Capacitary estimates of solutions of semilinear parabolic equations

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Abstract

We prove that any positive solution of $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\mathbb{R}^N \times (0, \infty)$ with initial trace $(F, 0)$, where F is a closed subset of $\mathbb{R}^N$ can be represented, up to two universal multiplicative constants, by a series involving the Bessel capacity $C_{2/q,q'}$. As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of $u(x,t)$ when $t \downarrow 0$, by using the "density" of F expressed in terms of the $C_{2/q,q'}$-capacity.

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Contents

1 Introduction

Let $T \in (0, \infty]$ and $Q_T = \mathbb{R}^N \times (0, T]$ $(N \geq 1)$. If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \tag{1.1}$$

it has been proven by Marcus and Véron [25] that there exists a unique outer-regular positive Borel measure ν in $\mathbb{R}^N$ such that

$$\lim_{t \to 0} u(., t) = \nu, \tag{1.2}$$

in the sense of Borel measures; the set of such measures is denoted by $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. To each of its element ν is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where $\mathcal{S}_\nu$, *the singular part* of ν, is a closed subset of $\mathbb{R}^N$ and μ_ν, *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2) has the following meaning :

$$\begin{array}{lll} (i) & \lim_{t\to 0} \int_{\mathcal{R}_\nu} u(.,t)\zeta dx = \displaystyle\int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) & \lim_{t\to 0} \displaystyle\int_{\mathcal{O}} u(.,t)dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open, } \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{array} \tag{1.3}$$

The measure ν is by definition *the initial trace* of u and denoted by $Tr_{\mathbb{R}^N}(u)$. It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [6], that if $q \geq q_c$, isolated singularities of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are removable. Conversely, if $1 < q < q_c$, it is proven by the same authors that for any $k > 0$, equation (1.1) admits a unique solution $u_{k\delta_0}$ with initial data $k\delta_0$. This existence and uniqueness results extends in a simple way if the initial data $k\delta_0$ is replaced by any Radon measure μ in $\mathbb{R}^N$ (see [5]). Furthermore, if $k \to \infty$, $u_{k\delta_0}$ increases and converges to a positive, radial and self-similar solution u_∞ of (1.1). Writing it under the form $u_\infty(x, t) = t^{-\frac{1}{q-1}} f(|x| / \sqrt{t})$, f is a positive solution of

$$\begin{cases} \Delta f + \frac{1}{2} y.Df + \frac{1}{q-1} f - f^q = 0 & \text{in } \mathbb{R}^N \\ \lim_{|y|\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{1.4}$$

The existence, uniqueness and the expression of the asymptotics of f has been studied thoroughly by Brezis, Peletier and Terman in [7]. Later on, Marcus and Véron proved in [25] that in the same range of exponents, for any $\nu \in \mathfrak{B}_+^{reg}(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, \end{cases} \tag{1.5}$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and $\mathfrak{B}_+^{reg}(\mathbb{R}^N)$. A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a| / \sqrt{t}) \leq u(x,t) \leq ((q-1)t)^{-\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{1.6}$$

valid for any $a \in \mathcal{S}_\nu$. As a consequence of Brezis and Friedman's result if $q \geq q_c$, i.e. in the *supercritical range*, Problem (1.5) may admit no solution at all. If $\nu \in \mathfrak{B}^{reg}_+(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \overline{u}_\nu$ to Problem (1.5) are obtained in [25] and expressed in terms of the the Bessel capacity $C_{2/q,q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [21]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and ν is simply the Radon measure μ_ν, the necessary and sufficient condition for solvability is that μ_ν does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. This result was already proven by Baras and Pierre [4] in the particular case of bounded measures and extended by Marcus and Véron [25] to the general case. We shall denote by $\mathfrak{M}^q_+(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which does not charge Borel subsets with zero $C_{2/q,q'}$-capacity. Notice that $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ is a subset of $\mathfrak{M}^q_+(\mathbb{R}^N)$; here $\mathfrak{M}^b_+(\mathbb{R}^N)$ is the cone of positive bounded Radon mesures in $\mathbb{R}^N$. For such measures, uniqueness always holds and we denote $\overline{u}_{\mu_\nu} = u_{\mu_\nu}$.

In view of the already known facts concerning the parabolic equation, it is useful to recal the much more advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

in a smooth bounded domain Ω of $\mathbb{R}^N$. This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on $\partial\Omega$, is proven by Le Gall [20], [21] in the case $q = N = 2$, by probabilistic methods, and by Marcus and Véron in [23], [24] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron [12] who shew that, if $q \geq q_e$ boundary isolated singularities of solutions of (1.7) are removable, which is not the case if $1 < q < q_e$. In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in Ω and outer regular positive Borel measures on $\partial\Omega$, which is not the case in the *supercritical case* $q \geq q_e$. In [10] Dynkin and Kuznetsov introduced the notion of σ-moderate solution which means that u is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q,q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\left\{\begin{array}{ll} -\Delta v + v^q = 0 & \text{in } \Omega \\ \qquad\quad v = \mu_n & \text{in } \partial\Omega \end{array}\right. \tag{1.8}$$

converges to u locally uniformly in Ω. This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a σ-moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if $q = 2$) [32] and then by Dynkin (if $q_e \leq q \leq 2$) [8], that all the positive solutions of (1.7) are σ-moderate. The key-stone element in their proof (partially probabilistic) is the fact that the maximal solution $\overline{u}_K$ of (1.7) with a boundary trace vanishing outside a compact subset $K \subset \partial\Omega$ is indeed σ-moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [32] in the case $q = 2$ and by purely analytic tools by Marcus and Véron [28], [29] in the case $q \geq q_e$. Defining $\underline{u}_K$ as the largest σ-moderate solution of (1.7) with a boundary trace concentrated on K, the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by $\overline{u}_K$ and $\underline{u}_K$

$$C^{-1}\rho(x)W_K(x) \leq \underline{u}_K(x) \leq \overline{u}_K(x) \leq C\rho(x)W_K(x). \tag{1.9}$$

In this expression $C = C(\Omega, q)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and $W_F(x)$ is the *capacitary potential* of K defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q,q'}(2^m K_m(x)), \tag{1.10}$$

where $K_m(x) = K \cap \{z : 2^{-m-1} \leq |z - x| \leq 2^{-m}\}$, the Bessel capacity being relative to $\mathbb{R}^{N-1}$. Note that, using a technique introduced in [24], inequality $\overline{u}_K \leq C^2 \underline{u}_K$ implies $\underline{u}_K = \overline{u}_K$.

Extending Dynkin's ideas to the parabolic case, we introduce the following notion

Definition 1.1 *A positive solution u of (1.1) is called σ-moderate if their exists an increasing sequence $\{\mu_n\} \subset W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of*

$$\left\{\begin{aligned} \partial_t u - \Delta u + u^q &= 0 \quad \text{in } Q_\infty \\ u(x,0) &= \mu_n \quad \text{in } \mathbb{R}^N, \end{aligned}\right. \tag{1.11}$$

converges to u locally uniformly in Q_∞.

If F is a closed subset of $\mathbb{R}^N$, we denote by $\overline{u}_F$ the maximal solution of (1.1) with an initial trace vanishing on F^c, and by $\underline{u}_F$ the maximal σ-moderate solution of (1.1) with an initial trace vanishing on F^c. Thus $\underline{u}_F$ is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in \mathfrak{M}_+^q(\mathbb{R}^N), \mu(F^c) = 0\}, \tag{1.12}$$

where $\mathfrak{M}_+^q(\mathbb{R}^N) := W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$. One of the main goal of this article is to prove that $\overline{u}_F$ is σ-moderate and more precisely,

Theorem 1.2 *For any $q > 1$ and any closed subset F of $\mathbb{R}^N$, $\overline{u}_F = \underline{u}_F$.*

We define below a set function which will play a fundamental role in the sequel.

Definition 1.3 *Let F be a closed subset of $\mathbb{R}^N$. The $C_{2/q,q'}$-capacitary potential W_F of F is defined by*

$$W_F(x,t) = t^{-\frac{1}{q-1}} \sum_{n=0}^{\infty} (n+1)^{\frac{N}{2}-\frac{1}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{\sqrt{(n+1)t}}\right) \qquad \forall (x,t) \in Q_\infty, \tag{1.13}$$

where $F_n = F_n(x,t) := \{y \in F : \sqrt{nt} \leq |x - y| \leq \sqrt{(n+1)t}\}$.

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwhile it reduces to (1.6);

Theorem 1.4 *For any $q \geq q_c$ there exist two positive constants $C_1 \geq C_2 > 0$, depending only on N and q such that for any closed subset F of $\mathbb{R}^N$, there holds*

$$C_2 W_F(x,t) \leq \underline{u}_F(x,t) \leq \overline{u}_F(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.14}$$

It is important to notice that the capacitary potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} W_F(\sqrt{\ell}x, \ell t) = W_{F/\sqrt{\ell}}(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.15}$$

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely what are the blow-up point of $\overline{u}_F = \underline{u}_F := u_F$. Introducing an integral expression comparable to W_F we show, in particular, the following results

$$\lim_{\tau \to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma \in [0,\infty) \Longrightarrow \lim_{t \to 0} t^{-\frac{1}{q-1}} u_F(x,t) = C\gamma \tag{1.16}$$

for some $C_\gamma = C(N,q,\gamma) > 0$, and

$$\limsup_{\tau \to 0} \tau^{\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty \Longrightarrow \limsup_{t \to 0} u_F(x,t) < \infty. \tag{1.17}$$

Our paper is organized as follows. In Section 2 we obtain estimates from above on $\overline{u}_F$. In Section 3 we give estimates from below on $\underline{u}_F$. In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

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2 Estimates from above

Some notations. Let Ω be a domain in $\mathbb{R}^N$ with a compact C^2 boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center a (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega \times (0,T), \quad \partial_\ell Q_T^\Omega = \partial\Omega \times (0,T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[.]$ (resp. $\mathbb{H}[.]$) denote the heat potential in Ω with zero lateral boundary data (resp. the heat potential in $\mathbb{R}^N$) with corresponding kernel

$$(x,y,t) \mapsto H^\Omega(x,y,t) \quad (\text{resp.}(x,y,t) \mapsto H(x,y,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by $q_c := 1 + \frac{N}{2}$, the parabolic critical exponent.

Theorem 2.1 *Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N,q)$ such that for any closed subset F of $\mathbb{R}^N$ and any $u \in C^2(Q_\infty) \cap C(\overline{Q_\infty} \setminus F)$ satisfying*

$$\left\{ \begin{array}{ll} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty \\ \lim_{t \to 0} u(x,t) = 0 & \text{locally uniformly in } F^c, \end{array} \right. \tag{2.1}$$

there holds
$$u(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty, \tag{2.2}$$
where W_F is the $(2/q,q')$-capacitary potential of F defined by (1.13).

First we shall consider the case where $F = K$ is compact and
$$K \subset B_r \subset \overline{B}_r, \tag{2.3}$$
and then we shall extend to the general case by a covering argument.

2.1 Global L^q-estimates

Let $\rho > 0$, we assume (2.3) holds and we put
$$\mathcal{T}_{r,\rho}(K) = \{\eta \in C_0^\infty(B_{r+\rho}), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}. \tag{2.4}$$
If $\eta \in \mathcal{T}_{r,\rho}(K)$, we set $\eta^* = 1 - \eta$, $\zeta = \mathbb{H}[\eta^*]^{2q'}$ and
$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]|. \tag{2.5}$$
We fix $T > 0$ and shall consider the equation on Q_T. Throughout this paper C will denote a generic positive constant, depending only on N, q and sometimes T, the value of which may vary from one ocurrence to another. We shall also use sometimes the notation $A \approx B$ for meaning that there exists a constant $C > 0$ independent of the data such that $C^{-1}A \leq B \leq CA$.

Except in Lemma 2.12 the only assumption on q is $q > 1$. In the sequel we shall obtain pointwise estimate on the solution expressed in terms of the $L^{q'}$-norm of $R[\eta]$ for $\eta \in \mathcal{T}_{r,\rho}(K)$. Although these estimates could have been immediately turned into capacitary estimates as in [29], the advantage of keeping them comes from the possibility of performing operations such as dilations or summations on them. The next lemma points out the connection between $R[\eta]$ and the the $C_{2/q,q'}$- capacity of K.

Lemma 2.2 *There exists $C = C(N,q) > 0$ such that*
$$C^{-1}\|\eta\|^{q'}_{W^{2/q,q'}} \leq \iint_{Q_\infty} (R[\eta])^{q'}\,dx\,dt := \|R[\eta]\|^{q'}_{L^{q'}} \leq C\|\eta\|^{q'}_{W^{2/q,q'}} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.6}$$
Therefore
$$\inf\left\{\|R[\eta]\|^{q'}_{L^{q'}} : \eta \in \mathcal{T}_{r,\rho}(K)\right\} \approx C^{B_{r+\rho}}_{2/q,q'}(K). \tag{2.7}$$

Proof. There holds $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, and
$$\iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'}\,dx\,dt = \int_0^\infty \left\| t^{1-1/q}\partial_t \mathbb{H}[\eta]\right\|^{q'}_{L^{q'}(\mathbb{R}^N)} \frac{dt}{t} \approx \|\eta\|^{q'}_{[W^{2,q'},L^{q'}]_{1/q,q'}} \tag{2.8}$$

where $\left[W^{2,q'}, L^{q'}\right]_{1/q,q'}$ indicates the real interpolation functor of degree $1/q$ between $W^{2,q'}(\mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N)$ [35]. Similarly, and using the Gagliardo-Nirenberg inequality,

$$\iint_{Q_\infty} |\nabla(\mathbb{H}[\eta])|^{2q'}\, dx\, dt \leq C\|\eta\|^{q'}_{W^{2/q,q'}} \|\eta\|^{q'}_{L^\infty} = C\|\eta\|^{q'}_{W^{2/q,q'}}. \tag{2.9}$$

Inequality (2.6) follows from (2.8) and (2.9), and (2.7) from the definition of the Bessel capacity relative to $B_{r+\rho}$. $\square$

Lemma 2.3 *There exists $C = C(N, q) > 0$ such that for any $T > 0$,*

$$\iint_{Q_\infty} u^q\zeta dx\, dt + \int_{\mathbb{R}^N} (u\zeta)(x, T)dx \leq C\|R[\eta]\|^{q'}_{L^{q'}} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.10}$$

Proof. We recall that there always hold

$$0 \leq u(x, t) \leq \left(\frac{1}{t(q-1)}\right)^{\frac{1}{q-1}} \qquad \forall (x, t) \in Q_\infty, \tag{2.11}$$

and (see [6] e.g.)

$$0 \leq u(x, t) \leq \left(\frac{C}{t + (|x| - r)^2}\right)^{\frac{1}{q-1}} \qquad \forall (x, t) \in Q_\infty \setminus B_r. \tag{2.12}$$

Since η^* vanishes in an open neighborhood $\mathcal{N}_1$, for any open subset $\mathcal{N}_2$ such that $K \subset \mathcal{N}_2 \subset \overline{\mathcal{N}}_2 \subset \mathcal{N}_1$ there exist $c_{\mathcal{N}_2} > 0$ and $C_{\mathcal{N}_2} > 0$ such that

$$\mathbb{H}[\eta^*](x, t) \leq C_{\mathcal{N}_2} \exp(-c_{\mathcal{N}_2} t), \qquad \forall (x, t) \in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t\to 0} \int_{\mathbb{R}^N} (u\zeta)(x, t)dx = 0.$$

Thus ζ is an admissible test function and one has

$$\iint_{Q_T} u^q\zeta dx\, dt + \int_{\mathbb{R}^N} (u\zeta)(x, T)dx = \iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\, dt. \tag{2.13}$$

Notice that the three terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned}
\partial_t\zeta + \Delta\zeta &= 2q'\mathbb{H}^{2q'-1}_{\eta^*}(\partial_t\mathbb{H}_{\eta^*} + \Delta\mathbb{H}_{\eta^*}) + 2q'(2q'-1)\mathbb{H}^{2q'-2}_{\eta^*}|\nabla\mathbb{H}_{\eta^*}|^2, \\
&= 2q'\mathbb{H}^{2q'-1}_{\eta^*}(\partial_t\mathbb{H}_{\eta} + \Delta\mathbb{H}_{\eta}) + 2q'(2q'-1)\mathbb{H}^{2q'-2}_{\eta}|\nabla\mathbb{H}_{\eta}|^2,
\end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t\zeta + \Delta\zeta) = u\mathbb{H}^{2q'/q}_{\eta^*}\left[2q'(2q'-1)\mathbb{H}^{2q'-2-2q'/q}_{\eta^*}|\nabla\mathbb{H}_\eta|^2 - 2q'\mathbb{H}^{2q'-1-2q'/q}_{\eta^*}(\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta)\right].$$

Since $2q' - 2 - 2q'/q = 0$ and $0 \leq \mathbb{H}_{\eta^*} \leq 1$,

$$\left|\iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\,dt\right| \leq C(q)\left(\iint_{Q_T} u^q\zeta dx\,dt\right)^{1/q}\left(\iint_{Q_T} R^{q'}(\eta)dx\,dt\right)^{1/q'},$$

where

$$R(\eta) = |\nabla\mathbb{H}_\eta|^2 + |\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta|.$$

Using Lemma 2.2 one obtains (2.10). $\square$

Proposition 2.4 *Let $r > 0$, $\rho > 0$, $T \geq (r+\rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x,t) : |x|^2 + t \leq (r+\rho)^2\}$$

and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. There exists $C = C(N,q,T) > 0$ such that

$$\iint_{Q_{r+\rho,T}} u^q dx\,dt + \int_{\mathbb{R}^N} u(x,T)dx \leq C\|R[\eta]\|^{q'}_{L^{q'}} \qquad \forall\eta \in \mathcal{T}_{r,\rho}(K). \tag{2.14}$$

Proof. Because $K \subset B_r$ and $\eta^* \equiv 1$ outside $B_{r+\rho}$ and takes value between 0 and 1,

$$\begin{aligned}\mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1-\chi_{B_{r+\rho}}](x,t) &= \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\geq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy,\\ &= 1 - \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\leq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy.\end{aligned}$$

For $(x,t) \in \mathcal{E}_{r+\rho}$, put $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in \mathcal{E}_1$ and

$$\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\leq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy = \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon.$$

We claim that

$$\max\left\{\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon : (\xi,\tau) \in \mathcal{E}_1\right\} = \ell, \tag{2.15}$$

and $\ell = \ell(N) \in (0,1]$. We recall that

$$\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon < 1 \qquad \forall\tau > 0. \tag{2.16}$$

If the maximum is achieved for some $(\bar\xi,\bar\tau) \in \mathcal{E}_1$, it is smaller that 1 and

$$\mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1-\chi_{B_{r+\rho}}](x,t) \geq 1-\ell > 0, \qquad \forall(x,t) \in \mathcal{E}_{r+\rho}. \tag{2.17}$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n, \tau_n)\}$ with $\tau_n \to 0$ and $|\xi_n| \downarrow 1$. We can assume that $\xi_n \to \bar{\xi}$ with $\left|\bar{\xi}\right| = 1$, then

$$\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{|\upsilon|\le 1} e^{-\frac{|\xi_n-\upsilon|^2}{4\tau_n}} d\upsilon = \left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{B_1(\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon.$$

But $B_1(\xi_n) \cap B_1(-\xi_n) = \emptyset$,

$$\int_{B_1(\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon + \int_{B_1(-\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon < \int_{\mathbb{R}^N} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon$$

and

$$\int_{B_1(\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon = \int_{B_1(-\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon.$$

This implies

$$\lim_{n\to\infty} \left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}} \int_{B_1(\xi_n)} e^{-\frac{|\upsilon|^2}{4\tau_n}} d\upsilon \le 1/2.$$

If the maximum were achieved with a sequence $\{(\xi_n, \tau_n)\}$ with $|\tau_n| \to \infty$, it would also imply (2.17), since the integral term in (2.16) is always bounded. Therefore (2.16) holds. Put $C = (1-\ell)^{-1}$, then

$$\iint_{Q_{r,T}} u^q dx\, dt + \int_{\mathbb{R}^N} u(.,T)dx \le C \left\| R[\eta] \right\|_{L^{q'}}^{q'}, \tag{2.18}$$

and (2.14) follows. □

2.2 Pointwise estimates

We give first a rough pointwise estimate.

Lemma 2.5 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$,*

$$u(x,(r+2\rho)^2) \le \frac{C \left\| R[\eta] \right\|_{L^{q'}}^{q'}}{(\rho(r+\rho))^{\frac{N}{2}}}, \qquad \forall x \in \mathbb{R}^N. \tag{2.19}$$

Proof. We observe first that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx\, dt + \int_{\mathbb{R}^N} u(x,T)dx = \int_{\mathbb{R}^N} u(x,s)dx \qquad \forall T > s > 0. \tag{2.20}$$

By the maximum principle u is dominated by the maximal solution v which has the indicatrix function I_{B_r} for initial trace. The function v is the limit, as $k \to \infty$, of the solutions v_k with initial data $k\chi_{B_r}$. Since $v_k \le k\mathbb{H}[\chi_{B_r}]$, it follows Hence

$$\int_{\mathbb{R}^N} u(.,s)dx \le C \left\| R[\eta] \right\|_{L^{q'}}^{q'} \qquad \forall T > s \ge (r+\rho)^2, \tag{2.21}$$

by Lemma 2.3. Using the fact that

$$u(x,\tau+s) \le \mathbb{H}[u(.,s)](x,\tau) \le \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(.,s)dx,$$

we obtain (2.19) with $s=(r+\rho)^2$ and $\tau=(r+2\rho)^2-(r+\rho)^2 \approx \rho(r+\rho)$. $\square$

The above estimate does not take into account the fact that $u(x,0)=0$ if $|x|\ge r$. It is mainly interesting if $|x|\le r$. In order to derive a sharper estimate which uses the localization of the singularity and not only the $L^{q'}$-norm of $R[\eta]$. For such a goal, we need some lateral boundary estimates.

Lemma 2.6 *Let $\gamma \ge r+2\rho$ and $c>0$ and either $N=1$ or 2 and $0\le t\le c\gamma^2$ for some $c>0$, or $N\ge 3$ and $t>0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} udSd\tau \le C_5\gamma \left\|R[\eta]\right\|^{q'}_{L^{q'}}. \tag{2.22}$$

where $C>0$ depends on N, q and c if $N=1$, 2 or depends only on N and q if $N\ge 3$.

Proof. Let us assume that $N=1$ or 2. Put $G^\gamma := B^c_\gamma \times (-\infty,0)$ and $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty,0)$. Set

$$h_\gamma(x) = 1-\frac{\gamma}{|x|},$$

and let ψ_γ be the solution of

$$\begin{aligned} \partial_\tau\psi_\gamma + \Delta\psi_\gamma = 0 \qquad & \text{in } G^\gamma, \\ \psi_\gamma = 0 \qquad & \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(.,0) = h_\gamma \qquad & \text{in } B^c_\gamma. \end{aligned} \tag{2.23}$$

Thus the function

$$\tilde\psi(x,\tau) = \psi_\gamma(\gamma x, \gamma^2\tau)$$

satisfies

$$\begin{aligned} \partial_t\tilde\psi + \Delta\tilde\psi = 0 \qquad & \text{in } G^1 \\ \tilde\psi = 0 \qquad & \text{on } \partial_\ell G^1 \\ \tilde\psi(.,0) = \tilde h \qquad & \text{in } B^c_1, \end{aligned} \tag{2.24}$$

and $\tilde h(x) = 1-|x|^{-1}$. By the maximum principle $0\le\tilde\psi\le 1$, and by Hopf Lemma

$$-\left.\frac{\partial\tilde\psi}{\partial\mathbf{n}}\right|_{\partial B_1\times[-c,0]} \ge \theta > 0, \tag{2.25}$$

where $\theta=\theta(N,c)$. Then $0\le\psi_\gamma\le 1$ and

$$-\left.\frac{\partial\psi_\gamma}{\partial\mathbf{n}}\right|_{\partial B_\gamma\times[-\gamma^2,0]} \ge \theta/\gamma. \tag{2.26}$$

Multiplying (1.1) by $\psi_\gamma(x,\tau-t)=\psi^*_\gamma(x,\tau)$ and integrating on $B^c_\gamma\times(0,t)$ yields to

$$\int_0^t\int_{B^c_\gamma}u^q\psi^*_r dxd\tau+\int_{B^c_\gamma}(uh_\gamma)(x,t)dx-\int_0^t\int_{\partial B_\gamma}\frac{\partial u}{\partial\mathbf{n}}\psi^*_\gamma dSd\tau=-\int_0^t\int_{\partial B_\gamma}\frac{\partial\psi^*_\gamma}{\partial\mathbf{n}}ud\sigma d\tau. \tag{2.27}$$

Since ψ^*_γ is bounded from above by 1, (2.22) follows from (2.26) and Proposition 2.4 (notice that $B^c_\gamma\times(0,t)\subset\mathcal{E}^c_\gamma$), first by taking $t=T=\gamma^2\geq(r+2\rho)^2$, and then for any $t\leq\gamma^2$.

If $N\geq 3$, we proceed as above except that we take

$$h_\gamma(x)=1-\left(\frac{\gamma}{|x|}\right)^{N-2}.$$

Then $\psi_\gamma(x,t)=h_\gamma(x)$ and $\theta=N-2$ is independent of the length of the time interval. This leads to the conclusion. □

Lemma 2.7 *I- Let M, $a>0$ and $\eta\in L^\infty(\mathbb{R}^N)$ such that*

$$0\leq\eta(x)\leq Me^{-a|x|^2}\qquad a.e.\ in\ \mathbb{R}^N. \tag{2.28}$$

Then, for any $t>0$,

$$0\leq\mathbb{H}[\eta](x,t)\leq\frac{M}{(4at+1)^{\frac{N}{2}}}e^{-\frac{a|x|^2}{4at+1}}\qquad\forall x\in\mathbb{R}^N. \tag{2.29}$$

II- Let M, a, $b>0$ and $\eta\in L^\infty(\mathbb{R}^N)$ such that

$$0\leq\eta(x)\leq Me^{-a(|x|-b)^2_+}\qquad a.e.\ in\ \mathbb{R}^N. \tag{2.30}$$

Then, for any $t>0$,

$$0\leq\mathbb{H}[\eta](x,t)\leq\frac{Me^{-\frac{a(|x|-b)^2_+}{4at+1}}}{(4at+1)^{\frac{N}{2}}}\qquad\forall x\in\mathbb{R}^N,\ \forall t>0. \tag{2.31}$$

Proof. For the first statement, put $a=\frac{1}{4}s$. Then

$$0\leq\eta(x)\leq M(4\pi s)^{\frac{N}{2}}\frac{1}{(4\pi s)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4s}}=C(4\pi s)^{\frac{N}{2}}\mathbb{H}[\delta_0](x,s).$$

By the order property of the heat kernel,

$$0\leq\mathbb{H}[\eta](x,t)\leq M(4\pi s)^{\frac{N}{2}}\mathbb{H}[\delta_0](x,t+s)=M\left(\frac{s}{t+s}\right)^{\frac{N}{2}}e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.29) follows by replacing s by $\frac{1}{4}a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$, and (2.31) implies

$$0 \leq \eta(x) \leq M e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x,t) \leq \frac{C e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t+1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.32}$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t+1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at+1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.31) follows from (2.32). □

Lemma 2.8 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$u(x,(r+2\rho)^2) \leq C \max\left\{\frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}}\right\} e^{-\frac{(|x|-(r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.33}$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. It is classical that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of B_1 satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x',y',t',s') \leq C_7(t'-s')^{-(N+2)/2}(|x'|-1)e^{-\frac{|x'-y'|^2}{4(t'-s')}}, \tag{2.34}$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r+2\rho)x'$, $t' \mapsto (r+2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x,t) \leq C(|x|-r-2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y,s)d\sigma(y)ds. \tag{2.35}$$

The right-hand side term in (2.35) is smaller than

$$\max\left\{\frac{C(|x|-r-2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0,t)\right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s)d\sigma(y)ds.$$

We fix $t = (r+2\rho)^2$ and $|x| \geq r+3\rho$. Since

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in \big(0,(r+2\rho)^2\big)\right\}$$
$$= (|x|-r-2\rho)^{-2-N} \max\left\{\frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\},$$

a direct computation gives

$$
\max\left\{\frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\}
$$

$$
=\begin{cases}(2N+4)^{1+\frac{N}{2}}e^{-(N+2)/2} & \text{if } r+3\rho\le|x|\le(r+2\rho)(1+\sqrt{4+2N}),\\ \left(\dfrac{|x|-r-2\rho}{r+2\rho}\right)^{2+N}e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} & \text{if } |x|\ge(r+2\rho)(1+\sqrt{4+2N}).\end{cases}
$$

Thus there exists a constant $C(N)>0$ such that

$$
\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}:s\in\left(0,(r+2\rho)^2\right)\right\}\le C(N)\rho^{-2-N}e^{-\left(\frac{|x|-(r+2\rho)}{2r+4\rho}\right)^2}. \tag{2.36}
$$

Combining this estimate with (2.22) with $\gamma=r+2\rho$ and (2.35), one derives (2.33). □

Lemma 2.9 *Under the assumptions of Lemma 2.8, there exists a constant $C=C(N,q)>0$ such that*

$$
0\le u(x,(r+2\rho)^2)\le C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{(r+\rho)^{N-1}\rho}\right\}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.37}
$$

for every $x\in\mathbb{R}^N\setminus B_{r+3\rho}$.

Proof. This is a direct consequence of the inequality

$$
(|x|-r-2\rho)e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}\le\frac{C(r+\rho)^2}{\rho}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2},\qquad\forall x\in B^c_{r+2\rho}, \tag{2.38}
$$

and Lemma 2.8. □

Lemma 2.10 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$, the following estimate holds*

$$
u(x,t)\le\frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}}\|R[\eta]\|_{L^{q'}}^{q'},\qquad\forall x\in\mathbb{R}^N,\ \forall t\ge(r+2\rho)^2, \tag{2.39}
$$

where

$$
\tilde{M}=\tilde{M}(x,r,\rho)=\begin{cases}\left(1+\frac{r}{\rho}\right)^{\frac{N}{2}} & \text{if } |x|<r+3\rho\\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \text{if } r+3\rho\le|x|\le C_N(r+2\rho)\\ 1+\frac{r}{\rho} & \text{if } |x|\ge C_N(r+2\rho)\end{cases} \tag{2.40}
$$

with $C_N=1+\sqrt{4+2N}$.

Proof. It follows by the maximum principle

$$u(x,t) \leq \mathbb{H}[u(.,(r+2\rho)^2)](x,t-(r+2\rho)^2).$$

for $t \geq (r+2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.5 and Lemma 2.9

$$u(x,(r+2\rho)^2) \leq C_{10}\tilde{M} e^{-\frac{(|x|-r-3\rho)^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r+\rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^3}{\rho}\left(|x|-r-2\rho)\right)^{N+2} & \text{if } r+3\rho \leq |x| \leq C_N(r+2\rho) \\ \frac{1}{(r+\rho)^{N-1}\rho} & \text{if } |x| \geq C_N(r+2\rho) \end{cases}$$

Applying Lemma 2.7 with $a = (2r+4\rho)^{-2}$, $b = r+3\rho$ and t replaced by $t-(r+2\rho)^2$ implies

$$u(x,t) \leq C\frac{(r+2\rho)^N\tilde{M}}{t^{\frac{N}{2}}} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.41}$$

for all $x \in B_{r+3\rho}^c$ and $t \geq (r+2\rho)^2$, which is (2.39). □

The next estimate gives a precise upper bound for u when t is not bounded from below.

Lemma 2.11 *Assume that $0 < t \leq (r+2\rho)^2$ for some $c > 0$, then there exists a constant $C = C(N,q) > 0$ such that the following estimate holds*

$$u(x,t) \leq C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}}\right\} e^{-\frac{(|x|-r-3\rho)^2}{4t}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.42}$$

for any $(x,t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0,(r+2\rho)^2]$.

Proof. By using (2.22) the following estimate is a straightforward variant of (2.33) for any $\gamma \geq r+2\rho$,

$$u(x,t) \leq C_8(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \leq t\right\} \|R[\eta]\|_{L^{q'}}^{q'}. \tag{2.43}$$

Clearly

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \leq t\right\}$$
$$= \begin{cases} (2N+4)^{1+\frac{N}{2}}(|x|-r-2\rho)^{-N-2}e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \leq r+2\rho+\sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases}$$

By elementary analysis, if $x \in B^c_{r+3\rho}$,

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \dfrac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x| - r - 2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

from which inequality (2.42) follows. $\square$

Lemma 2.12 *Assume $q \geq q_c$. Let $r > 0$, $\rho > 0$ and K be a compact subset of $B_{r+\rho}$. If $\eta \in \mathcal{T}_{r,\rho}(K)$, denote by η_r the function defined by $\eta_r(x) = \eta(rx)$ and*

$$R_r[\eta_r](x,t) = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta] + \Delta \mathbb{H}[\eta]| \, (rx, r^2 t) \qquad \forall (x,t) \in Q_\infty.$$

Then

$$\|R[\eta]\|^{q'}_{L^{q'}} = r^{N-\frac{2}{q-1}} \|R_r[\eta_r]\|^{q'}_{L^{q'}}. \tag{2.44}$$

Furthermore

$$C^{B_{r+\rho}}_{2/q,q'}(K) = r^{N-\frac{2}{q-1}} C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K/r), \tag{2.45}$$

and

$$r^{N-\frac{2}{q-1}} C_{2/q,q'}(K/r) \leq C^{B_{r+\rho}}_{2/q,q'}(K) \leq C r^{N-\frac{2}{q-1}} \left(1 + \frac{r}{\rho}\right)^{\frac{2}{q-1}} C_{2/q,q'}(K/r). \tag{2.46}$$

Proof. Estimate (2.44) follows from the change of variable $(rx, r^2 t) = (y,s)$. Thus it implies the scaling property (2.47), since there is a one to one correspondence between $\mathcal{T}_{r,\rho}(K)$ and $\mathcal{T}_{1,\frac{\rho}{r}}(K/r)$. In order to prove (2.44) set $K' = K/r \subset B_1$, thus

$$C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K') = \inf \left\{ \|\zeta\|^{q'}_{W^{2/q,q'}} : \zeta \in \mathcal{T}_{1,\frac{\rho}{r}}(K') \right\}.$$

Let $\phi \in C^2(\mathbb{R}^N)$ be a radial cut-off function such that $0 \leq \rho \leq 1$, $\rho = 1$ on B_1, $\rho = 0$ on $\mathbb{R}^N \setminus B_{1+\frac{\rho}{r}}$, $|\nabla \phi| \leq C r \rho^{-1} \chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$ and $|D^2 \phi| \leq C r^2 \rho^{-2} \chi_{B_{1+\frac{\rho}{r}} \setminus B_1}$, where C is independent of r and ρ. Let $\zeta \in C^2_0(\mathbb{R}^N)$. Then

$$\nabla(\zeta\phi) = \zeta \nabla \phi + \phi \nabla \zeta\,, \; D^2(\zeta\phi) = \zeta D^2 \phi + \phi D^2 \zeta + 2 \nabla \phi \otimes \nabla \zeta.$$

Thus $\|\zeta\phi\|_{L^{q'}(B_{1+\frac{\rho}{r}})} \leq \|\zeta\|_{L^{q'}(\mathbb{R}^N)}$,

$$\int_{B_{1+\frac{\rho}{r}}} |\nabla(\zeta\phi)|^{q'} dx \leq C\left(1+\frac{r}{\rho}\right)^{q'} \|\zeta\|^{q'}_{W^{1,q'}},$$

and

$$\int_{B_{r+\rho}} \left|D^2(\zeta\phi)\right|^{q'} dx \leq C\left(1+\frac{r^2}{\rho^2}\right)^{q'} \|\zeta\|^{q'}_{W^{2,q'}}.$$

Finally

$$\|\zeta\phi\|_{W^{2/q,q'}} \leq C\left(1+\frac{r^2}{\rho^2}\right)\|\zeta\|_{W^{2/q,q'}}.$$

Denote by $\mathcal{T}$ the linear mapping $\zeta \mapsto \zeta\phi$. Because

$$W^{2/q,q'} = \left[W^{2,q'}, L^{q'}\right]_{1/q,q'},$$

(here we use the Lions-Petree real interpolation notations and results from [22]), it follows

$$\|\mathcal{T}\|_{\mathcal{L}(W_0^{2/q,q'}(\mathbb{R}^N), W_0^{2/q,q'}(B_{1+\frac{\rho}{r}}))} \leq C(q)\left(1+\frac{r^2}{\rho^2}\right)^{1/q}.$$

Therefore

$$C^{B_{1+\frac{\rho}{r}}}_{2/q,q'}(K') \leq C\left(1+\frac{r^2}{\rho^2}\right)^{\frac{1}{q-1}} C_{2/q,q'}(K').$$

Thus we get the left-hand side of (2.46). The right-hand side is a straightforward consequence of (2.47). □

Remark. In the subcritical case $1 < q < q_c$, estimate (2.46) becomes

$$C^{B_{r+\rho}}_{2/q,q'}(K) \leq C\max\left\{r^N, \rho^N\right\}\left(1+\rho^{-\frac{2}{q-1}}\right) C_{2/q,q'}(K/r). \tag{2.47}$$

By using Lemma 2.11, it is easy to derive from this estimate that any positive solution u of (2.1), the initial trace of which vanishes outside 0, satisfies

$$u(x,t) \leq Ct^{-\frac{1}{q-1}} \min\left\{1, \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}}\right\} \qquad \forall (x,t) \in Q_\infty. \tag{2.48}$$

This upper estimate corresponds to the one obtained in [7]. If $F = \overline{B}_r$ the upper estimate is less esthetic. However, it is proved in [25] by a barrier method that, if the initial trace of positive solution u of (2.1), vanishes outside F, and if $1 < q < 3$, there holds

$$u(x,t) \leq t^{-\frac{1}{q-1}} f_1((|x|-r)/\sqrt{t}) \qquad \forall (x,t) \in Q_\infty,\ |x| \geq r, \tag{2.49}$$

where $f = f_1$ is the unique positive (and radial) solution of

$$\left\{\begin{array}{l} f'' + \dfrac{y}{2}f' + \dfrac{1}{q-1}f - f^q = 0 \quad \text{in } (0,\infty) \\ f'(0) = 0\,,\ \lim_{y\to\infty}|y|^{\frac{2}{q-1}}f(y) = 0. \end{array}\right. \tag{2.50}$$

Notice that the existence of f_1 follows from [7] since q is the critical exponent in 1 dim. Furthermore f_1 has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)}e^{-y^2/4t}(1+\circ(1))) \quad \text{as } y \to \infty.$$

2.3 The upper Wiener test

Definition 2.13 We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* δ_2 and δ_∞ by

$$\delta_2[(x,t),(y,s)] := \sqrt{|x-y|^2 + |t-s|}, \tag{2.51}$$

and

$$\delta_\infty[(x,t),(y,s)] := \max\{|x-y|, \sqrt{|t-s|}\}. \tag{2.52}$$

If $K \subset \mathbb{R}^N$ and $i = 2, \infty$,

$$\delta_i[(x,t),K] = \inf\{\delta_i[(x,t),(y,0)] : y \in K\} = \left\{\begin{array}{ll} \max\left\{\text{dist}\,(x,K), \sqrt{|t|}\right\} & \text{if } i = \infty, \\ \sqrt{\text{dist}^2(x,K) + |t|} & \text{if } i = 2. \end{array}\right.$$

For $\beta > 0$ and $i = 2, \infty$, we denote by $\mathcal{B}^i_\beta(m)$ the parabolic ball of center $m = (x,t)$ and radius β in the parabolic distance δ_i.

Let K be *any* compact subset of $\mathbb{R}^N$ and $\overline{u}_K$ the maximal solution of (1.1) which blows up on K. The function $\overline{u}_K$ is obtained as the decreasing limit of the $\overline{u}_{K_\epsilon}$ ($\epsilon > 0$) when $\epsilon \to 0$, where

$$K_\epsilon = \{x \in \mathbb{R}^N : \text{dist}\,(x,K) \le \epsilon\}$$

and $\overline{u}_{K_\epsilon} = \lim_{k\to\infty} u_{k,K_\epsilon} = \overline{u}_K$, where u_k is the solution of the classical problem,

$$\left\{\begin{array}{rll} \partial_t u_k - \Delta u_k + u_k^q = 0 & & \text{in } Q_T, \\ u_k = 0 & & \text{on } \partial_\ell Q_T, \\ u_k(.,0) = k\chi_{K_\epsilon} & & \text{in } \mathbb{R}^N. \end{array}\right. \tag{2.53}$$

If $(x,t) = m \in \mathbb{R}^N \times (0,T]$, we set $d_K = \text{dist}\,(x,K)$, $D_K = \max\{|x-y| : y \in K\}$ and $\lambda = \sqrt{d_K^2 + t} = \delta_2[m,K]$. We define a slicing of K, by setting $d_n = d_n(K,t) := \sqrt{nt}$ ($n \in \mathbb{N}$),

$$T_n = \overline{B}_{d_{n+1}}(x) \setminus B_{d_n}(x), \qquad \forall n \in \mathbb{N},$$

thus $T_0 = B_{\sqrt{t}}(x)$, and

$$K_n(x) = K \cap T_n(x) \ \text{ for } n \in \mathbb{N} \text{ and } \mathcal{Q}_n(x) = K \cap B_{d_{n+1}}(x).$$

When there is no ambiguity, we shall skip the x variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

Theorem 2.14 *Assume $q \geq q_c$. Then there exists $C = C(N,q,T) > 0$ such that*

$$\overline{u}_K(x,t) \leq \frac{C}{t^{\frac{N}{2}}} \sum_{n=0}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_T, \tag{2.54}$$

where a_t is the largest integer j such that $K_j \neq \emptyset$.

With no loss of generality, we can first assume that $x = 0$. Furthermore, in considering the scaling transformation $u_\ell(y,t) = \ell^{\frac{1}{q-1}} u(\sqrt{\ell}y, \ell t)$, with $\ell > 0$, we can assume $t = 1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we shall still denote K. We shall also set $a_{_K} = a_{_{K,1}}$ Since for each $n \in \mathbb{N}$,

$$\frac{1}{2\sqrt{n+1}} \leq d_{n+1} - d_n \leq \frac{1}{\sqrt{n+1}},$$

it is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{1/\sqrt{n+1}}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq 1/2\sqrt{n+1} \text{ and } \#\Theta_n \leq Cn^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{1/\sqrt{n+1}}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_{_K}} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities [2].

Lemma 2.15 *Let $q \geq q_c$. There exists a constant $C = C(N,q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q,q'}(K_{n,j}) \leq Cn^{\frac{N}{2} - \frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right) \qquad \forall n \in \mathbb{N}_*, \tag{2.55}$$

where $B_{n,j} = B_{2/\sqrt{n+1}}(a_{n,j})$ and $C_{2/q,q'}$ stands for the capacity taken with respect to $\mathbb{R}^N$.

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j^\theta}(b_j)$ are disjoint in $\mathbb{R}^N$ and G is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the ρ_j are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q,q'}(G) \leq \sum_j C_{2/q,q'}(G \cap B_{\rho_j}(b_j)) \leq AC_{2/q,q'}(G), \tag{2.56}$$

where $\theta = 1 - 2/N(q-1)$, for some A depending on N, q and ρ^*. This property is called quasi-additivity. We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = \sqrt{n+1}\, T_n, \quad \tilde{K}_n = \sqrt{n+1}\, K_n \quad \text{and} \quad \tilde{\mathcal{Q}}_n = \sqrt{n+1}\, \mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and the $C_{2/q,q'}$ capacities are taken with respect to the balls $B_{2/\sqrt{n+1}}(a_{n,j}) = B_{n,j}$. By Lemma 2.12 with $r = \rho = 1/\sqrt{n+1}$

$$C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \leq C(n+1)^{\frac{1}{q-1}-\frac{N}{2}} C_{2/q,q'}(\tilde{K}_{n,j}), \tag{2.57}$$

where $\tilde{K}_{n,j} = \sqrt{n+1}\,K_{n,j}$ and $\tilde{B}_{n,j} = \sqrt{n+1}\,B_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\tilde{a}_{n,j} = \sqrt{n+1}\,a_{n,j}$ such that the balls $B_{2^\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds in the following sense: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} K_{n,j}\ , \quad \tilde{K}_{n,\Lambda} = \sqrt{n+1}\,K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = \sqrt{n+1}\,K_n,$$

then

$$\sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq A C_{2/q,q'}(\tilde{K}_{n,\Lambda}). \tag{2.58}$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set $\tilde{T}_n$ is of finite order depending upon the dimension. Because $\tilde{K}_n$ is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j} \in \Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda} \sum_{a_{n,j} \in \Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \leq C\, C_{2/q,q'}(\tilde{K}_n). \tag{2.59}$$

Since, by Lemma 2.12,

$$C_{2/q,q'}(\tilde{K}_n) \leq C^{B_{2(n+1)}}_{2/q,q'}(\tilde{K}_n) = (n+1)^{N-\frac{1}{q-1}} C^{B_2}_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right) \leq C(n+1)^{N-\frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{\sqrt{n+1}}\right),$$

we obtain (2.55) by combining this last inequality with (2.57) and (2.59). □

Proof of Theorem 2.14. Step 1. We first notice that

$$\overline{u}_K \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \overline{u}_{K_{n,j}}. \tag{2.60}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$. Because a finite sum of positive solutions of (1.1) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \leq \sum_{0 \leq n \leq a_K} \sum_{a_{n,j} \in \Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{2.61}$$

Letting successively ϵ' and ϵ go to 0 implies (2.60).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{1/\sqrt{n+1}}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2 = (\sqrt{n+1} + \sqrt{n})/2$, we can apply the previous lemmas with $r = 1/\sqrt{n+1}$ and $\rho = r$. For $n \geq n_N$ there holds $t = 1 \geq (r+2\rho)^2 = 9/(n+1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \geq (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \geq 8$). Thus

$$u_{K_{n,j}}(0,1) \leq Ce^{(\sqrt{n}-3/\sqrt{n+1})^2/4} C^{B_{n,j}}_{2/q,q'}(K_{n,j}) \leq Ce^{3/2} e^{-\frac{n}{4}} C^{B_{n,j}}_{2/q,q'}(K_{n,j}). \tag{2.62}$$

Using Lemma 2.15 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0,1) \leq C \sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.63}$$

Finally, we apply Lemma 2.5 if $1 \leq n < n_N$ and get

$$\begin{aligned} \sum_{1}^{n_N-1} \sum_{a_{n,j} \in \Theta_n} u_{K_{n,j}}(0,1) &\leq C \sum_{1}^{n_N-1} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \\ &\leq C' \sum_{1}^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \end{aligned} \tag{2.64}$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (2.63) and (2.64), we derive

$$\overline{u}_K(0,1) \leq C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.65}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-\frac{1}{q-1}} \overline{u}_{K\sqrt{t}}(y\sqrt{t}, 1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$, we derive from (2.65) and the fact that $a_{K,t} = a_{K\sqrt{t},1}$

$$\overline{u}_K(0,t) \leq Ct^{-\frac{1}{q-1}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{2.66}$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (2.54) with $x = 0$, and a space translation leads to the final result. □

Proof of Theorem 2.1. Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B^c_m}$ the maximal solution of (1.1) in Q_∞ the initial trace of which vanishes on B_m. Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t \to 0} u(x,t) = \infty$$

uniformly on $B^c_{m'}$, for any $m' > m$: this can be checked by noticing that

$$U_{B^c_m}\,\ell(y,t) = \ell^{\frac{1}{q-1}} U_{B^c_m}(\sqrt{\ell}y, \ell t) = U_{B^c_{m/\sqrt{\ell}}}(y,t).$$

Furthermore

$$\lim_{m\to\infty} U_{B^c_m}(y,t) = \lim_{m\to\infty} m^{-\frac{2}{q-1}} U_{B^c_1}(y/m, t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B^c_m}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \leq W_F(x,t)$ and $\overline{u}_{F_m} \leq C_1 W_{F_m}(x,t)$, the result follows. □

Theorem 2.1 admits the following integral expression.

Theorem 2.16 *Assume $q \geq q_c$. Then there exists a positive constant $C^*_1 = C^*(N,q,T)$ such that, for any closed subset F of $\mathbb{R}^N$, there holds*

$$\overline{u}_F(x,t) \leq \frac{C^*_1}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{1}{s} F \cap B_1(x)\right) s\,ds, \tag{2.67}$$

where $a_t = \min\{n : F \subset B_{\sqrt{n+1)t}}(x)\}$.

Proof. We first use

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \leq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we denote

$$\Phi(s) = C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall s > 0. \tag{2.68}$$

Step 1. The following inequality holds (see [1] and [29])

$$c_1 \Phi(\alpha s) \leq \Phi(s) \leq c_2 \Phi(\beta s) \qquad \forall s > 0, \quad \forall 1/2 \leq \alpha \leq 1 \leq \beta \leq 2, \tag{2.69}$$

for some positive constants c_1, c_2 depending on N and q. If $\beta \in [1,2]$,

$$\Phi(\beta s) = C_{2/q,q'}\left(\frac{1}{\beta}\left(\frac{F}{s} \cap B_\beta\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\beta\right) \geq c_1 \Phi(s).$$

If $\alpha \in [1/2, 1]$,

$$\Phi(\alpha s) = C_{2/q,q'}\left(\frac{1}{\alpha}\left(\frac{F}{s} \cap B_\alpha\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\alpha\right) \leq c_2 \Phi(s).$$

Step 2. By (2.69)

$$C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \leq c_2 C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall\, s \in [d_{n+1}, d_{n+2}],$$

and $n \leq a_t$. Then

$$c_2 \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\,ds$$
$$\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s\,ds.$$

Using the fact that $N-\frac{2}{q-1}\geq 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}}e^{-\frac{s^2}{4t}}s\,ds \geq e^{-\frac{n+2}{4}}d_{n+1}^{N-\frac{2}{q-1}+1}(d_{n+2}-d_{n+1}) \tag{2.70}$$

$$\geq \frac{t}{4e^2}d_{n+1}^{N-\frac{2}{q-1}}e^{-\frac{n}{4}}. \tag{2.71}$$

Thus

$$\overline{u}_F(x,t)\leq \frac{C}{t^{1+\frac{N}{2}}}\int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}}e^{-\frac{s^2}{4t}}C_{2/q,q'}\left(\frac{1}{s}F\cap B_1\right)s\,ds, \tag{2.72}$$

which ends the proof. □

3 Estimate from below

If $\mu\in\mathfrak{M}^q_+(\mathbb{R}^N)\cap\mathfrak{M}^b(\mathbb{R}^N)$, we denote by $u_\mu=u_{\mu,0}$ the solution of

$$\left\{\begin{array}{rl}\partial_t u_\mu-\Delta u_\mu+u_\mu^q=0 & \text{in } Q_T,\\ u_\mu(.,0)=\mu & \text{in } \mathbb{R}^N.\end{array}\right. \tag{3.1}$$

The maximal σ-moderate solution of (1.1) which has an initial trace vanishing outside a closed set F is defined by

$$\underline{u}_F=\sup\left\{u_\mu:\mu\in\mathfrak{M}^q_+(\mathbb{R}^N)\cap\mathfrak{M}^b(\mathbb{R}^N),\ \mu(F^c)=0\right\}. \tag{3.2}$$

The main result of this section is the next one

Theorem 3.1 *Assume $q\geq q_c$. There exists a constant $C_2=C_2(N,q,T)>0$ such that, for any closed subset $F\subset\mathbb{R}^N$, there holds*

$$\underline{u}_F(x,t)\geq C_2W_F(x,t)\qquad\forall(x,t)\in Q_T. \tag{3.3}$$

We first assume that F is compact, and we shall denote it by K. The first observation is that if $\mu\in\mathfrak{M}^q_+(\mathbb{R}^N)$, $u_\mu\in L^q(Q_T)$ (see lemma below) and $0\leq u_\mu\leq\mathbb{H}[\mu]:=\mathbb{H}_\mu$. Therefore

$$u_\mu\geq\mathbb{H}_\mu-\mathbb{G}\left[\mathbb{H}_\mu^q\right], \tag{3.4}$$

where $\mathbb{G}$ is the Green heat potential in Q_T defined by

$$\mathbb{G}[f](t)=\int_0^t\mathbb{H}[f(s)](t-s)ds=\int_0^t\int_{\mathbb{R}^N}H(.,y,t-s)f(y,s)dyds.$$

Since the details of the proof are very technical, we shall present its main line. The key idea is to construct, for any $(x,t)\in Q_T$, a measure $\mu=\mu(x,t)\in\mathfrak{M}^q_+(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}_\mu(x,t)\geq CW_K(x,t)\qquad\forall(x,t)\in Q_T, \tag{3.5}$$

and

$$\mathbb{G}\left(\mathbb{H}_\mu\right)^q \leq C\,\mathbb{H}_\mu \quad \text{in } Q_T, \tag{3.6}$$

with constants C depends only on N, q, and T, then to replace μ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1}CW_K. \tag{3.7}$$

From this follows

$$\underline{u}_K \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1}CW_K. \tag{3.8}$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$.

We recall the following regularity result which actually can be used for defining the norm in negative Besov spaces [35]

Lemma 3.2 *There exists a constant $c > 0$ such that*

$$c^{-1}\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}_\mu\|_{L^q(Q_T)} \leq c\|\mu\|_{W^{-2/q,q}(\mathbb{R}^N)} \tag{3.9}$$

for any $\mu \in W^{-2/q,q}(\mathbb{R}^N)$.

3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on $\mathbb{R}^N \times \{0\}$ of an *extended slicing* in Q_T which is constructed as follows: if K is a compact subset of $\mathbb{R}^N$, $m = (x,t)$, we define d_K, λ, d_n and a_t as in Section 2.3. Let $\alpha \in (0,1)$ to be fixed later on, we define $\mathcal{T}_n$ for $n \in \mathbb{Z}$ by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}^2_{\sqrt{t(n+1)}}(m) \setminus \mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n \geq 1, \\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m) \setminus \mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n \leq 0, \end{cases}$$

and put

$$\mathcal{T}_n^* = \mathcal{T}_n \cap \{s : 0 \leq s \leq t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for $n \in \mathbb{N}_*$,

$$\mathcal{Q}_n = K \cap \mathcal{B}^2_{\sqrt{t(n+1)}}(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap \left(B_{d_{n+1}}(x) \setminus B_{d_n}(x)\right).$$

Let $\nu_n \in \mathfrak{M}_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ be the q-capacitary measure of the set K_n/d_{n+1} (see [1, Sec. 2.2]). Such a measure has support in K_n/d_{n+1} and

$$\nu_n(K_n/d_{n+1}) = C_{2/q,q'}(K_n/d_{n+1}) \text{ and } \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)} = \left(C_{2/q,q'}(K_n/d_{n+1})\right)^{1/q}. \tag{3.10}$$

We define μ_n as follows

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}}\nu_n(A/d_{n+1}) \qquad \forall A \subset K_n,\ A \text{ Borel}, \tag{3.11}$$

and set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \tag{3.12}$$

Proposition 3.3 *Let $q \geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x,t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.13}$$

in $\mathbb{R}^N \times (0,T)$.

Proof. Since

$$\mathbb{H}_{\mu_n}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \tag{3.14}$$

and

$$y \in K_n \Longrightarrow |x-y| \leq d_{n+1},$$

(3.13) follows because of (3.11) and (3.12). □

3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x,t) &\geq \sum_{n\in\mathbb{Z}} \mathbb{H}_{\mu_n}(x,t) - \int_0^t \int_{\mathbb{R}^N} H(x,y,t-s) \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= I_1 - I_2. \end{aligned} \tag{3.15}$$

since $\mu_n = 0$ if $n \notin A_K = \mathbb{N} \cap [1, a_t]$, and

$$\begin{aligned} I_2 &\leq \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^t \int_{\mathbb{R}^N} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\in A_K} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &\leq \frac{1}{(4\pi)^{\frac{N}{2}}} (J_\ell + J'_\ell), \end{aligned} \tag{3.16}$$

for some $\ell \in \mathbb{N}^*$ to be fixed later on, where

$$J_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_\ell = \sum_{p\in\mathbb{Z}} \iint_{\mathcal{T}^*_p} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n\geq p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds.$$

The next estimate will be used several times in the sequel.

Lemma 3.4 *Let $0<a<b$ and $t>0$, then,*

$$\max\left\{\sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}} : 0\leq\sigma\leq t,\ at\leq\rho^2+\sigma\leq bt\right\} = e^{\frac14}\begin{cases} t^{-\frac{N}{2}} e^{-\frac{a}{4}} & \text{if } \dfrac{a}{2N}>1,\\ \left(\dfrac{2N}{at}\right)^{\frac{N}{2}} e^{-\frac{N}{2}} & \text{if } \dfrac{a}{2N}\leq 1.\end{cases}$$

Proof. Set

$$\mathcal{J}(\rho,\sigma) = \sigma^{-\frac{N}{2}} e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \left\{(\rho,\sigma)\in[0,\infty)\times(0,t]:\ at\leq\rho^2+\sigma\leq bt\right\}.$$

We first notice that, for fixed σ, the maximum of $\mathcal{J}(.,\sigma)$ is achieved for ρ minimal. If $\sigma\in[at,bt]$ the minimal value of ρ is 0, while if $\sigma\in(0,at)$, the minimum of ρ is $\sqrt{at-s}$.

- Assume first $a\geq 1$, then $\mathcal{J}(\sqrt{at-\sigma},\sigma) = e^{\frac14}\sigma^{-\frac{N}{4}} e^{-\frac{at}{4\sigma}}$, thus, if $1\leq a/2N$ the minimal value of $\mathcal{J}(\sqrt{at-\sigma},\sigma)$ is $e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}$, while, if $a/2N<1\leq a$, the minimum is $e^{\frac14}t^{-\frac{N}{2}}e^{-\frac{a}{4}}$.
- Assume now $a\leq 1$. Then

$$\begin{aligned}\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}_{a,b,t}\} &= \max\left\{\max_{\sigma\in(at,t]}\mathcal{J}(0,\sigma),\ \max_{\sigma\in(0,at]}\mathcal{J}(\sqrt{at-\sigma},\sigma)\right\}\\ &= \max\left\{(at)^{-\frac{N}{2}},\ e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\}\\ &= e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}.\end{aligned}$$

Combining these two estimates, we derive the result. □

Remark. The following variant of Lemma 3.4 will be useful in the sequel: *For any $\theta\geq 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}(a,b,t)\}\leq e^{\frac14}\left(\frac{2N\theta}{t}\right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a\geq 1. \tag{3.17}$$

Lemma 3.5 *There exists a positive constant $C=C(N,\ell,q)$ such that*

$$J_\ell \leq Ct^{-\frac{N}{2}}\sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4}\, C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.18}$$

Proof. The set of p for the summation in J_ℓ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$ and we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n<p+\ell} \mathbb{H}_{\mu_n}(y,s)\right]^q.$$

If $p = 2-\ell,\ldots,0$,

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \le |x-y|^2 + t - s \le t\alpha^{-2p},$$

and, if $p\ge 1$

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow pt \le |x-y|^2 + t - s \le (p+1)t.$$

By Lemma 3.4 and (3.17), there exists $C = C(N,\ell,\alpha) > 0$ such that

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-\alpha^{2-2p}/4}, \tag{3.19}$$

if $p = 2-\ell,\ldots,0$, and

$$\max\left\{(t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} : (y,s)\in\mathcal{T}_p^*\right\} \le Ct^{-\frac{N}{2}} e^{-p/4}, \tag{3.20}$$

if $p\ge 1$. When $p = 2-\ell,\ldots,0$

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \le C \sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}^q(y,s). \tag{3.21}$$

for some $C = C(\ell,q) > 0$, thus

$$\begin{aligned}
J_{1,\ell} &\le Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\
&\le Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \sum_{p=n-\ell+1}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \\
&\le Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}.
\end{aligned} \tag{3.22}$$

If the set of p's is not upper bounded, we introduce $\delta > 0$ to be made precise later on. Then

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \le \left[\sum_{1}^{p+\ell-1} e^{\delta q' \frac{n}{4}}\right]^{q/q'} \sum_{1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}_{\mu_n}^q(y,s), \tag{3.23}$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > a_t$, we obtain, for some $C > 0$ which depends also on δ,

$$\begin{aligned} J_{2,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta qn}{4}} \left\| \mathbb{H}_{\mu_n} \right\|_{L^q(Q_t)}^q \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} \left\| \mathbb{H}_{\mu_n} \right\|_{L^q(Q_t)}^q e^{-\frac{\delta qn}{4}} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \left\| \mathbb{H}_{\mu_n} \right\|_{L^q(Q_t)}^q . \end{aligned} \tag{3.24}$$

Notice that we choose δ such that $\delta\ell q < 1$. Combining (3.22) and (3.24), we derive (3.18) from Lemma 3.2, (3.10) and (3.11). $\square$

The set of indices p for which the μ_n terms are not zero in J'_ℓ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds.$$

Lemma 3.6 *There exists a constant $C = C(N, q, \ell) > 0$ such that*

$$J'_{1,\ell} \leq Ct^{1-\frac{Nq}{2}} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C_{2/q,q'}^q \left(\frac{K_n}{d_{n+1}} \right), \tag{3.25}$$

where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.

Proof. Since

$$(y,s) \in \mathcal{T}_p^*, \text{ and } (z,0) \in K_n \Longrightarrow |y-z| \geq (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.26}$$

there holds

$$\mathbb{H}_{\mu_n}(y,s) \leq (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \leq Ct^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.4. Let $\epsilon_n > 0$ such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$
\begin{aligned}
J'_{1,\ell} &\leq CA_\epsilon^{q/q'}t^{-\frac{Nq}{2}}\sum_{p=-\infty}^{0}\iint_{\mathcal{T}^*_p}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\sum_{n=1\vee(p+\ell)}^{\infty}\epsilon_n^{-q}e^{-q\frac{(\sqrt{n}-\alpha^{-p})^2}{4}}\mu_n^q(K_n)ds\,dy \\
&\leq CA_\epsilon^{q/q'}t^{-\frac{Nq}{2}}\sum_{n=1}^{\infty}\epsilon_n^{-q}\mu_n^q(K_n)\sum_{-\infty}^{p=0\wedge(n-\ell)}e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}}\iint_{\mathcal{T}^*_p}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}ds\,dy \\
&\leq CA_\epsilon^{q/q'}t^{-\frac{Nq}{2}}\sum_{n=1}^{\infty}\epsilon_n^{-q}\mu_n^q(K_n)e^{-\frac{q(\sqrt{n}-1)^2}{4}}\iint_{\cup_{p\leq 0}\mathcal{T}^*_p}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}ds\,dy \\
&\leq CA_\epsilon^{q/q'}t^{1-\frac{Nq}{2}}\sum_{n=1}^{\infty}\epsilon_n^{-q}\mu_n^q(K_n)e^{-\frac{q(\sqrt{n}-1)^2}{4}}.
\end{aligned}
\tag{3.27}
$$

Set $h=2q(q+1)/(q-1)^2$ and $Q=(1+q)/2$, then $q(\sqrt{n}-1)^2\geq Q(n-h)_+$ for any $n\geq 1$. If we choose $\epsilon_n=e^{-\frac{(q-1)(n-h)_+}{16q}}$, there holds $\epsilon_n^{-q}e^{-\frac{q(\sqrt{n}-1)^2}{4}}\leq e^{\frac{(q+3)(n-h)_+}{16}}$. Finally

$$
J'_{1,\ell}\leq Ct^{1-\frac{Nq}{2}}\sum_{n=1}^{\infty}e^{\frac{(1+\epsilon_0)(n-h)_+}{4}}\mu_n^q(K_n),
$$

with $\beta_0=(q-1)/4$, which yields to (3.25) by the choice of the μ_n. $\square$

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

Lemma 3.7 *Assume $N=1$ and ℓ is an integer larger than 1. There exists a positive constant $C=C(q,\ell)>0$ such that*

$$
J'_{2,\ell}\leq Ct^{-1/2}\sum_{n=\ell}^{a_t}e^{-\frac{n}{4}}d_{n+1}^{\frac{q-3}{q-1}}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right).
\tag{3.28}
$$

Proof. If $(y,s)\in\mathcal{T}^*_p$ and $z\in K_n$ ($p\geq 1$, $n\geq p=\ell$) , there holds $|x-y|\geq\sqrt{t}\sqrt{p}$ and $|y-z|\geq\sqrt{t}(\sqrt{n}-\sqrt{p+1})$. Therefore

$$
J'_{2,\ell}\leq C\sqrt{t}\sum_{p=1}^{a_t-\ell}\frac{1}{\sqrt{p}}\int_0^t e^{-\frac{pt}{4(t-s)}}\left(\sum_{n=p+\ell}^{a_t}s^{-1/2}e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2t}{4s}}\mu_n(K_n)\right)^q.
$$

If $\epsilon\in(0,q)$ is some positive parameter which will be made more precise later on, there holds

$$
\begin{aligned}
&\left(\sum_{n=p+\ell}^{a_t}s^{-1/2}e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2t}{4s}}\mu_n(K_n)\right)^q \\
&\qquad\leq\left(\sum_{n=p+\ell}^{a_t}e^{-\epsilon q'\frac{(\sqrt{n}-\sqrt{p+1})^2t}{4s}}\right)^{q/q'}\sum_{n=p+\ell}^{a_t}s^{-\frac{q}{2}}e^{-(q-\epsilon)\frac{(\sqrt{n}-\sqrt{p+1})^2t}{4s}}\mu_n^q(K_n),
\end{aligned}
$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\le \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &= 2\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x+\sqrt{p+1}) dx \\ &\le \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1}\int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\ &\le C\sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\ &\le C\sqrt{\frac{(p+1)s}{t}}.\end{aligned}$$

If we set $q_\epsilon = q-\epsilon$, then

$$J'_{2,\ell} \le C\epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where $C = C(\epsilon, q) > 0$. Since

$$\begin{aligned}&\int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\ &\qquad = \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1})$. In this range of indices $B \ge \sqrt{q_\epsilon}(\sqrt{p+\ell}-\sqrt{p+1}) \ge \sqrt{q_\epsilon}(\ell-1)\sqrt{p}$, thus $\kappa = \sqrt{q_\epsilon}(\ell-1)$ and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \le p^{\frac{1}{4}} n^{-1/2} (\sqrt{n}-\sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \le \frac{Cp^{\frac{1}{4}}(\sqrt{n}-\sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \tag{3.29}$$

which implies

$$J'_{2,\ell} \le Ct^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{1/2} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \tag{3.30}$$

where C depends of ϵ, q and ℓ. By Lemma A.2

$$J'_{2,\ell} \le Ct^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \tag{3.31}$$

Because $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)$ (remember $N=1$) and diam $\frac{K_n}{d_{n+1}} \leq n^{-1}$, there holds

$$\mu_n^q(K_n) \leq C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} \mu_n(K_n) = C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}(K_n/d_{n+1}) \tag{3.32}$$

and inequality (3.28) follows. $\square$

Next we give the general proof. For this task we shall use again the quasi-additivity with separated partitions.

Lemma 3.8 *Assume $N \geq 2$ and ℓ is an integer larger than 1. There exist a positive constant $C_1 = C_1(q, N, \ell) > 0$ such that f*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.33}$$

Proof. As in the proof of Theorem 2.14, we know that there exists a finite number J, depending only on the dimension N, of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the rescaled sets $\tilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$ by the N-dim balls $B_2(\tilde{a}_{n,j})$ where $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$, $|a_{n,j}| = \dfrac{d_{n+1}+d_n}{2}$ and $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta_{t,n}^h \leq Cn^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$. We write $\mu_n = \sum_{h=1}^J \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^J J'^{\,h}_{2,\ell}$, where $\mu_n^h = \sum_{j\in\Theta_{t,n}^h} \mu_{n,j}$, and $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_n, j)$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \tag{3.34}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds.$$

We denote

$$J'^{\,h}_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds,$$

and clearly

$$J'_{2,\ell} \leq C \sum_{h=1}^{J} J'^{\,h}_{2,\ell}, \tag{3.35}$$

where C depends only on N and q. For integers n and p such that $n \geq \ell+1$, we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\sum_{n=p+\ell}^{a_t}\int_{K_n} e^{-\frac{|y-z|^2}{4s}}d\mu_n^h(z) = \sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}^h}\int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}}d\mu_{n,j}(z)$$
$$\leq \left(\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}^h} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}}\right)^{1/q'}\left(\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}^h} e^{-q\lambda_{n,j,y}^2\frac{1-\epsilon}{4s}}\mu_{n,j}^q(K_{n,j})\right)^{1/q}$$

where $\epsilon > 0$ will be made precise later on.

Step 1 We claim that

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}} \leq C\sqrt{\frac{ps}{t}} \tag{3.36}$$

where C depends on ϵ, q and N. If y is fixed in T_p, we denote by z_y the point of T_n which solves $|y - z_y| = \mathrm{dist}\,(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \leq |y - z_y| \leq t(\sqrt{n} - \sqrt{p}).$$

Let $Y = y\sqrt{t(p+1)}/|y|$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = Y/|Y|$, consider the points $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$ where $-n \leq k \leq n$ and denote by $G_{n,k}$ the spherical shell obtain by intersecting the spherical shell T_n with the domain $H_{n,k}$ which is the set of points in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$ and $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n + 1 - |k|)^{N-2}$, where C depends only on N, and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric considerations (Pythagore's theorem) imply that $\lambda_{n,j,y}^2$ is greater than $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\epsilon q'\frac{\lambda_{n,j,y}^2}{4s}} \leq C\sum_{n=p+\ell}^{a_t}\sum_{k=-n}^{n}(n+1-|k|)^{N-2}e^{-\frac{\epsilon q'(n+p+1-2k\sqrt{p+1}/)t}{4s\sqrt{n}}} \tag{3.37}$$

Case $N = 2$. Summing a geometric series and using the inequality $\frac{e^u}{e^u-1} \leq 1 + u^{-1}$ for $u > 0$, we obtain

$$\begin{aligned}\sum_{k=-n}^{n} e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}}\frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}-1} \\ &\leq e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}}\left(1+\frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right).\end{aligned} \tag{3.38}$$

Thus, by comparison between series and integrals,

$$\begin{aligned}\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda^2_{n,j,y}}{4s}} &\leq C\sum_{n=p+\ell}^{a_t}\left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right)e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2}{4s}}\\ &\leq C\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx \\ &\qquad\qquad + \frac{Cs}{t\sqrt{p}}\int_{p+1}^{\infty}\sqrt{x}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx.\end{aligned} \tag{3.39}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}}ydy\\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}ydy + 2\sqrt{p+1}\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}dy\\ &= \frac{2s}{t}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}zdz + 2\sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}dz,\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}\int_{p+1}^{\infty}\sqrt{x}e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}}dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}}y^2dy\\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}(y+\sqrt{p+1})^2dy\\ &\leq 4\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}y^2dy + 4(p+1)\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}}dy\\ &\leq 4\left(\frac{s}{t}\right)^{3/2}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}z^2dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}}dz\end{aligned} \tag{3.41}$$

Jointly with (3.39), these inequalities imply

$$\sum_{n=p+\ell}^{a_t}\sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q'\lambda^2_{n,j,y}}{4s}} \leq C\sqrt{\frac{ps}{t}}. \tag{3.42}$$

Case $N>2$. Because the value of the right-hand side of (3.37) is an increasing value of N, it is sufficient to prove (3.36) when N is even, say $(N-2)/2=d\in\mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n}(n+1-|k|)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} \leq 2\sum_{k=0}^{n}(n+1-k)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.43}$$

We set

$$\alpha = \epsilon q'\frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d=\sum_{k=0}^{n}(n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha}=\frac{e^{(k+1)\alpha}-e^{k\alpha}}{e^{\alpha}-1},$$

we use Abel's transform to obtain

$$
\begin{aligned}
I_d &= \frac{1}{e^\alpha - 1}\left(e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^{n} \left((n+2-k)^d - (n+1-k)^d \right) e^{k\alpha} \right) \\
&\le \frac{1}{e^\alpha - 1}\left((1-d)e^{(n+1)\alpha} - (n+1)^d + de^\alpha \sum_{k=1}^{n} \left((n+1-k)^{d-1} \right) e^{k\alpha} \right).
\end{aligned}
$$

Therefore the following induction holds

$$
I_d \le \frac{de^\alpha}{e^\alpha - 1} I_{d-1}. \tag{3.44}
$$

In (3.38), we have already used the fact that

$$
\frac{de^\alpha}{e^\alpha - 1} \le C\left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right),
$$

and

$$
I_d \le C\left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) I_0.
$$

Thus (3.39) is replaced by

$$
\begin{aligned}
\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\le C \sum_{n=p+\ell}^{a_t} \left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \\
&\le C \int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\
&\qquad + \left(\frac{Cs}{t\sqrt{p}}\right)^{d+1} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx.
\end{aligned} \tag{3.45}
$$

The first integral on the right-hand side has already been estimated in (3.40), for the second integral, there holds

$$
\begin{aligned}
\int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= \int_{0}^{\infty} (y + \sqrt{p+1})^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dx \\
&\le C\int_0^\infty y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^\infty e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\
&\le C\left(\frac{s}{t}\right)^{2+\frac{d}{2}} \int_0^\infty z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\
&\qquad + C\left(\frac{s}{t}\right)^{3/2} p^{1+\frac{d}{2}} \int_0^\infty e^{-\frac{\epsilon q' z^2}{4}} dz.
\end{aligned} \tag{3.46}
$$

Combining (3.40), (3.45) and (3.46), we derive (3.36).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y,s) \in \mathcal{T}_p^*$ implies that $|x-y|^2 \geq (p-1)t$, thus $J'^{\,h}_{2,\ell}$ satisfies

$$
\begin{aligned}
J'^{\,h}_{2,\ell} &\leq Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\
&\qquad\qquad \times \sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}^h} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} \mu_{n,j}^q(K_{n,j}) ds dy \\
&\leq Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j \in \Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}) \\
&\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} ds dy
\end{aligned} \tag{3.47}
$$

and the constant C depends on N, q and ϵ. Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$
|y-a_{n,j}|^2 = |x-y|^2 + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle \geq pt + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle,
$$

we get

$$
\int_{\Gamma_p} e^{-\frac{q_\epsilon |y-a_{n,j}|^2}{4s}} dy = e^{-\frac{q_\epsilon |x-a_{n,j}|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \langle y-x, a_{n,j}-x\rangle/4s} dS_r(y) dr.
$$

For estimating the value of the spherical integral, we can assume that $a_{n,j}-x = (0,\ldots,0,|a_{n,j}-x|)$, $y = (y_1,\ldots,y_N)$ and, using spherical coordinates with center at x, that the unit sphere has the representation $S^{N-1} = \{(\sin\phi.\sigma, \cos\phi) \in \mathbb{R}^{N-1} \times \mathbb{R} : \sigma \in S^{N-2},\ \phi \in [0,\pi]\}$. With this representation, $dS_r = r^{N-1} \sin^{N-2}\phi\, d\phi\, d\sigma$ and $\langle y-x, a_{n,j}-x\rangle = |a_{n,j}-x|\,|y-x|\cos\phi$. Therefore

$$
\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) = r^{N-1} \left|S^{N-2}\right| \int_0^\pi e^{2q_\epsilon \frac{|a_{n,j}-x| r\cos\phi}{4s}} \sin^{N-2}\phi\, d\phi.
$$

By Lemma A.3

$$
\begin{aligned}
\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) &\leq C \frac{r^{N-1} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}}{\left(1 + \frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\
&\leq C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}.
\end{aligned} \tag{3.48}
$$

Therefore

$$
\int_{\Gamma_p} e^{-q_\epsilon \frac{|y-a_{n,j}|^2}{4s}} dy \leq Ct^{\frac{N-1}{4}} p^{\frac{N-3}{4}} \frac{s^{\frac{N-1}{2}} e^{-q_\epsilon \frac{(|a_{n,j}-x| - \sqrt{t(p+1)})^2}{4s}}}{|a_{n,j}-x|^{\frac{N-1}{2}}}, \tag{3.49}
$$

and, since $|a_{n,j}-x| \geq \sqrt{tn}$,

$$\begin{aligned}
&\int_0^t\int_{\Gamma_p}(t-s)^{-\frac{N}{2}}s^{-(q(N-1)+1)/2}e^{-\frac{|x-y|^2}{4(t-s)}}e^{-q_\epsilon\frac{\lambda^2_{n,j,y}}{4s}}dy\,ds\\
&\qquad\leq C\frac{\sqrt{t}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\int_0^t(t-s)^{-\frac{N}{2}}s^{-\frac{(q-1)(N-1)+1}{2}}e^{-\frac{pt}{4(t-s)}}e^{-q_\epsilon\frac{(\sqrt{tn}-\sqrt{t(p+1)})^2}{4s}}ds\\
&\qquad\leq C\frac{t^{\frac{1-q(N-1)}{2}}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\int_0^1(1-s)^{-\frac{N}{2}}s^{-\frac{(q-1)(N-1)+1}{2}}e^{-\frac{p}{4(1-s)}}e^{-q_\epsilon\frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}}.
\end{aligned}\tag{3.50}$$

We apply Lemma A.1, with $A=\sqrt{p}$, $B=\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1})$, $b=\frac{(q-1)(N-1)+1}{2}$, $a=\frac{N}{2}$ and $\kappa=\sqrt{q_\epsilon}(\ell-1)/8$ as in the case $N=1$, and noticing that, for these specific values,

$$\begin{aligned}
A^{1-a}B^{1-b}(A+B)^{a+b-2} &= p^{\frac{2-N}{4}}(\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}}\\
&\qquad\times(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}}\\
&\leq C\left(\frac{n}{p}\right)^{\frac{N}{4}-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},
\end{aligned}$$

where C depends on N, q and κ. Therefore

$$\begin{aligned}
&\int_0^t\int_{\Gamma_p}(t-s)^{-\frac{N}{2}}s^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}e^{-q_\epsilon|y-z|^2/4s}dy\,ds\\
&\quad\leq C\frac{t^{(1-q(N-1))/2}p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}}\left(\frac{n}{p}\right)^{\frac{N}{4}-1/2}\left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}\\
&\quad\leq Ct^{\frac{1-q(N-1)}{2}}p^{-\frac{1}{4}}n^{\frac{(q-1)(N-1)-2}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned}\tag{3.51}$$

We derive from (3.47), (3.51),

$$\begin{aligned}
&J'^{\,h}_{2,\ell}\leq Ct^{1-\frac{Nq}{2}}\\
&\times\sum_{n=\ell+1}^{a_t}\sum_{j\in\Theta^h_{t,n}}n^{\frac{(q-1)(N-1)-2}{4}}\mu^q_{n,j}(K_{n,j})\sum_{p=1}^{n-\ell}p^{\frac{2q-3}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned}\tag{3.52}$$

By Lemma A.2 with $\alpha=\dfrac{2q-3}{4}$, $\beta=\frac{1-(q-1)(N-1)}{2}$, $\delta=\frac{1}{4}$ and $\gamma=q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell}p^{\frac{2q-3}{4}}(\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}}e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}\leq Cn^{\frac{N(q-1)+q-3}{4}}e^{-\frac{n}{4}},\tag{3.53}$$

thus

$$J'^{\,h}_{2,\ell}\leq Ct^{1-\frac{Nq}{2}}\sum_{n=\ell+1}^{a_t}n^{\frac{N(q-1)}{2}-1}e^{-\frac{n}{4}}\sum_{j\in\Theta^h_{t,n}}\mu^q_{n,j}(K_{n,j}).\tag{3.54}$$

Because

$$\mu_{n,j}(K_{n,j}) = C^{B_{n,j}}_{2/q,q'}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.15, except that the scale factor is $\sqrt{(n+1)t}$ instead of $\sqrt{n+1}$ so that the sets $\tilde{T}_n$, $\tilde{K}_n$, $\tilde{\mathcal{Q}}_n$ and $\tilde{K}_n$ remains unchanged Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\tfrac{K_n}{d_{n+1}}\right), \tag{3.55}$$

which implies (3.33). □

The proof of Theorem 3.1 follows from the previous estimates on J_1 and J_2. Furthermore the following integral expression holds

Theorem 3.9 *Assume $q \geq q_c$. Then there exists a positive constants C_2^* , depending on N,q and T, such that for any closed set F, there holds*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\, ds, \tag{3.56}$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We shall distinguish according $q = q_c$, or $q > q_c$, and for simplicity we shall denote $B_r = B_r(x)$ for the various values of r.
Case 1: $q = q_c \Longleftrightarrow N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right) = C_{2/q,q'}\left(\frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n}\right) \leq C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

thus

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

it follows

$$\begin{aligned}
\sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right) \\
&\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} \sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \\
&\geq (1 - e^{-\frac{1}{4}}) \sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - e^{-\frac{1}{4}} C_{2/q,q'}\left(\frac{F}{\sqrt{t}} \cap B_1\right).
\end{aligned}$$

Since, by (2.69),

$$C_{2/q,q'}\left(\frac{F}{s'}\cap B_1\right)\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\geq C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right),$$

for any $s'\in[d_{n+1},d_{n+2}]$ and $s\in[d_n,d_{n+1}]$, there holds

$$\begin{aligned}te^{-\frac{n}{4}}C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)&\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\int_{d_n}^{d_{n+1}}e^{-s^2/4t}s\,ds\\&\geq\int_{d_n}^{d_{n+1}}e^{-s^2/4t}C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right)s\,ds.\end{aligned}$$

This implies

$$W_F(x,t)\geq(1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}}e^{-s^2/4t}C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right)s\,ds.$$

Case 2: $q>q_c\Longleftrightarrow N-\frac{2}{q-1}>0$. In that case it is known [1] that

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right)\approx d_{n+1}^{\frac{2}{q-1}-N}C_{2/q,q'}\left(F_n\right)$$

thus

$$W_F(x,t)\approx t^{-1-\frac{N}{2}}\sum_{n=0}^{a_t}e^{-\frac{n}{4}}C_{2/q,q'}\left(F_n\right).$$

Since

$$C_{2/q,q'}\left(F_n\right)\geq C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right)-C_{2/q,q'}\left(F\cap B_{d_n}\right),$$

and again

$$\begin{aligned}t^{-\frac{N}{2}}\sum_{n=0}^{a_t}e^{-\frac{n}{4}}C_{2/q,q'}\left(F_n\right)&\geq(1-e^{-\frac{1}{4}})t^{-\frac{N}{2}}\sum_{n=0}^{a_t-1}e^{-\frac{n}{4}}C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right)\\&\geq(1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}}e^{-\frac{s^2}{4t}}C_{2/q,q'}\left(F\cap B_s\right)s\,ds.\end{aligned}$$

Because $C_{2/q,q'}\left(F\cap B_s\right)\approx s^{N-\frac{2}{q-1}}C_{2/q,q'}\left(s^{-1}F\cap B_1\right)$, (3.56) follows. $\square$

4 Applications

The first result of this section is the following

Theorem 4.1 *Assume $N\geq1$ and $q>1$. Then $\overline{u}_K=\underline{u}_K$.*

Proof. If $1 < q < q_c$, the result is already proved in [25]. The proof in the super-critical case is an adaptation that we shall recall, for the sake of completeness. By Theorem 2.16 and Theorem 3.9 there exists a positive constant C, depending on N, q and T such that

$$\overline{u}_F(x,t) \leq \underline{u}_F(x,t) \qquad \forall (x,t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \dfrac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$ if we assume that $\overline{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta \overline{u}_F$ is a subsolution. Therefore there exists a solution u_1 of (1.1) in Q_∞ such that $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1) which is above the subsolution θu_μ. Thus $u_{\theta\mu} \leq u_1 < \underline{u}_F$ and finally $\underline{u}_F \leq u_1 < \underline{u}_F$, a contradiction. □

If we combine Theorem 2.16 and Theorem 3.9 we derive the following integral approximation of the capacitary potential

Proposition 4.2 *Assume $q \geq q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \leq W_F(x,t) \\ \leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \end{aligned} \tag{4.57}$$

for any $(x,t) \in Q_T$.

Definition 4.3 *If F is a closed subset of $\mathbb{R}^N$, we define the $(2/q,q')$-integral capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \qquad \forall (x,t) \in Q_\infty, \tag{4.58}$$

where $D_F(x) = \max\{|x-y| : y \in F\}$.

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x,t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} 0 \leq t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds - \mathcal{W}_F(x,t) \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \end{aligned} \tag{4.60}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q,q'}\left(\frac{F}{s\sqrt{t}} \cap B_1(x)\right) s\, ds. \tag{4.61}$$

The following result gives a sufficient condition in order $\overline{u}_F$ has not a strong blow-up at some point x.

Proposition 4.4 *Assume $q \geq q_c$ and F is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau \to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma, \tag{4.62}$$

then

$$\lim_{t \to 0} t^{\frac{1}{q-1}} \overline{u}_F(x,t) = C\gamma, \tag{4.63}$$

for some $C = C(N,q) > 0$.

Proof. Clearly, condition (4.62) implies

$$\lim_{t \to 0} C_{2/q,q'}\left(\frac{F}{\sqrt{t}s} \cap B_1(x)\right) = \gamma$$

for any $s > 0$. Then (4.63) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q. □

In the next result we give a condition in order the solution remains bounded at some point x. The proof is similar to the previous one.

Proposition 4.5 *Assume $q \geq q_c$ and F is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau \to 0} \tau^{-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty, \tag{4.64}$$

then $\overline{u}_F(x,t)$ remains bounded when $t \to 0$.

A Appendix

The next estimate is crucial in the study of semilinear parabolic equations.

Lemma A.1 *Let a and b be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a,b,\kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.1}$$

Proof. We first notice that

$$\max\{e^{-A^2/4(1-x)}e^{-B^2/4x} : 0 \le x \le 1\} = e^{-(A+B)^2/4}, \tag{A.2}$$

and it is achieved for $x_0 = B/(A+B)$. Set $\Phi(x) = (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)}e^{-B^2/4x}$, thus

$$\int_0^1 \Phi(x)dx = \int_0^{x_0} \Phi(x)dx + \int_{x_0}^1 \Phi(x)dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{A.3}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{A.4}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a}x^{-b}e^{-A^2/4(1-x)-B^2/4x}dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a}x(u)^{-b}e^{-u}x'(u)du$$

Putting $x' = x'(u)$ and differentiating (A.4),

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{4u + B^2 - A^2 - 8ux(u)}. \tag{A.5}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x)dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1}x(u)^{-b+1}e^{-u}du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{A.6}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on A and B. It means that the two sided-inequalities up to multiplicative constants independent of A and B. Therefore

$$\int_0^{x_0} \Phi(x)dx = 2^{a-b-4}e^{-(A+B)^2/4}\int_0^{\infty} \tilde{\Phi}(v)dv \quad \text{where}$$

$$\tilde{\Phi}(v) = \frac{\left(v+(AB+B^2)/2-\sqrt{v(v+AB)}\right)^{1-b}\left(v+(A^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-a}}{\left(v+(A+B)^2/4\right)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.7}$$

Case 1: $a \geq 1$, $b \geq 1$. First

$$\frac{\left(v+(A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v+(A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v+(A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{A.8}$$

since $a+b-2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v+(A^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-a} \approx \left(v+A(A+B)\right)^{1-a}. \tag{A.9}$$

Furthermore

$$\begin{aligned} v+(AB+B^2)/2-\sqrt{v(v+AB)} &= B^2\frac{v+(A+B)^2/4}{v+B(A+B)/2+\sqrt{v(v+AB)}} \\ &\approx B^2\frac{v+(A+B)^2}{v+B(A+B)}. \end{aligned} \tag{A.10}$$

Then

$$\left(v+(AB+B^2)/2-\sqrt{v(v+AB)}\right)^{1-b} \approx B^{2-2b}\left(\frac{v+B(A+B)}{v+(A+B)^2}\right)^{b-1} \tag{A.11}$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(v+B(A+B)\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{v^{b-1}+(B^2+AB)^{b-1}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{A.12}$$

where C depends on a, b and κ. The function $v \mapsto (v+(A+B)^2)/(v+A(A+B))$ is decreasing on $(0,\infty)$. If we set

$$C_1 = \int_0^{\infty}\frac{v^{b-1}e^{-v}dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^{\infty}\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}$$

then

$$C_1 \leq K(B^2+AB)^{b-1}C_2$$

with $K = C_1\kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0}\Phi(x)dx \leq Ce^{-(A+B)^2/4}B^{1-b}A^{1-a}(A+B)^{a+b-2}. \tag{A.13}$$

The estimate of $J_{a,b}$ is obtained by exchanging (A,a) with (B,b) and replacing x by $1-x$. *Mutadis mutandis*, this yields directely to the same expression as in A.13 and finally

$$\int_0^1 \Phi(x)dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.14}$$

Case 2: $a \geq 1$, $b < 1$. Estimates (A.7), (A.8), (A.9), (A.10) and (A.11) are valid. Because $v \mapsto (v+B(A+B))^{b-1}$ is decreasing, (A.12) has to be replaced by

$$\tilde{\Phi}(v) \leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(AB+B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{A.15}$$

This implies (A.13) directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1-x$. If $x_1 = 1-x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a}(1-x)^{-b}e^{-A^2/4x}e^{-B^2/4(1-x)}dx = \int_0^{x_1}\Psi(x)dx.$$

Then

$$\int_0^{x_1}\Psi(x)dx = 2^{b-a-4}e^{-(A+B)^2/4}\int_0^{x_1}\tilde{\Psi}(v)dv \quad \text{where}$$
$$\tilde{\Psi}(v) = \frac{\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a}\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b}}{(v+(A+B)^2/4)^{2-a-b}\sqrt{v(v+AB)}}e^{-v}dv. \tag{A.16}$$

Equivalence (A.8) is unchanged; (A.9) is replaced by

$$\left(v+(B^2+AB)/2+\sqrt{v(v+AB)}\right)^{1-b} \approx (v+B(A+B))^{1-b}, \tag{A.17}$$

(A.10) by

$$v+(AB+A^2)/2-\sqrt{v(v+AB)} \approx A^2\frac{v+(A+B)^2}{v+A(A+B)}, \tag{A.18}$$

and (A.11) by

$$\left(v+(AB+A^2)/2-\sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a}\left(\frac{v+A(A+B)}{v+(A+B)^2}\right)^{a-1}. \tag{A.19}$$

Because $a > 1$, (A.12) turns into

$$\begin{aligned}\tilde{\Psi}(v) &\leq CA^{2-2b}(v+(A+B)^2)^{b-1}\frac{(v+A^2+AB)^{a-1}(v+B^2+AB)^{1-b}}{\sqrt{v(v+\kappa)}}\\ &\leq Ce^{-(A+B)^2/4}A^{2-2b}(A+B)^{2b-2}\\ &\qquad \times\frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}+A^{a-1}B^{1-b}(A+B)^{a-b}}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.20}$$

Because $AB \geq \kappa$, there exists a positive constant C, depending on κ, such that

$$\begin{aligned}\int_0^\infty \frac{v^{a-b} + (A^2+AB)^{a-1}v^{1-b} + (B^2+AB)^{1-b}v^{a-1}}{\sqrt{v(v+\kappa)}} e^{-v} dv \\ \leq CA^{a-1}B^{1-b}(A+B)^{a-b} \int_0^\infty \frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.21}$$

Combining (A.20) and (A.21) yields to

$$\int_0^{x_1} \Psi(x)dx \leq Ce^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.22}$$

This, again, implies that (A.1) holds.

Case 3: $\max\{a,b\} < 1$. Inequalities (A.7)-(A.11) hold, but (A.12) has to be replaced by

$$\begin{aligned}\tilde{\Phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1} \frac{\left(v+B^2+AB\right)^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{1-b}(A+B)^{2a+b-3} \frac{v^{1-a}+\left(A^2+AB\right)^{1-a}}{\sqrt{v(v+\kappa)}}\end{aligned} \tag{A.23}$$

Noticing that

$$\int_0^\infty \frac{v^{1-a}e^{-v}dv}{\sqrt{v(v+\kappa)}} \leq C\left(A^2+AB\right)^{1-a} \int_0^\infty \frac{e^{-v}dv}{\sqrt{v(v+\kappa)}},$$

it follows that (A.13) holds. Finally (A.14) holds by exchanging (A,a) and (B,b). □

Lemma A.2 . *Let α, β, γ, δ be real numbers and ℓ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell} p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq Cn^{\alpha-\beta/2}e^{-\delta n}. \tag{A.24}$$

Proof. The function $x \mapsto (\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1},\infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta \leq Cx^\alpha(\sqrt{n}-\sqrt{x+1}\,)^\beta$ for $x \in [p,p+1]$ If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$\begin{aligned}S = \sum_{p=1}^{n-\ell} p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2/4} &= \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ \leq \sum_{p=1}^{p_0-1} p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\ + C\int_{p_0}^{n+1-\ell} x^\alpha(\sqrt{n}-\sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2}dx,\end{aligned}$$

(notice that $\sqrt{n}-\sqrt{x} \approx \sqrt{n}-\sqrt{x+1}$ for $x \leq n-\ell$). Clearly

$$\sum_{p=1}^{p_0-1} p^\alpha(\sqrt{n}-\sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq C_0 n^\alpha(\sqrt{n}-\sqrt{n-\ell}\,)^\beta e^{-\delta n} \tag{A.25}$$

for some C_0 independent of n. We set $y=y(x)=\sqrt{x+1}-\sqrt{x}/\sqrt{\gamma}$. Obviously

$$y'(x)=\frac{1}{2}\left(\frac{1}{\sqrt{x+1}}-\frac{1}{\sqrt{\gamma}\sqrt{x}}\right) \qquad \forall x \geq p_0,$$

and their exists $\epsilon=\epsilon(\delta,\gamma)>0$ such that $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$ and $y'(x) \geq \epsilon/\sqrt{x}$. Furthermore

$$\sqrt{x}=\frac{\sqrt{\gamma}\left(y+\sqrt{\gamma y^2+1-\gamma}\right)}{\gamma-1},$$

$$\begin{aligned}\sqrt{n}-\sqrt{x} &= \frac{\sqrt{n}(\gamma-1)-\sqrt{\gamma}y-\sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}}{\gamma-1}\\ &= \frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}(\gamma-1)-\sqrt{\gamma}y+\sqrt{\gamma}\sqrt{\gamma y^2+1-\gamma}}\\ &\approx \frac{n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2}{\sqrt{n}}\end{aligned}$$

since $y(x) \leq \sqrt{n}$. Furthermore

$$\begin{aligned}n(\gamma-1)+\gamma-2y\sqrt{\gamma n}-\gamma y^2 &= \gamma(\sqrt{n+1}+\sqrt{n}/\sqrt{\gamma}+y)(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y)\\ &\approx \sqrt{n}(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y),\end{aligned}$$

because y ranges between $\sqrt{n+2-\ell}-\sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1}-\sqrt{p_0}\sqrt{\gamma}$. Thus

$$(\sqrt{n}-\sqrt{x}\,)^\beta \approx \left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta.$$

This implies

$$\begin{aligned}&\int_{p_0}^{n+1-\ell} x^\alpha(\sqrt{n}-\sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\gamma(\sqrt{n}-\sqrt{x+1}))^2}dx\\ &\quad\leq C\int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1}\left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2}dy\\ &\quad\leq Cn^{\alpha+\beta/2+1}\int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz.\end{aligned} \tag{A.26}$$

Moreover

$$\begin{aligned}1-\frac{y(p_0)}{\sqrt{n}} &= 1-\frac{1}{\sqrt{n}}\left(\sqrt{p_0+1}-\frac{\sqrt{p_0}}{\sqrt{\gamma}}\right),\\ 1-\frac{y(n-\ell+1)}{\sqrt{n}} &= 1-\frac{\sqrt{n-\ell+2}}{\sqrt{n}}+\frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}}\\ &= \frac{1}{\sqrt{\gamma}}\left(1+\frac{\sqrt{\gamma}\,(\ell-2)-\ell+1}{2n}+\frac{\sqrt{\gamma}\,(\ell-2)^2-(\ell-1)^2}{8n^2}\right)+O(n^{-3}).\end{aligned} \tag{A.27}$$

Let θ fixed such that $1 - \frac{y(n-\ell+1)}{\sqrt{n}} < \theta < 1 - \frac{y(p_0)}{\sqrt{n}}$ for any $n > p_0$. Then

$$\begin{aligned}\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz &\le C_\theta \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} e^{-\gamma\delta n z^2} dz \\ &\le C_\theta\, e^{-\gamma\delta n\theta^2} \int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1} dz \\ &\le C\, e^{-\gamma\delta n\theta^2} \max\{1, n^{-\alpha-1/2}\}.\end{aligned}$$

Because $\gamma\theta^2 > 1$ we derive

$$\int_\theta^{1-y(p_0)/\sqrt{n}} (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \le C n^{-\beta} e^{-\delta n}, \tag{A.28}$$

for some constant $C > 0$. On the other hand

$$\begin{aligned}&\int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz \\ &\qquad\qquad \le C'_\theta \int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2} dz.\end{aligned}$$

The minimum of $z \mapsto (z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta$ is achieved at $1-y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1)+1-\ell}{2n\sqrt{\gamma}} + O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1)) = C_\gamma e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n} = 1+1/\sqrt{\gamma} - \sqrt{1+1/n} \quad \text{and} \quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} dz.$$

Since $1-y(n+1-\ell) \ge 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned}I_\beta &\le \sqrt{2\gamma}\int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z-z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2} dz \\ &\le \frac{-\sqrt{2\gamma}}{2n\gamma\delta}\left[(z-z_{\gamma,n})^\beta e^{-\gamma\delta n z^2}\right]_{1-y(n+1-\ell)/\sqrt{n}}^\theta + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta}\int_{1-y(n+1-\ell)/\sqrt{n}}^\theta (z-z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz\end{aligned}$$

But $1-y(n+1-\ell)/\sqrt{n} - z_{\gamma,n} = (\ell-1)(1-1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \le C_1 n^{-\beta-1} e^{-\delta n} + \beta C'_1 n^{-1} I_{\beta-1}. \tag{A.29}$$

If $\beta \leq 0$, we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with (A.26) and (A.28), yields to (A.24). If $\beta > 0$, we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1' n^{-1}(C_1 n^{-\beta} e^{-\delta n} + (\beta-1)C_1' n^{-1} I_{\beta-2})$$

If $\beta - 1 \leq 0$ we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1 C_1' n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to (A.24). If $\beta - 1 > 0$, we continue up we find a positive integer k such that $\beta - k \leq 0$, which again yields to

$$I_\beta \leq C_k n^{-\beta-1} e^{-\delta n}$$

and to (A.24). □

The next estimate is fundamental in deriving the N-dimensional estimate.

Lemma A.3 *For any integer $N \geq 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta \leq c_N \frac{e^m}{(1+m)^{(N-1)/2}} \qquad \forall m > 0. \tag{A.30}$$

Proof. Put $\mathcal{I}_N(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta$. Then $\mathcal{I}_2'(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta$ and

$$\begin{aligned}
\mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta} \cos^2\theta \, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta} \sin^2\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{1}{m}\int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta \\
&= \mathcal{I}_2(m) - \frac{1}{m}\mathcal{I}_2'(m).
\end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus (A.30) holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta} \sin\theta \, d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m}\frac{d}{d\theta}(e^{m\cos\theta}) \sin^{N-3}\theta \, d\theta = \frac{N-3}{m}\int_0^\pi e^{m\cos\theta} \cos\theta \sin^{N-4}\theta \, d\theta. \tag{A.31}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta = \mathcal{I}_2'(m),$$

and, again (A.30) holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using (A.31), we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{A.32}$$

for $N \geq 6$. Since the estimate (A.30) for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigthforward induction yields to the general result. $\square$

Remark. Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

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