp. a $\overline{\mathbb{Q}}_\ell$-perverse sheaf on $\mathcal{X}$). We say that $\mathscr{F}$ is *semi-simple of geometric origin* if it is semi-simple, and every simple direct summand belongs to the smallest family of simple perverse sheaves on complex analytic stacks (resp. lisse-étale sites of $\mathbb{C}$-algebraic stacks) that

(a) contains the constant sheaf $\underline{\Omega}$ over a point, and is stable under the following operations:

(b) taking the constituents of ${}^p\mathscr{H}^iT$, for $T = f_*, f_!, f^*, f^!, R\mathscr{H}om(-,-)$ and $- \otimes^L -$, where f is an arbitrary *algebraic* morphism between stacks.

A complex $K \in D^b_c(\mathfrak{X}, \Omega)$ (resp. $D^b_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$) is said to be *semi-simple of geometric origin* if it is isomorphic to the direct sum of the $({}^p\mathscr{H}^iK)[-i]$'s, and each ${}^p\mathscr{H}^iK$ is semi-simple of geometric origin.

One can replace the constant sheaf E_λ by its ring of integers $\mathscr{O}_\lambda$, and deduce that every complex $K \in \mathscr{D}^b_c(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$ that is semi-simple of geometric origin has an integral structure. Then we can apply (5.5.4).

Lemma 6.2.3. *(stack version of ([3], 6.2.6)) Let $\mathscr{F}$ be a simple $\overline{\mathbb{Q}}_\ell$-perverse sheaf of geometric origin on $\mathcal{X}$. For $A \subset \mathbb{C}$ large enough, the equivalence (6.1.1)*

$$D^b_{\mathscr{S},\mathcal{L}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \leftrightarrow D^b_{\mathscr{S}_s,\mathcal{L}_s}(\mathcal{X}_s, \overline{\mathbb{Q}}_\ell)$$

takes $\mathscr{F}$ to a simple perverse sheaf $\mathscr{F}_s$ on $\mathcal{X}_s$, such that $(\mathcal{X}_s, \mathscr{F}_s)$ is deduced by base extension from a pair $(\mathcal{X}_0, \mathscr{F}_0)$ defined over a finite field $\mathbb{F}_q$, and $\mathscr{F}_0$ is ι-pure.

Proof. $\mathscr{F}_s$ is obtained by base extension from some simple perverse sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$, so it suffices to show $\mathscr{F}_0$ is ι-mixed (3.5). This is clear, since the six operations, the perverse truncation functors and taking subquotients in the category of perverse sheaves all preserve ι-mixedness, and the constant sheaf $\overline{\mathbb{Q}}_\ell$ on a point is punctually pure. $\square$

Finally, we are ready to prove the stack version of the decomposition theorem over $\mathbb{C}$.

Theorem 6.2.4. *(stack version of ([3], 6.2.5)) Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism with finite diagonal of $\mathbb{C}$-algebraic stacks with affine automorphism groups. If $K \in D_c^b(\mathfrak{X}, \mathbb{C})$ is semi-simple of geometric origin, then $f_*^{\mathrm{an}} K$ is also bounded, and is semi-simple of geometric origin on $\mathfrak{Y}$.*

Proof. By (5.5.4) we can replace $D_c^b(\mathfrak{X}, \mathbb{C})$ by $D_c^b(\mathfrak{X}, \overline{\mathbb{Q}}_\ell)$, and by (5.4.3) we can replace this by $D_c^b(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$.

From ([15], 5.17) we know that there is a canonical isomorphism $f_! \simeq f_*$ on $D_c^-(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$. For $K \in D_c^b$, we have $f_! K \in D_c^-$ and $f_* K \in D_c^+$, hence $f_* K \in D_c^b$.

Lemma 6.2.5. *We can reduce to the case where K is a simple perverse sheaf $\mathscr{F}$.*

Proof. There are two steps: firstly, we show that the statement for simple perverse sheaves of geometric origin implies the statement for semi-simple perverse sheaves of geometric origin. This is clear:

$$f_*(\bigoplus_i \mathscr{F}_i) = \bigoplus_i f_* \mathscr{F}_i = \bigoplus_i \bigoplus_j {}^p\mathscr{H}^j (f_* \mathscr{F}_i)[-j] = \bigoplus_j {}^p\mathscr{H}^j (f_*(\bigoplus_i \mathscr{F}_i))[-j].$$

Then we show that the statement for semi-simple perverse sheaves implies the general statement. If K is semi-simple of geometric origin, we have

$$f_* K = \bigoplus_i f_* {}^p\mathscr{H}^i(K)[-i] = \bigoplus_i \bigoplus_j {}^p\mathscr{H}^j f_* {}^p\mathscr{H}^i(K)[-i-j].$$

Taking ${}^p\mathscr{H}^n$ on both sides, we get

$${}^p\mathscr{H}^n(f_* K) = \bigoplus_{i+j=n} {}^p\mathscr{H}^j f_* {}^p\mathscr{H}^i(K),$$

therefore $f_* K = \bigoplus_n {}^p\mathscr{H}^n(f_* K)$ and each summand is semi-simple of geometric origin. $\square$

Now assume K is a simple perverse sheaf $\mathscr{F}$. By ([17], 3.4v) every bounded complex is stratifiable. By (6.2.3), $\mathscr{F}$ corresponds to a simple perverse sheaf $\mathscr{F}_s$ which is induced from an ι-pure perverse sheaf $\mathscr{F}_0$ by base change. By (4.5), the formation of f_* over $\mathbb{C}$ is the same as the formation of $f_{s,*}$ over $\mathbb{F}$ or $f_{0,*}$ over a finite field. By (3.8), $f_{0,*}\mathscr{F}_0$ is also ι-pure. By (3.10, 3.11), we have

$$f_{s,*}\mathscr{F}_s \simeq \bigoplus_{i \in \mathbb{Z}} {}^p\mathscr{H}^i(f_{s,*}\mathscr{F}_s)[-i],$$

and each ${}^p\mathscr{H}^i(f_{s,*}\mathscr{F}_s)$ is semi-simple of geometric origin. Therefore $f_*\mathscr{F}$ is semi-simple of geometric origin. $\square$

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