

HEAT KERNEL ESTIMATES FOR THE $\bar{\partial}$ -NEUMANN PROBLEM ON G -MANIFOLDS

JOE J PEREZ AND PETER STOLLMANN

ABSTRACT. We prove heat kernel estimates for the $\bar{\partial}$ -Neumann Laplacian \square acting in spaces of differential forms over noncompact manifolds with a Lie group symmetry and compact quotient. We also relate our results to those for an associated Laplace-Beltrami operator on functions.

1. INTRODUCTION

We are concerned with bounds on the heat kernel of the $\bar{\partial}$ -Neumann Laplacian on manifolds with boundary possessing a Lie group symmetry. Heat kernel bounds are an object of intensive study and an attempt to describe only the most important works would go well beyond the scope of the present article. Instead we refer to [25] and point out the peculiarities of the model we are dealing with before properly introducing the setup. The operator we deal with is the natural Laplacian coming from the PDE of several complex variables. It acts on complex-valued differential forms on a manifold with boundary and has non-coercive boundary conditions. Despite these differences from the usual situation, some techniques from the theory of Dirichlet forms remain applicable to obtain the bounds which frequently are the goal in studies of heat estimates in very different settings. Due to the complications in our model, it comes as a nice surprise that these usual tools, *e.g.* the intrinsic metric, come in so handy. Apart from the results that will soon be mentioned, this surprise is certainly a message we want to pass along. Since we would like to communicate our results to people in at least two communities, we will take our time to explain certain basics that might be obvious to some readers. We ask those to please bear with us.

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Let M be a complex manifold, $n = \dim_{\mathbb{C}} M$, and assume that M has a smooth boundary bM such that $\bar{M} = M \cup bM$. Assume further that \bar{M} is contained in a slightly larger complex manifold \widetilde{M} of the same dimension. The space of holomorphic functions on M under various complex-geometric conditions on $bM \subset \widetilde{M}$ has been investigated from various standpoints, beginning with Hartogs and Levi [30, 39, 40] and, with Stein theory and sheaf-theoretic methods, culminating in the Oka-Grauert theorem, [23].

An approach to problems in several complex variables using partial differential equations was also developed by Morrey, Spencer, Andreotti-Vesentini, Kohn, Nirenberg, Hörmander, and others ([17, 53, 56]) bearing fruit in Kohn's solution to the $\bar{\partial}$ -Neumann problem, [36, 37]. This method heavily involves the analysis of a self-adjoint Laplace operator \square on differential forms in $\Lambda^{p,q}$, the subject of this article, which we describe here.

For any integers p, q with $1 \leq p, q \leq n$ denote by $C^\infty(M, \Lambda^{p,q})$ the space of all C^∞ forms of type (p, q) on M . These are the differential forms which can be written in local complex coordinates (z_1, z_2, \dots, z_n) as

$$(1) \quad \phi = \sum_{|I|=p, |J|=q} \phi_{I,J} dz^I \wedge d\bar{z}^J$$

where $dz^I = dz^{i_1} \wedge \dots \wedge dz^{i_p}$, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$, $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, $i_1 < \dots < i_p$, $j_1 < \dots < j_q$, and the $\phi_{I,J}$ are smooth functions in local coordinates. For such a form ϕ , the value of the antiholomorphic exterior derivative $\bar{\partial}\phi$ is

$$\bar{\partial}\phi = \sum_{|I|=p, |J|=q} \sum_{k=1}^n \frac{\partial \phi_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J$$

so $\bar{\partial} = \bar{\partial}|_{p,q}$ defines a linear map $\bar{\partial} : C^\infty(M, \Lambda^{p,q}) \rightarrow C^\infty(M, \Lambda^{p,q+1})$.

With respect to a smooth measure on M and a smoothly varying Hermitian structure in the fibers of the tangent bundle, define the spaces $L^2(M, \Lambda^{p,q})$. Let us extend the above $\bar{\partial}$ to the corresponding maximal operator in L^2 (and still call it $\bar{\partial}$) and let $\bar{\partial}^*$ be its adjoint operator (the forms in the domain of $\bar{\partial}^*$ will have to satisfy certain boundary conditions). Then

$$(2) \quad \text{dom}(Q^{p,q}) := \text{dom}(\bar{\partial}) \cap \text{dom}(\bar{\partial}^*)$$

$$(3) \quad Q^{p,q}(\phi, \psi) := \langle \bar{\partial}\phi, \bar{\partial}\psi \rangle_{L^2(M, \Lambda^{p,q+1})} + \langle \bar{\partial}^*\phi, \bar{\partial}^*\psi \rangle_{L^2(M, \Lambda^{p,q-1})},$$

defines a closed form $Q^{p,q}$ on $L^2(M, \Lambda^{p,q})$; we will frequently omit the superscripts indicating the type of forms and simply write Q and $\text{dom}(Q)$

instead. By standard theory (see details in Section 2.3 below) there is a unique selfadjoint operator $\square = \square_{p,q}$ corresponding to $Q = Q^{p,q}$ that we can write as

$$\square = \square_{p,q} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*.$$

This Laplacian \square is elliptic but its natural boundary conditions are not coercive, thus, in the interior of M , the operator gains two degrees in the Sobolev scale, as a second-order operator, while in neighborhoods of the boundary it gains less. The gain at the boundary depends on the geometry of the boundary, and the best such situation is that in which the boundary is *strongly pseudoconvex*, a condition already seen to be important in [30, 39, 40]; see [53]. In that case, the operator gains one degree on the Sobolev scale and so global estimates including both interior and boundary neighborhoods gain only one degree. More precisely, one obtains *a priori* (called Kohn-type) estimates of the form

$$\|u\|_{H^{s+1}(M, \Lambda^{p,q})} \lesssim \|\square u\|_{H^s(M, \Lambda^{p,q})} + \|u\|_{L^2(M, \Lambda^{p,q})}, \quad (u \in \text{dom } \square \cap C^\infty)$$

when the boundary is strongly pseudoconvex and $q > 0$ ([46], Lemma 7.11). Such estimates are usually called *subelliptic* as the gain of the operator is less than its order.

Assuming for the moment that \bar{M} is compact, under various well-investigated conditions on bM , ([7, 6], *etc.*) the Laplacian satisfies a *pseudolocal estimate with gain $\epsilon > 0$ in $L^2(M, \Lambda^{p,q})$* . That is, if $U \subset \bar{M}$ is a neighborhood with compact closure, $\zeta, \zeta' \in C_c^\infty(U)$ for which $\zeta'|_{\text{supp}(\zeta)} = 1$, and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\square + \mathbf{1})^{-1}\alpha \in H^{s+\epsilon}(\bar{M}, \Lambda^{p,q})$ and there exists a constant $C_{s,\zeta,\zeta'} > 0$ such that

$$(4) \quad \|\zeta(\square + \mathbf{1})^{-1}\alpha\|_{H^{s+\epsilon}(M, \Lambda^{p,q})} \leq C_{s,\zeta,\zeta'} (\|\zeta'\alpha\|_{H^s(M, \Lambda^{p,q})} + \|\alpha\|_{L^2(M, \Lambda^{p,q})})$$

uniformly for all α satisfying the assumption. Since \bar{M} is assumed compact, Rellich's theorem provides that $(\square + \mathbf{1})^{-1}$ is a compact operator and thus there exists an orthonormal basis of $L^2(M, \Lambda^{p,q})$ consisting of eigenforms of \square , [17, Prop. 3.1.11]. With the eigenvalues and eigenforms of \square , one can construct the heat operator and study it. In our case of noncompact M we will take a different approach. Still, to us, the most important result from the PDE of several complex variables is that a pseudolocal estimate (4) holds even without assuming the compactness of M , as shown in [14].

Main results: We will assume throughout this article that we have a complex manifold M which is the total space of a principal bundle on which a Lie group G acts by holomorphic transformations with compact orbit space $\bar{X} = \bar{M}/G$:

$$G \longrightarrow M \longrightarrow X.$$

Throughout this article, we will assume for simplicity that our manifolds are strongly pseudoconvex. That implies that a pseudolocal estimate with gain $\epsilon = 1$ holds in $L^2(M, \Lambda^{p,q})$ for all $q > 0$. All the bundles constructed in [31] and are treated in [29] are strongly pseudoconvex. In our results, one can revert to the more general setting, in which $0 < \epsilon < 1$, making inessential changes.

The first of our principal results is a Nash-type inequality, *cf.* [45]:

Theorem 1. (Nash inequality) *Let M be a strongly pseudoconvex G -manifold on which G acts freely by holomorphic transformations with compact quotient \bar{M}/G . For integer $s > \dim_{\mathbb{C}} M$*

$$\|u\|_{L^2(M, \Lambda^{p,q})}^{2+\frac{1}{s}} \lesssim Q(u) \|u\|_{L^1(M, \Lambda^{p,q})}^{\frac{1}{s}}, \quad (u \in \text{dom}(Q^{p,q}) \cap L^1(M, \Lambda^{p,q})).$$

Defining the heat semigroup by $P_t = e^{-t\Box}$, we obtain operator norm estimates in L^p spaces as well as Sobolev spaces:

$$\|P_t\|_{L^2 \rightarrow L^\infty}, \quad \|P_t\|_{L^1 \rightarrow L^\infty}, \quad \|P_t\|_{H^r \rightarrow H^s},$$

valid for $t > 0$, $r, s \in \mathbb{R}$. This last property can be used to obtain that the Schwartz kernel of the heat operator is smooth for $t > 0$.

We also obtain an off-diagonal estimate for the heat semigroup in terms of the intrinsic metric d_\Box induced by $\bar{\partial} : C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \Lambda^{0,1})$ and a G -invariant Hermitian structure on $\Lambda^{0,1}$. It turns out that d_\Box is equivalent to the intrinsic metric d_{LB} induced by the Laplace-Beltrami operator of a Riemannian metric simply related to the metric on $\Lambda^{0,1}$.

The off-diagonal estimate is

Theorem 2. (Off-diagonal heat kernel estimate) *Let M be as above. For measurable subsets A, B of M it follows that the heat semigroup satisfies*

$$\|\mathbf{1}_B P_t \mathbf{1}_A\|_{2 \rightarrow 2} \leq \exp \left[-\frac{d_\Box(A; B)^2}{4t} \right].$$

Note the following peculiarity: as already pointed out, the $\bar{\partial}$ -Neumann problem is not elliptic in the sense that inverse of \Box does not gain two degrees in the Sobolev scale. This is due to the boundary conditions, which, even in the strictly pseudoconvex case, give a gain of only one order of differentiability. Our method of proof does not make use of the better estimates that are valid in the interior, where the gain is two as in [17, Thm. 2.2.9]. The resulting Sobolev estimates make our Nash inequality somewhat weaker than what would be true for an elliptic operator with coercive boundary conditions.

On the other hand, the off-diagonal bound is not affected at all by this. The intrinsic metric gives just the kind of decay that one would expect for an elliptic problem.

Part of what is happening here is that the pseudolocal estimate that we use is given in terms of isotropic Sobolev norms while the problem is inherently anisotropic. In the compact case, anisotropic estimates have been worked out [18, 24] and it happens that the Laplacian gains two orders of differentiability in all directions except one “bad” direction in the boundary.

The pseudolocal and Kohn-type estimates that we use here were developed in the noncompact case in [14, 46] and applied in [47] and [11] to construct L^2 holomorphic functions in some cases, in a manner analogous to that of Kohn and Gromov, Henkin, Shubin, [36, 37, 17, 29]. This last reference contains other examples (regular covering spaces of compact, strongly pseudoconvex complex manifolds and two nonunimodular G -manifolds) to which our methods here apply.

The spectral theory of the $\bar{\partial}$ -Neumann problem has been previously investigated in [43, 19, 20] in the compact situation and in [54, 57, 1, 2, 3], methods involving pseudodifferential operators are brought to bear on the problem, still in the compact case. In [12], heat kernel asymptotics are developed for subelliptic operators on noncompact groups. In [42], an asymptotic expansion is developed for the heat kernel of a general elliptic operator with noncoercive boundary conditions.

The contents of the rest of this article are as follows. In Section 2 we will describe the basic constructions on M and review the principal properties of the $\bar{\partial}$ -Neumann problem relevant to our investigation. Also, we will draw the more directly accessible conclusions of these properties. In Section 3 we describe the intrinsic geometry carried by M and derive the heat estimates for the $\bar{\partial}$ -Neumann Laplacian.

2. THE $\bar{\partial}$ -NEUMANN PROBLEM

2.1. The geometry. We will introduce some complex-geometric concepts in this section, basically following [17]; see also [34, 35]. On a real, $2n$ -dimensional C^∞ manifold M , an *almost complex structure on M* is a splitting of the complexification $TM \otimes_{\mathbb{R}} \mathbb{C}$ of the real tangent bundle TM ,

$$TM \otimes_{\mathbb{R}} \mathbb{C} = T_{1,0}M \oplus T_{0,1}M,$$

with the following property; denoting the projections onto $T_{1,0}M$ and $T_{0,1}M$ by $\Pi_{1,0}$ and $\Pi_{0,1}$, respectively:

$$(5) \quad \Pi_{0,1}\zeta = \overline{\Pi_{1,0}\bar{\zeta}},$$

where $\bar{\cdot}$ denotes complex conjugation.

We can also describe an almost complex structure by a fibrewise linear mapping $J : TM \rightarrow TM$ with $J^2 = -\mathbf{1}$. These two descriptions are related via:

$$(6) \quad T_{1,0}M = \{X - iJX \mid X \in TM\} = \ker(J - i)$$

and

$$(7) \quad T_{0,1}M = \{X + iJX \mid X \in TM\} = \ker(J + i),$$

see [34, Chapter I, §7]. For a vector field $X \in TM$, a complex vector field in $TM \otimes_{\mathbb{R}} \mathbb{C}$ of the form $X - iJX \in T_{1,0}$ is called a holomorphic vector field while one of the form $X + iJX \in T_{0,1}$ is called antiholomorphic.

Dually, the projections $\Pi_{0,1}, \Pi_{1,0}$ induce a splitting of the exterior powers of the complexified cotangent bundle, $\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C}$ into holomorphic and antiholomorphic parts so that $\Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q}$. The exterior derivative in $\Lambda^k T^*M$ can be combined with the splittings of the complexified cotangent bundle of M to obtain holomorphic and antiholomorphic exterior derivatives ∂ and $\bar{\partial}$, respectively. The relations among these operators are given by

$$\bar{\partial} : C^\infty(\bar{M}, \Lambda^{p,q}) \rightarrow C^\infty(\bar{M}, \Lambda^{p,q+1}), \quad \bar{\partial}\phi = \Pi_{p,q+1}d\phi$$

and

$$\partial : C^\infty(\bar{M}, \Lambda^{p,q}) \rightarrow C^\infty(\bar{M}, \Lambda^{p+1,q}), \quad \partial\phi = \Pi_{p+1,q}d\phi$$

for $\phi \in C^\infty(\bar{M}, \Lambda^{p,q})$.

On a complex manifold, it is true that $d = \partial + \bar{\partial}$, see [17, Prop. 1.2.1] and that $\bar{\partial}^2 = 0$, which gives rise to the $\bar{\partial}$ -complex,

$$0 \rightarrow C^\infty(\bar{M}, \Lambda^{p,0}) \xrightarrow{\bar{\partial}} C^\infty(\bar{M}, \Lambda^{p,1}) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} C^\infty(\bar{M}, \Lambda^{p,n}) \rightarrow 0$$

which is the starting point for various cohomology theories due to Dolbeault, Hodge-Kodaira, and unified by Spencer, *cf.* [38]. See also [46] for some results related to our current setting.

2.2. Sobolev spaces. We will have to describe smoothness of functions, forms, and sections of vector bundles using G -invariant Sobolev spaces which we describe here.

If E is a vector G -bundle over M , then we may introduce a G -invariant pointwise inner product structure $\langle \cdot, \cdot \rangle_E$ on E . Together with a G -invariant measure on M , we define the Hilbert space of sections of E which we denote $L^2(M, E)$. Note that, in particular, spaces of sections in natural tensor bundles on a G -manifold have natural, invariant

Hermitian structures associated to a Riemannian structure on the underlying manifold; below we will provide more detail. We denote by $C^\infty(M, \Lambda^{p,q})$ the space of smooth (p, q) -forms on M , by $C^\infty(\bar{M}, \Lambda^{p,q})$ the subspace of those forms that can be smoothly extended to \bar{M} and by $C_c^\infty(\bar{M}, \Lambda^{p,q})$ the subspace of the latter consisting of those smooth forms with compact support. Given any G -invariant, pointwise Hermitian structure

$$C^\infty(\bar{M}, \Lambda^{p,q}) \ni u, v \longmapsto \langle u(x), v(x) \rangle_{\Lambda_x^{p,q}} \in \mathbb{C}, \quad (x \in \bar{M}),$$

we define the L^p -spaces $L^p(M, \Lambda^{q,r})$ of forms as the completions of $C_c^\infty(\bar{M}, \Lambda^{q,r})$ in the norms

$$\|u\|_{L^p(M, \Lambda^{q,r})} = \left[\int_M \langle u, u \rangle_{\Lambda^{q,r}}^{p/2} \right]^{1/p}.$$

As in [26, 51] we may construct appropriate partitions of unity and, by differentiating componentwise with respect to local geodesic coordinates, assemble G -invariant integer Sobolev spaces $H^s(M, \Lambda^{p,q})$, for $s = 0, 1, 2, \dots$. Because $X = \bar{M}/G$ is compact, the spaces $H^s(M, \Lambda^{p,q})$ do not depend on the choices of an invariant metric on M or of an invariant inner product on $\Lambda^{p,q}$. The usual duality relations for L^p spaces hold (polarizing the above norm) as well as the Sobolev lemma, *etc.* Background on this is provided in [22]. There, the Hermitian structure is defined in terms of the Hodge operator so our $\langle u, v \rangle_\Lambda$ translates to $u \wedge \star \bar{v}$. See p82 of [17] and Lemma 3.10 below.

2.3. Operators and forms. As we said in the introduction, \square will be defined in terms of an associated quadratic form. Good references for background on the general concept of closed forms and their associated operators are [15, 32, 49], among others. Here we will give more details concerning the case at hand and also describe certain subsets of smooth forms that belong to the respective form and operator domains. We begin by collecting some information concerning the building blocks of \square , $\bar{\partial}$ and $\bar{\partial}^*$.

Remark 2.1. Let M be as above.

- (1) The maximal operator $\bar{\partial}$ in $L^2(M, \Lambda^{p,q})$ is given by: $\alpha \in \text{dom}(\bar{\partial})$ whenever $\bar{\partial}\alpha \in L^2(M, \Lambda^{p,q+1})$ in the distributional sense. It acts from $L^2(M, \Lambda^{p,q})$ to $L^2(M, \Lambda^{p,q+1})$ and is a closed operator.
- (2) The operator $\bar{\partial}^*$ in $L^2(M, \Lambda^{p,q})$ is the adjoint of $\bar{\partial}$ (in $L^2(M, \Lambda^{p,q-1})$); it is given by: $\alpha \in \text{dom}(\bar{\partial}^*)$ whenever there exists $\beta \in L^2(M, \Lambda^{p,q-1})$ so that

$$\langle \bar{\partial}^* \alpha, \gamma \rangle_{L^2(M, \Lambda^{p,q})} = \langle \gamma, \beta \rangle_{L^2(M, \Lambda^{p,q-1})}$$

for all $\gamma \in L^2(M, \Lambda^{p,q-1})$ and $\bar{\partial}^* \alpha = \beta$.

(3) Since $\bar{\partial}$ is closed, the form

$$\text{dom}(\bar{\partial}) \times \text{dom}(\bar{\partial}) \ni (\alpha, \beta) \mapsto \langle \bar{\partial}\alpha, \bar{\partial}\beta \rangle_{L^2(M, \Lambda^{p,q+1})}$$

is a closed form in $L^2(M, \Lambda^{p,q})$; cf [15, 32].

(4) Since $\bar{\partial}^*$ is closed, the form

$$\text{dom}(\bar{\partial}^*) \times \text{dom}(\bar{\partial}^*) \ni (\alpha, \beta) \mapsto \langle \bar{\partial}^*\alpha, \bar{\partial}^*\beta \rangle_{L^2(M, \Lambda^{p,q-1})}$$

is a closed form in $L^2(M, \Lambda^{p,q})$, provided, $q \geq 1$.

(5) $Q = Q^{p,q}$ is the sum of the closed forms defined in (3), (4) above and therefore a closed form as well for $q \geq 1$. $Q^{p,0}$ is the form defined in (3).

Recall that a closed operator is one whose graph is closed, while a form Q is *closed* whenever its domain $\text{dom}(Q)$ is a Hilbert space with respect to the *form inner product*

$$(\cdot | \cdot)_Q := Q(\cdot, \cdot) + \langle \cdot, \cdot \rangle.$$

The stage is now set for the first form representation theorem, cf. [32], that asserts that for every semibounded closed form there is a unique selfadjoint operator associated with the form. In our case, there is a unique selfadjoint operator $\square_{p,q}$ associated with $Q^{p,q}$, meaning that

$$\text{dom}(\square_{p,q}) \subset \text{dom}(Q^{p,q}) \text{ and } Q(\alpha, \beta) = \langle \square\alpha, \beta \rangle,$$

whenever $\alpha \in \text{dom}(\square_{p,q})$ and $\beta \in \text{dom}(Q^{p,q})$. In fact, more is known:

$$\text{dom}(\square_{p,q}) = \{\alpha \mid \exists \gamma \in L^2(M, \Lambda^{p,q}) \forall \beta \in \text{dom}(Q^{p,q}) : Q^{p,q}(\alpha, \beta) = \langle \gamma, \beta \rangle\}$$

and, obviously, $\gamma = \square_{p,q}\alpha$ is uniquely determined. Moreover, defining the square root $\square_{p,q}^{\frac{1}{2}}$ by the functional calculus, we have that

$$\text{dom}(Q^{p,q}) = \text{dom}(\square_{p,q}^{\frac{1}{2}}) \text{ and } Q(\alpha, \beta) = \langle \square^{\frac{1}{2}}\alpha, \square^{\frac{1}{2}}\beta \rangle.$$

We note that \square can be seen as the *form sum* of the operators $\bar{\partial}^*\bar{\partial}$ and $\bar{\partial}\bar{\partial}^*$. In fact, the former operator is the selfadjoint operator associated with the form in part (3) of the preceding remark and the latter is the selfadjoint operator associated with the form in part (4) of the preceding remark. In that sense, the formula

$$\square = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$$

has now a precise meaning, interpreting the + sign as the form sum, cf. [15, 32].

In principle, all domain questions are settled now and we have defined the forms and operators we will be dealing with. However, the results

above give a rather implicit description so it is quite useful to have explicit subspaces of the operator and form domains given above.

We speak of a *core* of a form meaning a subspace of its domain that is dense in the domain with respect to the form norm. Similarly, a core of an operator is a subspace of its domain that is dense with respect to the graph norm.

The following lemma is from [29, Lemma 1.1] and [17, Lemma 2.3.2]; serves the purpose to get our hands on the smooth elements of certain form and operator domains.

Lemma 2.2. *Let M be as above, let ϑ be the formal adjoint operator to $\bar{\partial}$, and denote by $\sigma = \sigma(\vartheta, \cdot)$ its principal symbol.*

- (i) $\{u \in C_c^\infty(\bar{M}, \Lambda^\bullet) \mid \sigma(\vartheta, d\rho)u|_{bM} = 0\}$ is a core for $\bar{\partial}^*$ and on this space $\bar{\partial}^*$ agrees with ϑ .
- (ii) $\mathcal{D}^{p,q} := \{u \in C^\infty(\bar{M}, \Lambda^\bullet) \mid \sigma(\vartheta, d\rho)u|_{bM} = 0\}$ is a core for $Q^{p,q}$.
- (iii) The domains of $\bar{\partial}^*$ and Q are preserved by multiplication by cutoff functions.

2.4. Estimates for the Laplacian. In this section we give our requirements on the boundary geometry and state the pseudolocal estimate in more precise language than in the introduction. As before, assume M to be a complex manifold with nonempty smooth boundary bM , $\bar{M} = M \cup bM$, so that M is the interior of \bar{M} , and $\dim_{\mathbb{C}}(M) = n$. We will also assume for simplicity that \bar{M} is a closed subset in \widetilde{M} , a complex neighborhood of \bar{M} so that the complex structure on \widetilde{M} extends that of M , and every point of \bar{M} is an interior point of \widetilde{M} . Let us choose a smooth function $\rho : \widetilde{M} \rightarrow \mathbb{R}$ so that

$$M = \{z \mid \rho(z) < 0\}, \quad bM = \{z \mid \rho(z) = 0\},$$

and for all $x \in bM$, we have $d\rho(x) \neq 0$. In local coordinates near any $x \in bM$ define the *holomorphic tangent plane* to the boundary at x by

$$T_x^{\mathbb{C}}(bM) = \left\{ w \in \mathbb{C}^n \mid \sum_{k=1}^n \frac{\partial \rho}{\partial z^k} \Big|_x w^k = 0 \right\}$$

and define the Levi form L_x by

$$L_x(w, \bar{w}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z^j \partial \bar{z}^k} \Big|_x w^j \bar{w}^k, \quad (w \in T_x^{\mathbb{C}}(bM)).$$

Then M is said to be *strongly pseudoconvex* if for every $x \in bM$, the form L_x is positive definite.

The following theorem will be our principal tool from the PDE of several complex variables.

Theorem 2.3. (Pseudolocal estimate) *Let M be strongly pseudoconvex, U an open subset of \bar{M} with compact closure, and $\zeta, \zeta' \in C_c^\infty(U)$ for which $\zeta'|_{\text{supp}(\zeta)} = 1$. If $q > 0$ and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\square + 1)^{-1}\alpha \in H^{s+1}(\bar{M}, \Lambda^{p,q})$ and there exist constants $C_s > 0$ so that*

$$(8) \quad \|\zeta(\square + 1)^{-1}\alpha\|_{H^{s+1}(M, \Lambda^{p,q})} \leq C_s(\|\zeta'\alpha\|_{H^s(M, \Lambda^{p,q})} + \|\alpha\|_{L^2(M, \Lambda^{p,q})}).$$

Proof. This is Prop. 3.1.1 from [17] extended to the noncompact case in [14]. \blacklozenge

Remark 2.4. Boundary geometries giving more general subelliptic estimates are harder to define, so we refer the interested reader to [7, 6] instead of pursuing this issue here. For completeness, we mention that the theorem holds when M satisfies these weaker estimates, *mutatis mutandis* [14].

A word on notation: For two norms $\|\cdot\|$ and $|\cdot|$, we write $|\phi| \lesssim \|\phi\|$ to mean that there exists a constant $C > 0$ such that $|\phi| \leq C\|\phi\|$ for ϕ in whatever set relevant to the context.

Corollary 2.5. *For $s \in \mathbb{N}$, $q > 0$, and $\zeta \in C_c^\infty(\bar{M})$,*

$$(9) \quad \|\zeta(\square + 1)^{-s}\alpha\|_{H^s(M, \Lambda^{p,q})} \lesssim \|\alpha\|_{L^2(M, \Lambda^{p,q})}, \quad (\alpha \in L^2(M, \Lambda^{p,q})).$$

Proof. By induction. Putting $s = 0$ in the theorem, we have

$$\|\zeta(\square + 1)^{-1}\alpha\|_{H^1} \lesssim \|\zeta'\alpha\|_{L^2} + \|\alpha\|_{L^2} \lesssim \|\alpha\|_{L^2}, \quad (\alpha \in L^2(M)).$$

Assuming the result for $s - 1$, it follows that $(\square + 1)^{1-s}\alpha \in H_{\text{loc}}^{s-1}(M)$ for all $\alpha \in L^2(M)$. Applying the theorem to this form, we have

$$\|\zeta(\square + 1)^{-1}(\square + 1)^{1-s}\alpha\|_{H^s} \lesssim \|\zeta'(\square + 1)^{1-s}\alpha\|_{H^{s-1}} + \|(\square + 1)^{1-s}\alpha\|_{L^2},$$

and

$$\|\zeta(\square + 1)^{-s}\alpha\|_{H^s} \lesssim \|(\square + 1)^{1-s}\alpha\|_{L^2} \lesssim \|\alpha\|_{L^2}. \quad \blacklozenge$$

Corollary 2.6. *Let M be a strongly pseudoconvex G -manifold on which G acts freely by holomorphic transformations with compact quotient \bar{M}/G . For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$ we have the estimate*

$$(10) \quad \|(\square + 1)^{-s}\alpha\|_{L^\infty(M, \Lambda^{p,q})} \lesssim \|\alpha\|_{L^2(M, \Lambda^{p,q})}, \quad (\alpha \in L^2(M, \Lambda^{p,q})).$$

Proof. Choose $B \subset \bar{M}$ compact and sufficiently large so that $B \cdot G$ covers \bar{M} . This is possible since \bar{X} is compact. Choose $\zeta \in C_c^\infty(\bar{M})$ such that $\text{supp } \zeta \supset B$ in (9). Now, the Sobolev lemma provides that if $s > k + m/2$, then $H^s(\mathbb{R}^m) \subset C^k(\mathbb{R}^m)$ and there is a constant $C = C_{s,k}$ such that

$$(11) \quad \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^m} |\partial^\alpha u(x)| \leq C \|u\|_{H^s(\mathbb{R}^m)},$$

thus, if we take $s > k + 1/2 \dim_{\mathbb{R}} M = k + \dim_{\mathbb{C}} M$, we have

$$\|(\square + \mathbf{1})^{-s}\alpha\|_{C^k(\bar{M})} \lesssim \|\zeta(\square + \mathbf{1})^{-s}\alpha\|_{H^s} \lesssim \|\alpha\|_{L^2}, \quad (\alpha \in L^2(M))$$

by the G -invariance of M and our choice of local geodesic coordinates. \blacklozenge

Remark 2.7. Note that the exact invariances furnished by the group action assumed here are not essential and can be relaxed to assumptions on the uniformity of the estimates in (4), *etc.*

3. HEAT KERNEL ESTIMATES AND INTRINSIC GEOMETRY

Definition 3.1. Let $\square = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of the Laplacian and for $t > 0$ put

$$P_t = \int_0^\infty e^{-t\lambda} dE_\lambda.$$

That is, $P_t = e^{-t\square}$, and we would write $P_t^{p,q} = e^{-t\square_{p,q}}$ to be completely explicit.

Remark 3.2. The semigroup $(e^{-tH}; t \geq 0)$ of a selfadjoint operator H contains a wealth of information about its generator H and satisfies the semigroup property $e^{-(t+s)H} = e^{-tH}e^{-sH}$, see [8, 21] for the general theory and [52] for the case of Schrödinger operators. In the case at hand, where $H \geq 0$, the semigroup consists of contractions, *i.e.*, $\|e^{-tH}\|_{2 \rightarrow 2} \leq 1$. The symbol $\|\cdot\|_{2 \rightarrow 2}$ denotes the operator norm of an operator from L^2 to L^2 . Similar to what is known for the Laplacian, the semigroup of the $\bar{\partial}$ -Neumann Laplacian \square is *ultracontractive*. That is, it maps L^2 into L^∞ continuously. This is equivalent to the validity of a Nash-type inequality and will be discussed below.

3.1. Ultracontractivity and Nash inequalities. The heat operator's ultracontractivity (*i.e.* boundedness from $L^2 \rightarrow L^\infty$) follows immediately from the Sobolev estimate in Cor. 2.6 above. The proof is formally very similar to that from Davies [9]. The difference between the two cases is that our basic spaces consist of vector-valued functions and so certain concepts and manipulations are not available. For example, we cannot identify nonnegative elements or take the absolute value in a naive way.

Proposition 3.3. *Let M be a strongly pseudoconvex G -manifold on which G acts freely by holomorphic transformations with compact quotient \bar{M}/G . For integer $s > \dim_{\mathbb{C}} M$ and $q > 0$, we have*

$$(12) \quad \|P_t\alpha\|_{L^\infty(M, \Lambda^{p,q})} \lesssim \max(1, t^{-s})\|\alpha\|_{L^2(M, \Lambda^{p,q})}, \quad (\alpha \in L^2(M, \Lambda^{p,q})).$$

Proof. We plug $(\square + \mathbf{1})^s P_t \alpha$ into the inequality (10) from Cor. 2.6 and obtain:

$$\begin{aligned} \|P_t \alpha\|_{L^\infty} &= \|(\square + \mathbf{1})^{-s} (\square + \mathbf{1})^s P_t \alpha\|_{L^\infty} \\ &\lesssim \|(\square + \mathbf{1})^s P_t \alpha\|_{L^2} \\ &\lesssim t^{-s} \|\alpha\|_{L^2} \end{aligned}$$

for any $0 < t \leq 1$, by functional calculus, since the maximum of the function $\lambda \mapsto (\lambda + 1)^s e^{-\lambda t}$ goes like t^{-s} for $t > 0$. This gives the result for arbitrary $t \geq 0$, as the semigroup is a contraction on L^2 . \blacklozenge

We mention here that the usual duality properties of the L^p spaces hold in our setting, [22].

Corollary 3.4. *Let M be as in the previous proposition. Then, for integer $s > \dim_{\mathbb{C}} M$ and $q > 0$ we have*

$$(13) \quad \|P_t \alpha\|_{L^\infty(M, \Lambda^{p,q})} \lesssim \max(1, t^{-2s}) \|\alpha\|_{L^1(M, \Lambda^{p,q})},$$

uniformly for $\alpha \in L^1 \cap L^2(M, \Lambda^{p,q})$.

Proof. Since P_t is symmetric, $\|P_t\|_{2 \rightarrow \infty} = \|P_t\|_{1 \rightarrow 2}$ by duality, and

$$\|P_t\|_{2 \rightarrow \infty} \lesssim \max(1, t^{-s}),$$

from the previous statement, we have

$$\|P_t\|_{1 \rightarrow \infty} \leq \|P_t\|_{2 \rightarrow \infty} \|P_t\|_{1 \rightarrow 2} \leq \|P_{t/2}\|_{2 \rightarrow \infty}^2 \lesssim \frac{1}{t^{2s}}$$

by the semigroup property. \blacklozenge

Remark 3.5. The basic tool in the estimates to come is the fundamental theorem of calculus applied to the function $t \mapsto \|P_t u\|_{L^2}^2$ or variants thereof. This rests on the following immediate consequence of functional calculus: For any $u \in \text{dom } \square$:

$$P_t u \in \text{dom } \square \text{ and } \frac{d}{dt} [P_t u] = -\square P_t u.$$

Proposition 3.6. *Let M be as in the previous proposition. For any real-valued function $w \in C^\infty(\bar{M}) \cap L^\infty(M)$ for which $\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}}$ is bounded in M and $u \in L^2(M, \Lambda^{p,q})$,*

$$\frac{d}{dt} \|e^w P_t u\|_{L^2(M, \Lambda^{p,q})}^2 = -2\Re Q(P_t u, e^{2w} P_t u).$$

In particular, for $w = 0$ we get:

$$\frac{d}{dt} \|P_t u\|_{L^2(M, \Lambda^{p,q})}^2 = -2Q(P_t u).$$

Proof. For any $t > 0$ we have

$$\begin{aligned}
\frac{d}{dt} \|e^w P_t u\|_{L^2}^2 &= \lim_{h \rightarrow 0} \frac{1}{h} [\langle P_{t+h} u, e^{2w} P_{t+h} u \rangle - \langle P_t u, e^{2w} P_t u \rangle] \\
&= \lim_{h \rightarrow 0} \left[\left\langle \frac{1}{h} (P_{t+h} u - P_t u), e^{2w} P_{t+h} u \right\rangle + \left\langle e^{2w} P_t u, \frac{1}{h} (P_{t+h} u - P_t u) \right\rangle \right] \\
&= \langle -\square P_t u, e^{2w} P_t u \rangle + \langle e^{2w} P_t u, -\square P_t u \rangle \\
&= -Q(P_t u, e^{2w} P_t u) - Q(e^{2w} P_t u, P_t u),
\end{aligned}$$

where, in the last step we used that $e^{2w} u$ is in the domain of Q , by part (iii) of Lemma 2.2. \blacklozenge

Proof of Theorem 1. From Prop. 3.3 and duality we get

$$t^{-2s} \|u\|_{L^1}^2 \geq \langle P_t u, P_t u \rangle_{L^2} = \|P_t u\|_{L^2}^2.$$

We use the fundamental theorem of calculus and the above Prop. 3.6 in

$$\begin{aligned}
(14) \quad \dots &= \|u\|_{L^2}^2 - 2 \int_0^t Q(P_s u) ds \\
&\geq \|u\|_{L^2}^2 - 2tQ(u)
\end{aligned}$$

where, in the last inequality we use the following straightforward consequence of functional calculus:

$$Q(P_s u) = \|\square^{\frac{1}{2}} e^{-s\square} u\|_{L^2}^2 \leq \|\square^{\frac{1}{2}} u\|_{L^2}^2.$$

Putting $t = Q(u)^{-\frac{1}{2s+1}} \|u\|_{L^1(M, \Lambda^{p,q})}^{\frac{2}{2s+1}}$ in (14) gives the assertion. \blacklozenge

3.2. The intrinsic metric. We will measure the bounds on off-diagonal terms in the heat kernel with respect to the metric given by

Definition 3.7. We define the G -invariant pseudo-metric d_\square on M by

$$d_\square(x, y) = \sup\{w(y) - w(x) \mid w \in L^\infty \cap C^\infty(\bar{M}, \mathbb{R}), \langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1\}.$$

The distance between sets is given by

$$d_\square(A; B) := \sup\{\inf_B w - \sup_A w \mid w \in L^\infty \cap C^\infty(\bar{M}, \mathbb{R}), \langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1\}$$

for arbitrary $A, B \subset \bar{M}$.

The definition above is geared to the intrinsic metric of Dirichlet forms, as used in slightly different versions, *e.g.* in [5, 10, 13, 58, 59, 55] as well as the metrics considered in [16, 33, 44] and see [27, 28] as well. Note however, that our application of this concept is somewhat nonstandard. We use this metric, defined on functions, to estimate the heat kernels acting on forms! We now show that the metric above is

equivalent to an associated Riemannian distance. To this end, let us describe the metric structure of M in more detail, in the notation of Sect. 2.1 above.

On the tangent bundle TM of the $2n$ -dimensional real G -manifold underlying M , we have a G -invariant almost-complex structure $J : TM \rightarrow TM$, induced by the complex structure on M . Assume that we also have a G -invariant Riemannian metric g on TM so that J is an isometry with respect to g ; $g(X, Y) = g(JX, JY)$. Note that with respect to any such metric, $X \perp JX$. Indeed,

$$g(X, JX) = g(JX, -X) = -g(JX, X) = -g(X, JX) = 0.$$

We may extend any Riemannian structure for which J is an isometry by complex sesquilinearity (linear in the first slot, conjugate-linear in the second slot) to obtain Hermitian inner products which we say are *associated to g* in $T_{1,0}, T_{0,1} \subset TM \otimes_{\mathbb{R}} \mathbb{C}$:

$$\langle X - iJX, Y - iJY \rangle_{T_{1,0}} := g(X, Y) + ig(X, JY),$$

$$\langle X + iJX, Y + iJY \rangle_{T_{0,1}} := g(X, Y) + ig(JX, Y).$$

By duality, these structures extend naturally to $\Lambda^{1,0}$ and $\Lambda^{0,1}$ and by tensoriality to each of the spaces $\Lambda^{p,q}$. We will also metrize the bundle of complex k -forms as an orthogonal sum

$$(15) \quad \Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q}, \quad (k = 0, 1, \dots, n).$$

Let us describe the $(0, 1)$ -forms in terms of J analogously to our vector fields in (6), (7). Since $\Lambda^{0,1}$ is the dual of $T_{0,1}$ in the Hermitian metric above, we have $\xi_X \in \Lambda^{0,1}$, the dual of $X + iJX \in T_{0,1}$, naturally of the form

$$(16) \quad \begin{aligned} \xi_X(Y + iJY) &= \langle Y + iJY, X + iJX \rangle_{T_{0,1}} \\ &= g(Y, X) + ig(JY, X) = g(X, Y) - ig(JX, Y). \end{aligned}$$

We compute the last term in coordinates. Since by assumption we have $g(X, Y) = g(JX, JY)$, it is true that

$$g_{kl} J_i^k J_j^l = g_{ij},$$

with the convention that repeated indices be summed over. Multiplying this identity by J and using $J_j^l J_k^j = -\delta_k^l$, the Kronecker δ , we get

$$g_{kj} J_i^j = -g_{ij} J_k^j,$$

from which it follows that $g(JX, \cdot) = -Jg(X, \cdot)$ since

$$g(JX, \cdot) = g_{ij} J_k^j X^k dx^i \quad \text{and} \quad Jg(X, \cdot) = J_i^j g_{jk} X^k dx^i.$$

Going back to (16) and writing $Jg(X, \cdot)|_Y$ too simply “ $Jg(X, Y)$,” we see that

$$\xi_X(Y + iJY) = g(X, Y) + iJg(X, Y)$$

thus $\Lambda^{0,1} \ni \xi_X = \phi_X + iJ\phi_X$ for the real 1-form $\phi_X = g(X, \cdot)$. Similarly, a form $\Lambda^{1,0} \ni \xi_X = \phi_X - iJ\phi_X$ again for the real 1-form $\phi_X = g(X, \cdot)$

Now we return to the description of the intrinsic metric. For $w \in C^\infty(\bar{M}, \mathbb{R})$, consider the following computation:

$$\langle dw, dw \rangle_{\Lambda^1} = \langle (\bar{\partial} + \partial)w, (\bar{\partial} + \partial)w \rangle_{\Lambda^1} = \langle \bar{\partial}w, \bar{\partial}w \rangle_{\Lambda^{0,1}} + \langle \partial w, \partial w \rangle_{\Lambda^{1,0}}$$

since $\bar{\partial}w \in \Lambda^{0,1}$ and $\partial w \in \Lambda^{1,0}$ are orthogonal by the decomposition (15).

Now, w is real so $\bar{\partial}w$ is the complex conjugate of ∂w by (5), thus there is a *single* real 1-form ϕ such that $\bar{\partial}w = \phi + iJ\phi$ and $\partial w = \phi - iJ\phi$. In fact, $\phi = \frac{1}{2}dw$ since $d = \partial + \bar{\partial}$. Computing the inner products,

$$\langle \bar{\partial}w, \bar{\partial}w \rangle_{\Lambda^{0,1}} = \langle \partial w, \partial w \rangle_{\Lambda^{1,0}} = 2g(\phi, \phi)$$

since $g(\phi, J\phi) = 0$. Thus $\langle dw, dw \rangle_{\Lambda^1} = 2\langle \bar{\partial}w, \bar{\partial}w \rangle_{\Lambda^{0,1}} = 4g(\phi, \phi)$ in our metric.

Since the Laplace-Beltrami operator on functions is induced by the quadratic form $w \mapsto \int \langle dw, dw \rangle_{\Lambda^1}$, cf. [50, 59], we have shown

Proposition 3.8. *For a J -invariant Riemannian structure g , let Δ_{LB} be the corresponding Laplace-Beltrami operator. Given the Hermitian structure on $\Lambda^{0,1}$ associated to g , the intrinsic metric d_\square is equivalent to the one induced by the intrinsic metric of $-\Delta_{LB}$ on functions.*

Remark 3.9. (1) At least in the case of complete manifolds without boundary it is well-known, cf. [59] that the intrinsic metric d_{LB} of the Laplace-Beltrami operator coincides with the Riemannian distance, i.e.,

$$d_{LB}(x, y) = \inf\{L(\gamma) \mid \gamma : I \rightarrow M \text{ a curve joining } x, y \in M\}.$$

In view of [55], the presence of a boundary should not change this picture.

(2) For Kähler manifolds, $\square = \frac{1}{2}\Delta$, cf. [34, Chap. III, §2], acting componentwise on forms, therefore it is clear in this case that we recover the intrinsic metric of the Laplacian up to a factor of $\sqrt{2}$.

3.3. Off-diagonal heat kernel estimates. Here, we basically use the proof from [13], pointing out once more that our setup is substantially different as our spaces are spaces of differential forms rather than functions. Let us also remind the reader that multiplication by functions preserves the domain of Q and this is crucial to our treatment.

Lemma 3.10. *For $w \in L^\infty \cap C^1(\bar{M}, \mathbb{R})$, we have*

$$Q(u, u) = Q(e^{-\epsilon w} u, e^{\epsilon w} u) - 2i\epsilon \Im \left\{ \langle \bar{\partial} u, \bar{\partial} w \wedge u \rangle_{L^2} + \langle \star(\partial w \wedge \star u), \bar{\partial}^* u \rangle_{L^2} \right\} \\ + \epsilon^2 \left\{ \|\bar{\partial} w \wedge u\|_{L^2}^2 + \|\partial w \wedge \star u\|_{L^2}^2 \right\}$$

for all $u \in \text{dom } Q$.

Proof. By definition,

$$Q(e^{-\epsilon w} u, e^{\epsilon w} u) = \langle \bar{\partial} e^{-\epsilon w} u, \bar{\partial} e^{\epsilon w} u \rangle + \langle \bar{\partial}^* e^{-\epsilon w} u, \bar{\partial}^* e^{\epsilon w} u \rangle.$$

The first term simplifies as follows

$$\langle \bar{\partial} e^{-\epsilon w} u, \bar{\partial} e^{\epsilon w} u \rangle = \langle \bar{\partial} u, \bar{\partial} u \rangle + 2i\epsilon \Im \langle \bar{\partial} u, \bar{\partial} w \wedge u \rangle - \epsilon^2 \langle \bar{\partial} w \wedge u, \bar{\partial} w \wedge u \rangle.$$

For the second term, note that $\bar{\partial}^* = -\star \partial \star$ where \star is the Hodge operator and $\partial = d - \bar{\partial}$, (cf. Prop. 5.1.1, [17]). Thus

$$\bar{\partial}^* e^{-w} u = -\star \partial \star (e^{-w} u) = -\star [\partial e^{-w} (\star u)] = -\star [\partial e^{-w} \wedge \star u + e^{-w} \partial \star u] \\ = -\star [\partial e^{-w} \wedge \star u] + e^{-w} \bar{\partial}^* u = e^{-w} \star [\partial w \wedge \star u] + e^{-w} \bar{\partial}^* u.$$

With the corresponding expression

$$\bar{\partial}^* e^w u = -e^w \star [\partial w \wedge \star u] + e^w \bar{\partial}^* u,$$

we obtain

$$\langle \bar{\partial}^* e^{-w} u, \bar{\partial}^* e^w u \rangle = \langle \bar{\partial}^* u, \bar{\partial}^* u \rangle + 2i \Im \langle \star(\partial w \wedge \star u), \bar{\partial}^* u \rangle \\ - \langle (\partial w \wedge \star u), (\partial w \wedge \star u) \rangle,$$

where we have used the fact that the Hodge \star is an isometry. \blacklozenge

Corollary 3.11. *Assuming $\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1$, we have*

$$-\Re Q(e^{-w} u, e^w u) \leq 2\|u\|_{L^2(M, \Lambda^{p,q})}^2.$$

Proof. The previous assertion implies

$$-\Re Q(e^{-\epsilon w} u, e^{\epsilon w} u) = \epsilon^2 \left\{ \|\bar{\partial} w \wedge u\|^2 + \|\partial w \wedge \star u\|^2 \right\} - Q(u, u)$$

and since we have assumed $\langle \partial w, \partial w \rangle_{\Lambda^{1,0}} = \langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1$, (see Sect. 3.2) we have the result by Cauchy-Schwarz and again the fact that the Hodge \star is an isometry. \blacklozenge

Proof of Theorem 2. For arbitrary $f \in \text{dom } Q$, the computation in Prop. 3.6 gives

$$\|e^w P_t f\|_{L^2}^2 - \|e^w f\|_{L^2}^2 = \int_0^t \frac{d}{ds} \|e^w P_s f\|_{L^2}^2 ds \\ (17) \quad = -2\Re \int_0^t ds Q(P_s f, e^{2w} P_s f).$$

Writing

$$Q(P_s f, e^{2w} P_s f) = Q(e^{-w} e^w P_s f, e^w e^w P_s f)$$

and applying Cor. 3.11, the integrand in (17) satisfies

$$(18) \quad -\Re Q(P_s f, e^{2w} P_s f) \leq \|e^w P_s f\|_{L^2}^2,$$

as usual, assuming that $\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1$. It follows that

$$\|e^w P_t f\|_{L^2}^2 - \|e^w f\|_{L^2}^2 \leq 2 \int_0^t ds \|e^w P_s f\|_{L^2}^2.$$

Gronwall's inequality implies that

$$\|e^w P_t f\|_{L^2}^2 \leq e^{2t} \|e^w f\|_{L^2}^2$$

and replacing w by δw we obtain $\|e^{\delta w} P_t f\|_{L^2} \leq e^{\delta^2 t} \|e^{\delta w} f\|_{L^2}$ by inspection in (18). This implies that

$$\|e^{\delta w} P_t e^{-\delta w}\|_{2 \rightarrow 2} \leq e^{\delta^2 t}$$

since f was arbitrary in the domain.

Now, for arbitrary $\alpha, \beta \in L^2$

$$\begin{aligned} |\langle \mathbf{1}_B P_t \mathbf{1}_A \alpha, \beta \rangle| &= |\langle e^{\delta w} P_t e^{-\delta w} e^{\delta w} \mathbf{1}_A \alpha, e^{-\delta w} \mathbf{1}_B \beta \rangle| \\ &\leq \|e^{\delta w} P_t e^{-\delta w} e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)} \\ &\leq \|e^{\delta w} P_t e^{-\delta w}\|_{2 \rightarrow 2} \|e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)}. \\ &\leq e^{\delta^2 t} \|e^{\delta w} \mathbf{1}_A \alpha\|_{L^2(M)} \|e^{-\delta w} \mathbf{1}_B \beta\|_{L^2(M)}. \end{aligned}$$

For $\varepsilon > 0$ choose a weight function w like in the definition of $d_{\square}(A; B)$ above, with $\langle \bar{\partial} w, \bar{\partial} w \rangle_{\Lambda^{0,1}} \leq 1$ and so that

$$d_{\square}(A; B) - \varepsilon \leq \inf_B w - \sup_A w \quad \text{and} \quad \sup_A w = 0$$

(we can achieve the latter by adding a suitable constant). This gives

$$\inf_B w \geq d_{\square}(A; B) - \varepsilon.$$

Inserting gives

$$|\langle \mathbf{1}_B P_t \mathbf{1}_A \alpha, \beta \rangle| \leq e^{\delta^2 t} e^{-\delta(d_{\square}(A; B) - \varepsilon)} \|\alpha\| \|\beta\|$$

so that (since ε is arbitrary)

$$\|\mathbf{1}_B P_t \mathbf{1}_A\| \leq e^{\delta^2 t} e^{-\delta d_{\square}(A; B)}.$$

For $d_{\square}(A; B) < \infty$, choose $\delta = d_{\square}(A; B)/(2t)$. \blacklozenge

Remark 3.12. In light of Prop. 3.8, we may replace d_{\square} with d_{LB} , making the necessary changes.

3.4. Sobolev estimates for the Heat Operator. Here we extend some L^p results from the preceding treatment to Sobolev spaces. First note that for $t > 0$ and $k \in \mathbb{N}$ arbitrary, we have $P_t : L^2 \rightarrow \text{dom } \square^k$.

Proposition 3.13. *For $t > 0$ and $q > 0$ we have*

$$P_t : L^2(M, \Lambda^{p,q}) \rightarrow C^\infty(\bar{M}, \Lambda^{p,q}).$$

Proof. We will proceed by induction and use the Sobolev lemma, (11) above. Fix $t > 0$. For any $\alpha \in L^2$, since $\text{im } P_t \subset \text{dom } \square$, and $(\square + 1)^{-1} : L^2 \rightarrow \text{dom } \square$ is onto, we may apply Thm. 2.3 to the form $\alpha = (\square + 1)P_t\beta$, $\beta \in \text{dom } \square$, to obtain

$$\|\zeta P_t\beta\|_{H^1} \lesssim \|\zeta'(\square + 1)P_t\beta\|_{L^2} + \|(\square + 1)P_t\beta\|_{L^2} \lesssim \|\beta\|_{L^2},$$

and conclude that $\text{im } P_t \in H_{\text{loc}}^1$. Furthermore, since P_t is a function of \square , they commute and we also have

$$(\square + 1)P_t\beta = P_t(\square + 1)\beta \in H_{\text{loc}}^1 \quad (\alpha \in L^2).$$

Assuming now that $(\square + 1)P_t\beta \in H_{\text{loc}}^{s-1}$, the same theorem provides

$$\|\zeta P_t\beta\|_{H^s} \lesssim \|\zeta'(\square + 1)P_t\beta\|_{H^{s-1}} + \|(\square + 1)P_t\beta\|_{L^2},$$

so $P_t\beta \in H_{\text{loc}}^s$. \blacklozenge

We will need the following *a priori* estimate for \square , proven in our setting by a variation on the methods in [38, 17], in Thm. 4.5 of [46].

Lemma 3.14. (Kohn inequality) *If M is a strongly pseudoconvex G -manifold on which G acts freely by holomorphic transformations with compact quotient \bar{M}/G and $q > 0$, then for every integer $s \geq 0$ there exists a positive constant C_s so that*

$$\|u\|_{H^{s+1}} \leq C_s(\|\square u\|_{H^s} + \|u\|_{L^2}), \quad (u \in \text{dom } \square \cap C^\infty(\bar{M}, \Lambda^{p,q}))$$

uniformly.

Corollary 3.15. *For $t > 0$ and $q > 0$ we have $\text{im } P_t \subset H^\infty(M, \Lambda^{p,q})$.*

Proof. Combining the results of Prop. 3.13 and Lemma 3.14, we have

$$\|u\|_{H^{s+1}} \leq C_s(\|\square u\|_{H^s} + \|u\|_{L^2}) \quad (u \in \text{im } P_t)$$

but $\text{im } P_t \subset \text{dom } \square^k$ for all powers of the Laplacian, so this estimate can be iterated. Thus the estimates

$$(19) \quad \|\square^{k-s}u\|_{H^{s+1}} \lesssim \|\square^{k-s+1}u\|_{H^s} + \|\square^{k-s}u\|_{L^2}, \quad (s = 1, 2, \dots, k)$$

hold for $u \in \text{im } P_t$. These imply the result. \blacklozenge

Proposition 3.16. *If M is as above, $t > 0$, and $q > 0$, then the heat operator P_t is bounded from $H^{-s}(\bar{M}, \Lambda^{p,q}) \rightarrow H^s(M, \Lambda^{p,q})$ for any positive integer s .*

Proof. First recall the following fact about Sobolev spaces on manifolds with boundary from Remark 12.5 of [41]. For $s > 0$, the dual space of $H^s(M)$, denoted $H^{-s}(\bar{M})$, consists of elements of $H^{-s}(\widetilde{M})$ whose support is in \bar{M} . Now, from Cor. 3.15 we have that for all $s > 0$, $P_t : L^2 \rightarrow H^s(M)$ continuously. Since P_t is self-adjoint, its domain can be extended to the dual of $H^s(M)$ so that $P_t : H^{-s}(\bar{M}) \rightarrow L^2(M)$. The semigroup law $P_t^2 = P_{2t}$ holds on $C_c^\infty(M) \subset L^2(M)$, a dense subspace of all the $H^s(M)$, ($s \in \mathbb{R}$) so we may conclude that $P_t : H^{-s}(\bar{M}) \rightarrow H^s(M)$ for all $s > 0$. \blacklozenge

Remark 3.17. These results have three easy consequences.

1) For an operator norm estimate, we can put $u = P_t\alpha$ in the estimates (19) and telescope them to find that for $s \in \mathbb{N}$,

$$\|P_t\alpha\|_{H^s} \lesssim \sum_{k=0}^s \|\square^k P_t\alpha\|_{L^2} \lesssim \sum_{k=0}^s t^{-k} \|\alpha\|_{L^2},$$

which yields an estimate analogous to that in Prop. 3.3.

2) Combining Cor. 3.15 with Gagliardo-Nirenberg-Sobolev embeddings, *e.g.*

$$H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \quad p = \frac{2n}{n-2s}, \quad 0 \leq s < \frac{n}{2},$$

[4], one obtains results overlapping those of the previous sections in L^p spaces. With other such embeddings can obtain results for L^p -Sobolev spaces.

3) One can continue the treatment in Sect. 6 of [46] to obtain that, for $t > 0$, the heat operator's Schwartz kernel $K_t \in C^\infty(\bar{M} \times \bar{M})$ and

$$(20) \quad \int_{\frac{M \times M}{G}} |K_t|^2 < \infty, \quad (t > 0),$$

noting that \square and thus K_t are G -invariant. When G is unimodular, (20) means that von Neumann's G -trace of P_{2t} is finite.

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, VIENNA, AUSTRIA
E-mail address: `joe_j_perez@yahoo.com`

FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT, CHEMNITZ, GERMANY
E-mail address: `peter.stollmann@mathematik.tu-chemnitz.de`