A scaling limit theorem for the parabolic Anderson model with exponential potential

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Abstract The parabolic Anderson problem is the Cauchy problem for the heat equation $\partial_t u(t,z) = \Delta u(t,z) + \xi(t,z)u(t,z)$ on $(0,\infty)\times\mathbb{Z}^d$ with random potential $(\xi(t,z)\colon z\in\mathbb{Z}^d)$ and localized initial condition. In this paper we consider potentials which are constant in time and independent exponentially distributed in space. We study the growth rate of the total mass of the solution in terms of weak and almost sure limit theorems, and the spatial spread of the mass in terms of a scaling limit theorem. The latter result shows that in this case, just like in the case of heavy tailed potentials, the mass gets trapped in a single relevant island with high probability.

1 Introduction and main results

1.1 Overview and background

We consider the heat equation with random potential on the integer lattice $\mathbb{Z}^d$ and study the Cauchy problem with localised initial datum,

$$\begin{aligned} \partial_t u(t,z) &= \Delta u(t,z) + \xi(t,z)\,u(t,z), && \text{for } (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ \lim_{t\downarrow 0} u(t,z) &= \mathbb{1}_0(z), && \text{for } z\in\mathbb{Z}^d, \end{aligned}$$

where

$$(\Delta f)(z) = \sum_{y\sim z}[f(y)-f(z)], \qquad \text{for } z\in\mathbb{Z}^d,\ f\colon\mathbb{Z}^d\to\mathbb{R},$$

is the discrete Laplacian, and the potential $(\xi(t,z)\colon t>0, z\in\mathbb{Z}^d)$ is a random field. This equation is known as the *parabolic Anderson model*.

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In the present paper we assume that the potential field is constant in time and independent, identically distributed in space according to some nondegenerate distribution. Under this hypothesis the solutions are believed to exhibit *intermittency*, which roughly speaking means that at any late time the solution is concentrated in a small number of *relevant islands* at large distance from each other, such that the diameter of each island is much smaller than this distance, see Figure 1 for a schematic picture. The relevant islands are located in areas where the potential has favourable properties, e.g. a high density of large potential values. As time progresses new relevant islands emerge in locations further and further away from the origin at places where the potential is more and more favourable, while old islands lose their relevance. The main aim of the extensive research in this model, which was initiated by Gärtner and Molchanov in [3, 4], is to get a better understanding of the phenomenon of intermittency for various choices of potentials.

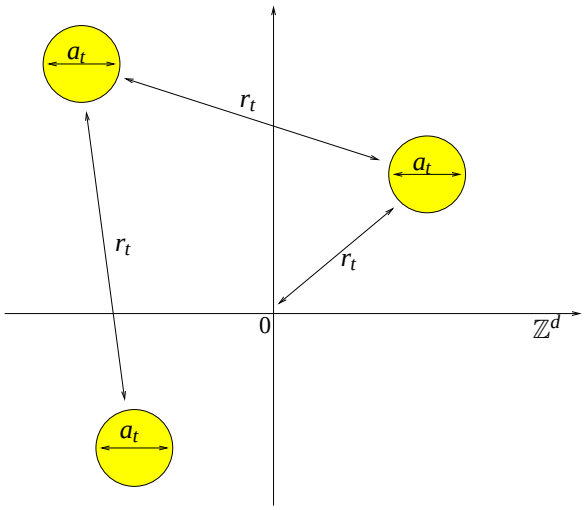

Fig. 1 A schematic picture of intermittency: the mass of the solution is concentrated on relevant islands (indicated by shaded balls) with radius of order a_t and distances of order $r_t \gg a_t$.

Natural questions about the nature of intermittency are the following:

- What is the diameter of the relevant islands? Are they growing in time?
- How much mass is concentrated in a relevant island?
  How big is the potential on a relevant island?
- Where are the relevant islands located? What is the distance of different islands?
- How many relevant islands are there?
- How do new relevant islands emerge? What is the lifetime of a relevant island?

Explicit answers to these questions and, more generally, results on the precise geometry of solutions to the parabolic Anderson model are typically very difficult to obtain. In the related context of Brownian motion among Poissonian obstacles, Sznitman [14] provides methodology to study properties of Brownian paths conditioned on survival, which offer a possible route to the geometry of solutions, at least in the case of bounded potentials. In a seminal paper Gärtner, König and Molchanov [2] follow a different route to analyse size and position of relevant islands in the case of double exponential potentials. Their results also offer some insight into potentials with heavier tails. In [7] and [9] a complete picture of the geometry of the solutions is given in the case of Pareto distributed potentials, building on the work of [2]. In this case of an extremely heavy tailed potential it can be shown that, for any $\varepsilon > 0$ at sufficiently late times, there exists a single point carrying a proportion of mass exceeding $1-\varepsilon$ with probability converging to one. This point constitutes the single relevant island and very precise results about the location, lifetime and dynamics of this island can be obtained, see also [10] for a survey of this research.

For more complicated potentials however, one has to rely on less explicit results. A natural way forward is to investigate the growth rates of the *total mass*

$$U(t) := \sum_{z\in\mathbb{Z}^d} u(t,z)$$

of the solution. If the potential is bounded from above we define the (quenched) *Lyapunov exponent* as

$$\lambda := \lim_{t\to\infty} L_t \text{ where } L_t := \frac{1}{t}\log U(t),$$

whenever this limit exists in the almost sure sense. If the potential is unbounded one expects superexponential growth and is interested in an asymptotic expansion of L_t. If the tails of the potential distribution are sufficiently light so that the logarithmic moment generating function

$$H(x) := \log E e^{x\xi(0)}$$

is finite for all $x \geq 0$, a large deviation heuristics suggests that,we get

$$L_t = \frac{H(\beta_t\alpha_t^{-d})}{\beta_t\alpha_t^{-d}} - \frac{1}{\alpha_t^2}\big(\kappa + o(1)\big), \text{ almost surely as } t\uparrow\infty,$$

where α,β are deterministic scale functions and κ is a deterministic constant. According to the heuristics, the quantity α_t can be interpreted as the diameter of the relevant islands at time t, and the leading term as the size of the potential values on the island. The constant κ is given in terms of a variational problem whose maximiser describes the shape of a vertically shifted and rescaled potential on an island. More details and a classification of light-tailed potentials according to this paradigm are given in [5].

If the potential is such that the moment generating functions do not always exist, this approach breaks down. Indeed, one can no longer expect the leading terms in an expansion of L_t to be deterministic. Instead, one should expect the solutions to be concentrated in islands consisting of single sites and the expansion of L_t to reflect fluctuations in the size of the potential on these sites. One would expect the sites of the islands to be those with the largest potential in some time-dependent centred box and the fluctuations to be similar to those seen in the order statistics of independent random variables. This programme is carried out in detail in [6] for potentials with Weibull (stretched exponential) and Pareto (polynomial) tails. In the present paper we add the case of standard exponential potentials and present weak (see Theorem 1) and almost sure (see Theorem 2) asymptotic expansions for L_t in this case. These results are taken from the first author's unpublished master thesis [8] and were announced without proof in [6].

Very little has been done so far to get a precise understanding of the number and position of the relevant islands, the very fine results for the Pareto case being the only exception. A natural idea to approach this with somewhat softer techniques is to prove a scaling limit theorem. To this end we define a probability distribution ν_t on $\mathbb{Z}^d$ associating to each site z a weight proportional to the solution $u(t,z)$, i.e.

$$\nu_t := \sum_{z\in\mathbb{Z}^d} \frac{u(t,z)}{U(t)}\,\delta(z), \text{ for any } t \geq 0,$$

where $\delta(z)$ denotes the Dirac measure concentrated at $z \in \mathbb{R}^d$. For $a > 0$, we also define the distribution of mass at the time t in the scale a as

$$\nu_t^a := \nu_t\big(\tfrac{\cdot}{a}\big) = \sum_{z\in\mathbb{Z}^d} \frac{u(t,z)}{U(t)}\,\delta\big(\tfrac{z}{a}\big),$$

which is considered as an element of the space $\mathscr{M}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$. Identifying the scale r_t of the distances between the islands and the origin, intermittency would imply that islands are contracted to points and that $\nu_t^{r_t}$ converges in law to a random probability measure, which is purely atomic with atoms representing intermittent islands and their weights representing the proportion of mass on the islands. In the case of Pareto potentials such a result follows easily from the detailed geometric picture, see [9, Proposition 1.4], but in principle could be obtained from softer arguments. It therefore seems viable that scaling limit theorems like the above can be obtained for a large class of potentials including some which are harder to analyse because they have much lighter tails.

In Theorem 3 of the present paper we show that in the case of exponential potentials for $r_t = t/\log\log t$ the random probability measures $\nu_t^{r_t}$ converge in distribution to a point mass in a nonzero random point. In particular this shows that for exponential potential we also have *only one relevant island*. Moreover, the solution of the parabolic Anderson problem spreads *sublinearly* in space. Our arguments can be adapted to the easier case of Weibull, or stretched exponential, potentials, where there is also only one relevant island but the solution has a *superlinear spread*. These results are new and open up possibilities for further research projects, which we briefly mention in our concluding remarks.

1.2 Statement of results

We now assume that $(\xi(z)\colon z \in \mathbb{Z}^d)$ is a family of independent random variables with

$$P\big(\xi(z) > x\big) = e^{-x} \text{ for } x \geq 0.$$

Suppose $(u(t,z)\colon t > 0, z \in \mathbb{Z}^d)$ is the unique nonnegative solution to the parabolic Anderson model with this potential, and let $(U(t)\colon t > 0)$ be the total mass of the solution. We recall that

$$L_t = \frac{1}{t} \log U(t)$$

and first ask for a weak expansion of L_t up to the first nondegenerate random term. This turns out to be the third term in the expansion, which is of constant order. In the following we use $\Rightarrow$ to indicate convergence in distribution.

Theorem 1 (Weak asymptotics for the growth rate of the total mass).
We have

$$L_t - d\log t + d\log\log\log t \Rightarrow X,$$

where X has a Gumbel distribution

$$P(X \leq x) = \exp\big\{-2^d e^{-x+2d}\big\} \qquad \textit{for } x \in \mathbb{R}.$$

In an almost sure expansion already the second term exhibits fluctuations.

Theorem 2 (Almost sure asymptotics for the growth rate of the total mass).
Almost surely,

$$\limsup_{t\uparrow\infty} \frac{L_t - d\log t}{\log\log t} = 1,$$

and

$$\liminf_{t\uparrow\infty} \frac{L_t - d\log t}{\log\log\log t} = -(d+1).$$

Remark 1. Note that neither of these almost sure asymptotics agree with the asymptotics

$$\lim_{t\uparrow\infty} \frac{L_t - d\log t}{\log\log\log t} = -d \quad \text{ in probability,}$$

which follows from Theorem 1. The almost sure results pick up fluctuations on both sides of the second term in the weak expansion, with those above being significantly stronger than those below the mean. This is different in the stretched exponential case studied in [6], where the liminf behaviour coincides with the weak limit behaviour. The limsup behaviour in the exponential case is included in the results of [6] and therefore not proved here.

Recall that the distribution of the mass of the solution at time $t > 0$ and on the scale $a > 0$ is defined as a (random) element of the space $\mathscr{M}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ by

$$v_t^a := v_t\left(\tfrac{\cdot}{a}\right) = \sum_{z\in\mathbb{Z}^d} \frac{u(t,z)}{U(t)}\,\delta\left(\tfrac{z}{a}\right).$$

The following theorem is the main result of this paper.

Theorem 3 (Scaling limit theorem). *Defining the sublinear scale function*

$$r_t = \frac{t}{\log\log t},$$

we have

$$\lim_{t\uparrow\infty} v_t^{r_t} = \delta(Y) \text{ in distribution,}$$

where $\delta(x)$ denotes the Dirac measure concentrated in $x \in \mathbb{R}^d$ and Y is a random variable in $\mathbb{R}^d$ with independent coordinates given by standard exponential variables with uniform random sign.

Remark 2. In the case of a Weibull potential with parameter $0 < \gamma < 1$ given by

$$P\big(\xi(z) > x\big) = e^{-x^\gamma} \text{ for } x \geq 0,$$

a variant of the proof gives convergence of $v_t^{r_t}$ for the superballistic scale function

$$r_t = \frac{t(\log t)^{\frac{1}{\gamma}-1}}{\log\log t},$$

to a limit measure $\delta(Y)$ where the components of Y are independent exponentially distributed with parameter $d^{1-1/\gamma}$ and uniform sign. Details are left to the reader.

2 Proof of the main results

2.1 Overview

The proofs are based on the Feynman-Kac formula

$$u(t,z) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\mathbb{1}\{X_t = z\}\Big],$$

where $(X_s\colon s \geq 0)$ is a continuous-time simple random walk on $\mathbb{Z}^d$ started at the origin and the probability $\mathbb{P}$ and expectation $\mathbb{E}$ refer only to this walk and not to the potentials. Recall that $(X_s\colon s \geq 0)$ is the Markov process generated by the discrete Laplacian Δ featuring in the parabolic Anderson problem. It is shown in [3] that the

Feynman-Kac formula gives the unique solution to the parabolic Anderson problem under a moment condition on the potential, which is satisfied in the exponential case. By summing over all sites the Feynman-Kac formula implies that the total mass is given by

$$U(t) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\Big].$$

An analysis of this formula allows us to approximate $L_t = \frac{1}{t}\log U(t)$ almost surely from above and below by variational problems for the potential. These variational problems have the structure that one optimizes over all sites $z \in \mathbb{Z}^d$ the difference between the potential value $\xi(z)$, corresponding to the reward for spending time in the site, and a term corresponding to the cost of getting to the site, which is going to infinity when $z \to \infty$ and thus ensure that the problem is well-defined.

We can use the result for the lower bound given in [6, Lemmas 2.1 and 2.3]. Here and throughout this paper we use $|\cdot|$ to denote the ℓ^1-norm on $\mathbb{R}^d$.

Lemma 1 (Lower bound on L_t). *Let*

$$\underline{N}(t) := \max_{z\in\mathbb{Z}^d}\Big\{\xi(z) - \frac{|z|}{t}\log\xi(z)\Big\},$$

then, almost surely, for all sufficiently large t, we have

$$L_t \geq \underline{N}(t) - 2d + o(1).$$

The appearance of $\xi(z)$ in the cost term can be explained by the fact that part of the cost arises from the fact that the optimal paths leading to z spend a positive proportion of the overall time traveling to the site and therefore miss out on the optimal potential value for some considerable time, see Section 1.3 in [6] for a heuristic derivation of this formula.

The corresponding upper bound will be our main concern here.

Lemma 2 (Upper bound on L_t). *For any $c > 0$ let*

$$\overline{N}_c(t) := \max_{t/(\log t)^2 \leq |z| \leq t\log t}\Big\{\xi(z) - \frac{|z|}{t}\big(\log\log|z| + c\big)\Big\}.$$

Then, for any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that, almost surely, for all sufficiently large t, we have

$$L_t \leq \overline{N}_{c(\varepsilon)}(t) - 2d + \varepsilon + o(1).$$

This lemma will be proved in two steps: We first remove paths that do not make an essential contribution from the average in the Feynman-Kac formula using an ad-hoc approach, see Lemma 7 and Lemma 8. Then we use the properties of the remaining paths to refine the argument and get an improved bound, see Proposition 1.

The variational problems for the upper and lower bound can then be studied using an extreme value analysis, which follows along the lines of [6]. It turns out that the weak and almost sure asymptotics of the two problems coincide up to the accuracy required to prove Theorem 1 and Theorem 2.

For the proof of the scaling limit we need to give an upper bound on the growth rate of the contribution of all those paths ending in a site at distance more than δr_t, for some $\delta > 0$, from the site with the largest potential among those sites that can be reached by some path with the same number of jumps. This bound needs to be strictly better than the lower bound on the overall growth rate. To this end, in a first step, we again use Lemma 7 and Lemma 8 to eliminate some paths using ad-hoc arguments. In the second step we remove paths that never hit the site with largest potential that is within their reach. This is done on the basis of the gap between the largest and the second largest value for the variational problem in the upper bound. In the third step it remains to analyse the contribution of paths that hit the optimal site but then move away by more than δr_t. Again it turns out that the rate of growth of the contribution of these paths is strictly smaller than the lower bound on the growth rate of the total mass. Proposition 1 is set up in such a way that it can deal with both the second and third step. We conclude from this that the solution is concentrated in a single island of diameter at most δr_t around the optimal site. An extreme value analysis characterizes the location of the optimal site and concludes the proof of Theorem 3.

The remainder of the paper is structured as follows: In Section 2.2 we give some notation and collect auxiliary results from [6]. Section 2.3 contains the required upper bounds and constitutes the core of the proof. Section 2.4 studies the variational problem arising in the upper bound. Using these approximations we complete the proof of Theorem 2 in Section 2.5 and of Theorem 1 in Section 2.6. The proof of the scaling limit theorem, Theorem 3, is completed in Section 2.7.

2.2 Auxiliary results

Let $B_r = \{|z| \le r\}$ be the ball of radius r centered at the origin in $\mathbb{Z}^d$. The number l_r of points in B_r grows asymptotically like r^d. More precisely, there exists a constant κ_d such that, $\lim_{r\to\infty} l_r r^{-d} = \kappa_d$. We define $M_r = \max_{|z|\le r} \xi(z)$ to be the maximal value of the potential on B_r. The behavior of M_r is described quite accurately in [6, Lemma 4.1], which we restate now.

Lemma 3 (Bounds for M_r). *Let $\delta \in (0,1)$ and $c > 0$. Then, almost surely,*

$$\begin{aligned} M_r &\le d\log r + \log\log r + (\log\log r)^{\delta} && \text{for all sufficiently large } r,\\ M_r &\ge d\log r - (1+c)\log\log\log r && \text{for all sufficiently large } r. \end{aligned}$$

In particular, for any pair of constant c_1 and c_2 satisfying $c_1 < d < c_2$, we have

$$c_1 \log r \le M_r \le c_2 \log r \quad \text{for all sufficiently large } r.$$

Let $M_r^{(i)}$ denote the i-th biggest value taken by the potential in the ball of radius r centered at the origin. The next lemma gives us estimates for upper order statistics for the potential.

Lemma 4 (Rough asymptotic behaviour for upper order statistics). *Let* $0 < \beta < 1$ *be a fixed constant. Then, almost surely,*

$$\lim_{n\to\infty} \frac{M_n^{(\lfloor n^\beta \rfloor)}}{\log n} = d - \beta.$$

Proof. Recalling that l_n is the number of points in a ball of radius n in $\mathbb{Z}^d$ we get

$$P\Big(M_n^{(\lfloor n^\beta \rfloor)} \le x\Big) = \sum_{i=0}^{\lfloor n^\beta \rfloor - 1} \binom{l_n}{i} e^{-xi} \left(1 - e^{-x}\right)^{l_n - i}. \tag{1}$$

We fix $\varepsilon > 0$ and infer that

$$\begin{aligned} P\Big(M_n^{(\lfloor n^\beta \rfloor)} \le (d-\beta-\varepsilon)\log n\Big) &\le \sum_{i=0}^{\lfloor n^\beta \rfloor} \left(l_n n^{-d+\beta+\varepsilon}\right)^i \left(1 - n^{-d+\beta+\varepsilon}\right)^{l_n - n^\beta} \\ &\le \left(n^\beta + 1\right) \left((\kappa_d + o(1)) n^{\beta+\varepsilon}\right)^{n^\beta} \exp\left[-(\kappa_d + o(1)) n^{\beta+\varepsilon}\right] \\ &= \exp\left[-n^{\beta+\varepsilon}(\kappa_d + o(1))\right]. \end{aligned}$$

Since this sequence is summable, we can use the Borel–Cantelli lemma to obtain the lower bound. Similarly, for the upper bound, we use (1) to get

$$P\Big(M_n^{(\lfloor n^\beta \rfloor)} \ge (d-\beta+\varepsilon)\log n\Big) \le \sum_{i=\lfloor n^\beta \rfloor}^{l_n} \binom{l_n}{i} n^{-(d-\beta+\varepsilon)i}. \tag{2}$$

We now use a rough approximation for the binomial coefficient, namely

$$\binom{l_n}{i} \le \frac{(l_n)^i}{i!} \le \left(\frac{e l_n}{i}\right)^i,$$

when i is big enough. Combining this with (2) and using that the first term in the ensuing sum is the largest, we obtain, for all sufficiently large n,

$$P\Big(M_n^{(\lfloor n^\beta \rfloor)} \ge (d-\beta+\varepsilon)\log n\Big) \le \sum_{i=\lfloor n^\beta \rfloor}^{l_n} \left(\frac{e l_n}{i n^{d-\beta+\varepsilon}}\right)^i \le l_n \left(\frac{e l_n}{n^{d+\varepsilon}}\right)^{n^\beta} \le e^{-n^\beta}.$$

Using the Borel–Cantelli lemma again we obtain an upper bound, completing the proof of our statement.

Let $0 < \sigma < \rho < \frac{1}{2}$ be some fixed constants. We define

$$k_n = \lfloor n^\sigma \rfloor \qquad \text{and} \qquad m_n = \lfloor n^\rho \rfloor$$

Combining Lemma 3 and Lemma 4, we get the following result.

Lemma 5. *For any constant $c > 0$, for all sufficiently large n, we have*

(i) $M_n^{(1)} - M_n^{(k_n)} > (\sigma - c)\log n$;
(ii) $M_n^{(k_n)} - M_n^{(m_n)} > (\rho - \sigma - c)\log n$.

Finally, we use Lemma 3 to give a lower bound for $\underline{N}(t)$.

Lemma 6 (Eventual lower bound for $\underline{N}(t)$). *For any small $\varepsilon > 0$, we have*

$$\underline{N}(t) \geq d\log t - (d+1+\varepsilon)\log\log\log t,$$

for all sufficiently large t, almost surely.

Proof. Using Lemma 3 we get, for any fixed $c > 0$ and $c_2 > d$,

$$\underline{N}(t) \geq \max_{r>0} \left[d\log r - (1+c)\log\log\log r - \frac{r}{t}\log\log r - \frac{r}{t}\log c_2 \right],$$

if the maximum of the expression in the square brackets (which we denote by $f_t(r)$) is attained at a point r_t, large enough so that Lemma 3 holds.

The solution $r = r_t$ of $f_t'(r) = 0$ satisfies

$$\frac{d}{r} = \frac{\log\log r}{t}\big(1 + o(1)\big).$$

Writing $r_t = t\varphi(r_t)$, where $\varphi(r) = d(\log\log r)^{-1}(1+o(1))$ we get that

$$\log\varphi(r) = -\log\log\log r + \log d + o(1) \tag{3}$$

and hence $\log r_t = \log t + \log\varphi(r_t) = \log t + o(\log r_t)$, which implies $\log r_t / \log t = 1 + o(1)$. Note that this implies $r_t \to \infty$ as $t \to \infty$, which justifies *a posteriori* the application of Lemma 3. Combining this with (3) we get,

$$\begin{aligned} f(r_t) &= d(\log(t\varphi(r_t))) - (1+c)\log\log\log r_t - \varphi(r_t)(\log\log r_t + \log c_2) \\ &= d\log t - (1+d+c)\log\log\log t + O(1). \end{aligned}$$

2.3 Upper bounds

We start by showing ad-hoc bounds for the growth rates of the contribution of certain families of paths. These can be compared to the lower bound for the growth rate of $U(t)$ showing that the paths can be be neglected. For a path $(X_s\colon s \geq 0)$ on the lattice $\mathbb{Z}^d$ we denote by J_t the number number of jumps up to time t. Recall that $M_n^{(k)}$ denotes the k^{th} largest potential value on the sites $z \in \mathbb{Z}^d$ with $|z| \leq n$.

Lemma 7. *Fix $0<\sigma<\frac{1}{2}$ and $k_n=n^\sigma$. Let*

$$U_2(t)=\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}\mathbb{1}\Big\{\tfrac{t}{(\log t)^2}\le J_t\le t\log t,\max_{0\le s\le t}\xi(X_s)\le M_{J_t}^{(k_{J_t})}\Big\}\Big].$$

Then

$$\lim_{t\uparrow\infty}\frac{1}{t}\log\frac{U_2(t)}{U(t)}=-\infty.$$

Proof. Simply replacing $\xi(X_s)$ in the integral by the maximum, we get

$$\begin{aligned}U_2(t)&=\sum_{t/(\log t)^2\le n\le t\log t}\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}\mathbb{1}\big\{J_t=n,\max_{0\le s\le t}\xi(X_s)\le M_n^{(k_n)}\big\}\Big]\\&\le\sum_{t/(\log t)^2\le n\le t\log t}e^{tM_n^{(k_n)}}\,\mathbb{P}(J_t=n)\le\max_{t/(\log t)^2\le n\le t\log t}e^{tM_n^{(k_n)}}.\end{aligned}$$

By Lemma 4 we have $M_n^{(k_n)}=(d-\sigma)\log n+o(\log n)$ and hence

$$\frac{1}{t}\log U_2(t)\le(d-\sigma)\log t+o(\log t),$$

so that the result follows by comparison with Lemma 1 and Lemma 6.

Lemma 8. *Let*

$$U_3(t)=\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}\Big(\mathbb{1}\big\{J_t<\tfrac{t}{(\log t)^2}\big\}+\mathbb{1}\big\{J_t>t\log t\big\}\Big)\Big].$$

Then

$$\lim_{t\uparrow\infty}\frac{1}{t}\log\frac{U_3(t)}{U(t)}=-\infty.$$

Proof. We first show that almost surely,

$$\frac{1}{t}\log U_3(t)\le\max_{n<t/(\log t)^2}\Big\{M_n-\frac{n}{t}\log\frac{n}{2det}\Big\}-2d+o(1).\tag{4}$$

Indeed, we have

$$\begin{aligned}U_3(t)&\le\sum_{\substack{\{n<t/(\log t)^2\}\\ \cup\{n>t\log t\}}}e^{tM_n}\mathbb{P}(J_t=n)=\sum_{\substack{\{n<t/(\log t)^2\}\\ \cup\{n>t\log t\}}}e^{tM_n}\frac{(2dt)^ne^{-2dt}}{n!}\\&\le\sum_{\substack{\{n<t/(\log t)^2\}\\ \cup\{n>t\log t\}}}\exp\big(tM_n-2dt+n\log 2dt-\log n!\big).\end{aligned}\tag{5}$$

To estimate $n!$ we use Stirling's formula,

$$n!=\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^ne^{\delta(n)},\qquad\text{with }\lim_{n\uparrow\infty}\delta(n)=0.$$

Fixing some $\varepsilon > 0$ we know from Lemma 3, that $M_n \le (d+\varepsilon)\log n$ for all sufficiently large n, so for t large enough, we obtain for all $n > t\log t$,

$$\begin{aligned} tM_n - 2dt + n\log 2dt - \log n! &\le t(d+\varepsilon)\log n - n\log\tfrac{n}{2edt} - \delta(n) \\ &\le t(d+\varepsilon)\log n\Big(1 - \tfrac{1+o(1)}{(d+\varepsilon)}\log\Big(\tfrac{\log t}{2ed}\Big) + o(1)\Big) \\ &\le -2\log n, \end{aligned}$$

by noticing that $n \mapsto \frac{n}{t\log n}\log\frac{n}{2edt}$ is decreasing on $(t\log t, \infty)$. Hence, almost surely,

$$\sum_{n > t\log t} \exp\left(tM_n - 2dt + n\log 2dt - \log n!\right) = o(1),$$

so that using (5) the following upper bound for U_3

$$\begin{aligned} U_3(t) &\le \frac{t}{(\log t)^2} \max_{n < t/(\log t)^2} \exp\left(tM_n - 2dt + n\log 2dt - \log n!\right) + o(1) \\ &\le \frac{t}{(\log t)^2} \max_{n < t/(\log t)^2} \exp\left(tM_n - 2dt - n\log\tfrac{n}{2edt} + o(t)\right) + o(1) \end{aligned}$$

and hence (4) follows. As a second step we show that

$$\frac{1}{t}\log U_3(t) \le d\log t - (2d-1)\log\log t + o(\log\log t). \tag{6}$$

Recall that $r \mapsto M_r$ is a non-decreasing function and check that

$$r \longmapsto \frac{r}{t}\log\frac{r}{2det} \text{ is decreasing on } (0, 2det),$$

hence, replacing r in the bracket by $t/(\log t)^2$

$$\max_{r < t/(\log t)^2}\left[M_r - \frac{r}{t}\log\frac{r}{2det}\right] = M_{t/(\log t)^2} + o(1).$$

By Lemma 3 we have $M_r \le d\log r + \log\log r + o(\log\log r)$ for all sufficiently large r, we get, for t large enough

$$\max_{r < t/(\log t)^2}\left[M_r - \frac{r}{t}\log\frac{r}{2det}\right] \le d\log t - (2d-1)\log\log t + o(\log\log t), \tag{7}$$

and combining (4) and (7), we have proved (6). Using Lemma 1 and Lemma 6,

$$\begin{aligned} \frac{1}{t}\log\frac{U_3(t)}{U(t)} &\le \frac{1}{t}\log U_3(t) - \underline{N}(t) - 2d + o(1) \\ &\le -(2d-1)\log\log t + o(\log\log t) \to -\infty, \end{aligned}$$

and hence our statement is proved.

The following versatile upper bound is the main tool in the proof of all our theorems and will be used repeatedly. Note for example that, together with Lemmas 7 and 8 it implies Lemma 2 if the parameters in (ii) are chosen as $k=1$ and $\delta=0$.

Proposition 1. *For a path $(X_s\colon s\geq 0)$ on the lattice $\mathbb{Z}^d$ we denote by J_t the number of jumps up to time t. We denote by $M_n^{(k)}$ the k^{th} largest potential value on the sites $z\in\mathbb{Z}^d$ with $|z|\leq n$, and let $Z_n^{(k)}$ be the site where this maximum is attained. Further fix $0<\sigma<\frac{1}{2}$ and let $k_n=\lfloor n^\sigma\rfloor$ and $a_t\downarrow 0$.*

(a) For $n\in\mathbb{N}$ let

$$U_1^{(n)}(t)=\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}\mathbb{1}\{J_t=n\}\,\mathbb{1}\big\{\max_{0\leq s\leq t}\xi(X_s)>M_n^{(k_n)}\big\}\Big].$$

Then, for all $\varepsilon>0$ there exists $C_\varepsilon>0$ such that uniformly for all $ta_t\leq n\leq t\log t$,

$$\frac{1}{t}\log U_1^{(n)}(t)\leq M_n^{(1)}-\frac{n}{2t}\big(\log\log n-C_\varepsilon\big)+\varepsilon-2d+o(1)\quad\text{ as } t\uparrow\infty.$$

(b) For fixed $\delta\geq 0$ and $k,n\in\mathbb{N}$ let

$$\begin{aligned}U_1^{(\delta,k,n)}(t)=\mathbb{E}\Big[\exp\Big\{\int_0^t\xi(X_s)\,ds\Big\}&\\ \mathbb{1}\{J_t=n\}\,\mathbb{1}\Big\{\sup_{0\leq s\leq t}\xi(X_s)\notin\{M_n^{(1)},\dots,M_n^{(k-1)}\}\Big\}&\\ \mathbb{1}\big\{Z_n^{(k)}\in\{X_s\colon 0\leq s\leq t\},|X_t-Z_n^{(k)}|\geq\delta r_t\big\}\Big].&\end{aligned}$$

Then, almost surely,

$$\text{uniformly in } k\leq k_n \text{ and } \tfrac{t}{(\log t)^2}\leq n\leq ta_t,$$

we have that

$$\frac{1}{t}\log U_1^{(\delta,k,n)}(t)\leq M_n^{(k)}-\frac{|Z_n^{(k)}|}{t}\log\log|Z_n^{(k)}|-2d-\delta+o(1)\quad\text{ as } t\uparrow\infty.$$

The first step in the proof is to integrate out the waiting times of the continuous time random walk paths. The following fact taken from [6] helps with this.

Lemma 9. *Let $\eta_0,\dots,\eta_n$ be fixed real numbers attaining their maximum only once, i.e. there is an index $0\leq k\leq n$ with $\eta_k>\eta_i$ for all $i\neq k$. Then, for all $t>0$,*

$$\int_{\mathbb{R}_+^n}\exp\Big\{\sum_{i=0}^{n-1}t_i\eta_i+\Big(t-\sum_{i=0}^{n-1}t_i\Big)\eta_n\Big\}\mathbb{1}\Big\{\sum_{i=0}^{n-1}t_i<t\Big\}\,dt_0\dots dt_{n-1}\leq e^{t\eta_k}\prod_{i\neq k}\frac{1}{\eta_k-\eta_i}.$$

Proof. First, we prove the result for the case $k = n$, i.e. $\eta_n > \eta_i$ for all $i < n$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\Big\{ \sum_{i=0}^{n-1} t_i\eta_i + \Big(t - \sum_{i=0}^{n-1} t_i\Big)\eta_n\Big\} \mathbb{1}\Big\{ \sum_{i=0}^{n-1} t_i < t, t_i \geq 0 \forall i \leq n-1\Big\} dt_0 \dots dt_{n-1} \\ = e^{t\eta_n} \int_{\mathbb{R}_+^n} \exp\Big\{ \sum_{i=0}^{n-1} t_i(\eta_i - \eta_n)\Big\} \mathbb{1}\Big\{ \sum_{i=0}^{n-1} t_i < t\Big\} dt_0 \dots dt_{n-1} \\ \leq e^{t\eta_n} \int_{\mathbb{R}_+^n} \exp\Big\{ \sum_{i=0}^{n-1} t_i(\eta_i - \eta_n)\Big\} dt_0 \dots dt_{n-1} = e^{t\eta_n} \prod_{i<n} \frac{1}{\eta_n - \eta_i}.\end{aligned}$$

Now we show that any permutation of the indices does not change the value of the integral above and this will be sufficient to prove the statement. First, it is obvious that transposition of i and j does not change the integral if $i, j \leq n-1$. Now we consider the case of a transposition τ on j and n, where $j < n$. We change variables such that $t_i' = t_i$ if $i \neq j, i \leq n-1$ and $t_j' = t - \sum_{i=0}^{n-1} t_i$, and get

$$\begin{aligned}&\int_{\mathbb{R}_+^n} \exp\Big\{ \sum_{i=0}^{n-1} t_i\eta_i + \Big(t - \sum_{i=0}^{n-1} t_i\Big)\eta_n\Big\} \mathbb{1}\Big\{ \sum_{i=0}^{n-1} t_i < t\Big\} dt_0 \dots dt_{n-1} \\ &= \int_{\mathbb{R}^n} \exp\Big\{ \sum_{i=0}^{n-1} t_i'\eta_{\tau(i)} - \Big(t - \sum_{i=0}^{n-1} t_i'\Big)\eta_{\tau(n)}\Big\} \mathbb{1} \\ &\qquad \times \Big\{ \sum_{i=0}^{n-1} t_i' < t, t_i' \geq 0 \forall i \leq n-1\Big\} dt_0' \dots dt_{n-1}',\end{aligned}$$

which completes the proof.

For the proof of Proposition 1 (b) denote by

$$\begin{aligned}\mathscr{P}^{(\delta,k,n)} = \Big\{ &y = (y_0, \dots, y_n) \colon y_0 = 0, |y_{i-1} - y_i| = 1, \\ &\{y_0, \dots, y_n\} \cap \{Z_n^{(1)}, \dots, Z_n^{(k-1)}\} = \emptyset, Z_n^{(k)} \in \{y_0, \dots, y_n\}, |y_n - Z_n^{(k)}| \geq \delta r_t\Big\}\end{aligned}$$

the set of all 'good' paths and let (τ_i) be a sequence of independent, exponentially distributed random variables with parameter $2d$.

Denote by E the expectation with respect to (τ_i). We have

$$\begin{aligned}U_1^{(\delta,k,n)}(t) = \sum_{y \in \mathscr{P}^{(\delta,k,n)}} (2d)^{-n} \mathsf{E}\Big[&\exp\Big\{ \sum_{i=0}^{n-1} \tau_i \xi(y_i) + \Big(t - \sum_{i=0}^{n-1} \tau_i\Big)\xi(y_n)\Big\} \\ &\times \mathbb{1}\Big\{ \sum_{i=0}^{n-1} \tau_i < t, \sum_{i=0}^{n} \tau_i > t\Big\}\Big].\end{aligned} \tag{8}$$

In the further proof we apply Lemma 9 to the values of the potential ξ along a path y. However, to do so we need the maximum of ξ along the path y to be attained only once. Therefore we have to modify the potential along the path slightly.

We fix $y \in \mathscr{P}^{(\delta,k,n)}$ and let

$$i(y) = \min\big\{i \in \{0,\dots,n\} \colon y_i = Z_n^{(k)}\big\}$$

be the index of the first instant where the maximum of the potential over the path is attained. Now we define a slight variation of ξ on y in the following way. Fix $\varepsilon > 0$ and define $\xi^y \colon \{0,\dots,n\} \to \mathbb{R}$ by $\xi_i^y = \xi(y_i)$ if $i \neq i(y)$, and $\xi_{i(y)}^y = \xi(y_{i(y)}) + \varepsilon$. We obtain, using $\xi(y_i) \le \xi_i^y$, that

$$\begin{aligned}
&\mathsf{E}\Big[\exp\Big\{\sum_{i=0}^{n-1}\tau_i\xi(y_i) + \Big(t - \sum_{i=0}^{n-1}\tau_i\Big)\xi(y_n)\Big\}\mathbb{1}\Big\{\sum_{i=0}^{n-1}\tau_i < t, \sum_{i=0}^{n}\tau_i > t\Big\}\Big] \\
&\le \mathsf{E}\Big[\exp\Big\{\sum_{i=0}^{n-1}\tau_i\xi_i^y + \Big(t - \sum_{i=0}^{n-1}\tau_i\Big)\xi_n^y\Big\}\mathbb{1}\Big\{\sum_{i=0}^{n-1}\tau_i < t, \sum_{i=0}^{n}\tau_i > t\Big\}\Big] \\
&= (2d)^{n+1}\int_{\mathbb{R}_+^d} \exp\Big\{\sum_{i=0}^{n-1}t_i\xi_i^y + \Big(t - \sum_{i=0}^{n-1}t_i\Big)\xi_n^y\Big\} \\
&\qquad\qquad \mathbb{1}\Big\{\sum_{i=0}^{n-1}t_i < t, \sum_{i=0}^{n}t_i > t\Big\} e^{-2d\sum_{i=0}^{n}t_i}\,dt_0\dots dt_{n-1}dt_n \\
&= (2d)^n e^{-2dt}\int_{\mathbb{R}_+^d} \exp\Big\{\sum_{i=0}^{n-1}t_i\xi_i^y + \Big(t - \sum_{i=0}^{n-1}t_i\Big)\xi_n^y\Big\}\mathbb{1}\Big\{\sum_{i=0}^{n-1}t_i < t\Big\}\,dt_0\dots dt_{n-1} \\
&\le (2d)^n e^{-2dt} e^{\xi_{i(y)}^y t}\prod_{i\neq i(y)}\frac{1}{\xi_{i(y)}^y - \xi_i^y}, \qquad (9)
\end{aligned}$$

where the last line follows from Lemma 9. Using the definition of our function ξ^y we get

$$\begin{aligned}
e^{\xi_{i(y)}^y t}\prod_{i\neq i(y)}\frac{1}{\xi_{i(y)}^y - \xi_i^y} &= e^{(\xi(y_{i(y)})+\varepsilon)t}\prod_{i\neq i(y)}\frac{1}{\varepsilon + \xi(y_{i(y)}) - \xi(y_i)} \\
&\le e^{(\xi(y_{i(y)})+\varepsilon)t}\varepsilon^{-n}\prod_{(\xi(y_{i(y)})-\xi(y_i))>1}\frac{1}{\xi(y_{i(y)}) - \xi(y_i)}. \qquad (10)
\end{aligned}$$

Next recall that ρ is fixed, and $m_n = \lfloor n^\rho \rfloor$. Let

$$G_n = \big\{Z_n^{(1)},\dots,Z_n^{(m_n)}\big\} \subset \{z \in \mathbb{Z}^d \colon |z| \le n\},$$

and call the complement G_n^{c} the set of sites with very low potential. Note that there are at least $|Z_n^{(k)}| + \lfloor \delta r_t \rfloor - m_n$ points in the path y that belong to G_n^{c}. Hence there are at least

$$|Z_n^{(k)}| + \lfloor \delta r_t \rfloor - m_n$$

terms in the product in the left hand side of (10) that are smaller than

$$\big(M_n^{(k_n)} - M_n^{(m_n)}\big)^{-1}$$

provided this is less than 1. Combining this with (8), (9) and (10), we get

$$\begin{aligned} U_1^{(\delta,k,n)}(t) &\le \sum_{y\in\mathscr{P}(\delta,k,n)} \varepsilon^{-n} e^{(M_n^{(k)}+\varepsilon-2d)t} \big(M_n^{(k_n)} - M_n^{(m_n)}\big)^{-|Z_n^{(k)}|-\lfloor \delta r_t\rfloor+m_n} \\ &\le (2d)^n \varepsilon^{-n} e^{(M_n^{(k)}+\varepsilon-2d)t} \big(\tfrac{\rho-\sigma}{2}\log n\big)^{-|Z_n^{(k)}|-\lfloor \delta r_t\rfloor+m_n}. \end{aligned}$$

Taking the log of the above and defining $C_\varepsilon := \log(\frac{2d}{\varepsilon}) - \log(\frac{\rho-\sigma}{2})$ we get

$$\begin{aligned} \frac{1}{t}\log U_1^{(\delta,k,n)}(t) &\le \tfrac{n}{t}\log\tfrac{2d}{\varepsilon} + M_n^{(k)} - 2d + \varepsilon - \tfrac{1}{t}\big(|Z_n^{(k)}| + \lfloor \delta r_t\rfloor - m_n\big)\log\big(\tfrac{\rho-\sigma}{2}\log n\big) \\ &\le M_n^{(k)} - \tfrac{1}{t}|Z_n^{(k)}|\log\log|Z_n^{(k)}| - 2d + \varepsilon + \tfrac{n}{t}C_\varepsilon - \delta\,\tfrac{\log\log n}{\log\log t} + o(1), \end{aligned}$$

where we use that $|Z_n^{(k)}| + \lfloor \delta r_t\rfloor \le n$. Observing that $\log\log n \ge (1+o(1))\log\log t$ and $\frac{n}{t}C_\varepsilon = o(1)$, uniformly for all n in the given range, concludes the proof of (b).

To prove part (a) we show that regardless of the distance travelled by the path, it hits a site with very low potential in every other step. Recall that a set H of vertices of $\mathbb{Z}^d$ is *totally disconnected* if there is no pair of vertices $(x,y)\in H^2$ such that $|x-y|=1$.

Lemma 10. *Almost surely, for sufficiently large n, the set G_n is totally disconnected.*

Proof. We prove the statement for $d\ge 2$ first. If i and j are distinct integers in $\{1,\dots,m_n\}$, the random pair of points $(Z_n^{(i)}, Z_n^{(j)})$ is uniformly distributed over all possible pairs of points in the ball of radius n. As no vertex has more than $2d$ neighbours, we have $P(Z_n^{(i)} - Z_n^{(j)}) \le 2d/l_n$. Summing over all possible pairs i, j we get

$$P\big(G_n \text{ not totally disconnected}\big) \le \sum_{i<j} P\big(Z_n^{(i)} - Z_n^{(j)}\big) \le \binom{m_n}{2}\frac{2d}{l_n} \le Cn^{2\rho-d}. \quad (11)$$

for some constant C. Since $\rho < \frac{1}{2}$ and $d\ge 2$ we can apply the Borel-Cantelli lemma and obtain the result. We now prove the the same result when $d=1$. We introduce a new quantity

$$m_n' = \big\lfloor n^{\rho'}\big\rfloor \text{ with } \rho<\rho'<\tfrac{1}{2}$$

Let G_n' be the set of the m_n' vertices in the ball of radius n where the biggest values of ξ are taken, and let p_n be the biggest integer power of 2, which is less than n. Note that, by (11), the set G_{p_n}' is totally disconnected for all sufficiently large n.

We now prove that

$$G_n \subseteq G_{2p_n}' \quad \text{for all sufficiently large } n. \quad (12)$$

For this it suffices to show that at least m_n points of G'_{2p_n} are in the ball of radius n. Indeed, if we assume this and also that $G_n \nsubseteq G'_{2p_n}$ we can find a vertex z_0 satisfying, $|z_0| \le n$, $z_0 \in G'_{2p_n}$ and $z_0 \notin G_n$. This implies that every $z \in G_n$ satisfies $\xi(z) > \xi(z_0)$, because G_n is the set where the largest values of ξ are achieved. Then, because $z_0 \in G'_{2p_n}$, we have

$$G_n \subseteq \{\xi(z) > \xi(z_0)\} \cap B_n \subseteq \{\xi(z) > \xi(z_0)\} \cap B_{2p_n} \subseteq G'_{2p_n},$$

which leads to a contradiction to our assumption.

In fact we will prove the slightly stronger statement that there are at least m_{2p_n} vertices of G'_{2p_n} in the ball of radius p_n, and we will now write p instead of p_n. We write

$$G'_{2p} = \{a'_0, \dots, a'_{m'_{2p}-1}\},$$

where a'_i is the vertex where $\xi(a'_i) = M_{2p}^{(i+1)}$ and introduce

$$X = (X_i)_{0 \le i \le m'_{2p}-1} \text{ with } X_i = \mathbb{1}_{\{|a'_i| \le p\}} \text{ and } |X| = \sum_{i=0}^{m'_{2p}-1} X_i.$$

Observing that $m'_{2p} = o(p)$ and that G'_{2p} is uniformly distributed over all possible ordered sets and recalling that the box of radius p contains $2p+1$ vertices, it is easy to see that for p big enough,

$$P\big(X_j = 1 \,\big|\, X_i = x_i, \forall i < j\big) < \tfrac{3}{4} \text{ and } P\big(X_j = 0 \,\big|\, X = x_i, \forall i < j\big) < \tfrac{3}{4},$$

for all $j \le m'_{2p} - 1$ and for all fixed $(x_0, ..., x_{j-1}) \in \{0,1\}^j$. Hence

$$\begin{aligned} P\big(|X| < m_{2p}\big) &= \sum_{i=0}^{m_{2p}-1} \sum_{|x|=i} P(X = x) \\ &\le \sum_{i=0}^{m_{2p}-1} \binom{m'_{2p}}{i} \left(\frac{3}{4}\right)^{m'_{2p}} \le m_{2p} \left(m'_{2p}\right)^{m_{2p}-1} \left(\frac{3}{4}\right)^{m'_{2p}} \\ &= \exp\left(-m'_{2p} \log(4/3) + (m_{2p}-1)\log m'_{2p} + \log m_{2p}\right) \\ &= e^{-(2p)^{\rho'}(1+o(1))} \le e^{-n^{\rho'}} \quad \text{as } n \le 2p_n. \end{aligned}$$

Using the Borel-Cantelli lemma we can prove (12), which implies the statement.

We define the set of paths $\mathscr{P}_n$ to be

$$\mathscr{P}_n = \Big\{ y = (y_0, \dots, y_n) \colon y_0 = 0,\, |y_{i-1} - y_i| = 1, \\ \{y_0, \dots, y_n\} \cap \{Z_n^{(1)}, \dots, Z_n^{(k_n-1)}\} \neq \emptyset \Big\},$$

so that

$$U_1^{(n)}(t) = \sum_{y\in\mathscr{P}_n} (2d)^{-n}\mathsf{E}\Big[\exp\Big\{\sum_{i=0}^{n-1}\tau_i\xi(y_i)+\Big(t-\sum_{i=0}^{n-1}\tau_i\Big)\xi(y_n)\Big\} \times \mathbb{1}\Big\{\sum_{i=0}^{n-1}\tau_i < t, \sum_{i=0}^{n}\tau_i > t\Big\}\Big].$$

We can now argue similarly as for part (b) but using this time the fact that for any path in $\mathscr{P}_n$ the number of step out of G_n is at least $\lfloor n/2\rfloor$. More precisely,

$$U_1^{(n)}(t) \le \sum_{y\in\mathscr{P}_n} \varepsilon^{-n} e^{(M_n^{(1)}+\varepsilon-2d)t}\big(M_n^{(k_n)} - M_n^{(m_n)}\big)^{-\lfloor n/2\rfloor},$$

and taking the log of the above and defining $C_\varepsilon := 2\log(\frac{2d}{\varepsilon}) - \log(\frac{\rho-\sigma}{2})$ we get

$$\begin{aligned}\frac{1}{t}\log U_1^{(n)}(t) &\le \tfrac{n}{t}\log\tfrac{2d}{\varepsilon} + M_n^{(1)} - 2d + \varepsilon - \tfrac{1}{t}\lfloor n/2\rfloor\log\big(\tfrac{\rho-\sigma}{2}\log n\big)\\ &= M_n^{(1)} - \tfrac{n}{2t}\big(\log\log n - C_\varepsilon\big) - 2d + \varepsilon + o(1),\end{aligned}$$

which concludes the proof of (a).

2.4 Analysis of the variational problem

We use the point process framework established in [6, Section 2.2] adapting the approach of [11, Chapter 3]. We only give an outline of the framework and sketched proofs here, see [6, Section 2.2] for more details.

Observe that $\mu(dy) := e^{-y}dy$ is a Radon measure on $G := (-\infty,\infty]$. For any $z\in\mathbb{Z}^d$, $x\in\mathbb{R}$ and $r>0$, we have

$$r^d P\big(\xi(z) - d\log r \ge x\big) = r^d e^{-d\log r - x} = e^{-x} = \mu\big([x,\infty]\big).$$

Define, for any $q,\tau>0$ the set $H_\tau^q := \{(x,y)\in\dot{\mathbb{R}}^d\times G\colon y\ge q|x|+\tau\}$, where $\dot{\mathbb{R}}^d$ is the one-point compactification of $\mathbb{R}^d$. As in [6, Lemma 4.3] we see that the point process

$$\zeta_r = \sum_{z\in\mathbb{Z}^d}\delta\big((\tfrac{z}{r},\xi(z)-d\log r)\big)$$

converges in law to the Poisson process ζ with intensity $\mathrm{Leb}_d\otimes\mu$ in the sense that, for any pairwise disjoint compact sets $K_1,\dots,K_n\subset H_\tau^q$ with $\mathrm{Leb}_{d+1}(\partial K)=0$, we have that $(\zeta_r(K_1),\dots,\zeta_r(K_n))$ converge in law to

$$\bigotimes_{i=1}^n \mathrm{Poiss}\big(\mathrm{Leb}_d\otimes\mu(K_i)\big).$$

We further note that for $z = t^{1+o(1)}$ we have

$$\psi_t(z) := \xi(z) - \frac{|z|}{t}\log\log|z| = \xi(z) - \frac{|z|}{r_t}\big(1+o(1)\big).$$

As in [6, Lemma 4.4] applied to $T_t(z,x) := (z, x-|z|)$ we infer from this the convergence of the point process

$$\varpi_t := \sum_{z\in\mathbb{Z}^d} \delta\big((\tfrac{z}{r_t}, \psi_t(z) - d\log r_t)\big)$$

in law to a Poisson process ϖ with intensity

$$\big(\mathrm{Leb}_d \otimes \mu\big)\circ T_t^{-1} = e^{-|z|-y}\,dz\,dy,$$

where now the compact sets $K_1,\dots,K_n$ can be chosen from the set $H_\tau := \dot{\mathbb{R}}^{d+1} \setminus (\mathbb{R}^d \times (-\infty,\tau))$. The form of these and the previous domains, and in particular the use of the compactification, ensure that we can use these convergence results to analyse the right hand side of the final formula in Proposition 1.

Lemma 11. *Let $X_t^{(1)}$ and $X_t^{(2)}$ be the sites corresponding to the largest and second largest value of $\psi_t(z)$, $z\in\mathbb{Z}^d$. Then $\psi_t(X_t^{(1)}) - \psi_t(X_t^{(2)})$ converges in law to a standard exponential random variable.*

Proof. Using careful arguments in the convergence step we obtain, for any $a \geq 0$,

$$\begin{aligned}
&P\big(\psi_t(X_t^{(1)}) - \psi_t(X_t^{(2)}) \geq a\big)\\
&\qquad = \sum_y P\big(\varpi_t\big(\mathbb{R}^d\times(y,\infty)\big) = 0, \varpi_t(\mathbb{R}^d\times\{y\}) = 1, \varpi_t\big(\mathbb{R}^d\times(y-a,y)\big) = 0\big)\\
&\qquad \to \int P\big(\varpi\big(\mathbb{R}^d\times(y,\infty)\big) = 0\big)P\big(\varpi\big(\mathbb{R}^d\times(y-a,y)\big) = 0\big)e^{-y}\,dy\\
&\qquad = \int \exp(-e^{-y+a})e^{-y}\,dy = e^{-a}.
\end{aligned}$$

Lemma 12. *Let $X_t^{(1)}$ be the site corresponding to the largest value of $\psi_t(z)$, $z\in\mathbb{Z}^d$. Then $X_t^{(1)}/r_t$ converges in law to a random variable in $\mathbb{R}^d$ with coordinates given by independent standard exponential variables with uniform random signs.*

Proof. As above we obtain, for any $A\subset\mathbb{R}^d$ Borel with $\mathrm{Leb}_d(\partial A) = 0$,

$$\begin{aligned}
P\Big(\frac{X_t^{(1)}}{r_t}\in A\Big) &= \sum_y P\big(\varpi_t\big(\mathbb{R}^d\times(y,\infty)\big) = 0, \varpi_t\big(A\times\{y\}\big) = 1\big)\\
&\to \int_A dz \int dy\, e^{-|z|-y} P\big(\varpi\big(\mathbb{R}^d\times(y,\infty)\big) = 0\big)\\
&= \int_A dz \int dy \exp(-e^{-y})e^{-y-|z|} = \int_A 2^{-d}\,e^{-|z|}\,dz.
\end{aligned}$$

Observe that this implies that the limit variable has the given distribution.

2.5 Proof of the almost sure asymptotics

Note that combining Lemma 1 and Lemma 6 establishes the almost sure lower bound for the liminf result in Theorem 2. To find a matching upper bound, recall from Lemma 2 that, for sufficiently large t,

$$L_t \leq N_\varepsilon(t) - 2d + \varepsilon$$

for $N_\varepsilon(t) := \overline{N}_{c(\varepsilon)}(t)$. We now approximate the distribution of $N_\varepsilon(t)$.

Lemma 13 (Approximation for the distribution of $N_\varepsilon(t)$). *Let $b_t \uparrow \infty$, then*

$$\log\big(P\big(N_\varepsilon(t) \leq b_t\big)\big) = -e^{-b_t} r_t^d 2^d\,(1+o(1))\,.$$

Proof. Observe that

$$P\left(N_\varepsilon(t) \leq b_t\right) = \prod_{t/(\log t)^2 \leq |z| \leq t\log t} F\left(b_t + \frac{|z|}{t}\left(\log\log|z| - C_\varepsilon\right)\right).$$

The values which $|z|$ can take are such that $\log\log|z| = \log\log t + o(1)$ uniformly for all z, and since $b_t \to \infty$, we have,

$$\begin{aligned}
&\log\big(P\big(N_\varepsilon(t) \leq b_t\big)\big)\\
&\quad= \sum_{t/(\log t)^2 \leq |z| \leq t\log t} \log\Big(1 - \exp\big(-b_t - \tfrac{|z|}{t}\big(\log\log t - C_\varepsilon + o(1)\big)\big)\Big)\\
&\quad= -\left(1+o(1)\right) \sum_{t/(\log t)^2 \leq |z| \leq t\log t} e^{-b_t - \frac{|z|}{r_t}(1+o(1))}\\
&\quad= -e^{-b_t} r_t^d\,(1+o(1)) \int_{\mathbb{R}^d} e^{-|x|(1+o(1))}\,\mathbb{1}_{\left\{\log\log t/(\log t)^2 \leq |x| \leq \log t\log\log t\right\}}\,\mathrm{d}x
\end{aligned}$$

To obtain our final result, we apply the dominated convergence theorem to the integral, which converges to 2^d.

We are now ready to prove the upper bound. We consider a sequence of times $t_n := \exp(n^2)$ for which $N_\varepsilon(t_n)$ are independent random variables, in order to use Borel-Cantelli.

Lemma 14 (Upper bound for lower envelope of $N_\varepsilon(t_n)$). *For any small $c > 0$, almost surely there are infinitely many n such that*

$$N_\varepsilon(t_n) \leq d\log t_n - (1+d-c)\log\log\log t_n.$$

Proof. Note that $(N_\varepsilon(t_n))_{n\geq N}$ is a sequence of independent variables if N is large enough. To see this it suffices to notice that the different $(N_\varepsilon(t_n))_{n\geq N}$ depend on the values of the potential on disjoints areas. Indeed

$$\frac{t_{n+1}}{(\log t_{n+1})^2} = \frac{\exp\left(n^2+2n+1\right)}{(n+1)^4} > n^2 \exp\left(n^2\right) = t_n \log t_n \text{ for all large } n.$$

Now we use Lemma 13 with $b_t = d\log t - (1+d-c)\log\log\log t$ and we get,

$$\log\left(P\left(N_\varepsilon(t_n) \le b_{t_n}\right)\right) = -2^d \left(\log\log t_n\right)^{1-c}(1+o(1)) \ge -\log n,$$

for all sufficiently large n. Hence the sum over the probabilities diverges and we obtain our result by applying the converse of the Borel-Cantelli lemma.

2.6 Proof of the weak asymptotics

To prove Theorem 1 we show that the upper and lower bounds we found earlier for L_t both satisfy the required limit statement. We first state the result of [6, Proposition 4.12], which describes the limit result for the lower bound $\underline{N}(t)$.

Lemma 15 (Weak asymptotics for $\underline{N}(t)$). *As t tends to infinity,*

$$\underline{N}(t) - d\log t + d\log\log\log t \Rightarrow X, \quad \textit{where} \quad P(X \le x) = \exp\left(-2^d e^{-x}\right).$$

Next we check the analogous limit theorem for the upper bound $N_\varepsilon(t)$ and thus complete the proof of Theorem 1.

Lemma 16 (Weak asymptotics for $N_\varepsilon(t)$). *As t tends to infinity,*

$$N_\varepsilon(t) - d\log t + d\log\log\log t \Rightarrow X, \quad \textit{where} \quad P(X \le x) = \exp\left(-2^d e^{-x}\right).$$

Proof. Fix $x \in \mathbb{R}$ and apply Lemma 13 with $b_t = d\log t - d\log\log\log t + x$ to get

$$\log\left(P\left(N_\varepsilon(t) - d\log t + d\log\log\log t \le x\right)\right) = -e^{-x}2^d\,(1+o(1)),$$

which proves our result.

2.7 Proof of the scaling limit theorem

We recall that $X_t^{(k)}$ $(k = 1,\ 2)$ is the site at which

$$\psi_t(z) = \xi(z) - \frac{|z|}{t}\log\log|z|$$

takes its k^{th} largest value. Fix $\delta > 0$ and write

$$U(t) = U_1(t) + U_2(t) + U_3(t) + U_4(t) + U_5(t) + U_6(t),$$

where U_2 and U_3 were defined in Lemma 7, resp. Lemma 8, and

$$U_1(t) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\mathbb{1}\big\{\tfrac{2t}{(\log t)^2} \le J_t \le ta_t, \max_{0\le s\le t}\xi(X_s) > M_{J_t}^{(k_{J_t})}\big\} \\ \mathbb{1}\big\{X_t^{(1)} \in \{X_s\colon 0\le s\le t\}, |X_t - X_t^{(1)}| \le \delta r_t\big\}\Big],$$

$$U_4(t) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\mathbb{1}\big\{ta_t < J_t \le t\log t, \max_{0\le s\le t}\xi(X_s) > M_{J_t}^{(k_{J_t})}\big\}\Big],$$

$$U_5(t) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\mathbb{1}\big\{\tfrac{2t}{(\log t)^2} \le J_t \le ta_t, \max_{0\le s\le t}\xi(X_s) > M_{J_t}^{(k_{J_t})}\big\} \\ \mathbb{1}\big\{X_t^{(1)} \notin \{X_s\colon 0\le s\le t\}\big\}\Big],$$

$$U_6(t) = \mathbb{E}\Big[\exp\Big\{\int_0^t \xi(X_s)\,ds\Big\}\mathbb{1}\big\{\tfrac{2t}{(\log t)^2} \le J_t \le ta_t, \max_{0\le s\le t}\xi(X_s) > M_{J_t}^{(k_{J_t})}\big\} \\ \mathbb{1}\big\{X_t^{(1)} \in \{X_s\colon 0\le s\le t\}, |X_t - X_t^{(1)}| > \delta r_t\big\}\Big].$$

Observe that our result follows if the contributions of $U_i(t)$ for $i = 2,\dots,6$ to the total mass are negligible, as $U_1(t)$ only contributes to the mass distributed on points close to $X_t^{(1)}$ on the r_t scale.

Lemma 17. *Suppose $a_t \downarrow 0$ and $a_t \log\log t \to \infty$. Then we have, in probability,*

$$\lim_{t\uparrow\infty}\frac{U_4(t)}{U(t)} = \lim_{t\uparrow\infty}\frac{U_5(t)}{U(t)} = \lim_{t\uparrow\infty}\frac{U_6(t)}{U(t)} = 0.$$

Proof. For the first statement we use Proposition 1 (a) to see that

$$\frac{1}{t}\log U_4(t) \le \sup_{n\ge ta_t}\Big\{M_n^{(1)} - \frac{n}{2t}\big(\log\log n - C_\varepsilon\big)\Big\} + \varepsilon - 2d + o(1).$$

By Lemmas 12 and 11 the limit of the right hand side is strictly smaller than the growth rate of $U(t)$, proving that the first limit in the statement equals zero.

Using Proposition 1 (b) with $\delta = 0$ and summing over all $1 \le k \le t^\sigma$ with $X_t^{(1)} \ne Z_n^{(k)}$, and over all n with $2t/(\log t)^2 \le n \le ta_t$ we get

$$\frac{1}{t}\log U_5(t) \le \max_{z\setminus\{X_t^{(1)}\}}\psi_t(z) - 2d + o(1) = \psi_t(X_t^{(2)}) - 2d + o(1) \quad \text{in probability.}$$

By Lemma 11 we find $\varepsilon > 0$ such that, with a probability arbitrarily close to one

$$\frac{1}{t}\log U_5(t) \le \psi_t(X_t^{(1)}) - 2d - \varepsilon + o(1),$$

and a comparison with the lower bound $\underline{N}(t)$ for the growth rate of $U(t)$ proves the second result.

For the third statement we use Proposition 1 (b) with the choice of $\delta > 0$ from the statement. Summing over all $1 \le k \le t^\sigma$ and n with $2t/(\log t)^2 \le n \le t a_t$ we get, as above,

$$\frac{1}{t} \log U_6(t) \le \psi_t(X_t^{(1)}) - 2d - \delta + o(1).$$

We can now argue as before that this rate is strictly smaller than the lower bound $\underline{N}(t)$ for $U(t)$, proving the final statement.

We can now complete the proof of Theorem 3. By definition we have

$$1 \ge \liminf_{t\uparrow\infty} \nu_t\big\{z \in \mathbb{Z}^d \;\big|\; |z - X_t^{(1)}| \le \delta r_t\big\} \ge \liminf_{t\uparrow\infty} \frac{U_1(t)}{U(t)} = 1 - \limsup_{t\uparrow\infty} \sum_{j=2}^{6} \frac{U_j(t)}{U(t)}.$$

Combining Lemmas 7, 8 and 17 we see that the limsup is zero, so that we get

$$\lim_{t\uparrow\infty} \nu_t\big\{z \in \mathbb{Z}^d \;\big|\; |z - X_t^{(1)}| \le \delta r_t\big\} = 1 \quad \text{in probability.}$$

Combining this with the convergence of $X_t^{(1)}/r_t$ given in Lemma 12 and recalling that $\delta > 0$ was arbitrary concludes the proof.

3 Concluding remarks

It would be interesting to study scaling limit theorems for potentials with lighter tails and thus shed further light on the number of relevant islands in these cases.

The techniques of the present paper appear suitable to treat cases where the relevant islands are single sites, which is the case for potentials heavier than the double-exponential distributions. For the double-exponential distribution itself and lighter tails, arguments related to classical order statistics of i.i.d. random variables need to be replaced by eigenvalue order statistics for the random Schrödinger operator $\Delta + \xi$ on $\ell^2(\mathbb{Z}^d)$, making the problem much more complex. Work in an advanced state of progress by Biskup and König [1] deals with the double-exponential case and strongly hints at localization in a single island of finite size in this and other cases of unbounded potentials.

For *bounded* potentials the question of the number of relevant islands and the formulation of a scaling limit theorem at present seems wide open and constitutes an attractive research project. Sznitman in [12] discusses an 'elliptic version' of the Anderson problem, describing Brownian paths in a Poissonian potential conditioned to reach a remote location. Sznitman's technique of enlargement of obstacles, described in [14], offers a possible approach to the scaling limit theorem, leading in [13] to a study of fluctuations of the principal eigenvalues of the operator $\Delta + \xi$ and moreover an analysis of variational problems somewhat similar to those that we expect to arise in the proof of a scaling limit theorem.

In the light of our result and this discussion it would be of particular interest to know whether there at all exist potentials which lead to more than one relevant island, and if so, to find the nature and location of the transition between phases of one and several islands.

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