Quasi-modular forms attached to Hodge structures [1]

Hossein Movasati
Instituto de Matemática Pura e Aplicada, IMPA,
Estrada Dona Castorina, 110,
22460-320, Rio de Janeiro, RJ, Brazil,
`www.impa.br/∼ hossein, hossein@impa.br`

Abstract

The space D of Hodge structures on a fixed polarized lattice is known as Griffiths period domain and its quotient by the isometry group of the lattice is the moduli of polarized Hodge structures of a fixed type. When D is a Hermition symmetric domain then we have automorphic forms on D, which according to Baily-Borel theorem, they give an algebraic structure to the mentioned moduli space. In this article we slightly modify this picture by considering the space U of polarized lattices in a fixed complex vector space with a fixed Hodge filtration and polarization. It turns out that the isometry group of the filtration and polarization, which is an algebraic group, acts on U and the quotient is again the moduli of polarized Hodge structures. This formulation leads us to the notion of quasi-automorphic forms which generalizes quasi-modular forms attached to elliptic curves.

Around seventies Griffiths in his article [6] introduced the period domain D and described a project to enlarge D to a moduli space of degenerating polarized Hodge structures. He asked also for the existence of a certain automorphic form theory for D, generalizing the usual notion of automorphic forms on Hermitian symmetric domains. Since then there have been many efforts in the first part of Griffiths's project (see [8, 13] and the references there). For the second part Griffiths himself introduced the theory of automorphic cohomology, however, the generating function role of automorphic forms is somewhat missing in this theory.

Some years ago, I was looking for some analytic spaces over D for which one may state Baily-Borel theorem on the unique algebraic structure of quotients of Hermitian symmetric domains by discrete arithmetic groups. I realized that even in the simplest case of Hodge structures, namely $h^{01} = h^{10} = 1$, such spaces are not well studied. This led me to the definition of a class of holomorphic functions on the Poincaré upper half plane which generalize the classical modular forms (see [14]). Since a differential operator acts on them I called them differential modular forms. Soon after I realized that such functions play a central role in mathematical physics and, in particular, in mirror symmetry (see [11] and the references within there). Inspired by such a special case of Hodge structures with its fruitful applications, I felt the necessity to develop as much as possible similar theories for an arbitrary type of Hodge structures.

In this note we construct an analytic variety U and an action of an algebraic group G_0 on U from the right such that U/G_0 is the moduli space of polarized Hodge structures of a fixed type. We may pose the following algebraization problem for U, in parallel to Baily-Borel theorem in [1]: construct functions on U which have some automorphic properties with respect to the action of G_0 and have some finite growth when a Hodge structure degenerates. They must be enough in order to enhance U with a canonical

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structure of an algebraic variety such that the action of G_0 is algebraic. In the case for which the Griffiths period domain is Hermitian symmetric, for instance for the Siegel upper half plane, this problem seems to be promising but needs a reasonable amount of work if one wants to construct such functions through the inverse of the generalized period maps (see §4.1). Among them are calculating explicit affine coordinates in certain moduli spaces and calculating Gauss-Manin connections. Some main ingredients of such a study for K3 surfaces endowed with N-polarizations is recently done in [2]. For the case in which the Griffiths period domain is not Hermitian symmetric, we reformulate the algebraization problem further (see §3.3) and we solve it for the Hodge numbers $h^{30} = h^{21} = h^{12} = h^{03} = 1$ (see §4.2 and [13]). This gives us a first example of quasi-automorphic forms theory attached to a period domain which is not Hermitian symmetric.

The realization of the algebraization problem in the case of elliptic curves and the corresponding Hodge numbers $h^{10} = h^{01} = 1$ clarifies many details of the previous paragraph, therefore, I explain it here (for more details see [14, 15]). In this case $U = \mathrm{SL}(2,\mathbb{Z})\backslash P$, where

$$P := \{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Im}(x_1\overline{x_3}) > 0\}.$$

The algebraic group

$$G_0 = \{\begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \mid \ k' \in \mathbb{C},\ k \neq 0\}$$

acts from the right on U by the usual multiplication of matrices. The period map gives us a biholomorphism:

$$T := \{(t_1,t_2,t_3) \in \mathbb{C}^3 \mid 27t_3^2 - t_2^3 \neq 0\} \cong U. \tag{1}$$

Under the above biholomorphism the action of G_0 is given by

$$t \bullet g = (t_1k^{-2} + k'k^{-1}, t_2k^{-4}, t_3k^{-6}),$$

$$t = (t_1,t_2,t_3) \in \mathbb{C}^3,\ g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G_0.$$

The biholomorphism (1) is given by the generalized period map

$$\mathsf{pm} : T \to U,\ t \mapsto \left[\frac{1}{\sqrt{2\pi i}} \begin{pmatrix} \int_{\delta_1} \frac{dx}{y} & \int_{\delta_1} \frac{xdx}{y} \\ \int_{\delta_2} \frac{dx}{y} & \int_{\delta_2} \frac{xdx}{y} \end{pmatrix}\right].$$

Here, $[\cdot]$ means the equivalence class and $\{\delta_1,\delta_2\}$ is a basis of the $\mathbb{Z}$-module $H_1(E_t,\mathbb{Z})$ with $\langle\delta_1,\delta_2\rangle = 1$, where E_t is the elliptic curve

$$y^2 - 4(x-t_1)^3 + t_2(x-t_1) + t_3 = 0, \quad 27t_3^2 - t_2^3 \neq 0.$$

In fact, T is the moduli space of the pairs $(E, \{\omega_1,\omega_2\})$, where E is an elliptic curve and $\{\omega_1,\omega_2\}$ is basis of $H^1_{\mathrm{dR}}(E)$ such that ω_1 is represented by a differential form of the first kind and $\frac{1}{2\pi i}\int_E \omega_1 \cup \omega_2 = 1$.

The algebra of quasi-modular forms arises in the following way: We consider the composition of maps

$$\mathbb{H} \overset{i}{\hookrightarrow} P \to U \overset{\mathsf{pm}^{-1}}{\to} T \hookrightarrow \tilde{T}, \tag{2}$$

where $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is the upper half plane,

$$i : \mathbb{H} \to P,\ i(\tau) = \begin{pmatrix} \tau & -1 \\ 1 & 0 \end{pmatrix},$$

$P \to U$ is the quotient map and $\tilde{T} = \mathbb{C}^3$ is the underlying complex manifold of the affine variety $\mathrm{Spec}(\mathbb{C}[t_1, t_2, t_3])$. The pull-back of the functions ring $\mathbb{C}[t_1, t_2, t_3]$ of $\tilde{T}$ by the composition $\mathbb{H} \to \tilde{T}$, is a $\mathbb{C}$-algebra which we call it the $\mathbb{C}$-algebra of quasi-modular forms for $\mathrm{SL}(2, \mathbb{Z})$. Three Eisenstein series

$$g_i(\tau) = a_k\Big(1 + b_k \sum_{d=1}^{\infty} d^{2k-1} \frac{e^{2\pi i d\tau}}{1 - e^{2\pi i d\tau}}\Big), \quad k = 1, 2, 3, \tag{3}$$

where

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = (\frac{2\pi i}{12}, 12(\frac{2\pi i}{12})^2, 8(\frac{2\pi i}{12})^3)$$

are obtained by taking the pull-back of t_i's. Our reformulation of the algebraization problem is based on (2) and the pull-back argument, see §3.3.

We fix some notations from linear algebra. For a basis $\omega_1, \omega_2, \ldots, \omega_h$ of a vector space we denote by ω a $h \times 1$ matrix whose entries are ω_i's. In this way we also say that ω is a basis of the vector space. If there is no danger of confusion we also use ω to denote an element of the vector space. We use A^{t} to denote the transpose of the matrix A. Recall that if δ and ω are two bases of a vector space, $\delta = p\omega$ for some $p \in \mathrm{GL}(h, \mathbb{C})$ and a bilinear form on V_0 in the basis δ (resp. ω) has the matrix form A (resp. B) then $pBp^t = A$. By $[a_{ij}]_{h\times h}$ we mean a $h \times h$ matrix whose (i, j) entry is a_{ij}.

1 Moduli of polarized Hodge structures

In this section we define the generalized period domain U and we explain its comparison with the classical Griffiths period domain.

1.1 The space of polarized lattices

We fix a $\mathbb{C}$-vector space V_0 of dimension h, a natural number $m \in \mathbb{N}$ and a $h \times h$ integer valued matrix Ψ_0 such that the associated bilinear form

$$\mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z},\ (a, b) \to a^{\mathsf{t}}\Psi_0 b$$

is non-degenerate, symmetric if m is even and skew if m is odd. Note that in the case of $\mathbb{Z}$-modules by non-degenerate we mean that the associated morphism

$$\mathbb{Z}^h \to (\mathbb{Z}^h)^\vee,\ a \to (b \to a^{\mathsf{t}}\Psi_0 b)$$

is a an isomorphism, where $\vee$ means the dual of a $\mathbb{Z}$-module.

A lattice $V_\mathbb{Z}$ in V_0 is a $\mathbb{Z}$-module generated by a basis of V_0. A polarized lattice $(V_\mathbb{Z}, \psi_\mathbb{Z})$ of type Ψ_0 is a lattice $V_\mathbb{Z}$ together with a bilinear map $\psi_\mathbb{Z} : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$ such that in a $\mathbb{Z}$-basis of $V_\mathbb{Z}$, $\psi_\mathbb{Z}$ has the form Ψ_0.

Let $\mathcal{L}$ be the space of polarized lattices of type Ψ_0 in V_0. Usually, we denote an element of $\mathcal{L}$ by $x, y, \ldots$ and the associated lattice (resp. bilinear form) by $V_\mathbb{Z}(x), V_\mathbb{Z}(y), \ldots$ (resp.

$\psi_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(y), \ldots)$. Let R be any subring of $\mathbb{C}$. For instance, R can be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$. We define

$$V_R(x) := V_{\mathbb{Z}}(x) \otimes_{\mathbb{Z}} R \text{ and } \psi_R(x) : V_R(x) \times V_R(x) \to R \text{ the induced map.}$$

Conjugation with respect to $x \in \mathcal{L}$ of an element $\omega = \sum_{i=1}^{h} a_i \delta_i \in V_0$, where $V_{\mathbb{Z}}(x) = \sum_{i=1}^{h} \mathbb{Z}\delta_i$, is defined by

$$\overline{\omega}^x := \sum_{i=1}^{h} \overline{a}_i \delta_i,$$

where $\overline{s},\ s \in \mathbb{C}$ is the usual conjugation of complex numbers.

1.2 Hodge filtration

We fix Hodge numbers

$$h^{i,m-i} \in \mathbb{N} \cup \{0\},\ h^i := \sum_{j=i}^{m} h^{j,m-j},\ i = 0, 1, \ldots, m,\ h^0 = h$$

a filtration

$$F_0^\bullet : \{0\} = F_0^{m+1} \subset F_0^m \subset \cdots \subset F_0^1 \subset F_0^0 = V_0,\ dim(F_0^i) = h^i \tag{4}$$

on V_0 and a bilinear form

$$\psi_0 : V_0 \times V_0 \to \mathbb{C}$$

such that in a basis of V_0 its matrix is Ψ_0 and it satisfies

$$\psi_0(F_0^i, F_0^j) = 0,\ \forall i, j,\ i + j > m.$$

A basis $\omega_i, i = 1, 2, \ldots, h$ of V_0 is compatible with the filtration $F_0^\bullet$ if $\omega_i,\ i = 1, 2, \ldots, h^i$ is a basis of F_0^i for all i. It is sometimes convenient to fix a basis $\omega_i,\ i = 1, 2, \ldots, h$ of V_0 which is compatible with the filtration $F_0^\bullet$ and such that the polarization matrix $[\psi_0(\omega_i, \omega_j)]$ is a fixed matrix Φ_0:

$$[\psi_0(\omega_i, \omega_j)] = \Phi_0.$$

The matrices Ψ_0 and Φ_0 are not necessarily the same. For any $x \in \mathcal{L}$ we define

$$H^{i,m-i}(x) := F_0^i \cap \overline{F_0^{m-i}}^x$$

and the following properties for $x \in \mathcal{L}$:

1. $\psi_{\mathbb{C}}(x) = \psi_0$;
2. $V_0 = \oplus_{i=0}^{m} H^{i,m-i}(x)$;
3. $(-1)^{i+\frac{m}{2}} \psi_{\mathbb{C}}(x)(\omega, \overline{\omega}^x) > 0,\ \forall \omega \in H^{i,m-i}(x),\ \omega \neq 0.$

Throughout the text we call these properties P1, P2 and P3. Fix a polarized lattice $x \in \mathcal{L}$. P1 implies that

$$\psi_0(H^{i,m-i}(x), H^{j,m-j}(x)) = 0 \text{ except for } i + j = m.$$

This is because if $i+j>m$ then $\psi_0(F_0^i, F_0^j)=0$ and if $i+j<m$ then $\psi_0(\overline{F_0^i}^x, \overline{F_0^j}^x)=0$. We have also $\sum_i H^{i,m-i}(x)=\oplus_i H^{i,m-i}(x)$ if and only if

$$F_0^i \cap \overline{F_0^j}^x = 0,\ \forall\ i+j>m. \tag{5}$$

If $a_{m-k,k}+\cdots+a_{0,m}=0,\ a_{i,m-i}\in H^{i,m-i}(x)$ for some $0\leq k\leq m$ with $a_{m-k,k}\neq 0$, then

$$-a_{m-k,k}=a_{m-k-1,k+1}+\cdots+a_{0,m}\in F_0^{m-k}\cap \overline{F_0^{k+1}}^x \Rightarrow a_{k,m-k}=0$$

which is a contradiction. The proof in other direction is a consequence of

$$F_0^i\cap \overline{F_0^j}^x = H^{i,m-i}(x)\cap H^{m-j,j}(x),\ i+j>m.$$

1.3 Period domain U

Define

$$X := \{x\in \mathcal{L} \mid x \text{ satisfies P1 }\},$$

$$U := \{x\in \mathcal{L} \mid x \text{ satisfies P1,P2, P3 }\}.$$

Proposition 1. *The set X is an analytic subset of $\mathcal{L}$ and U is an open subset of X.*

Proof. Take a basis $\omega_i,\ i=1,2,\ldots,h$ of V_0 compatible with the Hodge filtration. The property P1 is given by

$$\psi_{\mathbb{C}}(x)(\omega_r,\omega_s)=0,\ r\leq h^i,\ s\leq h^j,\ i+j>m$$

and so X is an analytic subset of $\mathcal{L}$.

Now choose a basis δ of $V_{\mathbb{Z}}(x)$ and write $\delta=p\omega$. Using ω we may assume that $V_0=\mathbb{C}^h$ and δ constitutes of the rows of p. We have

$$\omega=p^{-1}\delta \Longrightarrow \overline{\omega}^x=\overline{p}^{-1}\delta=\overline{p}^{-1}p\omega$$

Therefore, the rows of $\overline{p}^{-1}p$ are complex conjugate of the the entries of ω. Now it is easy to verify that if the property (5), $\dim(H^{i,m-i}(x))=h^{i,m-i}$ and P3 are valid for one x then they are valid for all points in a small neighborhood of x (for P3 we may first restrict ψ_0 to the product of sphere of radius 1 and center $0\in\mathbb{C}^h$). □

1.4 An algebraic group

Let G_0 be the algebraic group

$$G_0 := \mathrm{Iso}(F_0^\bullet,\psi_0) :=$$

$$\{g: V_0\to V_0 \text{ linear } \mid g(F_0^i)=F_0^i,\ \psi_0(g(\omega_1),g(\omega_2))=\psi_0(\omega_1,\omega_2),\omega_1,\omega_2\in V_0\}.$$

It acts from the right on $\mathcal{L}$ in a canonical way:

$$xg := g^{-1}(x),\ \psi_{\mathbb{Z}}(xg)(\cdot,\cdot) := \psi_{\mathbb{Z}}(g(\cdot),g(\cdot)),\ g\in G_0,\ x\in\mathcal{L}.$$

One can easily see that for all $\omega\in V_0$, $x\in\mathcal{L}$ and $g\in G$ we have

$$\overline{\omega}^{xg}=g^{-1}\overline{g(\omega)}^x.$$

Proposition 2. *The properties P1, P2 and P3 are invariant under the action of* G_0.

Proof. The property $P1$ for xg follows from the definition. Let $x \in \mathcal{L},\ g \in G_0$ and $\omega \in V_0$. We have

$$\begin{aligned} H^{i,m-i}(xg) &= F_0^i \cap \overline{F_0^{m-i}}^{xg} = F_0^i \cap g^{-1}\overline{g(F_0^{m-i})}^{x} = F_0^i \cap g^{-1}(\overline{F_0^{m-i}}^{x}) \\ &= g^{-1}(F_0^i \cap \overline{F_0^{m-i}}^{x}) = g^{-1}(H^{i,m-i}(x)) \end{aligned}$$

and

$$\psi_{\mathbb{C}}(xg)(\omega, \overline{\omega}^{xg}) = \psi_{\mathbb{C}}(x)(g(\omega), gg^{-1}\overline{g(\omega)}^{x}) = \psi_{\mathbb{C}}(x)(g(\omega), \overline{g(\omega)}^{x}).$$

These equalities prove the proposition. ☐

The above proposition implies that G_0 acts from the right on U. We fix a basis $\omega_i, i = 1, 2, \ldots, h$ of V_0 compatible with the Hodge filtration $F_0^\bullet$ and, if there is no danger of confusion, we identify each $g \in G_0$ with the $h \times h$ matrix $\tilde{g}$ given by:

$$[g^{-1}(\omega_1), g^{-1}(\omega_2), \ldots, g^{-1}(\omega_h)] = [\omega_1, \omega_2, \ldots, \omega_h]\tilde{g}. \tag{6}$$

1.5 Griffiths period domain

In this section we give the classical approach to the moduli of polarized Hodge structures due to P. Griffiths. The reader is referred to [9, 8] for more developments in this direction.

Let us fix the $\mathbb{C}$-vector space V_0 and the Hodge numbers as in §1.2. Let also F be the space of filtrations (4) in V_0. In fact, F has a natural structure of a compact smooth projective variety. We fix the polarized lattice $x_0 \in \mathcal{L}$ and define the Griffiths domain:

$$D := \{F^\bullet \in \mathrm{F} \mid (V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0), F^\bullet) \text{ is a polarized Hodge structure } \}.$$

The group

$$\Gamma_{\mathbb{Z}} := \mathrm{Aut}(V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0))$$

acts on V_0 from the right in a usual way and this gives us an action of $\Gamma_{\mathbb{Z}}$ on D. The space $\Gamma_{\mathbb{Z}}\backslash D$ is the moduli of polarized Hodge structure.

Proposition 3. *There is a canonical isomorphism*

$$\beta : U/G_0 \overset{\sim}{\to} \Gamma_{\mathbb{Z}}\backslash D.$$

Proof. We take $x \in U$ and an isomorphism $\gamma : (V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x)) \overset{\sim}{\to} (V_{\mathbb{Z}}(x_0), \psi_{\mathbb{Z}}(x_0))$. The push-forward of the Hodge filtration $F_0^\bullet$ under this isomorphism gives us a Hodge filtration on V_0 with respect to the lattice $V_{\mathbb{Z}}(x_0)$ and so it gives us a point $\beta(x) \in D$. Different choices of γ leads us to the action of $\Gamma_{\mathbb{Z}}$ on $\beta(x)$. Therefore, we have a well-defined map

$$\beta : U \to \Gamma_{\mathbb{Z}}\backslash D.$$

Since $G_0 = \mathrm{Aut}(V_0, F_0^\bullet, \psi_0)$, β induces the desired isomorphism. ☐

The Griffiths domain is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{Z}$-basis in which the polarization has a fixed matrix form. Our domain U is the moduli of polarized Hodge structures of a fixed type and with a $\mathbb{C}$-basis compatible with Hodge filtration and for which the polarization has a fixed matrix form.

2 Period domain

In this section we introduce Poincaré duals, period matrices and Gauss-Manin connections in the framework of polarized Hodge structures.

2.1 Poincaré dual

In this section we explain the notion of Poincaré dual. Let $(V_{\mathbb{Z}}(x), \psi_{\mathbb{Z}}(x))$ be a polarized lattice and $\delta \in V_{\mathbb{Z}}(x)^\vee$, where $\vee$ means the dual of a $\mathbb{Z}$-module. We will use the symbolic integral notation

$$\int_\delta \omega := \delta(\omega),\ \forall \omega \in V_0.$$

The equality

$$\int_\delta \overline{\omega}^x = \overline{\int_\delta \omega},\ \forall \omega \in V_0,\ \ \delta \in V_{\mathbb{Z}}(x)^\vee \tag{7}$$

follows directly from the definition. The Poincaré dual of $\delta \in V_{\mathbb{Z}}(x)^\vee$ is an element $\delta^{\mathsf{pd}} \in V_{\mathbb{Z}}(x)$ with the property:

$$\int_\delta \omega = \psi_{\mathbb{Z}}(x)(\delta^{\mathsf{pd}}, \omega),\ \forall \omega \in V_{\mathbb{Z}}(x).$$

It exists and is unique because $\psi_{\mathbb{Z}}$ is non-degenerate. Using the Poincaré duality one defines the dual polarization:

$$\psi_{\mathbb{Z}}(x)^\vee(\delta_i, \delta_j) := \psi_{\mathbb{Z}}(x)(\delta_i^{\mathsf{pd}}, \delta_j^{\mathsf{pd}}),\ \ \delta_i, \delta_j \in V_{\mathbb{Z}}(x)^\vee.$$

We have:

$$(A^\vee \delta)^{\mathsf{pd}} = A^{-1}\delta^{\mathsf{pd}},\ \forall A \in \Gamma_{\mathbb{Z}},\ \ \delta \in V_{\mathbb{Z}}(x_0)^\vee,$$

where $A^\vee : V_{\mathbb{Z}}(x_0)^\vee \to V_{\mathbb{Z}}(x_0)^\vee$ is the induced dual map. This follows from:

$$\int_{A^\vee \delta} \omega = \int_\delta A\omega = \psi_{\mathbb{Z}}(x_0)(\delta^{\mathsf{pd}}, A\omega) = \psi_{\mathbb{Z}}(x_0)(A^{-1}\delta^{\mathsf{pd}}, \omega),\ \forall \omega \in V_0.$$

We define

$$\Gamma_{\mathbb{Z}}^\vee := \mathrm{Aut}(V_{\mathbb{Z}}(x_0)^\vee, \psi_{\mathbb{Z}}(x_0)^\vee).$$

It follows that $\Gamma_{\mathbb{Z}} \to \Gamma_{\mathbb{Z}}^\vee,\ A \mapsto A^\vee$ is an isomorphism of groups.

2.2 Period matrix

Let $\omega_i, i = 1, 2, \ldots, h$ be a $\mathbb{C}$-basis of V_0 compatible with $F_0^\bullet$. Recall that ω means the $h \times 1$ matrix with entries ω_i. For $x \in U$, we take a $\mathbb{Z}$-basis $\delta_i,\ i = 1, 2, \ldots, h$ of $V_{\mathbb{Z}}(x)^\vee$ such that the matrix of $\psi_{\mathbb{Z}}(x)$ in the basis δ is Ψ_0. We define the period matrix in the following way:

$$\mathsf{pm} = \mathsf{pm}(x) = [\int_{\delta_i} \omega_j]_{h\times h} := \begin{pmatrix} \int_{\delta_1}\omega_1 & \int_{\delta_1}\omega_2 & \cdots & \int_{\delta_1}\omega_h \\ \int_{\delta_2}\omega_1 & \int_{\delta_2}\omega_2 & \cdots & \int_{\delta_2}\omega_h \\ \vdots & \vdots & \vdots & \vdots \\ \int_{\delta_h}\omega_1 & \int_{\delta_h}\omega_2 & \cdots & \int_{\delta_h}\omega_h \end{pmatrix}.$$

Instead of the period matrix it is useful to use the matrix

$$\mathsf{q} = \mathsf{q}(x), \quad \text{where } \delta^{\mathsf{pd}} = \mathsf{q}\omega.$$

Then we have:

$$\Psi_0 = \mathsf{pm} \cdot \mathsf{q}^{\mathsf{t}}.$$

If we identify V_0 with $\mathbb{C}^h$ through the basis ω then q is a matrix whose rows are the entries of δ. We define P to be the set of period matrices pm. We write an element A of $\Gamma_{\mathbb{Z}}$ in a basis of $V_{\mathbb{Z}}(x_0)$, and redefine $\Gamma_{\mathbb{Z}}$:

$$\Gamma_{\mathbb{Z}} := \{A \in \mathrm{GL}(h, \mathbb{Z}) \mid A\Psi_0 A^{\mathsf{t}} = \Psi_0\}.$$

The group $\Gamma_{\mathbb{Z}}$ acts on P from the left by the usual multiplication of matrices and

$$U = \Gamma_{\mathbb{Z}} \backslash P.$$

In a similar way, if we identity each element g of G_0 with the matrix $\tilde{g}$ in (6) then G_0 acts from the right on P by the usual multiplication of matrices.

2.3 A canonical connection on $\mathcal{L}$

We consider the trivial bundle $\mathcal{H} = \mathcal{L} \times V_0$ on $\mathcal{L}$. On $\mathcal{H}$ we have a well-defined integrable connection

$$\nabla : \mathcal{H} \to \Omega^1_{\mathcal{L}} \otimes_{\mathcal{O}_{\mathcal{L}}} \mathcal{H}$$

such that a section s of $\mathcal{H}$ in an small open set $V \subset \mathcal{L}$ with the property

$$s(x) \in \{x\} \times V_{\mathbb{Z}}(x), \ x \in V.$$

is flat. Let $\omega_1, \omega_2, \ldots, \omega_h$ be a basis of V_0 compatible with the Hodge filtration $F_0^\bullet$. We can consider ω_i as a global section of $\mathcal{H}$ and so we have

$$\nabla \omega = A \otimes \omega, \ A = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1h} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2h} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{h1} & \omega_{h2} & \cdots & \omega_{hh} \end{pmatrix}, \ \omega_{ij} \in H^0(\mathcal{L}, \Omega^1_{\mathcal{L}}). \tag{8}$$

A is called the connection matrix of ∇ in the basis ω. The connection ∇ is integrable and so $dA = A \wedge A$:

$$d\omega_{ij} = \sum_{k=1}^{h} \omega_{ik} \wedge \omega_{kj}, \ i, j = 1, 2, \ldots, h. \tag{9}$$

Let δ be a basis of flat sections. Write $\delta = \mathsf{q}\omega$. We have

$$\omega = \mathsf{q}^{-1}\delta \Rightarrow \nabla(\omega) = d(\mathsf{q}^{-1})\mathsf{q}\omega \Rightarrow$$

$$A = d\mathsf{q}^{-1} \cdot \mathsf{q} = d(\mathsf{pm}^{\mathsf{t}} \cdot \Psi_0^{-\mathsf{t}}) \cdot (\Psi_0^{\mathsf{t}} \cdot \mathsf{pm}^{-\mathsf{t}}) = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}}.$$

and so

$$A = d(\mathsf{pm}^{\mathsf{t}}) \cdot \mathsf{pm}^{-\mathsf{t}}. \tag{10}$$

We have used the equality $\Psi_0 = \mathsf{pm} \cdot \mathsf{q}^{\mathsf{t}}$. Note that the entries of A are holomorphic 1-forms on $\mathcal{L}$ and a fundamental system for the linear differential equation $dY = A \cdot Y$ in $\mathcal{L}$ is given by $Y = \mathsf{pm}^{\mathsf{t}}$:

$$d\mathsf{pm}^{\mathsf{t}} = A \cdot \mathsf{pm}^{\mathsf{t}}.$$

We define the Griffiths transversality distribution by:

$$\mathcal{F}_{gr} : \omega_{ij} = 0,\ i \leq h^{m-x},\ j > h^{m-x-1},\ x = 0, 1, \ldots, m-2. \tag{11}$$

A holomorphic map $f : V \to U$, where V is an analytic variety, is called a period map if it is tangent to the Griffiths transversality distribution, that is, for all ω_{ij} as in (11) we have $f^{-1}\omega_{ij} = 0$.

2.4 Some functions on $\mathcal{L}$

For two vectors $\omega_1, \omega_2 \in V_0$, we have the following holomorphic function on $\mathcal{L}$:

$$\mathcal{L} \to \mathbb{C},\ x \mapsto \psi_{\mathbb{C}}(x)(\omega_1, \omega_2)$$

We choose a basis ω of V_0 and δ of $V_{\mathbb{Z}}(x)$ for $x \in \mathcal{L}$ and write $\delta = \mathsf{q} \cdot \omega$. Then

$$F := [\psi_{\mathbb{C}}(x)(\omega_i, \omega_j)] = (\mathsf{q}^{-1})^t \Psi_0 \mathsf{q}^{-1} = \mathsf{pm}^t \Psi_0^{-\mathsf{t}} \mathsf{pm} \tag{12}$$

(we have used the identity $\Psi_0 = \mathsf{q} \cdot \mathsf{pm}^{\mathsf{t}}$). The matrix F satisfies the differential equation:

$$dF = A \cdot F + F \cdot A^{\mathsf{t}}, \tag{13}$$

where A is the connection matrix. The proof is a straightforward consequence of (12) and (10):

$$\begin{aligned} dF &= d(\mathsf{pm}^t \Psi_0^{-\mathsf{t}} \mathsf{pm}) \\ &= (d\mathsf{pm}^t)\Psi_0^{-\mathsf{t}} \mathsf{pm} + \mathsf{pm}^t \Psi_0^{-\mathsf{t}} (d\mathsf{pm}) \\ &= A \cdot F + F \cdot A^{\mathsf{t}} \end{aligned}$$

It is easy to check that every solution of the differential equation (13) is of the form $\mathsf{pm}^t \cdot C \cdot \mathsf{pm}$ for some constant $h \times h$ matrix C with entries in $\mathbb{C}$ (if F is a solution of (13) then $F \cdot \mathsf{pm}^{-1}$ is a solution of $dY = A \cdot Y$). We restrict F, A and pm to U and we conclude that

$$\Phi_0 = \mathsf{pm}^{\mathsf{t}} \Psi_0^{-\mathsf{t}} \mathsf{pm} \tag{14}$$

$$A \cdot \Phi_0 = -\Phi_0 \cdot A.$$

where by definition $F|_U$ is the constant matrix Φ_0.

We have a plenty of non holomorphic functions on $\mathcal{L}$. For two elements $\omega_1, \omega_2 \in V_0$ we define:

$$\mathcal{L} \to \mathbb{C},\ x \mapsto \psi_{\mathbb{C}}(x)(\omega_1, \overline{\omega_2}^x).$$

Let ω and δ be as before. We write $\delta = \overline{\mathsf{q}} \cdot \overline{\omega}^x$ and we have

$$G := [\psi_{\mathbb{C}}(x)(\omega_i, \bar{\omega}_j^x)] = \mathsf{pm}^t \Psi_0^{-\mathsf{t}} \overline{\mathsf{pm}} = (\mathsf{q}^{-1})^t \Psi_0 \overline{\mathsf{q}}^{-1} \tag{15}$$

The matrix G satisfies the differential equation:

$$dG = A \cdot G + G \cdot \overline{A}^{\mathsf{t}}, \tag{16}$$

where A is the connection matrix.

3 Quasi-modular forms attached to Hodge structures

In this section we explain what is a quasi-modular form attached to a given fixed data of Hodge structures and a full family of enhanced projective varieties.

3.1 Enhanced projective varieties

Let X be a complex smooth projective variety of a fixed topological type. This means that we fix a C^∞ manifold X_0 and assume that X as a C^∞-manifold is isomorphic to X_0 (we do not fix the isomorphism). Let n be the complex dimension of X and let m be an integer with $1 \leq m \leq n$. We fix an element $\theta \in H^{2n-2m}(X, \mathbb{Z}) \cap H^{n-m,n-m}(X)$. By $H^i(X, \mathbb{Z})$ we mean its image in $H^i(X, \mathbb{C}) = H^i_{\text{dR}}(X)$, therefore, we have killed the torsions. We consider the bilinear map

$$\langle \cdot, \cdot \rangle_{\mathbb{C}} : H^m_{\text{dR}}(X) \times H^m_{\text{dR}}(X) \to \mathbb{C},\ \langle \omega, \alpha \rangle = \frac{1}{(2\pi i)^m} \int_X \alpha \cup \omega \cup \theta.$$

The $2\pi i$ factor in the above definition ensures us that the bilinear map is the complexification of a bilinear map $\langle \cdot, \cdot \rangle_{\mathbb{Z}} : H^m(X, \mathbb{Z}) \times H^m(X, \mathbb{Z}) \to \mathbb{Z}$ (see for instance Deligne's lecture in [3]). We assume that it is non-degenerate. The cohomology $H^m_{\text{dR}}(X)$ is equipped with the so called Hodge filtration $F^\bullet$. We assume that the Hodge numbers $h^{i,m-i},\ i = 0, 1, 2 \dots, m$ coincide with those fixed in this article. We also fix an isomorphism

$$(H^m_{\text{dR}}(X), F^\bullet, \langle \cdot, \cdot \rangle_{\mathbb{C}}) \cong (V_0, F^\bullet_0, \psi_0).$$

From now on, by an enhanced projective variety we mean all the data described in the previous paragraph.

We also need to introduce families of enhanced projective varieties. Let V be an irreducible affine variety and $\mathcal{O}_V$ be the functions $\mathbb{C}$-algebra on V. By definition V is the underlying complex space of $\text{Spec}(\mathcal{O}_{\text{V}})$ and $\mathcal{O}_V$ is a finitely generated reduced $\mathbb{C}$-algebra without zero divisor. Let also $X \to V$ be a family of smooth projective varieties as in the previous paragraph. We will also use the notations $\{X_t\}_{t \in V}$ or X/V to denote $X \to V$. The de Rham cohomology $H^m_{\text{dR}}(X/V)$ and its Hodge filtration $F^\bullet H^m_{\text{dR}}(X/V)$ are $\mathcal{O}_V$-modules (see for instance [7]) and in a similar way we have $\langle \cdot, \cdot \rangle_{\mathcal{O}_V} : H^m_{\text{dR}}(X/V) \times H^m_{\text{dR}}(X/V) \to \mathcal{O}_V$. Note that we fix an element $\theta \in F^{n-m} H^{2n-2m}_{\text{dR}}(X/V)$ and assume that it induces in each fiber X_t an element in $H^{2n-2m}(X_t, \mathbb{Z})$. We say that the family is enhanced if we have an isomorphism

$$(17) \qquad (H^m_{\text{dR}}(X/V),\ F^\bullet H^m_{\text{dR}}(X/V),\ \langle \cdot, \cdot \rangle_{\mathcal{O}_V}) \cong (V_0 \otimes_{\mathbb{C}} \mathcal{O}_V,\ F^\bullet_0 \otimes_{\mathbb{C}} \mathcal{O}_V,\ \psi_0 \otimes_{\mathbb{C}} \mathcal{O}_V).$$

We fix a basis $\omega_i,\ i = 1, 2, \dots, h$ of V_0 compatible with the filtration $F^\bullet_0$. Under the above isomorphism we get a basis $\tilde{\omega}_i,\ i = 1, 2, \dots, h$ of the $\mathcal{O}_V$-module $H^m_{\text{dR}}(X/V)$ which is compatible with the Hodge filtration and the bilinear map $\langle \cdot, \cdot \rangle_{\mathcal{O}_V}$ written in this basis is a constant matrix. This gives us another formulation of enhanced family of projective varieties. An enhanced family of projective varieties $\{X_t\}_{t \in V}$ is full if we have an algebraic action of G_0 (defined in §1.4) from the right on V (and hence on $\mathcal{O}_V$) such that it is compatible with the isomorphism (17). This is equivalent to say that for X_t and $\tilde{\omega}_i,\ i = 1, 2, \dots, h$ as above, we have an isomorphism

$$(X_{tg}, [\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_h]) \cong (X_t, [\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_h] g),\ t \in V,\ g \in G_0,$$

(remember the matrix form of $g \in G_0$ in (6)). A morphism $Y/W \to X/V$ of two families of enhanced projective varieties is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X \arrow[d] \\
W \arrow[r] & V
\end{tikzcd}$$

such that

$$\begin{tikzcd}
H^m(X/V) \arrow[r] \arrow[d] & H^m(Y/W) \arrow[d] \\
V_0 \otimes_{\mathbb{C}} \mathcal{O}_V \arrow[r] & V_0 \otimes_{\mathbb{C}} \mathcal{O}_W
\end{tikzcd}$$

is also commutative.

3.2 Period map

For an enhanced projective variety X, we consider the image of $H^m(X,\mathbb{Z})$ in $H^m(X,\mathbb{C}) \cong H^m_{\mathrm{dR}}(X) \cong V_0$ and hence we obtain a unique point in U. Note that by this process we kill torsion elements in $H^m(X,\mathbb{Z})$. We fix bases ω_i and $\tilde{\omega}_i$ as in §3.1 and a basis $\delta_i,\ i = 1,2,\dots,h$ of $H_m(X,\mathbb{Z}) = H^m(X,\mathbb{Z})^\vee$ with $[\langle \delta_i, \delta_j\rangle] = \Psi_0$ and we see that the corresponding point in $U := \Gamma_{\mathbb{Z}}\backslash P$ is given by the equivalence class of the geometric period matrix $[\int_{\delta_i} \tilde{\omega}_j]$.

For any family of enhanced projective varieties $\{X_t\}_{t\in V}$ we get

$$\mathsf{pm} : V \to U$$

which is holomorphic. It satisfies the so called Griffiths transversality, that is, it is tangent to the Griffiths transversality distribution. It is called a geometric period map. The pull-back of the connection ∇ constructed in §2.3 by the period map pm is the Gauss-Manin connection of the family $\{X_t\}_{t\in V}$. If the family is full then the geometric period map commutes with the action of G_0:

$$\mathsf{pm}(tg) = \mathsf{pm}(t)g,\ g \in G_0,\ t \in V.$$

3.3 Quasi-modular forms

Let M be the set of enhanced projective varieties. We would like to prove that M is in fact an affine variety. The first step in developing a quasi-modular form theory attached to enhanced projective varieties is to solve the following conjectures. Recall that for a enhanced projective variety we have fixed the topological data explained in §3.1.

Conjecture 1. There is an affine variety T and a full universal family X/T of enhanced projective varieties. This mean that for any family of enhanced projective varieties Y/S we have a unique morphism of $Y/S \to X/T$ of enhanced projective varieties.

We would also like to find a universal family which describes the degeneration of projective varieties:

Conjecture 2. There is an affine variety $\tilde{T} \supset T$ of the same dimension as T and with the following property: for any family $f : Y \to S$ of projective varieties with a fixed prescribed topological data, but not necessarily enhanced and smooth, and with the discriminant variety $\Delta \subset S$, the map $Y\backslash f^{-1}(\Delta) \to S\backslash\Delta$ is an underlying morphism of an enhanced family, and hence, we have the map $S\backslash\Delta \to T$ which extends to $S \to \tilde{T}$.

Similar to Shimura varieties, we expect that T and $\tilde{T}$ are affine varieties defined over $\bar{\mathbb{Q}}$. Both conjectures are true in the case of elliptic curves (see the discussion in the Introduction). The function ring of T (resp. $\tilde{T}$) is $\mathbb{C}[t_1, t_2, t_3, \frac{1}{27t_3^2-t_2^3}]$ (resp. $\mathbb{C}[t_1, t_2, t_3]$). We have also verified the conjectures for a particular class of Calabi-Yau varieties (see §4.2 and [13]).

Now, consider the case in which both conjectures are true. We are going to explain the rough idea of the algebra of quasi-modular forms attached to all fixed data that we had. It is the pull-back of the $\mathbb{C}$-algebra of regular functions in $\tilde{T}$ by the composition:

$$\mathbb{H} \overset{i}{\hookrightarrow} P|_{\mathrm{Im}(\mathsf{pm})} \to U\,|_{\mathrm{Im}(\mathsf{pm})} \overset{\mathsf{pm}^{-1}}{\to} T \hookrightarrow \tilde{T}. \tag{18}$$

We need that the period map is local injective (local Torelli problem) and hence pm^{-1} is a local inverse map. The set $\mathbb{H}$ is a subset of the set of period matrices P and it will play the role of the Poincaré upper half plane. If the Griffiths period domain D is Hermitian symmetric then it is biholomorphic to D (see 4.1), however, in other cases it depends on the universal period map $T \to U$ and its dimension is the dimension of the deformation space of the projective variety. In this case we do no need to define $\mathbb{H}$ explicitly (see 4.2). More details of this discussion will be explained by two examples of the next section.

4 Examples

In this section we discuss two examples of Hodge structures and the corresponding quasi-modular form algebras: those attached to Calabi-Yau mirror quintic type and principally polarized Abelian varieties. The details of the first case is done in [13] and we will sketch the results which are related to the main stream of the present text. For the second case there are many works to be done and I only sketch some ideas. Much of the works for K3 surfaces endowed with N-polarizations is done in [2] and the generalization of the results obtained in this article to Siegel quasi-modular forms is a work for future.

4.1 Siegel quasi-modular forms

We consider the case in which the weight m is equal to 1 and the polarization matrix is:

$$\Psi_0 = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix. In this case $g := h^{10} = h^{01}$ and $h = 2g$. We take a basis $\omega_i,\ i = 1, 2, \ldots, 2g$ of V_0 compatible with $F_0^\bullet$, that is, the first g elements form a basis of F_0^1. We further assume that the polarization $\psi_0 : V_0 \times V_0 \to \mathbb{C}$ in the basis ω has the form $\Phi_0 := \Psi_0$. Because of the particular format of Ψ_0, both these assumptions are not in contradiction with each other. We take a basis δ of $V_{\mathbb{Z}}(x)^\vee$ such that the intersection form in this basis is of the form Ψ_0 and we write the associated period matrix in the form:

$$[\int_{\delta_i} \omega_j] = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

where $x_i, i = 1, \ldots, 4$ are $g \times g$ matrices. Since $\Psi_0^{-\mathsf{t}} = \Psi_0$, we have

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}}x_1 + x_1^{\mathsf{t}}x_3 & -x_3^{\mathsf{t}}x_2 + x_1^{\mathsf{t}}x_4 \\ -x_4^{\mathsf{t}}x_1 + x_2^{\mathsf{t}}x_3 & -x_4^{\mathsf{t}}x_2 + x_2^{\mathsf{t}}x_4 \end{pmatrix}$$

and

$$[\langle \omega_i, \bar{\omega}_j^x\rangle] = \begin{pmatrix} x_1^{\mathsf{t}} & x_3^{\mathsf{t}} \\ x_2^{\mathsf{t}} & x_4^{\mathsf{t}} \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 \\ \bar{x}_3 & \bar{x}_4 \end{pmatrix} = \begin{pmatrix} -x_3^{\mathsf{t}}\bar{x}_1 + x_1^{\mathsf{t}}\bar{x}_3 & -x_3^{\mathsf{t}}\bar{x}_2 + x_1^{\mathsf{t}}\bar{x}_4 \\ -x_4^{\mathsf{t}}\bar{x}_1 + x_2^{\mathsf{t}}\bar{x}_3 & -x_4^{\mathsf{t}}\bar{x}_2 + x_2^{\mathsf{t}}\bar{x}_4 \end{pmatrix}.$$

The properties P1, P2 and P3 are summarized in the properties

$$x_3^{\mathsf{t}}x_1 = x_1^{\mathsf{t}}x_3,\ -x_3^{\mathsf{t}}x_2 + x_1^{\mathsf{t}}x_4 = I_g,$$

$$x_1, x_2 \in \mathrm{GL}(g, \mathbb{C}),$$
$$-\sqrt{-1}(-x_3^{\mathsf{t}}\bar{x}_1 + x_1^{\mathsf{t}}\bar{x}_3) \text{ is a positive matrix.}$$

By definition P is the set of all $2g \times 2g$ matrices $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ satisfying the above properties: The matrix $x := x_1 x_2^{-1}$ is well-defined and invertible which satisfies the famous Riemann relations:

$$x^{\mathsf{t}} = x,\ \mathrm{Im}(x) \text{ is a positive matrix.}$$

The set of matrices $x \in \mathrm{Mat}^{g\times g}(\mathbb{C})$ with the above properties is called the Siegel upper half plane and is denoted by $\mathbb{H}$. We have $U = \Gamma_{\mathbb{Z}}\backslash P$, where

$$\Gamma_{\mathbb{Z}} = \mathrm{S}p(2g, \mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{Z}) \mid ab^{\mathsf{t}} = ba^{\mathsf{t}},\ cd^{\mathsf{t}} = dc^{\mathsf{t}},\ ad^{\mathsf{t}} - bc^{\mathsf{t}} = I_g\}.$$

We have also

$$G_0 = \{\begin{pmatrix} k & k' \\ 0 & k^{-\mathsf{t}} \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{C}) \mid kk'^{\mathsf{t}} = k'k^{\mathsf{t}}\}$$

which acts on P from the right. The group $\mathrm{S}p(2g, \mathbb{Z})$ acts on $\mathbb{H}$ by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax + b)(cx + d)^{-1},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{S}p(2g, \mathbb{Z}),\ x \in \mathbb{H}$$

and we have the isomorphism

$$U/G_0 \to \mathrm{S}p(2g, \mathbb{Z})\backslash\mathbb{H},$$

given by

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \to x_1 x_3^{-1}.$$

To each point x of P we associate a triple $(A_x, \theta_x, \alpha_x)$ as follows: We have $A_x := \mathbb{C}^g/\Lambda_x$, where Λ_x is a $\mathbb{Z}$-submodule of $\mathbb{C}^g$ generated by the rows of x_1 and x_3. We have cycles $\delta_i \in H_1(A_x, \mathbb{Z}),\ i = 1, 2, \ldots, 2g$ which are defined by the property $[\int_{\delta_i} dz_j] = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, where $z_j,\ j = 1, 2, \ldots, g$ are linear coordinates of $\mathbb{C}^g$. There is a basis $\alpha_x = \{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ of $H^1_{\mathrm{dR}}(A_x)$ such that

$$[\int_{\delta_i} \alpha_j] = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The polarization in $H_1(A_x, \mathbb{Z}) \cong \Lambda_x$ (which is defined by $[\langle\delta_i, \delta_j\rangle] = \Psi_0$) is an element $\theta_x : H^2(A_x, \mathbb{Z}) = \wedge_{i=1}^2 \mathrm{Hom}(\Lambda_x, \mathbb{Z})$. It gives the following bilinear map

$$\langle\cdot,\cdot\rangle : H^1_{\mathrm{dR}}(A_x) \times H^1_{\mathrm{dR}}(A_x) \to \mathbb{C},\ \langle\alpha, \beta\rangle = \frac{1}{(2\pi i)^2}\int_{A_x} \alpha \cup \beta \cup \theta_x^{g-1}$$

which satisfies $[\langle\alpha_i, \alpha_j\rangle] = \Psi_0$.

The triple $(A_x, \theta_x, \alpha_x)$ that we constructed in the previous paragraph does not depend on the action of $\mathrm{S}p(2g, \mathbb{Z})$ from the left on P, therefore, to each $x \in U$ we have constructed such a triple. In fact U is the moduli of the triples (A, θ, α) such that A is a principally polarized abelian variety with a polarization θ and α is a basis of $H^1_{\mathrm{dR}}(A)$ compatible with the Hodge filtration $F^1 \subset F^0 = H^1_{\mathrm{dR}}(A)$ and such that $[\langle\alpha_i, \alpha_j\rangle] = \Psi_0$.

We constructed the moduli space U in the framework of complex geometry. In order to introduce Siegel quasi-modular forms, we have to study the same moduli space in the framework of algebraic geometry. We have to construct an algebraic variety T over $\mathbb{C}$ such that the points of T are in one to one correspondence with the equivalence classes of the triples (A, θ, α). We also expect that T is an affine variety and it lies inside another affine variety $\tilde{T}$ which describes the degeneration of varieties (as it is explained in §3.3). The pull-back of the $\mathbb{C}$-algebra of regular functions on $\tilde{T}$ through the composition

$$\mathbb{H} \to P \to U \overset{\mathsf{pm}^{-1}}{\to} T \hookrightarrow \tilde{T}$$

is, by definition, the $\mathbb{C}$-algebra of Siegel quasi-modular forms. The first map is given by

$$z \to \begin{pmatrix} z & -I_g \\ I_g & 0 \end{pmatrix}$$

and the second is the canonical map. The period map in this case is a biholomorphism. If we put a functional property for f regarding the action of G_0 then this will be translated into a functional property of a Siegel quasi-modular form with respect to the action of $\mathrm{S}p(2g, \mathbb{Z})$. In this way we can even define a Siegel quasi-modular form defined over $\bar{\mathbb{Q}}$ (recall that we expect $\tilde{T}$ to be defined over $\bar{\mathbb{Q}}$). It is left to the reader to verify that the $\mathbb{C}$-algebra of Siegel quasi-modular forms contains the classical Siegel modular forms and it is closed under derivations with respect to z_{ij} with $z = [z_{ij}] \in \mathbb{H}$. For the realization of all these in the case of elliptic curves, $g = 1$, see the Introduction and [14]. See the books [10, 4, 12] for more information on Siegel modular forms.

4.2 Hodge numbers, 1,1,1,1

In this section we consider the case $m = 3$ and the Hodge numbers $h^{30} = h^{21} = h^{12} = h^{03} = 1,\ h = 4$. The polarization matrix written in an integral basis is given by:

$$\Psi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us fix a basis $\omega_1, \omega_2, \omega_3, \omega_4$ of V_0 compatible with the Hodge filtration $F_0^\bullet$, a basis $\delta_1, \delta_2, \delta_3, \delta_4 \in V_{\mathbb{Z}}(x)^\vee$ with the intersection matrix Ψ_0 and let us write the period matrix in the form $\mathsf{pm}(x) = [x_{ij}]_{i,j=1,2,\ldots,4}$. We assume that the polarization ψ_0 in the basis ω_i is given by the matrix:

$$\Phi_0 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The algebraic group G_0 is defined to be

$$G_0 := \left\{ g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{44} \end{pmatrix},\ g^{\mathsf{t}}\Phi_0 g = \Phi_0,\ g_{ij} \in \mathbb{C} \right\}.$$

We consider the subset $\tilde{\mathbb{H}}$ of P consisting of matrices:

$$\begin{pmatrix} \tau & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 \\ x_{41} & -\tau x_{32} + x_{31} & -\tau & 1 \end{pmatrix}$$

where τ is some variable in $\mathbb{C}$ defined in a neighborhood of $+\sqrt{-1}\infty$. The particular expressions for the $(4,2)$ and $(4,3)$ entries of the above matrix follow from the polynomial relations (14) between periods. The connection matrix A restricted to $\tilde{\mathbb{H}}$ is

$$d\mathsf{pm}^{\mathsf{t}} \cdot \mathsf{pm}^{-\mathsf{t}} \mid_{\tilde{\mathbb{H}}} = \begin{pmatrix} 0 & d\tau & -x_{32}d\tau + dx_{31} & -x_{31}d\tau + \tau dx_{31} + dx_{41} \\ 0 & 0 & dx_{32} & -x_{32}d\tau + dx_{31} \\ 0 & 0 & 0 & -d\tau \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Griffiths transversality distribution is given by $-x_{32}d\tau + dx_{31} = 0,\ -x_{31}d\tau + \tau dx_{31} + dx_{41} = 0$ and so, if we consider τ as an independent parameter and all other quantities x_{ij} depending on τ, then we have

$$x_{32} = x_{31}',\ x_{41}' = x_{31} - \tau x_{31}'. \tag{19}$$

In [13] we have checked the conjectures in §3.3 for the Calabi-Yau three-folds of mirror quintic type. In this case $\dim(T) = 7$ and hence we have constructed an algebra generated by seven functions in τ. We have

$$x_{31} = \frac{1}{2}(5(\tau + \tau^2) + \frac{1}{(2\pi i)^2}(\sum_{n=1}^{\infty}(\sum_{d|n} n_d d^3)\frac{e^{2\pi i \tau n}}{n^2})) \tag{20}$$

Here, n_d's are instanton numbers and the second derivative of x_{31} with respect to τ is the Yukawa coupling. The set $\mathbb{H}$ is a subset of $\tilde{\mathbb{H}}$ defined by (19) and (20). As far as I know this is the first case in which the Griffiths period domain is not Hermitian symmetric and we have an attached algebra of quasi-modular forms and even the Global Torelli problem is true, that is, the period map is globally injective (see [5]). However, note that in [13] we have only used the local injectivity of the period map.

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