SMALL BALL PROBABILITIES FOR SMOOTH GAUSSIAN FIELDS AND TENSOR PRODUCTS OF COMPACT OPERATORS

BY ANDREI I. KAROL' AND ALEXANDER I. NAZAROV[1]

St.Petersburg State University

We find the logarithmic L_2-small ball asymptotics for a class of zero mean Gaussian fields with covariances having the structure of "tensor product". The main condition imposed on marginal covariances is slow growth at the origin of counting functions of their eigenvalues. That is valid for Gaussian functions with smooth covariances. Another type of marginal functions considered as well are classical Wiener process, Brownian bridge, Ornstein–Uhlenbeck process, etc., in the case of special self-similar measure of integration. Our results are based on a new theorem on spectral asymptotics for the tensor products of compact self-adjoint operators in Hilbert space which is of independent interest. Thus, we continue to develop the approach proposed in the paper [10], where the regular behavior at infinity of marginal eigenvalues was assumed.

1 Introduction

The theory of small deviations of random functions is currently in intensive development. In this paper we address small deviations of Gaussian random fields in L_2-norm.

Suppose we have a real-valued Gaussian random field $X(x)$, $x \in \mathcal{O} \subset \mathbb{R}^d$, with zero mean and covariance function $G_X(x,y) = EX(x)X(y)$ for $x, y \in \mathcal{O}$. Let μ be a finite measure on $\mathcal{O}$. Set

$$\|X\|_\mu = \|X\|_{L_2(\mathcal{O};\mu)} = (\int\limits_{\mathcal{O}} X^2(x)\ \mu(dx))^{1/2}$$

(the subscript μ will be omitted when μ is the Lebesgue measure) and consider

$$Q(X, \mu\ ; \varepsilon) = \mathbf{P}\{\|X\|_\mu \le \varepsilon\}.$$

The problem is to evaluate the behavior of $Q(X, \mu\ ; \varepsilon)$ as $\varepsilon \to 0$. Note that the case $\mu(dx) = \psi(x)dx$, $\psi \in L_1(\mathcal{O})$, can be easily reduced to the Lebesgue case $\rho \equiv 1$ replacing X by the Gaussian field $X\sqrt{\rho}$. In general case, by scaling, one can assume that $\mu(\mathcal{O}) = 1$.

According to the well-known Karhunen-Loève expansion, we have

$$\|X\|_\mu^2 \stackrel{d}{=} \sum_{n=1}^{\infty} \lambda_n \xi_n^2,$$

where ξ_n, $n \in \mathbb{N}$, are independent standard normal r.v.'s, and $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum\limits_n \lambda_n < \infty$, are the eigenvalues of the integral equation

$$\lambda f(x) = \int\limits_{\mathcal{O}} G_X(x,y) f(y) \mu(dy), \qquad x \in \mathcal{O}.$$

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[1]Authors are partially supported by grants of RFBR No. 10-01-00154a and NSh. 4210.2010.1

Key words and phrases. Small deviations; slowly varying functions; smooth covariances; tensor product of operators; spectral asymptotics.

Thus we arrive to the equivalent problem of studying the asymptotic behavior of $\mathbf{P}\left\{\sum_{n=1}^{\infty}\lambda_n\xi_n^2 \le \varepsilon^2\right\}$ as $\varepsilon \to 0+$. The answer heavily depends on the available information about the eigenvalues λ_n.

The study of small deviation problem was initiated by Sytaya [1] and continued by a number of authors. See the history of the problem and the summary of main results in two excellent reviews [2] and [3]. The references to the latest results on L_2-small deviation asymptotics can be found on the site [4].

In this paper we continue to study the small deviation asymptotics of a vast and important class of Gaussian random fields having the covariance of "tensor product" type. It means that this covariance can be decomposed in a product of "marginal" covariances depending on different arguments. The classical examples of such fields are the Brownian sheet and the Brownian pillow. The notion of tensor products of Gaussian processes or Gaussian measures is known long ago. Such Gaussian fields are also studied in related domains, see, e.g., [5] and [6]. We recall briefly the construction of these fields.

Suppose we have two Gaussian random functions $X(x)$, $x \in \mathbb{R}^m$, and $Y(y)$, $y \in \mathbb{R}^n$, with zero means and covariances $G_X(x,u)$, $x,u \in \mathbb{R}^m$, and $G_Y(y,v)$, $y,v \in \mathbb{R}^n$, respectively. Consider the new Gaussian function $Z(x,y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, which has zero mean and the covariance

$$G_Z((x,y),(u,v)) = G_X(x,u)G_Y(y,v).$$

Such Gaussian function obviously exists, and the integral operator with the kernel G_Z is the tensor product of two "marginal" integral operators with the kernels G_X and G_Y. Therefore we use in the sequel the notation $Z = X \otimes Y$ and we call the Gaussian field Z *the tensor product* of the fields X and Y. The generalization to the multivariate case when obtaining the fields $\bigotimes_{j=1}^{d} X_j$ is straightforward.

The investigations of small deviations of Gaussian random functions of this class were started in a classical paper [7] where the logarithmic L_2-small ball asymptotics was obtained for the Brownian sheet $\mathbb{W}_d = \mathbb{W}_d(x_1,...,x_d)$ on the unit cube. This result was later extended by Li [8] to some other random fields.

In a very interesting paper [9] the *exact* asymptotics of small deviations in L_2-norm for the Brownian sheet was obtained using the Mellin transform. However, it is not clear if the method of [9] yields small deviation results for a more general class of Gaussian fields.

A new approach developed in the paper [10] is based on abstract theorems describing the spectral asymptotics of tensor products and of sums of tensor products for self-adjoint operators in Hilbert space. This approach gave the opportunity to consider quite general class of tensor products with the eigenvalues $\lambda_n^{(j)}$ of marginal covariances having the so-called *regular behaviour*:

$$\lambda_n^{(j)} \sim \frac{\varphi_j(n)}{n^{p_j}},\ n \to \infty,$$

where $p_j > 1$ and φ_j are some *slowly varying functions* (SVFs).

In this paper we consider the case where $\lambda_n^{(j)}$ have faster rate of decreasing. To be more precise, we assume that the so-called *counting functions*

$$\mathcal{N}_j(t) = \#\{n:\ \lambda_n^{(j)} > t\}$$

are SVFs. Such behavior of eigenvalues is typical for processes with smooth covariances, see, e.g., [11].

The structure of the paper is as follows. In §2 we present some auxiliary information about slowly varying functions. Next, in §3 we prove new results on the spectral asymptotics for tensor products of compact self-adjoint operators with slowly varying counting functions. Then, in §4, using the result of [11], we evaluate the small ball constants for the special rate of eigenvalues decay, namely,

$$\mathcal{N}(t) \sim \ln^p(1/t)\Phi(\ln(1/t))\,, \qquad t \to 0+,$$

with $p \geq 0$ and Φ being a SVF. Finally, we apply our theory to various specific examples of Gaussian random fields.

2 Auxiliary information on SVFs

We recall that a positive function $\varphi(\tau)$, $\tau > 0$, is called a *slowly varying* function (SVF) at infinity if for any $c > 0$

$$\varphi(c\tau)/\varphi(\tau) \to 1 \qquad \text{as} \quad \tau \to +\infty. \tag{1}$$

It is easily seen that any smooth positive function φ satisfying $\tau\varphi'(\tau)/\varphi(\tau) \to 0$ as $\tau \to +\infty$ is slowly varying. This test shows that the functions equal to $\ln^p(\tau)\ln^\varkappa(\ln(\tau))$ for $\tau >> 1$ are slowly varying.

We need some simple properties of SVFs. Their proofs can be found, for example, in [12].

Proposition 2.1. *Let φ be a SVF. Then following statements are true:*

1. *The relation (1) is uniform with respect to $c \in [a, b]$ for $0 < a < b < +\infty$.*

2. *There exists an equivalent SVF $\widehat{\varphi} \in \mathcal{C}^2(\mathbb{R}_+)$ (i.e. $\frac{\widehat{\varphi}(\tau)}{\varphi(\tau)} \to 1$ as $\tau \to +\infty$) such that*

$$\tau \cdot (\ln(\widehat{\varphi}))'(\tau) \to 0, \qquad \tau^2 \cdot (\ln(\widehat{\varphi}))''(\tau) \to 0, \qquad \tau \to +\infty. \tag{2}$$

3. *The function $\tau \mapsto \tau^p\varphi(\tau)$, $p > 0$, up to equivalence at infinity, is monotone for large τ, and its inverse function is $\tau \mapsto \tau^{1/p}\phi(\tau)$, where ϕ is a SVF.*

For two nondecreasing and unbounded SVFs φ and ψ, we define their *asymptotic convolution*

$$(\varphi \star \psi)(\tau) = \int\limits_1^\tau \varphi(\tau/\sigma)\, d\psi(\sigma).$$

Remark 2.1. It is easy to see that the asymptotic convolution is connected with the Mellin convolution (see [10, §2]) by the relation

$$(\varphi \star \psi)(\tau) = (\varphi * \psi_1)(\tau), \qquad \text{where} \qquad \psi_1(\tau) = \tau\psi'(\tau).$$

Therefore, basic properties of the asymptotic convolution could be extracted from [10, Theorem 2.2]. However, for the reader's convenience, we give them with full proofs.

Theorem 2.1. *The following statements are true:*

1. $(\varphi \star \psi)(\tau) \leq \varphi(\tau)\psi(\tau)$.

2. $\varphi(\tau) = o((\varphi \star \psi)(\tau))$ *as* $\tau \to +\infty$.

3. *Asymptotic symmetry:*

$$(\varphi \star \psi)(\tau) = (\psi \star \varphi)(\tau) + O(\varphi(\tau) + \psi(\tau)), \qquad \tau \to +\infty.$$

4. *If* $\varphi(\tau) = \widehat{\varphi}(\tau) \cdot (1 + o(1))$ *as* $\tau \to +\infty$*, then*

$$(\varphi \star \psi)(\tau) = (\widehat{\varphi} \star \psi)(\tau) \cdot (1 + o(1)), \qquad \tau \to +\infty.$$

5. $\varphi \star \psi$ *is a nondecreasing SVF.*

Remark 2.2. Note that, by the statement **3**, the statements **2** and **4** hold true with replacing φ by ψ.

Proof. **1**. This fact is trivial, since $\varphi(\tau/\sigma) \le \varphi(\tau)$ for all $\sigma \in [1, \tau]$.

2. By Proposition 2.1, part **1**, for any $a > 1$ we have

$$(\varphi \star \psi)(\tau) > \int\limits_1^a \varphi(\tau/\sigma)\, d\psi(\sigma) = \varphi(\tau)(\psi(a) - \psi(1)) \cdot (1 + o(1)), \qquad \tau \to +\infty.$$

Since ψ is unbounded, the statement follows.

3. Integrating by parts and changing the variable, we obtain

$$(\varphi \star \psi)(\tau) = \varphi(1)\psi(\tau) - \varphi(\tau)\psi(1) + (\psi \star \varphi)(\tau).$$

By **2**, the statement follows.

4. By **2** and **3**, for any $a > 1$

$$(\varphi \star \psi)(\tau) \sim (\psi \star \varphi)(\tau) \sim \int\limits_a^\tau \psi(\tau/\sigma)\, d\varphi(\sigma) \sim \int\limits_1^{\frac{\tau}{a}} \varphi(\tau/\sigma)\, d\psi(\sigma), \qquad \tau \to +\infty.$$

Given $\varepsilon > 0$, one can take a so large that $1 - \varepsilon < \frac{\widehat{\varphi}(\lambda)}{\varphi(\lambda)} < 1 + \varepsilon$ for $\lambda > a$, and the statement follows.

5. Due to **4** and to Proposition 2.1, part **2**, we can assume φ and ψ smooth. We have

$$\tau \cdot (\varphi \star \psi)'(\tau) = \tau\psi'(\tau)\varphi(1) + \int\limits_1^\tau \frac{\tau}{\sigma} \cdot \varphi'(\tau/\sigma)\, d\psi(\sigma).$$

By **2**,

$$\tau\psi'(\tau) = o(\psi(\tau)) = o((\varphi \star \psi)(\tau)), \qquad \tau \to +\infty.$$

Next, due to (2) we have for any $a > 1$ and $\tau > a$

$$\left| \int\limits_1^\tau \frac{\tau}{\sigma} \cdot \varphi'(\tau/\sigma)\, d\psi(\sigma) \right| \le \left| \int\limits_1^{\frac{\tau}{a}} \frac{\tau}{\sigma} \cdot \varphi'(\tau/\sigma)\, d\psi(\sigma) \right| + \left| \int\limits_{\frac{\tau}{a}}^\tau \frac{\tau}{\sigma} \cdot \varphi'(\tau/\sigma)\, d\psi(\sigma) \right| \le$$
$$\le \sup_{\lambda \ge a} \left| \frac{\lambda\varphi'(\lambda)}{\varphi(\lambda)} \right| \cdot (\varphi \star \psi)(\tau) + \sup_{\lambda \le a} |\lambda\varphi'(\lambda)| \cdot \psi(\tau).$$

By Proposition 2.1, part **2**, given $\varepsilon > 0$, one can take a so large that $\left|\frac{\lambda\varphi'(\lambda)}{\varphi(\lambda)}\right| < \varepsilon$ for $\lambda > a$. This gives, subject to **2**, $\tau \cdot (\varphi \star \psi)'(\tau) = o((\varphi \star \psi)(\tau))$ as $\tau \to +\infty$, and the statement follows. ☐

Example 1. Let

$$\varphi(\tau) = \ln^{\alpha}(\tau) \cdot \Phi(\ln(\tau)), \qquad \psi(\tau) = \ln^{\beta}(\tau) \cdot \Psi(\ln(\tau)),$$

where $\alpha,\ \beta \geq 0$ while Φ and Ψ are SVFs[2].

Then, as $\tau \to +\infty$,

$$(\varphi \star \psi)(\tau) \sim \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \cdot \varphi(\tau)\psi(\tau). \tag{3}$$

Proof. Changing the variable we obtain

$$(\varphi \star \psi)(\tau) = \int_0^{\ln(\tau)} (\ln(\tau)-s)^{\alpha}\, \Phi(\ln(\tau)-s)\, d\big[s^{\beta}\, \Psi(s)\big].$$

First, let at least one of exponents α and β be positive. By Theorem 2.1, part **3**, one can assume that $\beta > 0$. Then, substituting $s = \ln(\tau)\vartheta$ we get

$$(\varphi \star \psi)(\tau) = \varphi(\tau)\psi(\tau) \times$$
$$\times \int_0^1 (1-\vartheta)^{\alpha}\, \frac{\Phi(\ln(\tau)(1-\vartheta))}{\Phi(\ln(\tau))} \cdot \vartheta^{\beta-1}\, \frac{\beta\Psi(\ln(\tau)\vartheta) + \ln(\tau)\vartheta\Psi'(\ln(\tau)\vartheta)}{\Psi(\ln(\tau))}\, d\vartheta.$$

By Proposition 2.1, part **3**, for any $\varepsilon > 0$ the function $T^{\varepsilon}\Phi(T)$ increases for large s. Hence for $0 < z < 1$, $T > 0$ we have

$$\frac{\Phi(zT)}{\Phi(T)} = \frac{1}{z^{\varepsilon}} \cdot \frac{(zT)^{\varepsilon}\Phi(zT)}{T^{\varepsilon}\Phi(T)} \leq \frac{1}{z^{\varepsilon}}.$$

This estimate and a similar estimate for Ψ give us the majorant required in Lebesgue Dominated Convergence Theorem. Passing to the limit in the integral we obtain as $\tau \to +\infty$

$$(\varphi \star \psi)(\tau) \sim \varphi(\tau)\psi(\tau) \cdot \int_0^1 (1-\vartheta)^{\alpha} \cdot \beta\vartheta^{\beta-1}\, d\vartheta,$$

and we arrive at (3).

Now let $\alpha = \beta = 0$. Then for any $\delta \in]0,1[$

$$(\varphi \star \psi)(\tau) \geq \int_0^{\delta\ln(\tau)} \Phi(\ln(\tau)-s)\, d\Psi(s) \geq \Phi(\ln(\tau)(1-\delta)) \cdot [\Psi(\ln(\tau)\delta) - \Psi(0)].$$

By definition of SVF, we obtain

$$\liminf_{\tau\to+\infty} \frac{(\varphi \star \psi)(\tau)}{\varphi(\tau)\psi(\tau)} \geq 1.$$

Taking into account Theorem 2.1, part **1**, we arrive at (3). □

[2] If $\alpha = 0$ (respectively, $\beta = 0$), then we require in addition that Φ (respectively, Ψ) is nondecreasing and unbounded.

Example 2. Let

$$\varphi(\tau) = \exp(a^{\frac{1}{p'}} \ln^{\frac{1}{p}}(\tau)) \ln^{\alpha}(\tau) \cdot \Phi(\ln(\tau)), \qquad \psi(\tau) = \exp(b^{\frac{1}{p'}} \ln^{\frac{1}{p}}(\tau)) \ln^{\beta}(\tau) \cdot \Psi(\ln(\tau)),$$

where $a, b > 0$, $\alpha, \beta \geq 0$, $p > 1$, p' stands for the Hölder conjugate exponent, while Φ and Ψ are SVFs.

Then, as $\tau \to +\infty$,

$$(\varphi \star \psi)(\tau) \sim \sqrt{\frac{2\pi}{p-1}} \frac{a^{\alpha+\frac{1}{2}} b^{\beta+\frac{1}{2}}}{(a+b)^{\gamma+\frac{1}{2}}} \exp((a+b)^{\frac{1}{p'}} \ln^{\frac{1}{p}}(\tau)) \ln^{\gamma}(\tau) \cdot \Phi(\ln(\tau))\Psi(\ln(\tau)), \tag{4}$$

where $\gamma = \alpha + \beta + \frac{1}{2p}$.

Proof. Similarly to Example 1, we have

$$(\varphi \star \psi)(\tau) \sim \frac{b^{\frac{1}{p'}}}{p} T^{\alpha+\beta+\frac{1}{p}} \cdot \Phi(T)\Psi(T) \times$$
$$\times \int_0^1 \exp(T^{\frac{1}{p}} S(\vartheta)) \cdot (1-\vartheta)^{\alpha} \vartheta^{\beta - \frac{1}{p'}} \cdot \frac{\Phi(T(1-\vartheta))}{\Phi(T)} \cdot \frac{\Psi(T\vartheta)}{\Psi(T)} \, d\vartheta,$$

where $T = \ln(\tau)$, $S(\vartheta) = a^{\frac{1}{p'}}(1-\vartheta)^{\frac{1}{p}} + b^{\frac{1}{p'}} \vartheta^{\frac{1}{p}}$.

Denote by $\vartheta_\star$ the maximum point of $S(\vartheta)$. Then, using the Laplace method and Proposition 2.1, part **1**, we have

$$(\varphi \star \psi)(\tau) \sim \frac{b^{\frac{1}{p'}}}{p} T^{\alpha+\beta+\frac{1}{2p}} \cdot \Phi(T)\Psi(T) \sqrt{\frac{2\pi}{-S''(\vartheta_\star)}} \exp(T^{\frac{1}{p}} S(\vartheta_\star))(1-\vartheta_\star)^{\alpha} \vartheta_\star^{\beta - \frac{1}{p'}},$$

Direct calculation shows that

$$\vartheta^\star = \frac{b}{a+b}, \qquad S(\vartheta_\star) = (a+b)^{\frac{1}{p'}}, \qquad S''(\vartheta^\star) = -\frac{1}{pp'} \frac{(a+b)^{3-\frac{1}{p}}}{ab},$$

and we arrive at (4). □

3 Spectral asymptotics for tensor products of compact self-adjoint operators

We recall that for a compact self-adjoint nonnegative operator $\mathcal{T} = \mathcal{T}^* \geq 0$ in a Hilbert space H we denote by $\lambda_n = \lambda_n(\mathcal{T})$ its positive eigenvalues arranged in a nondecreasing sequence and repeated according to their multiplicity. Also we introduce the counting function

$$\mathcal{N}(t) = \mathcal{N}(t, \mathcal{T}) = \#\{n : \lambda_n(\mathcal{T}) > t\}.$$

Note that in view of Proposition 2.1, part **2**, any SVF arising in an asymptotic formula can be assumed smooth.

Theorem 3.1. *Let $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be compact self-adjoint nonnegative operators in Hilbert spaces H and $\widetilde{H}$, respectively. Let*

$$\mathcal{N}(t) \sim \varphi(1/t), \qquad \widetilde{\mathcal{N}}(t) \equiv \mathcal{N}(t, \widetilde{\mathcal{T}}) \sim \widetilde{\varphi}(1/t), \qquad t \to +0, \tag{5}$$

where φ and $\widetilde{\varphi}$ are unbounded SVFs at infinity.

Then for any $\varepsilon > 0$ the operator $\mathcal{T} \otimes \widetilde{\mathcal{T}}$ in the space $H \otimes \widetilde{H}$ satisfies the following estimates:

$$\mathcal{N}_\otimes(t) \equiv \mathcal{N}(t, \mathcal{T} \otimes \widetilde{\mathcal{T}}) \lesseqgtr \alpha_\pm(\varepsilon) \cdot \left[S(t,\varepsilon) + \widetilde{S}(t,\varepsilon) + \int\limits_{\alpha_\mp(\varepsilon)/\varepsilon}^{\varepsilon\tau} \varphi(\tau/\sigma)\, d\widetilde{\varphi}(\sigma) \right] \tag{6}$$

uniformly with respect to $t > 0$ (here τ stands for $\alpha_\pm(\varepsilon)/t$). For $\alpha_\mp(\varepsilon)/\varepsilon > \varepsilon\tau$ the integral in (6) should be omitted.

In (6) $\alpha_\pm(\varepsilon) \to 1$ as $\varepsilon \to +0$, and the functions S, $\widetilde{S}$ satisfy the following relations as $t \to +0$:

$$S(t,\varepsilon) \sim \varphi(1/t)\widetilde{\varphi}(1/\varepsilon); \qquad \widetilde{S}(t,\varepsilon) = o(\widetilde{\varphi}(1/t)). \tag{7}$$

Proof. We establish the upper bound for $\mathcal{N}_\otimes(t)$. The lower estimate can be proved in the same way. We have

$$\mathcal{N}_\otimes(t) = \sum_n \mathcal{N}(t/\widetilde{\lambda}_n) = \sum_{\widetilde{\lambda}_n < \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_n) + S(t,\varepsilon),$$

where

$$S(t,\varepsilon) = \sum_{\widetilde{\lambda}_n \geq \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_n).$$

The asymptotics (7) for the last sum follows from Proposition 2.1, part **1**.

Denote by $\widetilde{\theta}$ the inverse function for $\widetilde{\varphi}$. Then the second relation in (5) implies $\widetilde{\lambda}_n/\widetilde{\theta}(n) \to 1$ as $n \to \infty$, and we have

$$\alpha_-(\varepsilon)\widetilde{\theta}(n) \leq \widetilde{\lambda}_n \leq \alpha_+(\varepsilon)\widetilde{\theta}(n)$$

for $\widetilde{\lambda}_n < \varepsilon$, with $\alpha_\pm(\varepsilon) \to 1$ as $\varepsilon \to +0$.

Using monotonicity of $\mathcal{N}$ we get

$$\sum_{\widetilde{\lambda}_n < \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_n) \leq \sum_{\widetilde{\theta}(n) < \varepsilon\alpha_-^{-1}(\varepsilon)} \mathcal{N}(t/\alpha_+(\varepsilon)\widetilde{\theta}(n)),$$

and by monotonicity of the function $n \mapsto \mathcal{N}(t/\alpha_+(\varepsilon)\widetilde{\theta}(n))$ we estimate the sum by an integral:

$$\sum_{\widetilde{\lambda}_n < \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_n) \leq \mathcal{N}\left(\frac{\alpha_-(\varepsilon)t}{\alpha_+(\varepsilon)\varepsilon}\right) + \int\limits_0^{\varepsilon\alpha_-^{-1}(\varepsilon)} \mathcal{N}(t/\alpha_+(\varepsilon)\theta)(-d\widetilde{\varphi}(1/\theta)). \tag{8}$$

The first term in (8) is $O(\varphi(1/t))$. Therefore, adding it to the term $S(t,\varepsilon)$ we obtain the term $\alpha_+(\varepsilon)S(t,\varepsilon)$ with $\alpha_+(\varepsilon) \to 1$ as $\varepsilon \to 0+$. Further, splitting the integral in (8) and changing variables we obtain

$$\int\limits_\varepsilon^{+\infty} \mathcal{N}(s)\, d\widetilde{\varphi}(\alpha_+(\varepsilon)s/t) + \int\limits_{\alpha_-(\varepsilon)/\varepsilon}^{\varepsilon\tau} \mathcal{N}(\sigma/\tau)\, d\widetilde{\varphi}(\sigma). \tag{9}$$

The first integral in (9) gives us the term $\widetilde{S}(t,\varepsilon)$. Integrating by parts (note that for $s > \|T\|$ the integrand equals 0) we obtain

$$\widetilde{S}(t,\varepsilon) = \mathcal{N}(\varepsilon)\widetilde{\varphi}(\alpha_+(\varepsilon)\varepsilon/t) - \int\limits_{\varepsilon}^{\|T\|} \widetilde{\varphi}(\alpha_+(\varepsilon)s/t)\, d\mathcal{N}(s).$$

By Proposition 2.1, part **1**, $\frac{\widetilde{\varphi}(\alpha_+(\varepsilon)s/t)}{\widetilde{\varphi}(1/t)} \to 1$ as $t \to 0+$ uniformly with respect to $s \in [\varepsilon, \|T\|]$, and we arrive at (7).

By the first relation in (5) we can estimate $\mathcal{N}(\sigma/\tau)$ in the second integral by $\alpha_+(\varepsilon)\varphi(\tau/\sigma)$ that gives the integral term in (6). □

Theorem 3.2. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ be as in Theorem 3.1. Then*

$$\mathcal{N}_\otimes(t) \sim \phi(1/t) \equiv (\varphi \star \widetilde{\varphi})(1/t). \tag{10}$$

Proof. Fix arbitrary $\varepsilon > 0$ and consider the estimates (6). By Theorem 2.1, part **2**, we have

$$S(t,\varepsilon) = o((\varphi \star \widetilde{\varphi})(1/t)), \quad \widetilde{S}(t,\varepsilon) = o((\varphi \star \widetilde{\varphi})(1/t)), \qquad t \to +0.$$

Further, the integral in the right-hand side of (6) differs from the convolution $(\varphi \star \widetilde{\varphi})(\tau)$ by the integrals

$$\int\limits_{\varepsilon\tau}^{\tau} \varphi(\tau/\sigma)\, d\widetilde{\varphi}(\sigma) = O(\widetilde{\varphi}(\tau)) = o((\varphi \star \widetilde{\varphi})(\tau)), \qquad \tau \to +\infty,$$

$$\int\limits_{1}^{\alpha_\mp/\varepsilon} \varphi(\tau/\sigma)\, d\widetilde{\varphi}(\sigma) = O(\varphi(\tau)) = o((\varphi \star \widetilde{\varphi})(\tau)), \qquad \tau \to +\infty,$$

(we recall that $\tau = \alpha_\pm(\varepsilon)/t$).

Due to Theorem 2.1, part **5**, $(\varphi \star \widetilde{\varphi})(\tau) \sim (\varphi \star \widetilde{\varphi})(1/t)$, and hence

$$\limsup_{t\to 0+} \frac{\mathcal{N}_\otimes(t)}{\phi(1/t)} \le \alpha_+(\varepsilon), \qquad \liminf_{t\to 0+} \frac{\mathcal{N}_\otimes(t)}{\phi(1/t)} \ge \alpha_-(\varepsilon),$$

where ϕ is defined in (10). Taking into account that $\alpha_\pm(\varepsilon) \to 1$ as $\varepsilon \to 0+$, we arrive at (10). □

4 Small ball asymptotics. Examples

To connect the given asymptotic behavior of eigenvalues λ_n with the logarithmic decay rate for small deviations, we use the following statement, that is slightly reformulated [11, Theorem 2].

Proposition 4.1. *Let (λ_n), $n \in \mathbb{N}$, be a positive sequence with counting function $\mathcal{N}(t)$. Suppose that*

$$\mathcal{N}(t) \sim \varphi(1/t), \qquad t \to 0+,$$

where φ is a function slowly varying at infinity. Then, as $r \to 0+$,

$$\ln \mathbf{P}\left\{\sum_{n=1}^{\infty} \lambda_n \xi_n^2 \le r\right\} \sim -\frac{1}{2}\int\limits_{1}^{u} \varphi(z)\,\frac{dz}{z}, \tag{11}$$

where $u = u(r)$ satisfies the relation

$$\frac{\varphi(u)}{2u} \sim r, \qquad r \to 0+\,. \tag{12}$$

Example 3. Let $\varphi(\tau) = \ln^{\alpha}(\tau) \cdot \Phi(\ln(\tau))$, where $\alpha \geq 0$ while Φ is a SVF[3]. Then, as $\varepsilon \to 0+$,

$$\ln \mathbf{P} \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2 \right\} \sim -\frac{2^{\alpha}}{\alpha+1}\, \varphi(1/\varepsilon) \ln(1/\varepsilon). \tag{13}$$

Proof. Changing the variable $z = \frac{u}{\sigma}$ and using (3), we observe that

$$\int\limits_1^u \varphi(z)\, \frac{dz}{z} = (\varphi \star \ln)(u) \sim \frac{1}{\alpha+1}\, \varphi(u) \ln(u), \qquad u \to +\infty.$$

Next, direct calculation shows that $u = \frac{\varphi(1/r)}{2r}$ satisfies (12). Therefore, formula (11) gives, as $r \to 0+$,

$$\ln \mathbf{P} \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq r \right\} \sim -\frac{1}{2(\alpha+1)}\, \varphi(1/r) \ln(1/r).$$

Replacing r by ε^2, we arrive at (13). □

Example 4. Let $\varphi(\tau) = \exp(a \ln^{\frac{1}{p}}(\tau))\, \ln^{\alpha}(\tau) \cdot \Phi(\ln(\tau))$, where $\alpha \geq 0$, $p > 1$ while Φ is a SVF. Then, as $\varepsilon \to 0+$,

$$\ln \mathbf{P} \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2 \right\} \sim$$

$$\sim -\frac{2^{\alpha-\frac{1}{p}} p}{a} \exp\Big[2\ln(1/\varepsilon) \cdot \sum_{k=1}^{[p']} c_k \Big(\frac{a}{2^{\frac{1}{p'}}} \ln^{-\frac{1}{p'}}(1/\varepsilon) \Big)^k \Big] \ln^{\alpha+\frac{1}{p'}}(1/\varepsilon) \cdot \Phi(\ln(1/\varepsilon)), \tag{14}$$

where

$$c_1 = 1; \qquad c_k = \frac{1}{k!} \prod_{m=0}^{k-2} \Big(\frac{k}{p} - m \Big) \quad \text{for} \quad k \geq 2. \tag{15}$$

In particular, if $p > 2$ then

$$\ln \mathbf{P} \left\{ \sum_{n=1}^{\infty} \lambda_n \xi_n^2 \leq \varepsilon^2 \right\} \sim -\frac{2^{-\frac{1}{p}} p}{a}\, \varphi(\varepsilon^2) \ln^{\frac{1}{p'}}(1/\varepsilon).$$

Proof. Changing the variable we obtain

$$\int\limits_1^u \varphi(z)\, \frac{dz}{z} = T^{\alpha+1} \cdot \Phi(T) \int\limits_0^1 \exp(aT^{\frac{1}{p}} \vartheta^{\frac{1}{p}}) \cdot \vartheta^{\alpha} \cdot \frac{\Phi(T\vartheta))}{\Phi(T)}\, d\vartheta,$$

where $T = \ln(u)$.

[3]If $\alpha = 0$, then we require in addition that Φ is nondecreasing and unbounded.

The Laplace method and the Lebesgue dominated convergence theorem give us the asymptotics

$$\int_1^u \varphi(z)\,\frac{dz}{z} \sim \frac{p}{a}\exp(a\ln^{\frac{1}{p}}(u))\ \ln^{\alpha+\frac{1}{p'}}(u)\cdot\Phi(\ln(u)), \qquad u\to+\infty$$

(we recall that p' is the Hölder conjugate exponent for p).

Next, direct though cumbersome calculation shows that

$$u = \frac{1}{2r}\ \exp\Big[\ln(1/r)\cdot\sum_{k=1}^{[p']} c_k\big(a\ln^{-\frac{1}{p'}}(1/r)\big)^k\Big]\ \ln^{\alpha}(1/r)\cdot\Phi(\ln(1/r))$$

with c_k given by (15) satisfies (12). Therefore, formula (11) gives, as $r\to 0+$,

$$\ln\mathbf{P}\Big\{\sum_{n=1}^{\infty}\lambda_n\xi_n^2\le r\Big\} \sim -\frac{p}{a}\,ur\ln^{\frac{1}{p'}}(u) \sim$$

$$\sim -\frac{p}{2a}\ \exp\Big[\ln(1/r)\cdot\sum_{k=1}^{[p']} c_k\big(a\ln^{-\frac{1}{p'}}(1/r)\big)^k\Big]\ \ln^{\alpha+\frac{1}{p'}}(1/r)\cdot\Phi(\ln(1/r)).$$

Replacing r by ε^2, we arrive at (14). □

Turning to specific fields, we start with a stationary sheet

$$\mathcal{R}_d^{\mathcal{G}}(x) = \bigotimes_{j=1}^{d} R^{\mathcal{G}_j}(x_j), \qquad x=(x_1,\dots,x_d)\in[0,1]^d,$$

where $R^{\mathcal{G}_j}$ are stationary Gaussian processes with zero mean-values and the spectral densities

$$h_{R^{\mathcal{G}_j}}(\xi) = \exp(-\mathcal{G}_j(\xi)), \qquad \xi\in\mathbb{R}$$

(here $\mathcal{G}_j$ is even and $\mathcal{G}_j(\xi)\to+\infty$ as $\xi\to+\infty$).

Assume for simplicity that $\mathcal{G}$ is smooth and $\xi\mathcal{G}'(\xi)\to+\infty$ as $\xi\to+\infty$. Then it is easy to check that the corresponding covariances $G_{R^{\mathcal{G}}}$ are smooth functions. For instance, it is well known that

$$\mathcal{G}(\xi)=|\xi| \quad\Longrightarrow\quad G_{R^{\mathcal{G}}}(s,t)=\frac{1}{\pi(1+(s-t)^2)};$$

$$\mathcal{G}(\xi)=\xi^2 \quad\Longrightarrow\quad G_{\mathcal{G}}(s,t)=\frac{1}{2\sqrt{\pi}}\exp\Big(-\frac{(s-t)^2}{4}\Big).$$

Small deviations of such processes in various L_p-norms in the case $\mathcal{G}(\xi)=|\xi|^{\alpha}$ were considered in [11], [13], [14].

Proposition 4.2. 1. *Let $\mathcal{G}_j(\xi)\sim C_j\ln^p(\xi)$ as $\xi\to+\infty$, with $p>1$. Then, as $\varepsilon\to 0+$,*

$$\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\} \sim -\frac{p}{2}\Big(\pi^{d+1}\mathfrak{B}\mathfrak{C}\Big(\frac{p-1}{2}\Big)^{d-1}\Big)^{-\frac{1}{2}}\times$$

$$\times\exp\Big[2\ln(1/\varepsilon)\cdot\sum_{k=1}^{[p']} c_k\Big(\frac{2\ln(1/\varepsilon)}{\mathfrak{B}}\Big)^{-\frac{k}{p'}}\Big]\Big(\frac{2\ln(1/\varepsilon)}{\mathfrak{B}}\Big)^{1+\frac{d-3}{2p}}, \quad (16)$$

where

$$\mathfrak{B}=\sum_{j=1}^{d}C_j^{-\frac{1}{p-1}};\qquad \mathfrak{C}=\prod_{j=1}^{d}C_j^{\frac{1}{p-1}},$$

while c_k are given by (15). In particular, if $p>2$ then

$$\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{p}{2}\Big(\pi^{d+1}\mathfrak{B}\mathfrak{C}\Big(\frac{p-1}{2}\Big)^{d-1}\Big)^{-\frac{1}{2}}\exp\Big[\mathfrak{B}^{\frac{1}{p'}}\Big(2\ln(1/\varepsilon)\Big)^{\frac{1}{p}}\Big]\Big(\frac{2\ln(1/\varepsilon)}{\mathfrak{B}}\Big)^{1+\frac{d-3}{2p}}.$$

2. *Let $\mathcal{G}_j(\xi)\sim\xi^q\Phi_j(\xi)$ as $\xi\to+\infty$, with $0<q\le1$ and Φ_j being an SVF (if $q=1$ we require in addition that $\Phi_j(\xi)\to0$ as $\xi\to+\infty$). Then, as $\varepsilon\to0+$,*

$$\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{2^{\frac{d}{q}}\Gamma^d(\frac{q+1}{q})}{\pi^d\Gamma(\frac{d}{q}+2)}\cdot\ln^{\frac{d}{q}+1}(1/\varepsilon)\cdot\prod_{j=1}^{d}\phi_j(\ln(1/\varepsilon)),\tag{17}$$

where ϕ_j, $j=1,\dots,d$, are SVFs depending only on Φ_j and on q; for $q=1$ we have $\phi_j(t)\to+\infty$ as $t\to+\infty$. In particular,

$$\Phi_j(\xi)=C_j\ln^p(\xi)\quad\Longrightarrow\quad\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{\Gamma^d(\frac{q+1}{q})}{\pi^d\Gamma(\frac{d}{q}+2)}\cdot\Big(\frac{2q^p}{\mathfrak{C}}\Big)^{\frac{d}{q}}\cdot\frac{\ln^{\frac{d}{q}+1}(1/\varepsilon)}{\ln^{\frac{pd}{q}}(\ln(1/\varepsilon))},$$

where $\mathfrak{C}=\Big(\prod\limits_{j=1}^{d}C_j\Big)^{\frac{1}{d}}$ (we recall that $p<0$ for $q=1$).

3. *Let $\mathcal{G}_j(\xi)\sim C_j\xi$ as $\xi\to+\infty$. Then, as $\varepsilon\to0+$,*

$$\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{2^d}{\pi^d(d+1)!\,\mathfrak{C}}\cdot\ln^{d+1}(1/\varepsilon),\tag{18}$$

where

$$\mathfrak{C}=\prod_{j=1}^{d}\frac{\mathbf{K}(\operatorname{sech}(\pi/2C_j))}{\mathbf{K}(\tanh(\pi/2C_j))}$$

while $\mathbf{K}$ is the complete elliptic integral of the first kind, see, e.g., [15, 8.11].

4. *Let $\mathcal{G}_j(\xi)\sim\xi\Phi_j(\xi)$ as $\xi\to+\infty$, where Φ_j is an SVF. Suppose in addition that $\mathcal{G}_j(\xi)$ is convex for large ξ and $\Phi_j(\xi)\to+\infty$ as $\xi\to+\infty$. Then, as $\varepsilon\to0+$,*

$$\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{1}{(d+1)!}\cdot\frac{\ln^{d+1}(1/\varepsilon)}{\prod\limits_{j=1}^{d}\ln(\phi_j(\ln(1/\varepsilon)))},\tag{19}$$

where ϕ_j, $j=1,\dots,d$ are SVFs depending only on Φ_j, $\phi_j(t)\le\Phi_j(t)$ and $\phi_j(t)\to+\infty$ as $t\to+\infty$. In particular,

$$\Phi_j(\xi)=C_j\ln^p(\xi)\quad\Longrightarrow\quad\ln\mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\|\le\varepsilon\}\sim-\frac{1}{p^d(d+1)!}\cdot\frac{\ln^{d+1}(1/\varepsilon)}{\ln^d(\ln(\ln(1/\varepsilon)))}$$

(we recall that $p>0$).

5. *Let* $\ln(\mathcal{G}_j(\xi)) \sim q\ln(\xi)$ *as* $\xi \to +\infty$, *with* $q > 1$. *Then, as* $\varepsilon \to 0+$,

$$\ln \mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\| \le \varepsilon\} \sim -\frac{q^d}{(q-1)^d(d+1)!} \cdot \frac{\ln^{d+1}(1/\varepsilon)}{\ln^d(\ln(1/\varepsilon))}. \tag{20}$$

6. *Let* $\ln(\mathcal{G}_j(\xi))/\ln(\xi) \to +\infty$ *as* $\xi \to +\infty$. *Then, as* $\varepsilon \to 0+$,

$$\ln \mathbf{P}\{\|\mathcal{R}_d^{\mathcal{G}}\| \le \varepsilon\} \sim -\frac{1}{(d+1)!} \cdot \frac{\ln^{d+1}(1/\varepsilon)}{\ln^d(\ln(1/\varepsilon))}. \tag{21}$$

Remark 4.1. General formulas (16)–(21) are new even for $d = 1$. A particular case of purely power dependence of $\mathcal{G}_j$ with $d = 1$ was considered in [11]. The recent preprint [14] deals with $\mathcal{G}_j(\xi) = C_j\xi^2$ for arbitrary d but does not contain exact constant in (20).

Note that in the superexponential case (parts **4**–**6**) the logarithmic small ball asymptotics does not change if one multiplies $\mathcal{G}_j$ by a constant.

Proof. **1**. It is shown in the remarkable paper [16] that, if $\mathcal{G}_j(\xi)/\xi \to 0$ as $\xi \to +\infty$, then

$$\lambda_n^{(j)} \sim \exp(-\mathcal{G}_j(\pi n \cdot (1 + o(1)))) \qquad \text{as} \quad n \to \infty. \tag{22}$$

In our case (22) implies

$$\mathcal{N}_j(t) \sim \pi^{-1} \cdot \exp(C_j^{-\frac{1}{p}} \ln^{\frac{1}{p}}(1/t)), \qquad t \to 0+.$$

Using Theorem 3.2 and Example 2 successively $d-1$ times, we obtain that

$$\mathcal{N}(t) \sim \Big(\pi^{d+1}\mathfrak{C}\mathfrak{B}^{1+\frac{d-1}{p}}\Big(\frac{p-1}{2}\Big)^{d-1}\Big)^{-\frac{1}{2}} \exp(\mathfrak{B}^{\frac{1}{p'}} \ln^{\frac{1}{p}}(1/t)) \ln^{\frac{d-1}{2p}}(1/t).$$

By (14) we arrive at (16).

2. It follows from (22) that in this case

$$\mathcal{N}_j(t) \sim \pi^{-1} \cdot \ln^{\frac{1}{q}}(1/t)\phi_j(\ln(1/t)), \qquad t \to 0+,$$

and in a mentioned particular case

$$\mathcal{N}_j(t) \sim \pi^{-1} \left(\frac{q^p}{C_j} \cdot \frac{\ln(1/t)}{\ln^p(\ln(1/t))}\right)^{\frac{1}{q}}, \qquad t \to 0+.$$

Using Theorem 3.2 and Example 1 successively $d-1$ times, we obtain that

$$\mathcal{N}(t) \sim \frac{\Gamma^d(\frac{q+1}{q})}{\pi^d\Gamma(\frac{d}{q}+1)} \cdot \ln^{\frac{d}{q}}(1/t) \cdot \prod_{j=1}^{d} \phi_j(\ln(1/t)).$$

By (13) we arrive at (17).

3. It follows from [16], that in this case

$$\mathcal{N}_j(t) \sim \frac{\mathbf{K}(\tanh(\pi/2C_j))}{\pi\,\mathbf{K}(\operatorname{sech}(\pi/2C_j))} \cdot \ln(1/t), \qquad t \to 0+.$$

Similarly to the case **2**, we arrive at (18).

4. It is proved in [16], that, if $\mathcal{G}_j(\xi)$ is convex for large ξ and $\mathcal{G}_j(\xi)/\xi \to +\infty$ as $\xi \to +\infty$, then

$$\ln(\lambda_n^{(j)}) \sim -\mathcal{G}_j(\xi(n)) \qquad \text{as} \quad n \to \infty, \tag{23}$$

where $\xi(n) < n$ is a unique solution of the equation

$$\mathcal{G}_j(\xi) = 2n \ln\left(\frac{n}{\xi}\right).$$

In this case we obtain from (23)

$$\mathcal{N}_j(t) \sim \frac{\ln(1/t)}{2\ln(\phi_j(\ln(1/t)))}, \qquad t \to 0+.$$

and in a mentioned particular case

$$\mathcal{N}_j(t) \sim \frac{1}{2p} \cdot \frac{\ln(1/t)}{\ln(\ln(\ln(1/t)))}, \qquad t \to 0+.$$

Similarly to the case **2**, we arrive at (19).

5. It follows from (23), that in this case

$$\mathcal{N}_j(t) \sim \frac{1}{2-\frac{2}{q}} \cdot \frac{\ln(1/t)}{\ln(\ln(1/t))}, \qquad t \to 0+.$$

Similarly to the case **2**, we arrive at (20).

6. It follows from (23), that in this case

$$\mathcal{N}_j(t) \sim \frac{1}{2} \cdot \frac{\ln(1/t)}{\ln(\ln(1/t))}, \qquad t \to 0+.$$

Similarly to the case **2**, we arrive at (21). □

Now we consider a smooth homogeneous sheet

$$\mathcal{Z}_d^{(\mathfrak{a},\mathfrak{b})}(x) = \bigotimes_{j=1}^{d} Z^{(\mathfrak{a}_j,\mathfrak{b}_j)}(x_j), \qquad x \in [0,1]^d,$$

where $Z^{(\mathfrak{a}_j,\mathfrak{b}_j)}$ are Gaussian processes with zero mean-values and the covariances

$$G_{Z^{(\mathfrak{a}_j,\mathfrak{b}_j)}}(s,t) = \frac{s^{\mathfrak{b}_j} t^{\mathfrak{b}_j}}{(s+t)^{\mathfrak{a}_j}}.$$

Some properties of these processes were studied in [17] (for $\mathfrak{b} = \frac{1}{2}(\mathfrak{a}+1)$) and [18].

Proposition 4.3. *Let $\mathfrak{a}_j > 0$, and $\mathfrak{c}_j \equiv 2\mathfrak{b}_j - \mathfrak{a}_j + 1 > 0$, $j = 1, \dots, d$. Then, as $\varepsilon \to 0+$,*

$$\ln \mathbf{P}\{\|\mathcal{Z}_d^{(\mathfrak{a},\mathfrak{b})}\| \le \varepsilon\} \sim -\frac{2^{2d}}{\pi^{2d}(2d+1)!\,\mathfrak{C}} \cdot \ln^{2d+1}(1/\varepsilon), \tag{24}$$

where $\mathfrak{C} = \prod_{j=1}^{d} \mathfrak{c}_j$.

Proof. It is shown in [19], that

$$\mathcal{N}_j(t) \sim \left(2\pi^2\mathfrak{c}_j\right)^{-1} \cdot \ln^2(1/t), \qquad t \to 0+.$$

Similarly to the Proposition 4.2, case **2**, we arrive at (24). □

The next example deals with conventional Brownian sheet

$$\mathbb{W}_d(x) = \bigotimes_{j=1}^{d} W_j(x_j), \qquad x \in [0,1]^d, \tag{25}$$

where W_j are Wiener processes.

As mentioned in the Introduction, the logarithmic asymptotics of small deviations for the Brownian sheet in L_2-norm with respect to the Lebesgue measure was obtained in [7]. In [10], as a particular case of Proposition 5.1 (see also Theorem 8.2), this result was generalized to the case of absolutely continuous measure with arbitrary continuous density. Now we are going to obtain the logarithmic L_2-small ball asymptotics in the case of the discrete, degenerately self-similar measure (see [20], [21]). We recall briefly the construction of this measure (for $d = 1$).

Let $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_M < \alpha_{M+1} = 1$, $M \geq 2$, be a partition of the segment $[0,1]$. For some $m \in \{1, \ldots, M\}$, denote by $a = \alpha_{m+1} - \alpha_m$ the length of $[\alpha_m, \alpha_{m+1}]$. Also we introduce a real number δ and a real vector (β_k), $k = 1, \ldots, M$, such that

1. $0 < \delta < 1$;
2. $\beta_1 = 0; \qquad \chi_{\{m=M\}} \cdot \delta + \beta_M = 1$;
3. $\beta_k < \beta_{k+1}, \quad k = 1, \ldots, M-1; \qquad \delta\beta_M + \beta_m < \beta_{m+1}$

(for $m = M$ the last inequality is irrelevant).

It is shown in [21, §2] that under these conditions there exists a unique function f such that $\mathcal{S}(f) = f$, where $\mathcal{S}$ is the simplest *similarity operator*

$$\mathcal{S}[f](t) = \delta \cdot f(a^{-1}(t - \alpha_m)) + \sum_{k=1}^{M} \beta_k \cdot \chi_{]\alpha_k, \alpha_{k+1}[}(t).$$

Moreover, f increases on $[0,1]$, and $f(0) = 0$, $f(1) = 1$. The derivative f' in the sence of distributions is a discrete probability measure μ on $[0,1]$ with a unique singular point $\widehat{x} = \frac{\alpha_m}{1-a}$. It is called *degenerately self-similar measure*, generated by the set of parameters M, m, δ, (α_k) and (β_k).

Proposition 4.4. *Let the measure μ on $[0,1]^d$ be the tensor product:*

$$\mu(dx) = \bigotimes_{j=1}^{d} \mu_j(dx_j),$$

where μ_j, $j = 1, \ldots, d$, are degenerately self-similar measures, generated, respectively, by the sets of parameters

$$M_j, \quad m_j, \quad \delta_j, \quad (\alpha_k^{(j)}), \quad (\beta_k^{(j)}), \qquad j = 1, \ldots, d.$$

Then, as $\varepsilon \to 0+$,

$$\ln \mathbf{P}\{\|\mathbb{W}_d\|_\mu \le \varepsilon\} \sim -\frac{2^d\,\mathfrak{C}}{(d+1)!} \cdot \ln^{d+1}(1/\varepsilon), \tag{26}$$

where

$$\mathfrak{C} = \prod_{j=1}^{d} \frac{M_j - 1}{\ln(a_j \delta_j)}$$

(we recall that $a_j = \alpha^{(j)}_{m_j+1} - \alpha^{(j)}_{m_j}$).

Proof. It is shown in [21], that

$$\mathcal{N}_j(t) \sim \frac{M_j - 1}{\ln(a_j\delta_j)} \cdot \ln(1/t), \qquad t \to 0+.$$

Similarly to the Proposition 4.2, case **2**, we arrive at (26). □

Remark 4.2. Note, analogously to Remark 6 in [10], that the replacement of any factor in (25) by the Brownian bridge, by the Ornstein–Uhlenbeck process or by similar process does not influence on the relation (26). For instance, the Brownian pillow $\mathbb{B}_d(x) = \bigotimes_{j=1}^{d} B(x_j)$ satisfies, as $\varepsilon \to 0$, the relation

$$\ln \mathbf{P}\{\|\mathbb{B}_d\|_\mu \le \varepsilon\} \sim \ln \mathbf{P}\{\|\mathbb{W}_d\|_\mu \le \varepsilon\}.$$

Now we consider the *isotropically integrated* Brownian sheet

$$(\mathbb{W}_d)_\mathfrak{s}(x) = \bigotimes_{j=1}^{d} W_\mathfrak{s}(x_j),$$

where

$$W_\mathfrak{s}(t) \equiv W_\mathfrak{s}^{[b_1,\,\dots,\,b_\mathfrak{s}]}(t) = (-1)^{b_1+\,\dots\,+b_\mathfrak{s}} \underbrace{\int_{b_\mathfrak{s}}^{t} \dots \int_{b_1}^{t_1}}_{\mathfrak{s}} W(s)\; ds\; dt_1 \dots$$

(here any b_k equals either zero or one, $t \in [0,1]$; for various $j = 1, \dots, d$ the multi-indices $[b_1,\, \dots,\, b_\mathfrak{s}]$ can differ).

Proposition 4.5. *Let a measure μ on $[0,1]^d$ be as in Proposition 4.4. Then, as $\varepsilon \to 0+$,*

$$\ln \mathbf{P}\{\|(\mathbb{W}_d)_\mathfrak{s}\|_\mu \le \varepsilon\} \sim -\frac{2^d\,\mathfrak{C}}{(d+1)!} \cdot \ln^{d+1}(1/\varepsilon), \tag{27}$$

where

$$\mathfrak{C} = \prod_{j=1}^{d} \frac{M_j - 1}{\ln(a_j \delta_j^{2\mathfrak{s}+1})}.$$

Proof. This statement can be proved in the same way as Proposition 4.4. □

Finally, we can consider the fields-products corresponding to essentially different marginal processes. We restrict ourselves to a single example. On $[0,1]^2$ consider the Gaussian field $\mathfrak{R}(x_1) \otimes Z^{(\mathfrak{a},\mathfrak{b})}(x_2)$, where $\mathfrak{R}$ is a stationary Gaussian process with zero mean-value and the spectral density $h_\mathfrak{R}(\xi) = \frac{1}{\Gamma(|\xi|)}$.

Proposition 4.6. *Let* $\mathfrak{a} > 0$*, and* $\mathfrak{c} \equiv 2\mathfrak{b} - \mathfrak{a} + 1 > 0$*. Then*

$$\ln \mathbf{P}\{\|\mathfrak{R} \otimes Z^{(\mathfrak{a},\mathfrak{b})}\| \le \varepsilon\} \sim -\frac{1}{6\pi^2\mathfrak{c}} \cdot \frac{\ln^4(1/\varepsilon)}{\ln(\ln(\ln(1/\varepsilon)))}.$$

Proof. The asymptotics of marginal counting functions are calculated in Proposition 4.2, part **4**, and in Proposition 4.3. The result follows from Example 1 and (13). $\square$

We are grateful to M.A. Lifshits for useful comments and to Ya.Yu. Nikitin for the constant encouragement. We also thank T. Kühn who kindly provided us with the text of preprint [14].

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