arXiv:1009.5014v1 [math.AC] 25 Sep 2010

A GLIMPSE AT SUPERTROPICAL VALUATION THEORY

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To Serban A. Basarab with admiration,
on occasion of his seventieth birthday.

Abstract. We give a short tour through major parts of a recent long paper [IKR1] on supertropical valuation theory, leaving aside nearly all proofs (to be found in [IKR1]). In this way we hope to give easy access to ideas of a new branch of so called "supertropical algebra".

1. Introduction

We will be much concerned with semirings. Recall that a **semiring** R is a set R equipped with addition and multiplication such that both $(R, +)$ and $(R \setminus \{0\}, \cdot\,)$ are monoids, i.e., semigroups with a unit element, 0 and 1 respectively, such that multiplication distributes over addition in the usual way. In the present paper we always assume that multiplication (and, of course, addition) is commutative. A **semifield** is a semiring such that $(R \setminus \{0\}, \cdot\,)$ is a group. We give two examples of semifields.

Example 1.1. *If F is field then the set $R := \sum F^2$ consisting of all sums of squares in R is a subsemiring of F, and in fact a subsemifield since, if*

$$q := a_1^2 + \cdots + a_n^2$$

Date: September 28, 2010.

Key words and phrases. Supertropical algebra, monoids, bipotent semirings, valuation theory, supervaluations, lattices.

The research of the first and third authors has been supported by the Israel Science Foundation (grant No. 448/09).

The research of the second author was supported in part by the Gelbart Institute at Bar-Ilan University, the Minerva Foundation at Tel-Aviv University, the Department of Mathematics of Bar-Ilan University, and the Emmy Noether Institute at Bar-Ilan University.

The research of the first author has been supported by the Oberwolfach Leibniz Fellows Programme (OWLF), Mathematisches Forschungsinstitut Oberwolfach, Germany.

This article is based on a lecture of the second author at the International Conference and Humboldt Kolleg "Fundamental Structures of Algebra", at Constanta Romania, April 14–18, 2010, in honor of the 70th birthday of Professor Serban Basarab.

with $a_i \in F$ and $q \neq 0$, then

$$\frac{1}{q} = \left(\frac{a_1}{q}\right)^2 + \cdots + \left(\frac{a_n}{q}\right)^2.$$

For the second example of semifields we need some preparation. We call a semiring M **bipotent** if for any $a, b \in M$ the sum $a + b$ is either a or b. In this case we have a total ordering $\leq$ on the set M, defined by

$$a \leq b \;\Leftrightarrow\; a + b = b,$$

as is easily checked. Clearly 0 is the smallest element of M. The ordering is compatible with multiplication.

$$a \leq b \;\Rightarrow\; ac \leq bc,$$

and also with addition

$$a \leq b \;\Rightarrow\; a + c \leq b + c,$$

for any $a, b, c \in M$. We may state that

$$a + b = \max(a, b).$$

Notice that bipotent semirings are very far away from those semirings where addition is cancellative, i.e., $a + c = b + c \Rightarrow a = b$.

Example 1.2. *Let Γ be a (totally) ordered abelian group, in multiplicative notation. We add to Γ a new element 0 and extend the ordering of Γ to $M := \Gamma \cup \{0\}$ by declaring $0 < \gamma$ for all $\gamma \in \Gamma$. We define addition and multiplication on the set M as follows:*

$$\begin{aligned} x + y &= \max(x, y) \quad \text{for } x, y \in M, \\ 0 \cdot y &= y \cdot 0 = 0 \quad \text{for } y \in M, \\ x \cdot y &= \text{the given product in } \Gamma, \ \text{if } x, y \in \Gamma. \end{aligned}$$

Clearly, M is a bipotent semifield.

It is an easy exercise to check that in this way we obtain all bipotent semifields M from ordered abelian groups Γ in a unique way ($M = \Gamma \cup \{0\}$, $\Gamma = M \setminus \{0\}, \dots$). In short, *bipotent semifields are the same objects as ordered abelian groups.*

Subexample 1.3. *Take $\Gamma = (\mathbb{R}, +)$, the additive group of the real numbers with the standard ordering. Since we switched to an additive notation, we denote the zero element of the associated bipotent semiring M now by $-\infty$. Thus $M = \mathbb{R} \cup \{-\infty\}$. Addition and multiplication on M are given by*

$$x \oplus y := \max(x, y), \qquad x \odot y := x + y.$$

We refer to this bipotent semifield $\mathbb{R} \cup \{-\infty\}$ and related structures (e.g. the subsemiring $\mathbb{R}_{\geq 0} \cup \{-\infty\}$) as the "max-plus setting". It is used in tropical geometry (e.g. [G], [IMS]). {In some papers (e.g. [SS]) an equivalent "min-plus setting" is used}

The present authors feel that the max-plus setting is rather weak for the needs of tropical geometry, and thus are driven by the idea to develop a "supertropical algebra", which should serve tropical geometry better. Here the **supertropical semirings**, to be defined below, occupy a central place. The prefix "super" alludes to the fact that they are a sort of cover of bipotent semirings.

There exist already supertropic results on polynomials ([IR1], [IR5]), matrices ([IR2], [IR3], [IR4]) and, based on the supertropical matrix theory, first steps of a supertropical linear algebra [IKR2]. And now supertropical valuation theory [IKR1], to which we refer here.

2. Supertropical predomains with pregiven ghost map

Definition 2.1. *Let R be a semiring. A **valuation** v **on** R is a map $v : R \to M$ into a bipotent semifield M with $v(0) = 0$, $v(1) = 1$, and, for any $a, b \in M$,*

$$\begin{aligned} v(ab) &= v(a) \cdot v(b), \\ v(a+b) &\leq v(a) + v(b) \quad [= \max(v(a), v(b))]. \end{aligned}$$

If R is a ring, this definition can be found in [B, §3 No.1], up to our change from ordered abelian groups (additively written in [B]) to bipotent semirings. If R is a field, we meet the classical Krull valuations.

Definition 2.2. *We call a valuation v on the semiring R **strict**, if*

$$\forall a, b \in R : \qquad v(a+b) = v(a) + v(b),$$

(i.e., v is a semiring homomorphism).

*We call v **strong**, if*

$$\forall a, b \in R : \qquad v(a) \neq v(b) \;\Rightarrow\; v(a+b) = v(a) + v(b).$$

As is well known (at least for R a field), every valuation v on a ring R is strong, but no valuation on R is strict. But if R is a semiring which is not ring, v may be very well strict.

Example 2.3. *(For readers with experience in real algebra.) If $R = \sum F^2$ with F a formal real field (cf. Example 1.1) and w is a valuation on F, then the restriction $w|R$ is strict iff the valuation w is "real", i.e., w has a formally real residue class field. In this way the real valuations w on F correspond uniquely to the strict valuation v on R, provided the group $\Gamma := M \setminus \{0\}$ is 2-divisible; we obtain w back from v by the formula*

$$w(a) = v(a^2)^{\frac{1}{2}}.$$

Since any ordered abelian group can be enlarged to a 2-divisible ordered abelian group (even to a divisible ordered abelian group) in a unique way, it is essentially a question of preference, whether we study real valuations on fields or strict valuations on sub-semifields. With the second route we leave the cadre of classical algebra but have the possibility of transit to semirings which cannot be embedded into rings. For example we can study the image of the "total strict valuation map"

$$R \to \prod M_v, \qquad a \mapsto (v(a_1)),$$

with v running through all strict valuations $v : R \to M_v$ on R. We do not pursue this line here, but only point out that a "semiring–approach" is reasonable even for Krull valuations on fields.

3. Supertropical semirings

We now define **supertropical semirings**. Such semirings have first been constructed in a special case in [I], and then defined in general in [IR1], [IR2], [IKR1]. We follow the approach of [IKR1], which has the advantage of being short, but we refer the reader to the other papers to understand more on the intuition behind these semirings.

Definition 3.1. *A semiring U is* ***supertropical*** *if the following four axioms ST1-ST4 hold, which we state together with comments and side definitions.*

ST1: *The element $e := 1 + 1$ is idempotent, i.e., $1 + 1 = 1 + 1 + 1 + 1$.*
Thus eU is an ideal of U and is by itself a semiring.

ST2: *The semiring eU is bipotent.*
Then the elements of

$$\mathcal{G} := \mathcal{G}(U) := eU \setminus \{0\}$$

are called the ***ghost elements*** *of U. (In some sense also 0 is considered as a ghost element.) The map*

$$\nu_U : U \to eU, \qquad x \mapsto ex$$

is called the ***ghost map*** *of U. It associates to each $x \in U$ its* ***ghost*** *ex. (If $x \in eU$, then x is it own ghost.)*

ST3: *If $ex < ey$ then $x + y = y$.*
$\{$Recall that eU is totaly ordered, due to Axiom ST2.$\}$

ST4: *If $ex = ey$ then $x + y = ex$.*
With Axiom ST4 we meet a principal idea of supertropical algebra: While in the usual tropical geometry the semirings are ***idempotent****, i.e., $x+x = x$ for each x in the semiring, here $x + x$ is the ghost of x.*
If U is a supertropical semiring we call the elements of

$$\mathcal{T} := \mathcal{T}(U) := U \setminus eU$$

tangible*. We then have a partition*

$$U = \mathcal{T} \,\dot\cup\, \mathcal{G} \,\dot\cup\, \{0\},$$

and we remark that $\mathcal{G} + \mathcal{G} \subset \mathcal{G}$.

In the present paper we require for supertropical semirings one more axiom, namely

ST5: $\mathcal{T} \cdot \mathcal{T} \subset \mathcal{T}, \quad \mathcal{G} \cdot \mathcal{G} \subset \mathcal{G}.$

By this assumption we exclude only supertropical semirings which are rather pathological and seldom of interest. (They are sometimes needed for categorical reasons.)

We add three remarks for a supertropical semiring U.

(1) U is bipotent iff $\mathcal{T}$ is empty,

(2) $\forall x \in U: \ ex = 0 \ \Rightarrow \ x = 0.$
This is a consequence of ST4. We have $ex = e0$, hence $x + 0 = 0$.

(3) If $x_1, \dots, x_n \in U$ and $x_1 + \dots + x_n = 0$, then all $x_i = 0$.
Indeed, we have
$$ex_1 + \dots + ex_n = 0,$$
and $ex_i \geq 0$ for each i. Since U is totally ordered, it follows that each $ex_i = 0$, hence $x_i = 0$.

The second remark indicates a special role of the zero element of U. Informally it may be considered as both tangible and ghost.

We mention that there exists a completely explicit construction which gives us all supertropical semirings (with ST5), cf. [IKR1, Construction 3.16].

A basic intuition about ghost elements is that they are "noise" perturbing the tangible elements. This can be formulated as follows:

Definition 3.2. *Given $x, y \in U$ we say that x* ***surpasses*** *y* ***by ghost****, and write $x \underset{\text{gs}}{\models} y$, if there exists some $z \in eU$ with $x = y + z$.*

We call the relation $\underset{\text{gs}}{\models}$ the ***ghost surpassing relation****, or* ***GS-relation****, for short.*

We state two remarkable properties of the GS-relation.

(1) $\underset{\text{gs}}{\models}$ is a partial ordering of the set U, which is compatible with multiplication, i.e., $x \underset{\text{gs}}{\models} y$ implies $xz \underset{\text{gs}}{\models} yz$ for any $z \in U$. (The remarkable thing here is that $\underset{\text{gs}}{\models}$ is antisymmetric.)

(2) If $x \in \mathcal{T} \cup \{0\}$, $y \in U$, then $x \underset{\text{gs}}{\models} y$ implies $x = y$.

Thus if an element of U is perturbed by adding a ghost, the resulting element can never be tangible.

4. Supervaluatons

We now introduce *supervaluations.*

Definition 4.1. *A* ***supertropical semifield*** *is a supertropical semiring U for which the monoids $(\mathcal{T}(U), \cdot\ )$ and $(\mathcal{G}(U), \cdot\ )$ are groups.*

Here we have to apologize for an inconsistency of language: The ghost elements of a supertropical semifield are not invertible in U but only in $\mathcal{G}(U)$. Thus U is not a semifield as defined in §1, only $M := eU$ is a semifield.

Definition 4.2. *A **supervaluation** on a semiring R is a map $\varphi : R \to U$ from R to a supertropical semifield U with $\varphi(0) = 0$, $\varphi(1) = 1$, and, for any $a, b \in R$,*

$$\begin{aligned} \varphi(ab) &= \varphi(a) \cdot \varphi(b), \\ e\varphi(a+b) &\leq e\varphi(a) + e\varphi(b). \end{aligned}$$

If $\varphi : R \to U$ is a supervaluation, then the map

$$v : R \to M = eU, \qquad v(a) := e\varphi(a)$$

clearly is a valuation (as defined in §2). We say that φ **covers** the valuation v, and write $v = e\varphi$.

Starting with a valuation $v : R \to U$ with values in some bipotent semifield M we usually have very many supervaluations $\varphi : R \to U$ covering v, where U runs through the class of all supertropical semifields with $M \subset U$ and $eU = M$. We obtain a hierarchy between these supervaluations by a relation of "dominance", to be explained now.

Lemma 4.3. *If $\varphi : R \to U$ is a supervaluation, then the set*

$$\langle \varphi(R) \rangle := \varphi(R) \cup e\varphi(R)$$

is a subsemiring of U (and hence a supertropical semiring itself).

This can be easily verified.

Definition 4.4.

(a) *Given supervaluations $\varphi : R \to U$ and $\psi : R \to V$ we say that φ **dominates** ψ, and write $\varphi \geq \psi$, if there exists a semiring homomorphism*

$$\alpha : \langle \varphi(R) \rangle \to \langle \psi(R) \rangle,$$

necessarily surjective, such that $\psi(a) = \alpha(\varphi(a))$ for every $a \in R$.

(b) *We call φ and ψ **equivalent**, and write $\varphi \sim \psi$, of both $\varphi \geq \psi$ and $\psi \geq \varphi$.*

(c) *We denote the equivalence class of a supervaluation φ covering v by $[\varphi]$, and denote the set of all these classes by $\mathrm{Cov}(v)$.*

We obtain on the set $\mathrm{Cov}(v)$ a partial ordering by declaring that

$$[\varphi] \geq [\psi] \quad \text{iff} \quad \varphi \geq \psi.$$

We now have a fairly remarkable fact:

Theorem 4.5. *The partially ordered set $\mathrm{Cov}(v)$ is a complete lattice.*

As every complete lattice $\mathrm{Cov}(v)$ has a top element and a bottom element. The top element can be described explicitly, cf. [IKR1, Example 4.5]. The bottom element is the class $[v]$ of the supervaluation $v : R \to M$, viewed as a supervaluation. {N.B. We regard M is as supertropical semifield without tangible elements.}

Starting from now, until the end of the paper, *we assume that $v : R \to M$ is a strong valuation* (e.g. R is a ring), and we focus on a particularly good natured class of supervaluations covering v, to be defined as follows.

Definition 4.6. *A supervaluation $\varphi : R \to U$ covering v is **strong** if*

$$\forall a, b \in R : \qquad \varphi(a) + \varphi(b) \in \mathcal{T}(U) \;\Rightarrow\; \varphi(a+b) = \varphi(a) + \varphi(b).$$

The strong supervaluations turn out to be "nearly" semiring homomorphisms in the GS-sense. More precisely

Proposition 4.7. *A supervaluation $\varphi : R \to U$ covering v is strong iff for all $a, b \in R$*

$$\varphi(a) + \varphi(b) \underset{\mathrm{gs}}{\models} \varphi(a+b).$$

We call a supervaluation $\varphi : R \to U$ **tangible** if all its values are tangible or zero; $\varphi(R) \subset \mathcal{T}(U) \cup \{0\}$.

In the next section the strong valuations which are also tangible will play a useful role. We quote the following important fact, to be found in [IKR1, §11].

Theorem 4.8. *The subset $\mathrm{Cov}_{t,s}(v)$ of $\mathrm{Cov}(v)$ consisting of all classes $[\varphi] \in \mathrm{Cov}(v)$ with φ tangible and strong is a complete sublattice of $\mathrm{Cov}(v)$. In particular $\mathrm{Cov}_{t,s}(v)$ is not empty.*

Again the top and the bottom elements of $\mathrm{Cov}_{t,s}(v)$ can be described explicitly, cf. [IKR1, Theorem 11.8 and Example 10.16].

5. A supertropical version of Kapranov's lemma

Assume that R is a semiring, $\varphi : R \to U$ is a strong supervaluation covering $v : R \to M$, and $\lambda = (\lambda_1, \dots, \lambda_n)$ a set of variables. We start out to extend φ to a supervaluation on the polynomial semiring $R[\lambda]$ in various ways.

We first extend φ to a map

$$\widetilde{\varphi} : R[\lambda] \to U[\lambda]$$

by the formula

$$\widetilde{\varphi}\Bigg(\sum_i c_i \lambda^i\Bigg) := \sum_i \varphi(c_i)\lambda^i.$$

Here we use that standard monomial notation: i runs through the set of tuples $i = (i_1, \dots, i_n)$ with $i_1, \dots, i_n$ in $\mathbb{N}_0$; λ^i means $\lambda_i^{i_1} \cdots \lambda_n^{i_n}$; only finitely many c_i are not zero. In the same way we have a map $\tilde{v} : R[\lambda] \to M[\lambda]$,

$$\tilde{v}\Bigg(\sum_i c_i \lambda^i\Bigg) := \sum_i v(c_i)\lambda^i.$$

Now we choose a tuple $a = (a_1, \dots, a_n)$ in R^n.

It gives us tuples $\varphi(a) = (\varphi(a_1), \dots, \varphi(a_n))$ in U^n and $v(a) = (v(a_1), \dots, v(a_n))$ in M^n. Associated with these tuples we obtain evaluation maps

$$\varepsilon_a : R[\lambda] \to R, \qquad \varepsilon_{\varphi(a)} : U[\lambda] \to U, \qquad \varepsilon_{v(a)} : M[\lambda] \to M,$$

by inserting the tuples for the variables into the polynomials. For example

$$\varepsilon_{\varphi(a)}\left(\sum_i \gamma_i \lambda^i\right) := \sum_i \gamma_i \varphi(a)^i \qquad (\gamma_i \in U).$$

These maps are semiring homomorphisms.

It is then fairly obvious that the map $v \circ \varepsilon_a : R[\lambda] \to M$ is a valuation and $\varphi \circ \varepsilon_a : R[\lambda] \to U$ is a supervaluation covering $v \circ \varepsilon_a$. With some work it can be seen that also $\varepsilon_{v(a)} \circ \tilde{v} : R[\lambda] \to M$ is a valuation and $\varepsilon_{\varphi(a)} \circ \tilde{v} : R[\lambda] \to U$ is a supervaluation covering $\varepsilon_{v(a)} \circ \tilde{v}$. {Here it is important to assume that φ is strong.}[1]

Now most often the diagram

$$\begin{array}{ccc} R[\lambda] & \xrightarrow{\varepsilon_a} & R \\ {\scriptstyle \widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ U[\lambda] & \xrightarrow{\varepsilon_{\varphi(a)}} & U \end{array}$$

does not commute. Instead we have

Theorem 5.1. [IKR1, §13] *For any $f \in R[\lambda]$*

$$\varepsilon_{\varphi(a)}\widetilde{\varphi}(f) \underset{\text{gs}}{\models} \varphi\varepsilon_a(f).$$

The theorem says in more imaginative terms that the supervaluation $\varepsilon_{\varphi(a)}\widetilde{\varphi}$ is a perturbation of $\varphi\varepsilon_a$ by noise.

Theorem 5.1 has close relation to an initial key observation of tropical geometry, Kapranov's Lemma. Let us briefly indicate what is says.

Assume that R is a field and $f = \sum c_i \lambda^i$ is a polynomial over R. It gives us the hypersurface

$$Z(f) := \{a \in R^n \mid f(a) = 0\}.$$

In tropical geometry one relates $Z(f)$ to the so called "corner locus", or "tropical hypersurface", of the polynomial

$$\tilde{v}(f) = \sum_i v(c_i)\lambda^i \in M[\lambda].$$

Notice that, if a tuple $\xi \in M^n$ is given, then

$$\tilde{v}(f)(\xi) = \max_i (v(c_i)\xi^i).$$

[1] The valuations and supervaluations on $R[\lambda]$ ocuring here are again strong, but this will not matter for the following.

The corner locus $Z_0(\tilde{v}(f))$ is defined as the set of all tuples $\xi \in M^n$, where this maximum is attained at least at two indices. Kapranov's lemma states that

$$v(Z(f)) \subset Z_0(\tilde{v}(f)),$$

cf. [EKL, Lemma 2.1.4].

It can be deduced from Theorem 5.1 as follows.

We choose a tangible strong supervaluation $\varphi : R \to U$ covering v, which is possible by Theorem 4.8. Let $a \in Z(f)$. Then $\varphi\varepsilon_a(f) = \varphi(f(a)) = 0$. Theorem 5.1 tells us that

$$\varepsilon_{\varphi(a)}\widetilde{\varphi}(f) = \sum_i \varphi(c_i)\varphi(a)^i \underset{\text{gs}}{\models} 0,$$

i.e., this sum is ghost. But each summand $\varphi(c_i)\varphi(a)^i$ is tangible or zero. From the law ST3 in §3 we infer that the maximum of the values

$$e\varphi(c_i)\varphi(a)^i = (e\varphi(c_i))(e\varphi(a)^i) = v(c_i)v(a)^i$$

is attained by more than one index. In other words, $v(a)$ is an element of the corner locus $Z_0(\tilde{v}(f))$. Thus indeed $v(Z(f)) \subset Z_0(\tilde{v}(f))$.

Theorem 5.1 says more than the classical Kapranov lemma, not only since a semiring R instead of a field R is admitted, but also since it contains a statement about points $a \in R^n$ with $f(a) \neq 0$.

Finally, if φ and ψ are strong supervaluations covering v with $\varphi \geq \psi$, the statement of Theorem 5.1 for φ formally implies the same statement for ψ. Thus Theorem 5.1 seems to be "best", if φ is the top element of $\mathrm{Cov}_{t,s}(v)$, at least if we focus on tangible supervaluations.

To exploit all this, more work will be needed than what has been done in [IKR1].

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