Cup products in Hopf cyclic cohomology with coefficients in contramodules

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Dedicated to Henri Moscovici, with highest admiration and appreciation .

ABSTRACT. We use stable anti Yetter-Drinfeld contramodules to improve the cup products in Hopf cyclic cohomology. The improvement fixes the lack of functoriality of the cup products previously defined and show that the cup products are sensitive to the coefficients.

1. Introduction

Hopf cyclic cohomology was invented by Alain Connes and Henri Moscovici as a computational tool for computing the index cocycle of the hypoelliptic operators on manifolds [**6**]. One of the object of the theory was to study the cyclic cocycles generated by a symmetric system, in the sense of noncommutative geometry, which is usually given by an action or a coaction of a Hopf algebra on an algebra or a coalgebra. The main tool for transferring such cocycles to the cyclic complex of algebras is a characteristic map defined in [**6**]. The characteristic map is based on an invariant trace on the algebra of functions on the manifold in question. However in many situations the invariant trace does not exist, for example see [**4**]. For such cases the invariant cyclic cocycles play the role of invariant trace and one defines a higher version of the characteristic map [**7, 10**]. By the generalization of Hopf cyclic cohomology [**12, 11**] that allows one to take advantage of coefficients for Hopf cyclic cohomology, the invariant cyclic cocycles are understood as examples of Hopf cyclic cocycles. As a result, one generalizes the characteristic map to a cup product [**16**]. Similarly, the ordinary cup product in algebras was also generalized to another type of cup product in Hopf cyclic cohomology by replacing cycles and their characters with twisted cycles and their twisted characters. In [**19, 14**], by a straight application of cyclic Eilenberg-Zilber theorem (c.f. [**17, 9**]), the cup products was reconstructed and simplified. Finally, it is shown that all cup products defined in [**16, 19, 14, 10**] are the same in the level of cohomology [**15**].

The suitable coefficients for Hopf cyclic cohomology mentioned above is called stable anti Yetter-Drinfeld (SAYD) module [**11, 13**]. It has both module and comodule structure, over the Hopf algebra in question, with two compatibilities made of composition of action and coaction. However it is proved that Hopf cyclic cohomology works with a generalization of SAYD modules called SAYD contramodules [**1**]. Contramodules for a coalgebras was introduced in [**8**]. A right contramodule of a coalgebra C is a vector space $\mathcal{M}$ together with a $\mathbb{C}$-linear map $\alpha : \hom(C, \mathcal{M}) \to \mathcal{M}$ makes the diagrams (2.2) commutative.

An SAYD contramodule $\mathcal{M}$ is a module and contramodule together with two compatibilities made of α and the action of H on $\mathcal{M}$. As an example if M is a SAYD module over H then $\hom_k(M, \mathbb{C})$ is an SAYD contramodule over H.

In this paper, building on the methods we developed in [**19**], we generalize the cup products defined in the same paper by using SAYD contramodules coefficients. By Theorem 4.2 and Theorem 4.3 we show that the cup products is sensitive to coefficients. In Section 2 we recall Hopf cyclic cohomology with coefficients in SAYD modules and contramodules. In Section 3 we define the cup products for compatible pair of SAYD modules and contramodules. Here a compatible pair reads a pair of SAYD module and contramodule endowed with a pairing with values in the ground field and compatible with respect to actions and coactions. Finally, in Section 4 we generalize the results of Section 3 for arbitrary coefficients without any compatibility between them. The range of new cup products are ordinary cyclic cohomology of algebras with coefficients in vector spaces.

In this note a Hopf algebra is denoted by a sextuple $(H, \mu, \eta, \Delta, \varepsilon, S)$, where μ, η, Δ, ε, and S are multiplication, unit, comultiplication, counit, and antipode respectively. We use the Sweedler notation for comultiplications and coactions i.e., for coalgebras we use $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for comodules we use $\blacktriangledown(a) = a_{<0>} \otimes a_{<1>}$ and for coefficients we use $\blacktriangledown(m) = m_{<-1>} \otimes m_{<0>}$. All algebras, coalgebras and Hopf algebras are over the field of complex numbers $\mathbb{C}$. The unadorn tensor product $\otimes$ reads $\otimes_{\mathbb{C}}$.

2. Hopf cyclic cohomology with coefficients

2.1. Stable anti Yetter-Drinfeld-module. For the reader's convenience, we briefly recall the definition of Hopf cyclic cohomology of coalgebras and algebras under the symmetry of Hopf algebras with coefficients in SAYD modules [**12**], and with coefficients in SAYD contramodules [**1**].

Let us recall the definition of SAYD modules over a Hopf algebra from [**11**]. Given a Hopf algebra H, we say that M is a right-left SAYD module over H if M is a right module and left module over H with the following compatibilities.

$$(2.1) \qquad \blacktriangledown_M(m \cdot h) = S(h_{(3)}) m_{<-1>} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}$$

$$(2.2) \qquad m_{<0>} \cdot m_{<-1>} = m.$$

The other three flavors i.e, left-left, left-right, and right-right are defined similarly [**11**].

Let C be a H-module coalgebra, that is a coalgebra endowed with an action, say from left, of H such that its comultiplication and counit are H-linear, i.e,

$$(2.3) \qquad \Delta(h \cdot c) = h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}, \quad \varepsilon(h \cdot c) = \varepsilon(h)\varepsilon(c).$$

Having the datum (H, C, M), where C is an H-module coalgebra and M an right-left SAYD over H, one defines in [**12**] a cocyclic module $\{C_H^n(C, M), \partial_i, \sigma_j, \tau\}_{n \geq 0}$ as follows.

$$(2.4) \qquad C_H^n(C, M) := M \otimes_H C^{\otimes n+1}, \quad n \geq 0,$$

with the following cocyclic structure,

$$(2.5) \qquad \partial_i : C_H^n(C, M) \to C_H^{n+1}(C, M), \qquad 0 \leq i \leq n+1$$

$$(2.6) \qquad \sigma_j : C_H^n(C, M) \to C_H^{n-1}(C, M), \qquad 0 \leq j \leq n-1,$$

$$(2.7) \qquad \tau : C_H^n(C, M) \to C_H^n(C, M),$$

defined explicitly as follows, where we abbreviate $\tilde{c} := c^0 \otimes \cdots \otimes c^n$,

$$(2.8) \qquad \partial_i(m \otimes_H \tilde{c}) = m \otimes_H c^0 \otimes \cdots \otimes \Delta(c^i) \otimes \cdots \otimes c^n,$$

$$(2.9) \qquad \partial_{n+1}(m \otimes_H \tilde{c}) = m_{<\bar{0}>} \otimes_H c^0_{(2)} \otimes c^1 \otimes \cdots \otimes c^n \otimes m_{<\overline{-1}>} \cdot c^0_{(1)},$$

$$(2.10) \qquad \sigma_i(m \otimes_H \tilde{c}) = m \otimes_H c^0 \otimes \cdots \otimes \epsilon(c^{i+1}) \otimes \cdots \otimes c^n,$$

$$(2.11) \qquad \tau(m \otimes_H \tilde{c}) = m_{<\bar{0}>} \otimes_H c^1 \otimes \cdots \otimes c^n \otimes m_{<\overline{-1}>} \cdot c^0.$$

It is checked in [**12**] that the above graded module defines a cocyclic module. Similarly an algebra which is a H-module and its algebra structure is H-linear is called H-module algebra. In other words, for any $a, b \in A$ and any $h \in H$ we have

$$(2.12) \qquad h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad h \cdot 1_A = \varepsilon(h)1_A.$$

Let A be a H-module algebra. One endows $M \otimes A^{\otimes n+1}$ with the diagonal action of H and forms $C_H^n(A, M) := \mathrm{Hom}_H(M \otimes A^{\otimes n+1}, \mathbb{C})$ as the space of H-linear maps. It is checked in [**12**] that the following defines a cocyclic module structure on $C^n(A, M)$.

$$(\partial_i \varphi)(m \otimes \tilde{a}) = \varphi(m \otimes a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^{n+1}),$$

$$(\partial_{n+1}\varphi)(m \otimes \tilde{a}) = \varphi(m_{<\bar{0}>} \otimes (S^{-1}(m_{<\overline{-1}>}) \cdot a^{n+1})a^0 \otimes a^1 \otimes \cdots \otimes a^n),$$

$$(\sigma_i \varphi)(m \otimes \tilde{a}) = \varphi(m \otimes a^0 \otimes \cdots \otimes a^i \otimes 1 \otimes \cdots \otimes a^{n-1}),$$

$$(\tau \varphi)(m \otimes \tilde{a}) = \varphi(m_{<\bar{0}>} \otimes S^{-1}(m_{<\overline{-1}>}) \cdot a^n \otimes a^0 \otimes \cdots \otimes a^{n-1}).$$

The cyclic cohomology of this cocyclic module is denoted by $HC_H^*(A, M)$.

An algebra is called a H-comodule algebra if it is a H comodule and its algebra structure are H colinear, which means that

$$(2.13) \qquad (h \cdot a)_{<0>} \otimes (h \cdot a)_{<1>} = h_{(1)} \cdot a_{<0>} \otimes h_{(2)} \cdot a_{<1>}.$$

Similar to the other case, one defines $^H C^n(A, M)$ to be the space of all colinear maps from $A^{\otimes n+1}$ to M. One checks that the following defines a cocyclic module structure on $^H C^n(A, M)$.

$$(2.14) \qquad (\partial_i \varphi)(\tilde{a}) = \varphi(a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^{n+1}),$$
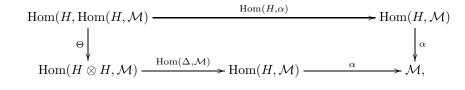
$$(2.15) \qquad (\partial_{n+1}\varphi)(\tilde{a}) = \varphi(a^{n+1}{}_{<0>} a^0 \otimes a^1 \cdots \otimes a^{n-1} \otimes a^n) \cdot a^{n+1}{}_{<-1>},$$
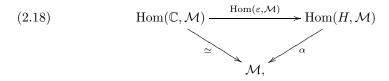
$$(2.16) \qquad (\sigma_i \varphi)(\tilde{a}) = \varphi(a^0 \otimes \cdots \otimes a^i \otimes 1 \otimes \cdots \otimes a^{n-1}),$$

$$(2.17) \qquad (\tau\varphi)(a^0 \otimes \cdots \otimes a^n) = \varphi(a^n{}_{<0>} \otimes a^0 \otimes \cdots \otimes a^{n-1} \otimes a^{n-1}) \cdot a^n{}_{<-1>}.$$

The cyclic cohomology of this cocyclic module is denoted by ${}^H HC^*(A, M)$.

2.2. SAYD contramodule. Let us recall SAYD contramodules from [**1**]. A right contramodule of a coalgebra H is a vector space $\mathcal{M}$ together with a $\mathbb{C}$-linear map $\alpha : \mathrm{Hom}(H, \mathcal{M}) \to \mathcal{M}$ making the following diagrams commutative

$$
\begin{array}{ccc}
\mathrm{Hom}(H, \mathrm{Hom}(H, \mathcal{M})) & \xrightarrow{\mathrm{Hom}(H,\alpha)} & \mathrm{Hom}(H, \mathcal{M}) \\
\Theta \downarrow & & \downarrow \alpha \\
\mathrm{Hom}(H \otimes H, \mathcal{M}) \xrightarrow{\mathrm{Hom}(\Delta,\mathcal{M})} \mathrm{Hom}(H, \mathcal{M}) & \xrightarrow{\alpha} & \mathcal{M},
\end{array}
$$

$$
(2.18) \qquad
\begin{array}{ccc}
\mathrm{Hom}(\mathbb{C}, \mathcal{M}) & \xrightarrow{\mathrm{Hom}(\varepsilon,\mathcal{M})} & \mathrm{Hom}(H, \mathcal{M}) \\
& \simeq \searrow \quad \swarrow \alpha & \\
& \mathcal{M}, &
\end{array}
$$

where Θ is the standard isomorphism given by $\Theta(f)(h \otimes h') = f(h)(h')$.

Definition 2.1 ([**1**])**.** A left-right anti-Yetter-Drinfeld (AYD) contramodule $\mathcal{M}$ is a left H-module (with the action denoted by a dot) and a right H-contramodule with the structure map α, such that, for all $h \in H$ and $f \in \mathrm{Hom}(H, \mathcal{M})$,

$$h \cdot \alpha(f) = \alpha\left(h_{(2)} \cdot f\left(S(h_{(3)})(-)h_{(1)}\right)\right).$$

M is said to be *stable*, provided that, for all $m \in \mathcal{M}$, $\alpha(r_m) = m$, where $r_m : H \to \mathcal{M}$, $h \mapsto h \cdot m$.

We refer the reader to [**1**] for more details on SAYD contramodules. If M is an AYD module, then its dual $\mathcal{M} = M^*$ is an AYD contramodule (with the sides interchanged) and SAYD modules correspond to SAYD contramodules. For example, let M be a right-left AYD module (2.1), the dual vector space $\mathcal{M} = M^*$ is a right H-module by $m \otimes h \mapsto m \cdot h$,

$$(h \cdot f)(m) = f(m \cdot h),$$

for all $h \in H$, $f \in \mathcal{M} = \mathrm{Hom}(M, \mathbb{C})$ and $m \in M$, and a right H-contramodule with the structure map $\alpha(f)(m) = f(m_{<-1>})(m_{<0>})$, $f \in \mathrm{Hom}(H, \mathcal{M})$, and $m \in M$ [**1**]. Let A be a left H-module algebra and $\mathcal{M}$ be a left-right SAYD contramodule over H. We let $C_H^n(A, \mathcal{M})$ to be the space of left H-linear maps

$$(2.19) \qquad \mathrm{Hom}_H(A^{\otimes n+1}, \mathcal{M}),$$

and, for all $0 \leq i, j \leq n$, define $\partial_i : C_H^{n-1}(A, M) \to C_H^n(A, \mathcal{M})$, $\sigma_j : C_H^{n+1}(A, \mathcal{M}) \to C_H^n(A, \mathcal{M})$, $\tau : C_H^n(A, \mathcal{M}) \to C_H^n(A, \mathcal{M})$, by

$$(2.20) \qquad \partial_i(\varphi)(a^0 \otimes \cdots \otimes a^n) = \varphi(a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^n), \qquad 0 \leq i < n,$$

$$(2.21) \qquad \partial_n(\varphi)(a^0 \otimes \cdots \otimes a^n) = \alpha \left(\varphi \left(\left(S^{-1}(-) \cdot a^n \right) a^0 \otimes a^1 \otimes \cdots \otimes a^{n-1} \right) \right),$$

$$(2.22) \qquad \sigma_j(\varphi)(a^0 \otimes \cdots \otimes a^n) = \varphi(a^0 \otimes \cdots \otimes a^j \otimes 1_A \otimes a^{j+1} \otimes \cdots \otimes a^n),$$

$$(2.23) \qquad \tau(\varphi)(a^0 \otimes \cdots \otimes a^n) = \alpha \left(\varphi \left(S^{-1}(-) \cdot a^n \otimes a^0 \otimes \cdots \otimes a^{n-1} \right) \right).$$

It is shown in [**1**] that the above operators define a cocyclic module on $C_H^*(A, \mathcal{M})$. We denote the cyclic cohomology of $C_H^*(A, \mathcal{M})$ by $HC_H^*(A, \mathcal{M})$. For $\mathcal{M} = M^*$, where M is a SAYD module over H, it is easy to see that $C_H^*(A, \mathcal{M}) \simeq C_H^*(A, M)$. Indeed let M be a right-left SAYD module and $\mathcal{M} := \mathrm{Hom}(M, \mathbb{C})$ be the corresponding right-left SAYD contramodule. We define the following maps

$$\mathcal{I} : C_H^n(A, M) \to C_H^n(A, \mathcal{M}), \quad \mathcal{J} : C_H^n(A, \mathcal{M}) \to C_H^n(A, M),$$

$$\mathcal{I}(\phi)(a^0 \otimes \cdots \otimes a^n)(m) = \phi(m \otimes a^0 \otimes \cdots \otimes a^n),$$

$$\mathcal{J}(\phi)(m \otimes a^0 \otimes \cdots \otimes a^n) = \phi(a^0 \otimes \cdots \otimes a^n)(m).$$

Proposition 2.2. *The above map $\mathcal{I}$ is an isomorphism of cocyclic modules.*

PROOF. It is obvious that $\mathcal{I}$ and $\mathcal{J}$ are inverse to one another. We shall check that $\mathcal{I}$ commutes with cyclic structures. It is easy to see that faces, except possibly the very last one, and degeneracies commutes with $\mathcal{I}$. So it is suffices to check that $\mathcal{I}$ commutes with the cyclic operators. Indeed,

$$\mathcal{I} \circ \tau(\phi)(a^0 \otimes \cdots \otimes a^n)(m) = \tau(\phi)(m \otimes a^0 \otimes \cdots \otimes a^n)$$

$$= \phi(m_{<\bar{0}>} \otimes S^{-1}(m_{<\overline{-1}>}) \cdot a^n \otimes a^0 \otimes \cdots \otimes a^{n-1})$$

$$= \mathcal{I}(\phi)(S^{-1}(m_{<\overline{-1}>}) \cdot a^n \otimes a^0 \otimes \cdots \otimes a^{n-1})(m_{<\bar{0}>})$$

$$= \tau \circ \mathcal{I}(\phi)(a^0 \otimes \cdots \otimes a^n)(m).$$

$\square$

3. Cup products in Hopf cyclic cohomology

In this section we use the same strategy as in [**19, 14**] to generalize the cup products constructed in the same references. Via these new cup products one has the luxury to construct cyclic cocycles by using a compatible pair of SAYD modules and contramodules rather than only a SAYD module.

3.1. Module algebras paired with module coalgebras. Let A be an H module algebra and C be a H module coalgebra acting on A in the sense that there is a map

$$(3.1) \qquad\qquad\qquad C \otimes A \to A,$$

such that for any $h \in H$, any $c \in C$ and any $a, b \in A$ one has

$$(3.2) \qquad\qquad\qquad (h \cdot c) \cdot a = h \cdot (c \cdot a)$$

$$(3.3) \qquad\qquad\qquad c \cdot (ab) = (c_{(1)} \cdot a)(c_{(2)} \cdot b)$$

$$(3.4) \qquad\qquad\qquad c(1) = \epsilon(c)1$$

One constructs a convolution algebra $B = \mathrm{Hom}_H(C, A)$, which is the algebra of all H-linear maps from A to C. The unit of this algebra is given by $\eta \circ \epsilon$, where $\eta : \mathbb{C} \to A$ is the unit of A. The multiplication of $f, g \in B$ is given by

$$(3.5) \qquad\qquad (f * g)(c) = f(c_{(1)})g(c_{(2)})$$

Definition 3.1. Let $(\mathcal{M}, \alpha)$ be a left-right SAYD contramodule and N be a right-left SAYD module over H. We call $(N, \mathcal{M})$ compatible if there is a pairing between $\mathcal{M}$ and N such that

$$(3.6) \qquad\qquad < n \cdot h \mid m > = < n \mid h \cdot m >,$$

$$(3.7) \qquad\qquad < n \mid \alpha(f) > = < n_{<\bar{0}>} \mid f(n_{<\overline{-1}>}) >,$$

for all $m \in \mathcal{M}$, $n \in N$, $f \in \mathrm{Hom}(H, \mathcal{M})$, and $h \in H$.

Let $(N, \mathcal{M})$ be compatible as above. We have the following cocyclic modules defined in $(2.19) \ldots (2.23)$, and $(2.4) \ldots (2.8)$ respectively.

$$(3.8) \qquad\qquad (C_H^*(A, \mathcal{M}), \partial_i, \sigma_j, \tau), \quad \text{and} \quad (C_H^*(C, N), \partial_i, \sigma_j, \tau).$$

We define a new bicocyclic module by tensoring these cocycle module over $\mathbb{C}$. The new bigraded module has in its bidegree (p, q)

$$(3.9) \qquad\qquad C_{\mathrm{a-c}}^{p,q} := \mathrm{Hom}_H(A^{\otimes p+1}, \mathcal{M}) \otimes (N \otimes_H C^{\otimes q+1}),$$

with horizontal structure $\vec{\partial}_i = \mathrm{Id} \otimes \partial_i$, $\vec{\sigma}_j = \mathrm{Id} \otimes \sigma_j$, and $\vec{\tau} = \mathrm{Id} \otimes \tau$ and vertical structure $\uparrow\partial_i = \partial_i \otimes \mathrm{Id}$, $\uparrow\sigma_j = \sigma_j \otimes \mathrm{Id}$, and $\uparrow\tau = \tau \otimes \mathrm{Id}$. Obviously $(C_{\mathrm{a-c}}^{p,q}, \vec{\partial}, \vec{\sigma}, \vec{\tau}, \uparrow\partial, \uparrow\sigma, \uparrow\tau)$ defines a bicocyclic module.
Now let us define the map

(3.10)

$\Psi : D^q(C_{\mathrm{a-c}}^{*,*}) \to \mathrm{Hom}(B^{\otimes q+1}, \mathbb{C}),$

$\Psi(\phi \otimes (n \otimes c^0 \otimes \cdots \otimes c^q))(f^0 \otimes \cdots \otimes f^q) = < n \mid \phi(f^0(c^0) \otimes \cdots \otimes f^q(c^q)) > .$

Here $D(C_{\mathrm{a-c}}^{*,*})$ denotes the diagonal of the bicocyclic module $C_{\mathrm{a-c}}^{*,*}$. It is a cocyclic module whose qth component is $C^{q,q}$ and its cocyclic structure morphisms are $\partial_i := \vec{\partial}_i \circ \uparrow\partial_i$, $\sigma_j := \vec{\sigma}_j \circ \uparrow\sigma_j$, and $\tau := \vec{\tau} \circ \uparrow\tau$.

Proposition 3.2. *The map Ψ is a well-defined map of cyclic modules.*

PROOF. First let us show that Ψ is well-defined. Indeed, by using the facts that $\mathcal{M}$ and N are compatible, f^i are H-linear, ϕ is equivariant and (3.2) holds, we see that,

$$\Psi(\phi \otimes (n \otimes h_{(1)} c^0 \otimes \cdots \otimes h_{(n+1)} c^n))(f^0 \otimes \cdots \otimes f^n)$$
$$= < n \mid \phi(f^0(h_{(1)} \cdot c^0) \otimes \cdots \otimes f^n(h_{(n+1)} \cdot c^n))$$
$$= < n \mid h \cdot \phi(f^0(c^0) \otimes \cdots \otimes f^n(c^n)) >$$
$$= < nh \mid \phi(f^0(c^0) \otimes \cdots \otimes f^n(c^n)) >$$
$$= \Psi(\phi \otimes (n \cdot h \otimes c^0 \otimes \cdots \otimes c^n))(f^0 \otimes \cdots \otimes f^n).$$

Next, we show that Ψ commutes with cocyclic structure morphisms. To this end, we need only to show the commutativity of Ψ with zeroth cofaces, the last codegeneracies and the cyclic operators because these operators generate all cocyclic

structure morphisms. We check it only for the cyclic operators and leave the rest to the reader. Let τ_B denote the cyclic operator of the ordinary cocyclic module of the algebra B.

$$
\begin{aligned}
&\Psi(\tau(\varphi) \otimes \tau(n \otimes c^0 \otimes \cdots \otimes c^q))(f^0 \otimes \cdots \otimes f^q) \\
&= \Psi(\tau\varphi \otimes (n_{<\bar{0}>} \otimes c^1 \otimes \cdots \otimes c^q \otimes n_{<\overline{-1}>} c^0))(f^0 \otimes \cdots \otimes f^q) \\
&= <n_{<\bar{0}>} \mid \tau(\varphi)(f^0(c^1) \otimes \cdots \otimes f^{q-1}(c^n) \otimes f^q(n_{<\overline{-1}>} c^0)) > \\
&= <n_{<\bar{0}>} \mid \alpha\left(\varphi\left(S^{-1}(-) \cdot f^q(n_{<\overline{-1}>} c^0) \otimes f^0(c^1) \otimes \cdots \otimes f^{q-1}(c^n)\right)\right) > \\
&= <n_{<\bar{0}>} \mid \varphi\left(S^{-1}(n_{<\overline{-1}>}) \cdot f^q(n_{<\overline{-2}>} c^0) \otimes f^0(c^1) \otimes \cdots \otimes f^{q-1}(c^n)\right) > \\
&= <n \mid \varphi\left(f^q(c^0) \otimes f^0(c^1) \otimes \cdots \otimes f^{q-1}(c^q)\right) > \\
&= \tau_B \Psi(\varphi \otimes (n \otimes c^0 \otimes \cdots \otimes c^q))(f^0 \otimes \cdots \otimes f^q).
\end{aligned}
$$

Here in the passage from fourth line to the fifth one we use (3.7). $\qquad \square$

Let $C := \bigoplus_{p,q \geq 0} C^{p,q}$ be a bicocyclic module. With $Tot(C)$ designating the total mixed complex $Tot(C)^n = \bigoplus_{p+q=n} C^{p,q}$, we denote by $\mathrm{Tot}(C)$ the associated *normalized* subcomplex, obtained by retaining only the elements annihilated by all degeneracy operators. Its total boundary is $b_T + B_T$, with b_T and B_T defined as follows:

$$
\overrightarrow{b}_p = \sum_{i=0}^{p+1}(-1)^i \overrightarrow{\partial}_i, \qquad\qquad \Uparrow b_q = \sum_{i=0}^{q+1}(-1)^i \Uparrow \partial_i,
$$

$$
(3.11) \qquad b_T = \sum_{p+q=n} \overrightarrow{b}_p + \Uparrow b_q,
$$

$$
\overrightarrow{B}_p = (\sum_{i=0}^{p-1}(-1)^{(p-1)i} \overrightarrow{\tau}^i) \overrightarrow{\sigma}_{p-1} \overrightarrow{\tau}, \qquad \Uparrow B_q = (\sum_{i=0}^{q-1}(-1)^{(q-1)i} \Uparrow \tau^i) \overrightarrow{\sigma}_{q-1} \Uparrow \tau,
$$

$$
(3.12) \qquad B_T = \sum_{p+q=n} \overrightarrow{B}_p + \Uparrow B_q.
$$

The total complex of a bicocyclic module C is a mixed complex, i.e, $b_T^2 = B_T^2 = b_T B_T + B_T b_T = 0$. As a result its cyclic cohomology is well-defined. By means of the analogue of the Eilenberg-Zilber theorem for bi-paracyclic modules [**9, 17**], the diagonal mixed complex $(D(C), b_D, B_D)$ and the total mixed complex $(\mathrm{Tot}\, C, b_T, B_T)$ can be seen to be quasi-isomorphic in both Hochschild and cyclic cohomology. Here $D(C) := \bigoplus_{q \geq 0} C^{q,q}$ is a cocyclic module and therefore a mixed complex with (co)boundaries,

$$
b_D := \sum_{i=0}^{q+1}(-1)^q \Uparrow \partial_i \circ \overrightarrow{\partial}_i,
$$

$$
(3.13)
$$

$$
B_D := \left(\sum_{i=0}^{q-1}(-1)^{(q-1)i} \overrightarrow{\tau}^i \Uparrow \tau^i\right) \overrightarrow{\sigma}_{q-1} \Uparrow \sigma_{q-1} \overrightarrow{\tau} \Uparrow \tau.
$$

At the level of Hochschild cohomology the quasi-isomorphism is implemented by the Alexander-Whitney map $AW := \bigoplus_{p+q=n} AW_{p,q} : \mathrm{Tot}(C)^n \to D(C)^n$,

$$AW_{p,q} : C^{p,q} \longrightarrow C^{p+q,p+q}$$

$$(3.14) \qquad AW_{p,q} = (-1)^{p+q} \underbrace{\uparrow\partial_0 \uparrow\partial_0 \ldots \uparrow\partial_0}_{q\,\text{times}} \vec{\partial}_n \vec{\partial}_{n-1} \ldots \vec{\partial}_{q+1}\,.$$

Using a standard homotopy operator H, this can be supplemented by a cyclic Alexander-Whitney map $AW' := AW \circ B \circ H : D^n \to \mathrm{Tot}(C)^{n+2}$, and thus upgraded to an S-map $\overline{AW} = (AW, AW')$, of mixed complexes. The inverse quasi-isomorphims are provided by the shuffle maps $Sh := D(C)^n \to \mathrm{Tot}(C)^n$, resp. $\overline{Sh} = (Sh, Sh')$, which are discussed in detail in [**9**, **17**].

Let c be (b, B) cocycle in $\mathrm{Tot}(C)^n$, $n = p + q$. Hence the class of $\overline{AW}(c)$ in $HC^n(D(C))$ is well defined.

Now we consider the inclusion $\iota : A \to B = \mathrm{Hom}_H(C, A)$, defined by $\iota(a)(c) = c \cdot a$. We see that $\iota(ab)(c) = c \cdot (ab) = (c_{(1)} \cdot a)(c_{(2)} \cdot b) = (\iota(a) * \iota(b))(c)$, and $\iota(1_A)(c) = c \cdot 1_A = \varepsilon(c)1_A = 1_B(c)$. Hence ι is an algebra map and in turn induces a map in the level of cyclic cohomology groups:

$$\iota : HC^*(B) \to HC^*(A).$$

Theorem 3.3. *Let H be a Hopf algebra, A be an H-module algebra, C be an H-module coalgebra acting on A, and $(N, \mathcal{M})$ be a compatible pair of SAYD module and contramodule over H. Then $\hat{\Psi} := \iota \circ \Psi \circ \overline{AW}$ defines a cup product in the level of cyclic cohomology groups:*

$$(3.15) \qquad \hat{\Psi} := \iota \circ \Psi \circ \overline{AW} : HC_H^p(A, \mathcal{M}) \otimes HC_H^q(C, N) \to HC^{p+q}(A).$$

PROOF. Let $[\phi] \in HC_H^p(A, \mathcal{M})$ and $[\omega] \in HC_H^q(C, N)$. Without loss of generality one assumes that ϕ and ω are both cyclic cocycles, i.e,

$$\vec{b}(\phi) = \uparrow b(\omega) = 0, \quad \vec{\tau}(\phi) = (-1)^p \phi, \quad \uparrow\tau(\omega) = (-1)^q \omega.$$

This implies that $\phi \otimes \omega$ is a (b, B) cocycle in $\mathrm{Tot}(C_{\mathrm{a-c}}^{*,*})^{p+q}$. Hence $\overline{AW}(\phi \otimes \omega)$ defines a class in $HC^{p+q}(D(C_{\mathrm{a-c}}^{*,*}))$. Finally, since ι and Ψ both are cyclic map, the transferred cochain $\iota \circ \Psi(\overline{AW}(\phi \otimes \omega))$ defines a class in $HC^{p+q}(A)$. $\square$

3.2. Module algebras paired with comodule algebras. Let H be a Hopf algebra, A a left H-module algebra, B a left H-comodule algebra, and $(N, \mathcal{M})$ be a compatible pair of SAYD module and contramodule over H. One constructs a crossed product algebra whose underlying vector space is $A \otimes B$ with the $1 \bowtie 1$ as its unit and the following multiplication:

$$(3.16) \qquad (a \bowtie b)(a' \bowtie b') = a\,(b_{<-1>} \cdot a') \bowtie b_{<0>}\,b'$$

Now consider the two cocyclic modules

$$(3.17) \qquad (C_H^*(A, \mathcal{M}), \partial_i, \sigma_j, \tau), \quad \text{and} \quad ({}^H C^*(B, N), \partial_i, \sigma_j, \tau)$$

introduced in [**1**] and [**12**] respectively and are recalled in $(2.14) \ldots (2.17)$ and $(2.19) \ldots (2.23)$. We define a bicocyclic module by tensoring these cocyclic modules over $\mathbb{C}$. The (p, q)-bidegree component $C_{\mathrm{a-a}}^{p,q}$ of this new bicocyclic module is given by

$$(3.18) \qquad {}^H \mathrm{Hom}(B^{\otimes q+1}, N) \otimes \mathrm{Hom}_H(A^{\otimes p+1}, \mathcal{M}),$$

with horizontal structure morphisms $\overrightarrow{\partial}_i = \mathrm{Id}\otimes\partial_i$, $\overrightarrow{\sigma}_j = \mathrm{Id}\otimes\sigma_j$, and $\overrightarrow{\tau} = \mathrm{Id}\otimes\tau$ and vertical structure morphisms $\uparrow\partial_i = \partial_i \otimes \mathrm{Id}$, $\uparrow\sigma_j = \sigma_j \otimes \mathrm{Id}$, and $\uparrow\tau = \tau \otimes \mathrm{Id}$. Now we define a new morphism

$$(3.19) \qquad \Phi : D(C^{*,*}_{\mathrm{a-a}})^n \to C^n(A \bowtie B),$$

define by

$$(3.20) \qquad \Phi(\psi \otimes \phi)(a^0 \bowtie b^0 \otimes \cdots \otimes a^n \bowtie b^n) =$$
$$=< \psi(b^0_{<0>} \otimes \cdots \otimes b^n_{<0>}) \mid \phi(S^{-1}(b^0_{<-1>} \cdots b^n_{<-1>}) \cdot a^0 \otimes \cdots$$
$$(3.21) \qquad \cdots \otimes S^{-1}(b^n_{<-n-1>}) \cdot a^n) > .$$

Proposition 3.4. *The map Φ defines a cyclic map between the diagonal of $C^{*,*}_{\mathrm{a-a}}$ and the cocyclic module $C^*(A \bowtie B)$.*

PROOF. We show that Φ commutes with the cyclic structure morphisms. We shall check it for the first face operator and the cyclic operator and leave the rest to the reader. Let us denote the cyclic structure morphisms of the algebra $A \bowtie B$ by $\partial_i^{A\bowtie B}, \sigma_j^{A\bowtie B}$ and $\tau^{A\bowtie B}$. First we show that Φ commutes with the zeroth cofaces.

$$\Phi(\overrightarrow{\partial}_0 \uparrow\partial_0(\psi \otimes \phi))(a^0 \bowtie b^0 \otimes \cdots \otimes a^{n+1} \bowtie b^{n+1})$$
$$= \Phi(\partial_0\phi \otimes \partial_0\psi))(a^0 \bowtie b^0 \otimes \cdots \otimes a^{n+1} \bowtie b^{n+1})$$
$$=< \partial_0\psi(b^0_{<0>} \otimes \cdots \otimes b^{n+1}_{<0>}) \mid \partial_0\phi(S^{-1}(b^0_{<-1>} \cdots b^{n+1}_{<-1>}) \cdot a^0 \otimes \cdots$$
$$\cdots \otimes S^{-1}(b^{n+1}_{<-n-2>}) \cdot a^{n+1}) >$$
$$=< \psi(b^0_{<0>} b^1_{<0>} \otimes \cdots \otimes b^{n+1}_{<0>})$$
$$\mid \phi(S^{-1}(b^0_{<-1>} \cdots b^{n+1}_{<-1>}) \cdot a^0 S^{-1}(b^1_{<-2>} \cdots b^{n+1}_{<-2>}) \cdot a^1 \otimes \cdots$$
$$\cdots \otimes S^{-1}(b^{n+1}_{<-n-2>}) \cdot a^{n+1}) >$$
$$=< \psi(b^0_{<0>} b^1_{<0>} \otimes \cdots \otimes b^{n+1}_{<0>})$$
$$\mid \phi(S^{-1}(b^0_{<-1>} b^1_{<-1>} \cdots b^{n+1}_{<-1>}) \cdot (a^0 b^0_{<-2>} \cdot a^1) \otimes \cdots$$
$$\cdots \otimes S^{-1}(b^{n+1}_{<-n-1>}) \cdot a^{n+1}) >$$
$$= \Phi(\psi \otimes \phi)(a^0 b^0_{<-1>} a^1 \bowtie b^0_{<0>} b^1 \otimes a^2 \bowtie b^2 \otimes \cdots \otimes a^{n+1} \bowtie b^{n+1})$$
$$= \partial_0^{A\bowtie B}\Phi(\psi \otimes \phi)(a^0 \bowtie b^0 \otimes \cdots \otimes a^{n+1} \bowtie b^{n+1}).$$

Now we show that Φ commutes with cyclic operators.

(3.22)
$$\Phi(\overrightarrow{\tau} \uparrow\tau(\psi \otimes \phi))(a^0 \bowtie b^0 \otimes \cdots \otimes a^n \bowtie b^n)$$
$$= \Phi(\tau\psi \otimes \tau\phi)(a^0 \bowtie b^0 \otimes \cdots \otimes a^n \bowtie b^n)$$
$$=< \tau\psi(b^0_{<0>} \otimes \cdots \otimes b^n_{<0>}) \mid \tau\phi(S^{-1}(b^0_{<-1>} \cdots b^n_{<-1>})a^0 \otimes \cdots$$
$$\cdots \otimes S^{-1}(b^n_{<-n-1>})a^n) >$$
$$=< \psi(b^n_{<0>} \otimes b^0_{<0>} \otimes \cdots$$
$$\cdots \otimes b^{n-1}_{<0>}) \cdot b^n_{<-1>} \mid \alpha(\phi(S^{-1}(-)S^{-1}(b^n_{<-n-2>}) \cdot a^n \otimes$$
$$S^{-1}(b^0_{<-1>} \cdots b^{n-1}_{<-1>} b^n_{<-2>}) \cdot a^0 \otimes \cdots \otimes S^{-1}(b^{n-1}_{<-n>} b^n_{<-n-1>}) \cdot a^{n-1})) > .$$

Using (3.7), and the fact that ψ is H-colinear, one has:

(3.23)

$$(3.22) = < [\psi(b^n{}_{<0>} \otimes b^0{}_{<0>} \otimes \cdots \otimes b^{n-1}{}_{<0>}) \cdot b^n{}_{<-1>}]_{<0>} \mid$$

$$\phi(S^{-1}([\psi(b^n{}_{<0>} \otimes b^0{}_{<0>} \otimes \cdots \otimes b^{n-1}{}_{<0>}) \cdot b^n{}_{<-1>}]_{<-1>})(S^{-1}(b^n{}_{<-n-2>}) \cdot a^n) \otimes$$

$$\otimes S^{-1}(b^0{}_{<1>} \cdots b^{n-1}{}_{<-1>} b^n{}_{<-2>}) \cdot a^0 \otimes \cdots \otimes S^{-1}(b^{n-1}{}_{<-n+1>} b^n{}_{<-n-1>}) \cdot a^{n-1}) > .$$

Using the fact that N is AYD module we have,

(3.24)

$$(3.23) = < \psi(b^n{}_{<0>} \otimes b^0{}_{<0>} \otimes \cdots \otimes b^{n-1}{}_{<0>}) \cdot b^n{}_{<-3>} \mid$$

$$\phi((S^{-1}(S(b^n{}_{<-2>})b^n{}_{<-1>} b^0{}_{<-1>} \cdots b^{n-1}{}_{<-1>})b^n{}_{<-4>})(S^{-1}(b^n{}_{<-n-2>}) \cdot a^n) \otimes$$

$$\otimes S^{-1}(b^0{}_{<1>} \cdots b^{n-1}{}_{<-1>} b^n{}_{<-5>}) \cdot a^0 \otimes \cdots \otimes S^{-1}(b^{n-1}{}_{<-n+1>} b^n{}_{<-n-4>}) \cdot a^{n-1}) > .$$

Using (3.6) and the facts that ϕ is H-linear and $\mathcal{M}$ is AYD contramodule we see

(3.25)

$$(3.24) = < \psi(b^n{}_{<0>} \otimes b^0{}_{<0>} \otimes \cdots \otimes b^{n-1}{}_{<0>})b^n{}_{<-1>} \mid$$

$$\phi(S^{-1}(b^0{}_{<-1>} \cdots b^{n-1}{}_{<-1>})b^n{}_{<-2>} S^{-1}(b^n{}_{<-n-2>})a^n \otimes$$

$$\otimes S^{-1}(b^0{}_{<1>} \cdots b^{n-1}{}_{<-1>} b^n{}_{<-3>})a^0 \otimes \cdots \otimes S^{-1}(b^{n-1}{}_{<-n+1>} b^n{}_{<-n-2>})a^{n-1}) >$$

$$= < \psi(b^n{}_{<0>} \otimes b^0{}_{<0>} \otimes \cdots \otimes b^{n-1}{}_{<0>}) \mid \phi(S^{-1}(b^n{}_{<-1>} b^0{}_{<-1>} \cdots b^{n-1}{}_{<-1>})a^n \otimes$$

$$\otimes S^{-1}(b^0{}_{<1>} \cdots b^{n-1}{}_{<-1>})a^0 \otimes \cdots \otimes S^{-1}(b^{n-1}{}_{<-n+1>})a^{n-1}) >$$

$$= \Phi(\phi \otimes \psi)(a^n \bowtie b^n \otimes a^0 \bowtie b^0 \otimes \cdots \otimes a^{n-1} \bowtie b^{n-1})$$

$$= \tau^{A \bowtie B} \Phi(\phi \otimes \psi)(a^0 \bowtie b^0 \otimes \cdots \otimes a^n \bowtie b^n).$$

$\square$

Theorem 3.5. *Let H be a Hopf algebra, A be an H-module algebra, B be an H-comodule algebra, $(N, \mathcal{M})$ be a compatible pair of SAYD module and contramodule. Then the map $\hat{\Phi} := \Phi \circ \overline{AW}$ defines a cup product:*

$$(3.26) \qquad \hat{\Phi} := \Phi \circ \overline{AW} : \; {}^H HC^q(B, N) \otimes HC^p_H(A, \mathcal{M}) \to HC^{p+q}(A \bowtie B).$$

PROOF. The proof is similar to the proof of Theorem 3.3. Let $[\phi] \in {}^H HC^p(A, \mathcal{M})$ and $[\psi] \in HC^q_H(B, N)$. Without loss of generality one assumes that ϕ and ψ are both cyclic cocycle, i.e, $\overrightarrow{b}(\phi) = \uparrow b(\psi) = 0$, $\overrightarrow{\tau}(\phi) = (-1)^p \phi$ and $\uparrow \tau(\psi) = (-1)^q \psi$. This implies that $\psi \otimes \phi$ is a (b, B) cocycle in $\mathrm{Tot}(C^{*,*}_{\mathrm{a-a}})^{p+q}$. Hence $\overline{AW}(\psi \otimes \phi)$ defines a class in $HC^{p+q}(D(C^{*,*}_{\mathrm{a-a}}))$. Finally, since Φ is cyclic map, the transferred cochain $\Phi(\overline{AW}(\psi \otimes \phi))$ defines a class in $HC^{p+q}(A \bowtie B)$. $\square$

4. Cup products for incompatible pairs

In this section we generalize the cup products defined in (3.15) and (3.26) to the case of incompatible coefficients. The target of the cup product in the new case is the ordinary cyclic cohomology of algebras with coefficients in a module produced out of the two incompatible coefficients.

Let $\mathcal{M}$ be a SAYD contramodule and let N be a SAYD module over a Hopf algebra H. We define $L(N, \mathcal{M})$ to be the coequalizer

$$(4.1) \qquad N \otimes_H \mathrm{Hom}(H, \mathcal{M}) \rightrightarrows N \otimes_H \mathcal{M} \longrightarrow L(N, \mathcal{M}),$$

where the equalized maps are $n \otimes m \mapsto n \otimes \alpha(f)$, and $n \otimes m \mapsto n_{<0>} \otimes f(n_{<-1>})$.

Remark 4.1. In fact $L(N, \mathcal{M})$ is the usual (contra)tensor product defined by Positselski [**18**, page 96]. One considers H-coring $\mathcal{C} := H \otimes H$ with the usual coring structure and identifies N with a left $H \otimes H$-comodule [**2**]. In the same fashion one identifies $\mathcal{M}$ with a right $\mathcal{C}$-contramodule. Then $L(N, \mathcal{M})$ is identified with the (contra)tensor $N \otimes_{\mathcal{C}} \mathcal{M}$.

Now let A and C satisfy (3.1)...(3.4). We recall that the algebra B is $\mathrm{Hom}_H(C, A)$ with the convolution multiplication and that $C^{*,*}_{\mathrm{a-c}}$ is the bicyclic module defined in (3.9). We define

$$(4.2) \quad \tilde{\Psi} : D(C^{*,*}_{\mathrm{a-c}})^q \to \mathrm{Hom}(B^{\otimes q+1}, L(N, \mathcal{M})),$$

$$\tilde{\Psi}(\phi \otimes (n \otimes c^0 \otimes \cdots \otimes c^q))(f^0 \otimes \cdots \otimes f^q) = n \otimes_H \phi(f^0(c^0) \otimes \cdots \otimes f^q(c^q)).$$

By a similar argument as in the proof of Proposition 3.2 one shows that $\tilde{\Psi}$ is a cyclic map. One proves the following theorem with a similar proof as of Theorem 3.3

Theorem 4.2. *Let H be a Hopf algebra, A be an H-module algebra, C be an H-module coalgebra acting on A, and $(N, \mathcal{M})$ be a not necessarily compatible pair of SAYD module and contramodule over H. Then $\iota \circ \tilde{\Psi} \circ \overline{AW}$ defines a cup product in the level of cyclic cohomology:*

$$\iota \circ \tilde{\Psi} \circ \overline{AW} : HC^p_H(A, \mathcal{M}) \otimes HC^q_H(C, N) \to HC^{p+q}(A, L(N, \mathcal{M})).$$

One notes that the range of this cup product is the ordinary cyclic cohomology of the algebra A with coefficients in the vector space $L(N, \mathcal{M})$. One also notes that if $(N, \mathcal{M})$ is compatible then $E : L(N, \mathcal{M}) \to \mathbb{C}$ defined by $E(n, m) = < n, m >$ is a map of vector spaces. As a result we get a cyclic map

$$\tilde{E} : \mathrm{Hom}(A^{\otimes *}, L(N, \mathcal{M})) \to \mathrm{Hom}(A^{\otimes *}, \mathbb{C}), \quad \tilde{E}(\varphi) = E \circ \varphi.$$

So we cover the old cup product as $\Psi = \tilde{\Psi} \circ \tilde{E}$, where Ψ is defined in (3.10).
Now let us generalize the other cup product for algebra-algebra in a similar fashion as the case of algebra-coalgebra. Let A be a left H-module algebra, B be a left H-comodule algebra, $(N, \mathcal{M})$ be a pair of SAYD module and contramodule over H, and $A \bowtie B$ be the crossed product algebra defined in 3.16. Let also $C^{*,*}_{\mathrm{a-a}}$ be the bicyclic module defined in (3.18) . We define

$$(4.3)$$
$$\tilde{\Phi} : D(C^{*,*}_{\mathrm{a-a}})^q \to \mathrm{Hom}((A \bowtie B)^{\otimes q+1}, L(N, \mathcal{M})),$$
$$\tilde{\Phi}(\psi \otimes \phi)(a^0 \bowtie b^0 \otimes \cdots \otimes a^q \bowtie b^q)$$
$$= \psi(b^0{}_{<0>} \otimes \cdots \otimes b^q{}_{<0>}) \otimes_H \phi(S^{-1}(b^0{}_{<-1>} \cdots b^q{}_{<-1>})a^0 \otimes \cdots \otimes S^{-1}(b^q{}_{<-q-1>})a^q).$$

Similarly we prove that $\tilde{\Phi}$ is cyclic and induces a map on the level of cyclic cohomologies:

Theorem 4.3. *Let H be a Hopf algebra, A be an H-module algebra, B be an H-comodule algebra, $(N, \mathcal{M})$ be a compatible pair of SAYD module and contramodule. Then the map $\tilde{\Phi} \circ \overline{AW}$ defines a cup product in the level of cyclic cohomology:*

$$\tilde{\Phi} \circ \overline{AW} : {}^H HC^p(B, N) \otimes HC^q_H(A, \mathcal{M}) \to HC^{p+q}(A \bowtie B, L(N, \mathcal{M})).$$

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