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AN EXPRESSION FOR THE HOMFLYPT POLYNOMIAL AND SOME RELATED PROPERTIES

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ABSTRACT

Associated with each oriented link is the two variable Homflypt polynomial. Formulas for the coefficient polynomials of the three lowest v-degrees are presented that shows they are determined by the writhe of any braid diagram for the link, the Conway polynomial for the link, and the remaining coefficient polynomials.

These Homflypt coefficient polynomials in z satisfy a system of linear equations with coefficients in $\mathbb{Z}[z]$. The Conway polynomial is essentially the unique Laurent polynomial that represents such a linear combination and is also a link invariant; any other is merely the product of the Conway polynomial and an arbitrary second polynomial. Two other independent functions that represent such a linear combination are determined by the writhe and are not link invariants.

Properties of the coefficient polynomials and Conway polynomial are also described. These include upper and lower bounds on their degree for various classes of links, the sign of the leading coefficients, and some conditions that ensure all coefficients have the same sign as the highest degree term. An explicit formula is presented for a subclass of links generated by alternating braids that are analogs to torus links.

Keywords: Homflypt polynomial, Conway polynomial, skein relation, braids and braid groups, Markov stabilization

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

The focus of this paper is to develop and display a relationship between the coefficient polynomials of the Homflypt polynomial, and to establish some properties of the the coefficient polynomials for various classes of braid words. The primary tools used are the defining skein relation and properties of the braid group.

The background required for this paper is a very basic understanding of skein relations, skein polynomials, and braid groups. Such material may be found in any introductory textbook, as for example K. Murasugi's, [17], and may also be found in many excellent surveys, [3]. However, nothing beyond this minimal background is assumed.

The attraction in using a braid word as the starting point for naming a link is that it not only immediately informs the reader of the exact link under discussion, but the braid word itself may be manipulated using the braid relations for the braid group. Of course, the downside in this approach is that there are (too) many braid words that describe the same link. Nevertheless, there is still much to be gained from the braid approach, as this paper hopes to demonstrate.

The immediate reaction of a student upon first encountering the skein polynomials is probably one of utter amazement that such entities exist; at least that was the significance for the author. However a few exercises in using a skein tree to calculate a skein polynomial quickly show how difficult this technique is to use, in practice, for an arbitrary link. Since finding useful closed form expressions to calculate skein polynomials for an arbitrary link is probably unattainable, it is natural to try to understand what are their properties and which invariants determine their form. Questions about the structure of these polynomials and how their coefficients relate to a braid representation are steps to better understand what information the skein polynomials convey about links.

The remainder of this section will introduce the terms and notation used in the paper. The second section will introduce the primary results, whose proofs will appear in the third section. The third section will also introduce the necessary tools and formulas to facilitate the proofs. The fourth, and final, section will introduce a number of open questions and conjectures.

1.1. *Definitions and Notation*

The conventions used in this paper largely follow those in, [17], to which the reader is referred for expanded discussion. A brief review of the standard symbols and terminology used in the paper is given here for reference. Some further definitions that only apply to Section 2.2 appear in Section 2.2.1.

The braid group on n strands is designated B_n and has $n-1$ standard generators, σ_i. Borrowing terminology from A. Stoimenow, [20], the term edge generator refers to either σ_1 or σ_{n-1} . This paper refers to the other generators as interior generators. There is a key distinction to be made between a braid word, typically called β, in B_n, and its use as a representative of some braid in the braid group, denoted $[\beta]$, that represents the equivalence class of braid words under the defining braid relations. Typically a braid word is presented, or arises from some process, such as skein tree calculation. It is often convenient to find a "better" representative of the same braid (or for the same link) under some criterion related to induction or some attribute. The following definitions present a number of such braid word attributes.

Definition 1.1. If a braid word, $\beta \in B_n$, has the expression $\prod_{k=1}^{m} \sigma_{i_k}^{\epsilon_k}$, with $\epsilon_k = \pm 1$ for each subscript, k,

(i) the braid length is m and is denoted $|\beta|$,
(ii) the positive (negative) crossing count, $x_p(\beta)$ or just x_p (for negative read $x_n(\beta)$ or x_n) is the number of $\epsilon_k = 1$ (for negative read $\epsilon_k = -1$). Hence the braid length is the sum of the positive and negative crossing counts,

(iii) the exponent sum, or writhe, denoted w or $w(\beta)$, is $\sum_{k=1}^{m} \epsilon_k$. Hence the writhe is the difference of the positive and negative crossing counts,
(iv) a syllable of a braid word is a maximal subword of identical subscripts (compare [21]). A syllable is reduced when all its exponents have the same sign. A braid word is in standard form when all syllables are reduced.

The writhe is actually associated with the braid diagram, rather than the braid word itself, but this imprecision in usage causes no problems.

Definition 1.2. If a braid word, $\beta \in B_n$, is in standard form and has the expression $\prod_{k=1}^{m} \sigma_{i_k}^{\epsilon_k}$, with $\epsilon_k = \pm 1$ for each subscript, k,

(i) a row of β is a maximal subword with non-increasing subscripts; e.g. $\sigma_3\sigma_2\sigma_2\sigma_1^{-1}\sigma_5$ has two rows, i.e. maximal subwords $\sigma_3\sigma_2\sigma_2\sigma_1^{-1}$, and σ_5,
(ii) the rank of β is the number of its rows, written $\rho(\beta)$.
(iii) a generator in β whose exponents are all of the same sign is called a homogeneous generator; otherwise it is termed a mixed generator. The number of generators whose corresponding exponents are all positive, all negative, mixed is denoted respectively: $\nu_p(\beta)$ or ν_p; $\nu_n(\beta)$ or ν_n; $\nu_m(\beta)$ or ν_m,
(iv) any generator of B_n absent from β is called a null generator. The number of null generators for β is denoted $\nu_0(\beta)$ or ν_0,
(v) any generator that appears in a unique syllable is called a clustered generator,
(vi) any generator that appears exactly once in β is called a trivial generator. The number of trivial generators whose corresponding exponent is one (minus one) is denoted $\nu_1(\beta)$ or ν_1 ($\nu_{-1}(\beta)$ or ν_{-1}),
(vii) a set of generators with subscripts in a range, (a, b), with $0 \le a \le b \le n$, yields a new braid word, $\Psi(a, b, \beta) = \prod_{i_k \in (a,b)} \sigma_{i_k - a}^{\epsilon_k}$, that belongs to B_{b-a}. $\Psi(a, b, \beta)$ is defined as the identity for B_1 when $b = a$ or $b = a + 1$.

An n-braid word whose twist exponents are all zero is defined to have zero for each of the properties in the prior definitions, except that $\nu_0 = n - 1$.

The link associated with the standard closure of a braid, β, is denoted $\widehat{\beta}$. The mirror image of a link, L, is denoted $\overline{L}$; similarly for the mirror image of a braid word, β, denoted $\overline{\beta}$. The number of link components in L is denoted $\mu(L)$. O_n denotes the trivial link with n components.

The link diagrams referenced in this paper will typically be associated with a braid word, but the skein relation, (1.1), for the Homflypt polynomial, P, is defined more generally to allow any valid oriented diagram, D, for the link. The diagrams, D_+, D_0, and D_- below refer to the usual diagrams for the link with positive crossing, null (smoothed) crossing, and negative crossing, respectively.

$$P_{D_+}(v, z) = vzP_{D_0}(v, z) + v^2 P_{D_-}(v, z). \tag{1.1}$$

The importance of the Homflypt polynomial derives from the fact it is the same for all diagrams of a given link, and so is an invariant of the oriented link.

The major skein polynomials and their definitions are: the Conway polynomial, $\nabla_D(z) = P_D(1, z)$, the Jones polynomial, $V_D(t) = P_D(t, (t-1)/\sqrt{t})$, and the Alexander polynomial, $\Delta_D = P_D(1, (t-1)/\sqrt{t})$.

The torus links are much studied due to their uniformity, symmetry, and other properties. The torus links are characterized by the number of full rows of twists, p, together with the number of strands, q, denoted $K(p, q)$. Each row may be realized by a braid word, $\prod_{i=q-1}^{1} \sigma_i = \sigma_{q-1} \cdots \sigma_1$, hereafter called α_{q-1}. Hence $K(p, q)$, may be represented by the braid word, α_{q-1}^p. The elementary torus links, denoted T_p, are those with two strands. The corresponding Conway polynomial is so central to the results of this paper that the lengthy expression, $\nabla_{K(p,2)}(z)$, will be shortened to $C_p(z)$ or even C_p. Among many other interesting properties of the Conway polynomial (Section 3.3), $C_p(1)$ is the p-th Fibonacci number.

A class of links with some resemblance to the torus links are those in which there are p full rows of twists on the q strands, but the twists alternate in sign. When the positive crossings are associated with the odd subscript generators, the link is denoted $K_\pm(p, q)$ with corresponding braid word description, $\alpha_{q-1,\pm}^p$.

2. The Primary Results

Given that the Homflypt polynomial is a function of two variables, one obvious way to organize its expression is as a polynomial in a single variable, with coefficients that are simply polynomials in the second variable. The question as to which variable should be the primary organizing factor may depend on circumstances, but a powerful motivation exists to chose v from (1.1). This is due to the remarkable results, [5], [15], known as the Morton-Franks-Williams inequality, that give bounds on the possible values for the highest and lowest powers of v. This paper builds on this inequality, as have papers by T. Kalman, [9], and A. Stoimenow, [20], [21].

The first step is to observe that the Homflypt polynomial for a link with a given braid representation ($\beta \in B_n$) may be organized in a standard form, (2.1), in which the p_j are ordinary polynomials with integer coefficients. An equivalent form appears in (2.2), in which the h_j, are Laurent polynomials with integer coefficients, and $h_j = p_j/z^{n-1}$. When multiple braid words are under discussion, the symbols $p_{j,\widehat{\beta}}$, or $h_{j,\widehat{\beta}}$, may be used for clarity; note that the argument, z, may be omitted. Both forms highlight the number of strands in the braid, and the writhe, w, of the braid diagram (exponent sum of the braid word). Theorem 2.6 describes the relations among the coefficient polynomials, p_j, defined by (2.1).

$$P_{\widehat{\beta}}(v, z) = \frac{v^w \sum_{j=0}^{n-1} p_j\,(z)\, v^{2j}}{(vz)^{n-1}}\,, \tag{2.1}$$

$$P_{\widehat{\beta}}(v, z) = v^{w-n+1} \sum_{j=0}^{n-1} h_j(z)\, v^{2j}\,. \tag{2.2}$$

The Homflypt formula for the elementary torus links is much simpler:

$$P_{K(p,2)}(v,z) = \frac{v^p\{\, C_{p+1}(z) - C_{p-1}(z)\, v^2\}}{(vz)^1}\,. \tag{2.3}$$

The Conway polynomial for the elementary torus links has an expression:

$$C_p(z) = \sum_{j=0}^{\lfloor\frac{p-1}{2}\rfloor} \binom{p-1-j}{j} z^{p-2j-1}\,, \text{ for } p > 0. \tag{2.4}$$

When p is negative, we have $C_p(z) = (-1)^{p+1}\, C_{-p}(z)$, and $C_0(z) = 0$.

In 1987, V.F.R. Jones published a number of results relating properties of Hecke algebras to knot theory, [8]. This landmark research displayed, among other insights, how the two variable skein polynomial could be calculated from the Burau representation of a braid. Unfortunately for knot theorists, the expression for the two variable skein polynomial is of nontrivial complexity. A further wrinkle in using the methodology is it requires knowledge of the number of link components, or at least whether this number is even or odd. For three-braid links, this paper first established the Homflypt polynomial is dependent only on the writhe and the Alexander polynomial. An equivalent result can be derived using only skein and braid properties, without knowledge or use of Theorem 2.6 or Theorem 2.7:

Lemma 2.1. *The Homflypt polynomial for a link, $\widehat{\beta}$, with $\beta \in B_3$ is*

$$P_{\widehat{\beta}} = P_{T_w}\, P_{O_2} - \nabla_{\widehat{\beta}}\; v^w\, (P_{O_3} - 1)\,. \tag{2.5}$$

It is readily apparent that substitution of $v = 1$ in Eq. (2.5), yields the identity $P_{\widehat{\beta}}(1,z) = \nabla_{\widehat{\beta}}(z)$. Three immediate consequences of (2.5) follow. For three-braid knots Eq. 8.4, p. 356, [8], is equivalent to Prop. 2.3, which is valid for all three-braid links. In relation to Prop. 2.4, Prop. 2.9 describes when $\nabla_{\widehat{\beta}} = C_{w(\beta)-1}$, and Cor. 2.14 shows $\nabla_{\widehat{\beta}} = 1$ only for the trivial knot.

Proposition 2.2. *If three-braid words, β and γ, have the same writhe, and $\widehat{\beta}$ and $\widehat{\gamma}$ have the same Conway, Jones, or Alexander polynomials, we have $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$.*

Proposition 2.3. *Three-braid links, $\widehat{\beta}$, with $\beta \in B_3$, satisfy (2.1) with*

(i) $p_0 = C_{w+1} - \nabla_{\widehat{\beta}}$,
(ii) $p_1 = z^2\,\nabla_{\widehat{\beta}} - p_0 - p_2$,
(iii) $p_2 = C_{w-1} - \nabla_{\widehat{\beta}}$.

Proposition 2.4. *When three-braid words, β, γ, satisfy $w(\beta) \geq w(\gamma)$, we have $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$ exactly when $\nabla_{\widehat{\beta}} = \nabla_{\widehat{\gamma}}$ and one of the following is true:*

(i) $w(\beta) = w(\gamma)$, *or*
(ii) $w(\beta) = w(\gamma) + 2$, *and* $\nabla_{\widehat{\beta}} = C_{w(\beta)-1}$ *or*
(iii) $w(\beta) = 2$, $w(\gamma) = -2$, *and* $\nabla_{\widehat{\beta}} = 1$.

The Jones polynomial for three-braid links may be derived from Eq. 2.5. Eq. 2.6 agrees with the knot formula in Prop. 11.10, p. 366, [8], but has an extra factor, $(-1)^w$, that handles all three-braid links. Eq. 2.6 also implies Prop. 2.16.

$$V_{\widehat{\beta}}(t) = t^{(w-2)/2}\{t^{w+1}+(-\mathbf{1})^{\mathbf{w}}(1+t+t^2)\}-(1+t+t^2)t^{w-1}\Delta_{\widehat{\beta}}(t)\,, \text{ for } \beta \in B_3. \quad (2.6)$$

An interesting pattern in the Homflypt coefficient polynomials is that in plotting the degree of p_j on the vertical axis, and the subscript, j, on the horizontal axis $(0 \leq j \leq n-1)$, the points outline what looks like the profile of a landscape. In this paper, the convention for the constant zero is that it has negative degree. In the analogy at hand, $p_j = 0$ corresponds to a point below sea level. The remarkable feature of this landscape is that there are, apparently, no valleys. For the case of three-braids, this is a consequence of Prop. 2.3, and has the following form that suggests a more general unproven result:

Proposition 2.5. *For each braid word, $\beta \in B_3$, the subscripts of p_j (zero, one, two) are the union of two disjoint subranges, L and R, with the following property:*

(i) the degree of p_j is non-decreasing on L and when L is non-empty, $0 \in L$,
(ii) the degree of p_j is non-increasing on R and when R is non-empty, $2 \in R$.

The prior decomposition can be made to be unique with a requirement that one of the subranges always includes all subscripts corresponding to the maximum degree among all p_j. Many knots have plateaus in the maximum degree among the p_j.

Generalizing the results of Proposition 2.3 to higher strand number shows the lowest v-degree coefficient polynomials, $p_0(z)$, $p_1(z)$, and $p_2(z)$ are determined by the writhe, the Conway polynomial and the $p_j(z)$ for $j \geq 3$. Before going to the general formula, the four-braid result is presented as a prelude:

(i) $p_0 = z\{\, C_w - \nabla_{\widehat{\beta}}\} - p_3\,,$
(ii) $p_1 = z^3\,\nabla_{\widehat{\beta}} - p_0 - p_2 - p_3\,,$
(iii) $p_2 = z\{\, C_{w-2} - \nabla_{\widehat{\beta}}(z)\} - (z^2+3)\,p_3\,.$

Now on to one of the main results, which is an immediate corollary to Thm. 2.7:

Theorem 2.6. *For an arbitrary braid, β, of $n \geq 1$ strands, the Homflypt polynomial for $\widehat{\beta}$ is given by (2.1) with:*

(i) $p_0 = z^{n-3}\{\, C_{w+4-n} - \nabla_{\widehat{\beta}}\} - q_0\,,$ *with* $q_0 = \sum_{j=3}^{n-1}\, z^{-2}(C_{2j-3}-1)\,p_j\,,$
(ii) $p_1 = z^{n-1}\,\nabla_{\widehat{\beta}} - p_0 - p_2 - \sum_{j=3}^{n-1}\, p_j\,,$
(iii) $p_2 = z^{n-3}\{\, C_{w+2-n} - \nabla_{\widehat{\beta}}(z)\} - q_2\,,$ *with* $q_2 = \sum_{j=3}^{n-1}\, z^{-2}(C_{2j-1}-1)\,p_j\,.$

In the formula for q_0, we have:

$$\frac{C_{2j-3}-1}{z^2} = \sum_{i=0}^{j-3} \binom{2j-4-i}{i} z^{2j-6-2i}, \textit{ for } j \geq 3\,.$$

Similarly, in the formula for q_2*, we have:*

$$\frac{C_{2j-1}-1}{z^2} = \sum_{i=0}^{j-2} \binom{2j-2-i}{i} z^{2j-4-2i}, \text{ for } j \geq 1\,.$$

A few comments are in order about this theorem. First, the formula for the trivial knot on one strand depends on a writhe of zero. Second, the formula for two strands is equivalent to (2.3). Third, p_1 is exactly what it has to be in order for the identity, $P_{\widehat{\beta}}(1,z) = \nabla_{\widehat{\beta}}(z)$, to be satisfied.

Unfortunately, Theorem 2.6 leaves open the question of whether there are any relations among the higher order coefficient polynomials, p_j, for $j \geq 3$. Additionally, and more critically, there is no hint of the information content within each such p_j.

However, a closer inspection of the forms for p_j, q_0, and q_2, shows a new relation is satisfied. It is expected that the sum of the Laurent coefficient polynomials will be the Conway polynomial, but more is true. The following theorem introduces a system of linear equations, with coefficients in the ring of polynomials over the integers, that is satisfied by the Laurent coefficient polynomials, h_j .

Theorem 2.7. *The Laurent coefficient polynomials for a link,* $\widehat{\beta}$*, with* $\beta \in B_n$*, satisfy the following relations. These are independent for* $n > 1$.

$$\sum_{j=0}^{n-1} C_{2j-3}\, h_j = C_{4+w-n}\,, \tag{2.7}$$

$$\sum_{j=0}^{n-1} C_{2j-1}\, h_j = C_{2+w-n}\,. \tag{2.8}$$

Any set of n functions, $\{f_j\}_0^{n-1}$*, that satisfy both equations, will also satisfy the following family of equations for each integer,* κ*:*

$$\sum_{j=0}^{n-1} C_{2j-1-\kappa}\, f_j = (-1)^{\kappa}\, C_{\kappa+2+w-n}\,. \tag{2.9}$$

In the prior theorem, independent means that for a fixed choice of w and n, an arbitrary set of Laurent polynomials with integer coefficients, $\{f_j\}$, that satisfy one of the equations need not satisfy the other equation. In fact, these equations are independent over the smaller set of ordinary polynomials with integer coefficients.

There are thus three independent equations satisfied by the n Laurent coefficient polynomials; two from the Theorem 2.7, and the simple equation relating the sum of the coefficient polynomials to the Conway polynomial. It would clearly be desirable to find any further such relations.

Suppose some linear equation exists in the form of Theorem 2.7 for each value of n, with coefficients that are Laurent polynomials over the integers. As each new strand is added, a new coefficient (A_{n-1} below) arises to match the new Homflypt coefficient polynomial, h_{n-1}. This assumption yields a relation for each n:

$$\sum_{j=0}^{n-1} A_j\, h_j = \Omega_{n,\beta}\,, \text{ with } A_j \in \mathbb{Z}[z, z^{-1}]\,. \tag{2.10}$$

As any two conjugate braid words in B_n produce the same set of h_j, the value of $\Omega_{n,\beta}$ must be identical for them. Any pair of braid words that generate the same braid element in B_n also have this same property, so $\Omega_{n,\beta}$ is a function on the conjugacy classes of braid elements. In fact, by (2.2), any two braid elements, even in different braid groups, with identical $w_i - n_i$ and whose closure is the same link, will have $\Omega_{n_1,\beta_1} = \Omega_{n_2,\beta_2}$. As the first $n-1$ coefficient polynomials, h_j, are identical for $\beta \in B_n$ and $\beta\sigma_n \in B_{n+1}$, and $h_{n,\widehat{\beta\sigma_n}} = 0$, the left side of (2.10) is invariant under positive Markov stabilization, so $\Omega_{n,\beta} = \Omega_{n+1,\beta\sigma_n}$.

If $\Omega_{n,\beta}$ is also invariant under negative Markov stabilization, $\Omega_{n,\beta}$ is a link invariant. In this case, observe that for the trivial knot, $A_0 = \Omega_{2,\sigma_1}$, and $A_1 = \Omega_{2,\sigma_1^{-1}}$, so $A_0 = A_1$. This quickly leads to the conclusion all A_j are equal to A_0, so (2.10) becomes $A_0\, \nabla_{\widehat{\beta}} = \Omega_{n,\beta}$, i.e. $\Omega_{n,\beta}$ is a multiple of $\nabla_{\widehat{\beta}}$.

If $\Omega_{n,\beta}$ is not invariant under negative Markov stabilization, a new family of relations is created. First observe that by Prop. 3.6, $\sum_{j=0}^{n-1} A_{j+1}\, h_j = \Omega_{n+1,\beta\sigma_n^{-1}}$, for every $\beta \in B_n$. By repeating the negative Markov stabilization κ times, we have

$$\sum_{j=0}^{n-1} A_{j+\kappa}\, h_j = \Omega_{n+\kappa,\beta \prod_{i=n}^{n-1+\kappa} \sigma_i^{-1}}\,.$$

Also observe that the functions, $\Omega_{n,\beta}$, obey the same skein relation as the Conway polynomial due to linearity of the relations, and the skein property for h_j.

The trivial knot on n strands has a well-known representation when the length of the braid word matches the writhe, i.e. α_{n-1}, and in this case $A_0 = \Omega_{n,\alpha_{n-1}}$. However, there are multiple representations of the trivial knot in B_n with length $n-1$ and constant writhe. Critically, all these representations with the same writhe have the same set of coefficient polynomials, h_j. Observe that any such choice of braid word, ω_j, with j trivial negative generators, and $n-1-j$ trivial positive generators, satisfies $A_j = \Omega_{n,\omega_j}$, since $h_{k,\widehat{\omega_j}} = \delta_{j,k}$.

It is instructive to investigate the behavior of $\Omega_{n,\beta}$ for the elementary torus links. Eq. 2.3 shows that for $T_{\kappa+2}$, we have $h_0 = C_{\kappa+3}/z$ and $h_1 = -C_{\kappa+1}/z$, and

$$\{A_0\, C_{\kappa+3} - A_1\, C_{\kappa+1}\}/z = \Omega_{2,\,\sigma_1^{\kappa+2}}\,. \tag{2.11}$$

Suppose $\Omega_{n,\beta}$ depends only on $w-n$, as is true for the braid words of length $n-1$ with all trivial generators. Then $\Omega_{n,\beta} = \Omega_{2,\,\sigma_1^{\kappa+2}}$ with $\kappa = w-n$ and $A_j = \Omega_{n,\omega_j} = \{A_0\, C_{-2j+2} - A_1\, C_{-2j}\}/z$. Substitution of these values in (2.10) and use of $\kappa = 1, -1$ in Eq. 2.9 reveals an identity, i.e. any $A_0, A_1 \in \mathbb{Z}[z, z^{-1}]$ are valid. In other words, $\Omega_{n,\beta}$ is merely a linear combination (with weights $A_0/z, -A_1/z$) of functions in Thm. 2.7, and the same is true for each A_j.

The prior discussion is summarized in:

Theorem 2.8. *Any function, $\Omega_{n,\beta}$, defined by a linear relation in the $\{h_j\}$ as in (2.10), satisfies one of the following three conditions:*

(i) when $\Omega_{n,\beta}$ is a link invariant, $\Omega_{n,\beta}$ must be a multiple of the Conway polynomial for $\widehat{\beta}$; in this case, $\Omega_{n,\beta} = A_0\ \nabla_\beta$, and each A_j equals $A_0 \in \mathbb{Z}[z, z^{-1}]$,
(ii) when $\Omega_{n,\beta}$ is not a link invariant, but depends only on $w - n$, $\Omega_{n,\beta}$ and A_j are linear combinations of the Conway polynomials described in Thm. 2.7,
(iii) $\Omega_{n,\beta}$ is not a link invariant and depends on parameters other than, or in addition to, $w - n$. Some parameter must not be a link invariant.

2.1. *Skein polynomials for three-braid links*

The prior three-braid results may be combined with the extensive analysis by K. Murasugi, [16], to determine all cases when the Homflypt polynomial of a three-braid link matches that of an elementary torus link, or when the v-span has a given value. Research in [21] suggests the Jones polynomial may distinguish the same three-braid links as the Homflypt polynomial. This is confirmed by Prop. 2.17.

Among several other topics, A. Stoimenow explored the Xu normal form for three-braids in [21] and ascertained (Thm. 4.1, p. 15) when the Alexander polynomial could vanish in terms of the band representation. This is mirrored in Props. 2.9, 2.10, and 2.12 using the Artin braid representation. Additionally, Cor. 4.5, , p. 17 [21], observes "No non-trivial 3-braid knot has trivial Alexander polynomial." Prop. 2.14 is an equivalent result from an entirely different perspective.

In [16], K. Murasugi defines a collection of seven disjoint sets, Ω_i, of three-braid words and shows (Prop. 2.1 p. 7) that each three-braid word is conjugate to exactly one element in some Ω_i. As the numbering and definition of the Artin braid generators differs from that in the later text, [17], we present a similar partitioning, Ω_i^*, consistent with the latter. The Alexander polynomial for a typical member may be derived by Prop. 3.4 and is included below (here $G = t^2 + t + 1$).

(i) $\Omega_0^* = \{\alpha_2^{3d} : d \in \mathbb{Z}\}$, with $\Delta_{\widehat{\alpha_2^{3d}}} = t(t^{3d} - 1)^2/(Gt^{3d})$,
(ii) $\Omega_1^* = \{\alpha_2^{3d+1} : d \in \mathbb{Z}\}$, with $\Delta_{\widehat{\alpha_2^{3d+1}}} = t(t^{6d+2} + t^{3d+1} + 1)/(Gt^{3d+1})$,
(iii) $\Omega_2^* = \{\alpha_2^{3d+2} : d \in \mathbb{Z}\}$, with $\Delta_{\widehat{\alpha_2^{3d+2}}} = t(t^{6d+4} + t^{3d+2} + 1)/(Gt^{3d+2})$,
(iv) $\Omega_3^* = \{\alpha_2^{3d+1}\sigma_2 : d \in \mathbb{Z}\}$, with $\Delta_{\widehat{\alpha_2^{3d+1}\sigma_2}} = t(t^{6d+3} - 1)/(Gt^{3d+1}\sqrt{t})$,
(v) $\Omega_4^* = \{\alpha_2^{3d}\sigma_2^{-e} : d, e \in \mathbb{Z}, e > 0\}$,
with $\Delta_{\widehat{\alpha_2^{3d}\sigma_2^{-e}}} = t(t^{3d} - 1)(t^{3d-e} - (-1)^e)/(Gt^{3d}t^{-e/2})$,
(vi) $\Omega_5^* = \{\alpha_2^{3d}\sigma_1^{E} : d, E \in \mathbb{Z}, E > 0\}$,
with $\Delta_{\widehat{\alpha_2^{3d}\sigma_1^{E}}} = t(t^{3d} - 1)(t^{3d+E} - (-1)^E)/(Gt^{3d}t^{E/2})$,
(vii) $\Omega_6^* = \{\alpha_2^{3d}\eta : d \in \mathbb{Z}, \eta \in B_3, \text{with } \eta = \prod_{k=1}^{r} \sigma_2^{-e_k}\sigma_1^{E_k} \text{ and } r, e_k, E_k > 0\}$,
with $\Delta_{\widehat{\alpha_2^{3d}\eta}} = \Delta_{\widehat{\eta}} + t(t^{3d} - 1)(t^{3d+w(\eta)} - (-1)^{w(\eta)})/(Gt^{3d}t^{w(\eta)/2})$.

Proposition 2.9. *When a three-braid word, β, has $\nabla_{\widehat{\beta}} = C_{w-1}$, then β is conjugate to one of the following. This collection and their conjugates is called Υ_{w-1}.*

(i) α_2 *and* α_2^2, *which represent the knots* O_1 *and* T_3 ,
(ii) σ_1 , $\sigma_2\sigma_1\sigma_2$, *and* $\alpha_2^3\sigma_2^{-1}$ *which represent the two-links,* O_2, T_2, *and* T_4 ,
(iii) $\sigma_2^{-e}\sigma_1$ *and* $\alpha_2^3\sigma_2^{-1}\sigma_1^E$, *with* $e, E > 0$, *which represent* T_{-e} *and* T_{E+4} ,
(iv) $\alpha_2^{3d}\sigma_2^{2-3d}\sigma_1$, *and* $d \geq 3$ *is odd, which represents a knot with* $w \equiv 0$ *mod* 6,
(v) $\alpha_2^{-3d}\sigma_2^{-1}\sigma_1^{3d+1}$, *and* $d \geq 2$ *is even, which represents a knot with* $w \equiv 0$ *mod* 6.

Proposition 2.10. *When a three-braid word, β, has $\nabla_{\widehat{\beta}} = C_{w+1}$, then β is conjugate to one of the following. This collection and their conjugates is called Υ_{w+1}.*

(i) α_2^{-1} *and* α_2^{-2}, *which represent the knots* O_1 *and* T_{-3} ,
(ii) σ_2^{-1} , $\alpha_2^{-2}\sigma_2$, *and* $\alpha_2^{-3}\sigma_1$ *which represent the two-links,* O_2, T_{-2}, *and* T_{-4} ,
(iii) $\sigma_2^{-1}\sigma_1^E$ *and* $\alpha_2^{-3}\sigma_2^{-e}\sigma_1$, *with* $e, E > 0$, *which represent* T_E *and* T_{-e-4} ,
(iv) $\alpha_2^{3d}\sigma_2^{-1-3d}\sigma_1$, *and* $d \geq 2$ *is even, which represents a knot with* $w \equiv 0$ *mod* 6,
(v) $\alpha_2^{-3d}\sigma_2^{-1}\sigma_1^{3d-2}$, *and* $d \geq 3$ *is odd, which represents a knot with* $w \equiv 0$ *mod* 6.

The following is an immediate consequence of Prop. 2.3 and the prior two results.

Corollary 2.11. *A three-braid link, $\widehat{\beta}$, has v-span equal to four exactly when $\beta \notin \Upsilon_{w-1} \cup \Upsilon_{w+1}$.*

Proposition 2.12. *When a three-braid word, β, has $\nabla_{\widehat{\beta}} = C_x$, then w is even with $-x = w \pm 1$, or $\beta \in \Upsilon_{w-1} \cup \Upsilon_{w+1}$, or β is conjugate to one of the following. This collection and their conjugates is called Υ_x, and is disjoint from $\Upsilon_{w-1} \cup \Upsilon_{w+1}$.*

(i) α_2^0, σ_2^{-e}, *and* σ_1^E *with* e, $E > 1$, *which represent the split links* O_3, $T_{-e} \coprod O_1$, *and* $T_E \coprod O_1$; *here* $x = 0$,
(ii) $\alpha_2^{-3d}\sigma_1^{3d}$, *with* $d \geq 2$ *is even, which represents a three-link; here* $x = 0$,
(iii) $\alpha_2^{3d}\sigma_2^{-3d}$, *and* $d \geq 2$ *is even, which represents a three-link; here* $x = 0$,
(iv) $\alpha_2^{-3}\sigma_1^5$ *and* $\alpha_2^3\sigma_2^{-5}$, *which represent two-links; here* $x = -4$ *and* $x = 4$.

Prop. 12.5, p. 57 [16], shows that $\alpha_2^{-3}\sigma_1^5$ and $\alpha_2^3\sigma_2^{-5}$ are not elementary torus links. The next result, which follows from Props. 2.3, 2.9, and 2.10, answers Question 5, p. 3950 [20]: "Are there P polynomials of v-span ≤ 2 other than those of the (2, n)-torus knots and links?". Prop. 12.6, p. 58 [16], shows that the three-braid words listed in Cor. 2.13 create links of braid index three. Cor. 2.14 easily follows.

Corollary 2.13. *The following are equivalent for a three-braid link, $\widehat{\beta}$:*

(i) $\widehat{\beta}$ *has v-span equal to two,*
(ii) $\beta \in \Upsilon_{w-1} \cup \Upsilon_{w+1}$ *with* $\nabla_{\widehat{\beta}} \neq 1$,
(iii) $\beta \in \Upsilon_{w-1} \cup \Upsilon_{w+1}$ *with* $\widehat{\beta} \neq O_1$.

When the braid index of $\widehat{\beta}$ is three, and $\widehat{\beta}$ has v-span equal to two, β is conjugate to one of the following. These all represent knots with $w \equiv 0$ mod 6:

(i) $\alpha_2^{3d}\sigma_2^{2-3d}\sigma_1$ *, and $d \geq 3$ is odd, with $w = 3d+3$,*
(ii) $\alpha_2^{-3d}\sigma_2^{-1}\sigma_1^{3d+1}$ *, and $d \geq 2$ is even, with $w = -3d$,*
(iii) $\alpha_2^{3d}\sigma_2^{-1-3d}\sigma_1$ *, and $d \geq 2$ is even, with $w = 3d$,*
(iv) $\alpha_2^{-3d}\sigma_2^{-1}\sigma_1^{3d-2}$ *, and $d \geq 3$ is odd, with $w = -3d-3$.*

Corollary 2.14. *The following are equivalent for a three-braid link, $\widehat{\beta}$:*

(i) $\widehat{\beta}$ *has v-span equal to zero,*
(ii) $\widehat{\beta} = O_1$,
(iii) $\nabla_{\widehat{\beta}} = 1$.

A. Stoimenow asks in Question 4.1, p. 18 [21], whether "any two 3-braid links with the same V (or Δ) have also equal P? ". Ex. 2.15 shows Δ doesn't have this property. Prop. 2.16 provides some partial results for the Jones polynomial based on (2.6) and fundamental properties of the Alexander polynomial alone. Explicit formulas for $V_{\widehat{\beta}}$, $\Delta_{\widehat{\beta}}$, $\Delta_{\widehat{\gamma}}$ are derived from (2.6, 3.20) when $V_{\widehat{\beta}} = V_{\widehat{\gamma}}$. However, it is the Murasugi classification that is decisive in proving that the Jones and Homflypt polynomials distinguish the same three-braid links (Prop. 2.17).

Example 2.15. Set $\beta = \alpha_2^{3x}\sigma_2^{-3x-3y-1}\sigma_1$, $\gamma = \alpha_2^{3y}\sigma_2^{-3x-3y-1}\sigma_1$, with $x > y > 0$ and $x \equiv y$ mod 2 . Then $\nabla_{\widehat{\beta}} = \nabla_{\widehat{\gamma}} \neq 0$, by Prop. 3.4, but $P_{\widehat{\beta}} \neq P_{\widehat{\gamma}}$ by Prop. 2.4.

Proposition 2.16. *Assume $\beta, \gamma, \eta \in B_3$, with $V_{\widehat{\beta}} = V_{\widehat{\gamma}}$. We have $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$ in the following cases. Additional consequences are listed for each case.*

(i) $w(\beta) = w(\gamma)$ *(this is a subset of Prop. 2.2),*
(ii) $\mu(\widehat{\beta}) = 3$*, also implies* $w(\beta) = w(\gamma)$,
(iii) $w(\beta) = w(\gamma) \pm 2$*, also implies* $\nabla_{\widehat{\beta}} = C_{w(\beta)\mp 1}$,
(iv) $\nabla_{\widehat{\beta}} = C_{w(\beta)\mp 1}$*, also implies* $\nabla_{\widehat{\gamma}} = C_{w(\gamma)\pm 1}$*, or* $w(\beta) = 0$ *and* $w(\gamma) = \pm 2$ *and* $\nabla_{\widehat{\beta}} = 1 = \nabla_{\widehat{\gamma}}$,

For knots with $w(\beta) - w(\gamma) = 2k > 2$, we have $w(\beta) \equiv 2$ mod 8 and $k = 2$. When $w(\beta) = 2$, we have $\nabla_{\widehat{\beta}} = 1 = \nabla_{\widehat{\gamma}}$, so $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$.

For two-links with $w(\beta) - w(\gamma) = 2k > 2$, we have $w(\beta) = k(3+4y)$ with k odd, $k \geq 3$, and $y \in \mathbb{Z}$.

When $V_{\widehat{\eta}} = 1$ we have $\Delta_{\widehat{\eta}} = 1$, hence $\nabla_{\widehat{\eta}} = 1$.

Proposition 2.17. *If $\beta, \gamma \in B_3$ have $V_{\widehat{\beta}} = V_{\widehat{\gamma}}$, we have $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$.*

While the prior results display a number of global properties, they provide little insight into specific characteristics: how large or small is the degree for a given link, what are the integer coefficients, or at least what is the sign of the coefficients, and is the sign uniform? There also remains the goal to find explicit formulas for the coefficient polynomials for classes of links, even when no single formula suffices for all links. These are the types of questions addressed in the following section. The interested reader should note that some number theoretic results have been published in this general area by A. Kawauchi, [13].

2.2. *General Attributes of the Conway and Homflypt coefficient polynomials*

All braid words in this section are assumed to be in standard form. Theorem 2.22 and Corollary 2.23 describe bounds on the degrees of the coefficient polynomials and Conway polynomial that apply to all links. Later subsections address special classes of links for which more precise statements may be made.

2.2.1. *Definitions*

Definition 2.18. If a braid word, $\beta \in B_n$, is in standard form and has the expression $\prod_{k=1}^{m} \sigma_{i_k}^{\epsilon_k}$, with $\epsilon_k = \pm 1$ for each subscript, k,

(i) the normalized length. denoted $\| \beta \|$, is the length less the number of generators in B_n, i.e. $\| \beta \| \equiv |\beta| - (n-1)$,
(ii) the porosity of a row is the number of null generators in the row considered as a braid word in its own right,
(iii) the porosity of a braid word is the sum of the porosity of all rows, written $o(\beta)$. A braid word is called nonporous when $o(\beta) = 0 = \nu_0(\beta)$,
(iv) the braid word is called polarized when all its generators are homogeneous; null generators are allowed,
(v) a polarized braid word is called homogeneous when it has no null generators (this is essentially the definition in [1]),
(vi) a polarized braid word is called alternating when the subscripts of the positive generators all have the same parity (congruence class modulo two) and the subscripts of the negative generators all have the opposite parity,
(vii) a polarized braid word is called positive (negative) when all generators in β are positive (negative); null generators are allowed.

An n-braid word whose twist exponents are all zero is defined to have $1 - n$ for its normalized length and zero porosity. Such a braid word is not called nonporous.

The following braid word properties become link invariants:

Definition 2.19. The following properties of a link are defined to be the minimum value of the same property among all braid words whose closure yields the link: the length, $|L|$; the normalized length, $\| L \|$; the rank, $\rho(L)$.

It is known, [14], [10] and [11], that the minimal possible degree of a term in h_j is $1 - \mu(L)$, so the eligible terms in h_j have exponents between this value and the degree of h_j, inclusive, and have the same parity modulo two. The minimum degree of h_j is the lowest degree attained by any term. It is convenient to introduce some terminology for results, such as Proposition 2.40, prior to their statement.

Definition 2.20. Suppose L is a link with a nonzero coefficient polynomial h_j: h_j is said to have uniform sign when all its nonzero integer coefficients have the same sign; h_j achieves the minimal degree provided the term with exponent $1-\mu(L)$ has a nonzero coefficient; h_j is complete when all terms with nonzero coefficients have exponents that represent an arithmetic sequence with increment two; h_j is fully populated when all eligible terms have nonzero coefficients.

When a braid word β generates a link for which each nonzero $h_{j,\widehat{\beta}}$ has uniform sign $(-1)^{j+x_n}$, the braid word, link, and h_j are each said to be h-uniform. When h_j is h-uniform and complete, the braid word, link, and h_j are said to have USc.

Positive, negative, and some alternating braid words are h-uniform. The connected sum, disjoint union, and mirror image of two links are h-uniform when this is true for each. When $h_{j,\widehat{\gamma}}$ and $h_{k,\widehat{\eta}}$ have USc, so does their product. When $h_{j\pm\delta,\widehat{\gamma}}$ and $h_{k\mp\delta,\widehat{\eta}}$ have USc and the degree of any term of $h_{j\pm\delta,\widehat{\gamma}}h_{k\mp\delta,\widehat{\eta}}$ lies in the range $[\min\deg h_{j,\widehat{\gamma}}+\min\deg h_{k,\widehat{\eta}}-2, \deg h_{j,\widehat{\gamma}}+\deg h_{k,\widehat{\eta}}+2]$, the sum of $h_{j,\widehat{\gamma}}h_{k,\widehat{\eta}}$ and $h_{j\pm\delta,\widehat{\gamma}}h_{k\mp\delta,\widehat{\eta}}$ also has USc. This latter observation shows that in favorable circumstances, the connected sum and disjoint union of USc links will also have USc.

2.2.2. *Universal coefficient polynomial and Conway polynomial attributes*

The following proposition establishes the basic upper bound for the degree of any Homflypt coefficient polynomial and for the Conway polynomial in terms of the length or normalized length of any braid word whose closure is the link, and thus in terms of the link invariants for length or normalized length.

Proposition 2.21. *For an n-braid link, $\widehat{\beta}$, we have:*

(i) $\deg p_j \le |\beta|$, *hence* $\deg p_j \le |\widehat{\beta}|$,
(ii) $p_j = 0$ *for* $j \notin [\nu_{-1}, n-1-\nu_1]$,
(iii) $\deg \nabla_{\widehat{\beta}} \le \|\beta\|$, *hence* $\deg \nabla_{\widehat{\beta}} \le \|\widehat{\beta}\|$,
(iv) the link $\widehat{\beta}$ *is generated by a braid word,* $\gamma \in B_m$ *, with* $m = n-\nu_{-1}-\nu_1$ *, and*
  (a) $w(\gamma) = w(\beta)+\nu_{-1}-\nu_1$,
  (b) $\deg p_{j,\widehat{\gamma}} \le |\gamma| = |\beta|-\nu_{-1}-\nu_1$.

The next two results, Thm. 2.22 and Cor. 2.23, establish sharp upper bounds on the degree of each Homflypt coefficient polynomial, and for the Conway polynomial for arbitrary braid words or links. Here, the effect of the number of mixed generators in a braid representation for a link is clearly visible.

Theorem 2.22. *For a link, $\widehat{\beta}$, represented by any n-braid word β, we have:*

(i) $\deg p_j \le |\beta| - 2(\nu_n + \nu_m - j)$, *for* $j \in [0, \nu_n + \nu_m]$,
(ii) $\deg p_j \le |\beta|$, *for* $j \in [\nu_n + \nu_m, \nu_n + \nu_0]$,
(iii) $\deg p_j \le |\beta| - 2(j - \nu_n - \nu_0)$, *for* $j \in [\nu_n + \nu_0, n - 1]$,
(iv) $p_j = 0$ *for* $j \notin [\nu_{-1}, n - 1 - \nu_1]$.

The second range is empty when $\nu_m > \nu_0$. *In this case, a simpler estimate follows:*

(i) $\deg p_j \le |\beta| - 2(\nu_n + \nu_m - j)$, *for* $j \in [0, \nu_n + (\nu_m + \nu_0)/2]$,
(ii) $\deg p_j \le |\beta| - 2(j - \nu_n - \nu_0)$, *for* $j \in [\nu_n + (\nu_m + \nu_0)/2, n - 1]$,
(iii) $\deg p_j \le |\beta| + \nu_0 - \nu_m$, *for all* j ,
(iv) *when* $\nu_m + \nu_0$ *is odd,* $\deg p_j \le |\beta| + \nu_0 - \nu_m - 1$, *for all* j ,
(v) $p_j = 0$ *for* $j \notin [\nu_{-1}, n - 1 - \nu_1]$.

These estimates are sharp for links represented by alternating braid words (compare Prop. 2.33). The second set of estimates applies to non-split links with mixed generators (see Example 2.41 for the 6-1 knot).

Corollary 2.23. *For a link,* $\widehat{\beta}$, *represented by any n-braid word,* β, *the degree of the Conway polynomial has the following upper bounds. In particular this is true when* β *achieves the link value,* $\| \widehat{\beta} \| = \| \beta \|$.

(i) $\deg \nabla_{\widehat{\beta}} \le \| \beta \| - \nu_m(\beta)$,
(ii) *when* $\nu_m(\beta)$ *is odd,* $\deg \nabla_{\widehat{\beta}} \le \| \beta \| - \nu_m(\beta) - 1$.

2.2.3. *Links generated by rank one braid words*

The following are useful benchmarks for the strength of the skein polynomials:

Proposition 2.24. *The Conway polynomial is a complete invariant for the class of connected sums of elementary torus links with all positive twists, i.e. for links generated by positive braid words of rank one with no null generators. The same statement is true when "positive" is replaced by "negative".*

The meaning of "complete invariant" in this context is that when the Conway polynomials for links generated by two braid words are equal, the links are equal.

Proposition 2.25. *The Jones polynomial and the Homflypt polynomial are each complete invariants for the class of connected sums of elementary torus links.*

When β *is a rank one n-braid word, with no trivial and no null generators, there is a unique subscript,* $j = \nu_n$, *for which the maximum degree is achieved among the* h_j *(or* p_j*). The maximum degree is* $\| \beta \|$ *(or* $|\beta|$*). The coefficient of the term in* h_{ν_n} *(or* p_{ν_n}*) of degree two below the maximum is* $(-1)^{\nu_n + x_n} \, \| \beta \|$.

2.2.4. *Links generated by rank two n-braid words*

It is noteworthy that the Homflypt polynomial cannot distinguish the links generated by two braid words, each of rank two, when one is obtained from the other by interchanging the twist exponents of any (number of) generator(s) between the two rows. This is a consequence of Equations 3.10, 3.11, 3.13, and 3.14. Thus it is meaningful to speak of an exponent pair for a generator with respect to the value of its Homflypt polynomial. For example, a pair might have an even exponent and the value one, leading to a so called even-one pair.

It is useful to describe a procedure and formula to calculate the number of link components in the closure of a braid word, β, at this point. For each generator, σ_i, form a set consisting of its syllable length from each row of β; this set has $\rho(\beta)$ members. Now define $L(i)$ to be the number of odd values in this set and form the rank one braid word, $\Lambda(\beta) = \prod_{i=n-1}^{1} \sigma_i^{L(i)}$.

It is clear that when the rank of β is one, the number of link components of $\widehat{\beta}$ is one more than the number of null generators for $\Lambda(\beta)$. When the rank of β is two, the number of components of $\widehat{\beta}$ is also related to the number and length of maximal subwords within $\Lambda(\beta)$ consisting of generators whose twist exponent is two. As this construct will appear frequently, the following definitions are provided.

Definition 2.26. When a rank two n-braid word, β, has a range of subscripts for generators with a common attribute, and the range is maximal, the set of generators is called a block. The name and attributes of interest are: eeblock for $L(i) = 0$; eoblock for $L(i) = 1$; ooblock for $L(i) = 2$; npblock when no generator is clustered.

When a block has an odd number of members it is said to have odd width; an odd width ooblock is called a band. When a block has only a single member it is said to be thin. Any generator whose subscript is one below the low subscript in the block or one above the high subscript in the block (eeblock, etc.) is called a block (eeblock, etc.) neighbor, as appropriate.

The number of bands is written as $\phi(\beta)$; the number of bands whose minimal subscript generator is even (odd) is $\phi_e(\beta)$ $(\phi_o(\beta))$. The number of generators in all bands (bands with even/odd minimal subscript) is $\phi_\sharp(\beta)$ $(\phi_{e,\sharp}(\beta), \phi_{o,\sharp}(\beta))$. The number of band neighbors with an exponent of one (minus one) is $\phi_{\partial,1}(\beta)$ $(\phi_{\partial,-1}(\beta))$.

The number of npblocks is written as $\tau(\beta)$; the number of generators in all npblocks is $\tau_\sharp(\beta)$.

Lemma 2.27. *Given the preceding terminology and definitions, with β of rank two, the number of link components of $\widehat{\beta}$ is $1 + \nu_0(\Lambda(\beta)) + \phi(\beta)$.*

Example 2.28. $\beta = (\sigma_{10}\sigma_8\sigma_7\sigma_6^2\sigma_5\sigma_4\sigma_3^2\sigma_2\sigma_1)(\sigma_{10}\sigma_7\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1)$ in B_{11} gives rise to $\Lambda(\beta) = \sigma_{10}^2\sigma_8\sigma_7^2\sigma_5^2\sigma_4^2\sigma_3\sigma_2^2\sigma_1^2$, with $\nu_0(\Lambda(\beta)) = 2$, and $\phi(\beta) = 2$. The two null generators in $\Lambda(\beta)$ are σ_9 and σ_6, and the two bands in β are $\{\sigma_{10}\}$ and $\{\sigma_7\}$.

The following result shows that band neighbors with an exponent of ± 1 cause the high/low subscript h_j to have minimum degree greater than $1-\mu(\widehat{\beta})$. Prop. 2.29 identifies constraints on which h_j can achieve the minimal degree, and shows that n-component links always achieve the minimal degree in each h_j.

Proposition 2.29. *When β is a rank two n-braid word we have:*

(i) *when all generators of β belong to a single eeblock, h_j achieves the minimal degree for all $j \in [0, n-1]$, and $\widehat{\beta}$ has n components,*
(ii) *h_j can only achieve the minimal degree when $j \in [\phi_{\partial,-1}(\beta), n-1-\phi_{\partial,1}(\beta)]$,*
(iii) *suppose β is nonporous and has a band with a range of generator subscripts, $[x, x+2\kappa]$, for some $\kappa \geq 0$. Any exponent of any σ_{x+2i}, for any $0 \leq i \leq \kappa$, may be changed to ± 1 without changing the minimum degree of any h_j,*
(iv) *when β has a band with generator subscripts in a range, $[x, y]$, and no $h_{j,\, \Psi(x-\widehat{1,y}+1,\beta)}$ achieves the minimal degree, the same is true for each $h_{j,\, \widehat{\beta}}$,*
(v) *assume all generators belong to bands or eeblocks, each band has a range of generator subscripts $[x_k, y_k]$, and some $h_{j_k,\, \Psi(x_k-\widehat{1,y_k}+1,\beta)}$ achieves the minimal degree for each $k \in [1, \phi(\beta)]$. Denote the minimal and maximal subscripts for which this happens as l_k and r_k, respectively. It follows that $h_{l,\, \widehat{\beta}}$ and $h_{r,\, \widehat{\beta}}$ achieve the minimal degree, for $l = \sum_{i=1}^{\phi(\beta)} l_k$ and $r = \nu_0(\Lambda(\beta)) + \sum_{i=1}^{\phi(\beta)} r_k$. Furthermore, $h_{j,\, \widehat{\beta}}$ does not achieve the minimal degree for $j \notin [l, r]$,*
(vi) *when all bands are thin, and all generators belong to bands or eeblocks, every h_j achieves the minimal degree, and $\widehat{\beta}$ has n components,*
(vii) *when $\widehat{\beta}$ has n components, every h_j achieves the minimal degree.*

Example 2.30. Prop. 2.29 and Prop. 2.38 imply only h_{2m-2} and h_{2m-1} achieve the minimal degree for any rank two braid word in B_{2m} that consists of a single band, and whose even subscript generators all have exponents equal to minus one.

The rank two braid words in B_{2m} whose edge generators are thin eoblocks, either of which has an exponent of one, and whose remaining generators belong to a single band, also generate two-links, but are distinct from those described above.

2.2.5. *Links generated by braid words with special properties*

A cursory scan of a knot table, [4], reveals many knots are generated by simple extensions of braid words on fewer strands. Apart from knots of braid index two, and those generated by alternating braid words, many knots of very low crossing number have sub-maximal v-span. This, and other properties of the h_j, are readily explained by their braid word form, as seen in Prop. 2.32. The following definition encapsulates one class of extensions.

Definition 2.31. Suppose x, y are integers, $\min\{x, y\} = 1$, and $\epsilon = \pm 1$. An edge prefix in B_n is a braid word of the form $\sigma_{n-1}^{\epsilon x}\sigma_{n-2}^{-\epsilon}\sigma_{n-1}^{\epsilon y}$. An edge suffix in B_n is a braid word of the form $\sigma_1^{\epsilon x}\sigma_2^{-\epsilon}\sigma_1^{\epsilon y}$. The edge exponent is the value, $\epsilon \max\{x, y\}$.

Suppose an n-braid word, β, has an expression as $\eta\gamma$, in which η is an edge prefix and all subscripts of generators in γ are less than $n-1$. In this case β is called an edge extension of γ, or simply an edge extension; γ is called the base. The related braid words $\sigma_{n-1}^{\epsilon}\beta$ and $\sigma_{n-2}^{-\epsilon}\gamma$ are denoted β_* and γ_*. The same terminology applies when $\beta = \gamma\eta$, in which η is an edge suffix and γ has generator subscripts greater than one, with $\beta_* = \beta\sigma_1^{\epsilon}$ and $\gamma_* = \gamma\sigma_2^{-\epsilon}$. By an abuse of notation, $\widehat{\gamma}$ is written for $\Psi(\widehat{0, n-1}, \gamma)$ or $\Psi(\widehat{1, n}, \gamma)$, so that the base, γ, is treated as a member of B_{n-1}; the same convention applies to γ_*.

The links for the edge extension and base have the same number of components exactly when the edge exponent is odd. Prop. 2.42 implies alternating braid words are h-uniform and complete when they are iterated edge extensions of rank two alternating braid words. Prop. 2.43 shows positive braid words are h-uniform. Prop. 2.32 follows from the Prop. 3.6 formula: $h_{j,\widehat{\beta}} = C_{e+1}h_{j,\widehat{\gamma}_*} + C_e h_{j-1,\widehat{\gamma}}$.

Proposition 2.32. *Suppose β is an n-braid word with no trivial or null generators, and is an edge extension of γ with edge exponent e. Let M represent the property to achieve the minimal degree.*

When γ and γ_* are both h-uniform, so are β and β_*

When the edge exponent is positive, $h_{j,\widehat{\beta}} = C_{e+1}h_{j,\widehat{\gamma}_*} + C_e h_{j-1,\widehat{\gamma}}$, with $h_{n-1,\widehat{\gamma}_*} = 0$ and $h_{-1,\widehat{\gamma}} = 0$. The following are consequences:

(i) $h_{0,\widehat{\beta}}$ is h-uniform (,complete, or 0) when $h_{0,\widehat{\gamma}_*}$ is h-uniform (,complete, or 0),
(ii) $h_{n-1,\widehat{\beta}}$ is h-uniform (,complete, or 0) when $h_{n-2,\widehat{\gamma}}$ is h-uniform (,complete, or 0),
(iii) $h_{j,\widehat{\beta}}$ is h-uniform when both $h_{j,\widehat{\gamma}_*}$ and $h_{j-1,\widehat{\gamma}}$ are h-uniform, for any $j \in (0, n-1)$,
(iv) $h_{0,\widehat{\beta}}$ has M exactly when $h_{0,\widehat{\gamma}_*}$ has M, and either e is even or $\mu(\widehat{\gamma}_*) > \mu(\widehat{\beta})$,
(v) assume $h_{j-1,\widehat{\gamma}}$ does not achieve minimal degree, for some $j \in (0, n-1)$. Then $h_{j,\widehat{\beta}}$ has M exactly when $h_{j,\widehat{\gamma}_*}$ has M, and either e is even or $\mu(\widehat{\gamma}_*) > \mu(\widehat{\beta})$,
(vi) $h_{n-1,\widehat{\beta}}$ has M exactly when $h_{n-2,\widehat{\gamma}}$ has M, and either e is odd or $\mu(\widehat{\gamma}) > \mu(\widehat{\beta})$,
(vii) assume $h_{j,\widehat{\gamma}_*}$ does not achieve minimal degree, for some $j \in (0, n-1)$. Then $h_{j,\widehat{\beta}}$ has M exactly when $h_{j-1,\widehat{\gamma}}$ has M, and either e is odd or $\mu(\widehat{\gamma}) > \mu(\widehat{\beta})$,
(viii) assume $h_{j-1,\widehat{\gamma}}$ and $h_{j,\widehat{\gamma}_*}$ have M, for some $j \in (0, n-1)$. Then $h_{j,\widehat{\beta}}$ has M when any of the following are true:

  (a) the minimum degree terms in $h_{j-1,\widehat{\gamma}}$ and $h_{j,\widehat{\gamma}_*}$ have the same sign, or
  (b) e is odd and $\mu(\widehat{\gamma}_*) < \mu(\widehat{\beta})$, or
  (c) e is even and $\mu(\widehat{\gamma}) < \mu(\widehat{\beta})$.

When the edge exponent is negative, $h_{j,\widehat{\beta}} = C_{e-1}h_{j-1,\widehat{\gamma}_*} + C_e h_{j,\widehat{\gamma}}$, with $h_{-1,\widehat{\gamma}_*} = 0$ and $h_{n-1,\widehat{\gamma}} = 0$. The results above may be applied to the mirror image.

2.2.6. *Links generated by alternating braid words*

The following proposition reinforces why alternating braids and their closures are so widely studied, for it provides the exact value for the degree of each Homflypt coefficient polynomial, and for the Conway polynomial, in terms of the length or normalized length of any alternating braid word whose closure is the link. In fact, any alternating braid word fulfills the values of the link invariants for length and normalized length. The result further shows the absolute value of the leading coefficients of p_j are bounded below by binomial coefficients determined by the number of positive, negative, and null generators and the number of trivial generators of each sign. The sign of these leading coefficients alternate by subscript as $(-1)^{j+x_n}$.

The relation of alternating braid words to Proposition 2.5 is as follows. First, L includes a first range, $[0, \nu_n]$, followed by a plateau in the next range, $[\nu_n, n-1-\nu_p]$, followed by a third range $[n-1-\nu_p, n-1]$, that falls inside R. Observe that the plateau in the degree of the coefficient polynomials that occurs in the "middle" is actually single point when, and only when, there are no null generators.

Proposition 2.33. *For a link, $\widehat{\beta}$, represented by an alternating n-braid word, β, let a_j be the leading coefficient of p_j. We have*

(i) *the sign of a_j is $(-1)^{j+x_n}$ for $j \in [\nu_{-1}, n-1-\nu_1]$,*

(ii) *for $j \in [\nu_{-1}, \nu_n]$:*

(a) $\deg p_j = |\beta| - 2(\nu_n - j) \geq 0$,

(b) $|a_j| \geq \binom{\nu_n - \nu_{-1}}{j - \nu_{-1}}$, *with equality when $\rho(\beta) = 1$,*

(iii) *for $j \in [\nu_n, n-1-\nu_p] = [\nu_n, \nu_n + \nu_0]$:*

(a) $\deg p_j = |\beta|$, *hence* $|\beta| = |\widehat{\beta}|$,

(b) $|a_j| \geq \binom{\nu_0}{j - \nu_n}$, *with equality when $\rho(\beta) = 1$,*

(iv) *for $j \in [n-1-\nu_p, n-1-\nu_1] = [\nu_n + \nu_0, n-1-\nu_1]$:*

(a) $\deg p_j = |\beta| - 2(j + \nu_p - (n-1)) = |\beta| - 2(j - \nu_n - \nu_0) \geq 0$,

(b) $|a_j| \geq \binom{\nu_p - \nu_1}{j - \nu_n - \nu_0}$, *with equality when $\rho(\beta) = 1$,*

(v) *when $\nu_0 = 0$,* $\deg \nabla_{\widehat{\beta}} = \| \beta \| = \| \widehat{\beta} \|$.

While links generated by alternating three-braid words are not generally of uniform sign, at least an eligible term in h_j whose degree is two lower than the top term has the same sign as the leading coefficient (and is nonzero), with three exceptions. The exceptional cases are the three-link, $K_{\pm}(3,3)$, and the closures of two alternating braid words with rank two and length five, that represent a two-link and its mirror image. Proposition 2.34 provides a complete description.

Proposition 2.34. *For a link, $\widehat{\beta}$, represented by an alternating three-braid word with no trivial generators, $\beta = \prod_{i=1}^{r} \sigma_2^{x_i} \sigma_1^{y_i}$, with all x_i, y_i nonzero, we have:*

(i) the value of the leading coefficients for p_0 and p_2 is $(-1)^{x_n(\beta)}$,
(ii) the value of the leading coefficient for p_1 is $(-1)^{1+x_n(\beta)}$,
(iii) the term of degree two lower than the highest degree term in p_j has the following values with the same sign as the leading coefficient. The values are all nonzero when $r \geq 4$, or $r = 1$, or $r = 2$ with $|\beta| \geq 6$, or $r = 3$ with $|\beta| \geq 7$:
  (a) for p_0, $(-1)^{x_n}(|\beta| - r - 3 + \delta)$, with $\delta = 1$ when $x_n = 2$, otherwise $\delta = 0$,
  (b) for p_1, $(-1)^{x_n+1}(|\beta| - r - 1)$,
  (c) for p_2, $(-1)^{x_n}(|\beta| - r - 3 + \delta)$, with $\delta = 1$ when $x_p = 2$, otherwise $\delta = 0$.
(iv) the only cases in which the term of the prior item is both zero and eligible are:
  (a) $|\beta| = 5$ with $r = 2$: the two-link $L = \widehat{\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^2}$ and its mirror image. For L we have: $p_0 = z^3 + z$, $p_1 = -z^5 - 2z^3 - z$ and $p_2 = z^3$,
  (b) $|\beta| = 6$ with $r = 3$: the three-link $K_{\pm}(3,3)$ with $p_0 = p_2 = -z^4 + 1$ and $p_1 = z^6 + 2z^4 - 2$.

When $|\beta| = 4$ and $r = 1$, the three-link $\widehat{\beta}$ is the connected sum of T_2 and T_{-2}, with $p_0 = p_2 = z^2 + 1$ and $p_1 = -z^4 - 2z^2 - 2$. When $|\beta| = 4$ and $r = 2$, the knot, $\widehat{\beta}$, has $p_0 = p_2 = z^2$ and $p_1 = -z^4 - z^2$.

Example 2.35. The prime three-braid knots, 8-18 and 10-123, generated by the braid words, $\alpha^4_{2,\pm}$, $\alpha^5_{2,\pm}$, have the following coefficient polynomials, h_j $(=p_j/z^2)$, [4]. This shows Prop. 2.34 cannot be unconditionally extended to three terms.

(i) $h_0 = z^4 + z^2 - 1$ and $h_0 = -z^6 - 2z^4 + z^2 + 2$,
(ii) $h_1 = -z^6 - 3z^4 - z^2 + 3$ and $h_1 = z^8 + 4z^6 + 3z^4 - 4z^2 - 3$,
(iii) $h_2 = z^4 + z^2 - 1$ and $h_2 = -z^6 - 2z^4 + z^2 + 2$.

It is convenient to introduce the following notation for some expressions that represent the integer coefficients of the h_j for $K_{\pm}(2,n)$. Recall $K_{\pm}(2,n)$ is generated by the alternating braid word of rank two, $\alpha^2_{n-1,\pm}$.

Definition 2.36. For integers j and k, define the functions $f_k(j)$ as follows:

(i) $f_k(j) = \lfloor j/2 \rfloor$, when k is even,
(ii) $f_k(j) = \lceil j/2 \rceil$, when k is odd.

Proposition 2.37. *For the knot, $K_{\pm}(2, 2m+1)$ with $m \geq 1$, the coefficient of z^{2k} in h_j, denoted $a_{j,k,m}$, is a product of two binomial coefficients, one of which is independent of m. The binomial coefficient, $\binom{0}{0} = 1$, while $\binom{0}{k} = 0$, for $k \neq 0$.*

$$a_{j,0,m} = (-1)^j \text{ for } 0 \leq j \leq m \,,$$

$$a_{j,k,m} = (-1)^j \binom{f_k(j) + f_0(k)}{k} \binom{m - f_1(j-k)}{k}$$
$$\textit{for } 1 \le k \le j \le m\,, \tag{2.12}$$

$$a_{j,k,m} = (-1)^j \binom{f_k(2m-j) + f_0(k)}{k} \binom{f_0(j+k)}{k}$$
$$\textit{for } j \in [m, 2m]\,, \textit{ and } k \in [0, 2m-j]\,, \tag{2.13}$$

$$a_{j,k,m} = 0 \textit{ otherwise.}$$

The Conway polynomial is related to the elementary torus links as follows:

$$\nabla_{K_\pm(2,2m+1)}(z) = C_{2m+1}(\imath z)\,, \textit{ with } \imath = \sqrt{-1}\,.$$

Thus for the knot, $K_\pm(2, 2m+1)$ with $m \ge 1$, each h_j is h-uniform and fully populated. There is a symmetry wherein $h_j = h_{2m-j}$, since $a_{j,k,m} = a_{2m-j,k,m}$. Observe also each coefficient of the central coefficient polynomial for the knot, $K_\pm(2, 2m+1)$, i.e. $a_{m,k,m}$, is (plus or minus) the square of a binomial coefficient. Finally, the outer two layers of coefficients, i.e. those of the two highest degrees of h_j are binomial coefficients: $(-1)^j \binom{m}{k}$ and $(-1)^j \binom{m-1}{k}$.

Table 1. $K_\pm(2, 11)$ coefficients, $a_{j,k,5}$, for h_j

k											
5						-1					
4					5	-1	5				
3				-10	4	-16	4	-10			
2			10	-6	18	-9	18	-6	10		
1		-5	4	-8	6	-9	6	-8	4	-5	
0	1	-1	1	-1	1	-1	1	-1	1	-1	1
j =	0	1	2	3	4	5	6	7	8	9	10

Proposition 2.38. *For the two-component link, $K_\pm(2, 2m)$ with $m \ge 2$, the coefficient of z^{2k} in zh_j, denoted $b_{j,k,m}$, is a product of two binomial coefficients, one of which is independent of m. The constant term, $b_{j,0,m}$ is zero for $j \in [0, 2m-3]$, while $b_{2m-2,0,m} = 1$ and $b_{2m-1,0,m} = -1$.*

For the left half of the subscript range, we have

$$b_{j,k,m} = (-1)^j \binom{f_k(j+1) + f_0(k)}{k} \binom{m - 1 - f_1(j+1-k)}{k-1}$$
$$\textit{for } 1 \le k \le j+1 \le m-1\,. \tag{2.14}$$

The following two expressions are for the coefficients of the two highest powers in zh_j for the right half of the subscript range. The values are identical to the values for $-a_{j+1,k,m}$, i.e. $b_{j,k,m} = -a_{j+1,k,m}$ for the top two values of k in this range:

$$b_{j,2m-j-1,m} = (-1)^j \binom{m}{2m-j-1} \text{ for } j \in [m-1, 2m-2]\,, \tag{2.15}$$

$$b_{j,2m-j-2,m} = (-1)^j \binom{m-1}{2m-j-2} \text{ for } j \in [m-1, 2m-3]\,. \tag{2.16}$$

The following expression is for the coefficients of the powers below the two highest powers in zh_j for the right half of the subscript range.

$$b_{j,k,m} = (-1)^j \binom{f_k(2m-3-j) + f_0(k)}{k-1} \binom{f_0(j+1+k)}{k}$$
$$\text{for } j \in [m-1, 2m-4]\,, k \in [1, 2m-j-3]\,. \tag{2.17}$$

The Conway polynomial is related to the elementary torus links as follows:

$$\nabla_{K_\pm(2,2m)}(z) = -\imath C_{2m}(\imath z)\,, \text{ with } \imath = \sqrt{-1}\,.$$

For the two component link, $K_\pm(2,2m)$ with $m \geq 2$, the coefficients of all eligible powers in each h_j are nonzero and have the same sign, except that the z^{-1} term is zero for $j \in [0, 2m-3]$. The coefficient of the z^1 term in h_j is $(-1)^j(1 + \lfloor j/2 \rfloor)$. A similarity to the odd strand result for $K_\pm(2, 2m+1)$ is that the outer two layers of coefficients, i.e. those of the two highest degrees of h_j are binomial coefficients. However on the left side they are for $m-1$ and $m-2$, while on the right side they are for m and $m-1$: $(-1)^j \binom{m-1}{k-1}$ and $(-1)^j \binom{m-2}{k-1}$ on the left, and $(-1)^j \binom{m}{k}$ and $(-1)^j \binom{m-1}{k}$ on the right.

Table 2. $K_\pm(2,10)$ coefficients, $b_{j,k,5}$, for zh_j

k										
5					1					
4				-4	1	-5				
3			6	-3	12	-4	10			
2		-4	3	-9	6	-12	6	-10		
1	1	-1	2	-2	3	-3	4	-4	5	
0	0	0	0	0	0	0	0	0	1	-1
j =	0	1	2	3	4	5	6	7	8	9

Thus there are nonporous rank two alternating braid words that generate links for which some h_j do not achieve their minimal degree. This condition appears in a more general setting, and is related to the presence of a band that is not thin. As explained in Prop. 2.40, the range on which h_j achieves the minimal degree is decreased by one for each band neighbor with an exponent of ± 1. For each band that is not thin, the range is decreased by the number of excess band generators above the minimal number.

Among links generated by alternating braid words, the following is a minimal interesting example in which coefficient polynomials are not complete, and/or do not achieve the minimal degree, and/or are not of uniform sign. All h_j are h-uniform and fully populated for links of braid index two, or links generated by any braid word of rank one with no trivial generators. Proposition 2.40 shows each h_j is fully populated and h-uniform when a knot, or link with n components, is generated by an alternating braid word of rank two with no trivial generators.

Example 2.39. The prime four-braid knot, 9-40 $= K_{\pm}(3,4)$, generated by the braid word, $\alpha^3_{3,\pm} = (\sigma_3\sigma_2^{-1}\sigma_1)^3$, has the following h_j $(=p_j/z^3)$, [4].

(i) $h_0 = -z^4 + 0z^2 + 2\,,$
(ii) $h_1 = z^6 + 2z^4 + 0z^2 - 2\,,$
(iii) $h_2 = -2z^4 - 2z^2 + 1\,,$
(iv) $h_3 = z^2\,.$

Proposition 2.40. *When β is a rank two alternating n-braid word with no trivial or null generators and the positive generators have odd subscript, we have:*

(i) each h_j is complete and h-uniform,
(ii) each h_j achieves the minimal degree, and thus is fully populated, exactly when $j \in [\phi_{\partial,-1}(\beta) + \phi_{o,\sharp}(\beta) - \phi_o(\beta),\, n-1-\phi_{\partial,1}(\beta) - \phi_{e,\sharp}(\beta) + \phi_e(\beta)]\,,$
(iii) the coefficient of $z^{\|\beta\|-2}$ in h_{ν_n} is $(-1)^{\nu_n + x_n}(\|\beta\| + \tau(\beta) - \tau_\sharp(\beta))$,
(iv) the rank of $\widehat{\beta}$ is two exactly when some npblock is not thin,
(v) when β is nonporous, $\widehat{\beta}$ is distinct from any link generated by any rank two alternating braid word with a non-trivial clustered generator.

Under the assumptions of Prop. 2.40, knots, and links of n components all have fully populated h_j. For a two-link, each h_j is fully populated when there is one thin eeblock with no bands. Otherwise, a two-link has a fully populated h_{n-1} exactly when there are no eeblocks, and one band, but when the band has even subscript for its minimal member, the band must be thin and have no band neighbor with an exponent of one. A complementary result applies to h_0 for a two-link.

Example 2.41. The 6-1 knot, generated by $\sigma_3^2\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}$, has the following coefficient polynomials, h_j $(=p_j/z^3)$, [4] and so has rank two. Thus Prop. 2.40 cannot be extended to all rank two alternating knots.

(i) $h_0 = 1\,,$
(ii) $h_1 = -z^2 + 0\,,$

(iii) $h_2 = -z^2 - 1\,,$
(iv) $h_3 = 1\,.$

Many knots are generated by edge extensions of rank two alternating braid words, e.g. the 10-42 knot, [4], is generated by, [2], [18]: $\sigma_4^2\sigma_3^{-1}\sigma_4(\sigma_3^{-1}\sigma_2\sigma_1^{-1})^2$. Prop. 2.42 shows each h_j for the 10-42 knot has USc, and this is true for any alternating braid word that is an iterated edge extension of a rank two alternating braid word. The proof for Prop. 2.42 is as for Prop. 2.32 and uses Prop. 3.6.

Proposition 2.42. *Suppose β is an alternating n-braid word with no trivial or null generators, and that β is an edge extension of γ with edge exponent e.*

When $\{h_{j,\widehat{\gamma}}\}_0^{n-2} \cup \{h_{j,\widehat{\gamma}_*}\}_0^{n-2}$ has USc, so does $\{h_{j,\widehat{\beta}}\}_0^{n-1} \cup \{h_{j,\widehat{\beta}_*}\}_0^{n-1}$.

Any $h_{j,\widehat{\beta}}$ has USc for $j \in (0, n-1)$ when:

(i) $e > 0$ and both $h_{j,\widehat{\gamma}_*}$ and $h_{j-1,\widehat{\gamma}}$ have USc,
(ii) $e < 0$ and both $h_{j-1,\widehat{\gamma}_*}$ and $h_{j,\widehat{\gamma}}$ have USc.

2.2.7. *Links generated by positive braid words*

The following proposition for positive braid links cannot be as simple as the result for alternating braid links, but does establish sharp upper and lower bounds on the degree of each Homflypt coefficient polynomial, and the exact degree of the Conway polynomial. These are expressed in terms of the length or normalized length of any positive braid word whose closure is the link. In fact, any positive braid word fulfills the values of the link invariants for length and normalized length. The result further shows each h_j is h-uniform.

Proposition 2.43. *For a link, $\widehat{\beta}$, represented by a positive n-braid word β we have:*

(i) p_0 is a monic polynomial with $\deg p_0 = |\beta|$, hence $|\beta| = |\widehat{\beta}|$,
(ii) $\deg p_j = |\beta|$, for $j \in [0, \nu_0]$,
(iii) each nonzero p_j (or h_j) is of uniform sign $(-1)^j$, and hence h-uniform,
(iv) for j in the range $(\nu_0, n-1-\nu_1]$, the following are true:
  - *(a) $\deg p_j \le |\beta| - 2\,(j - \nu_0)$,*
  - *(b) when $p_j \neq 0$, $\deg p_j > \deg p_{j+1}$, and $\deg p_j \ge |\beta| - (j-\nu_0)(j+1-\nu_0)$,*
  - *(c) when $p_j = 0$, it follows that $p_{j+1} = 0$,*

(v) when $\nu_0 = 0$, $\nabla_{\widehat{\beta}}$ is a monic polynomial with $\deg \nabla_{\widehat{\beta}} = \|\beta\| = \|\widehat{\beta}\|$.

The relation of positive braid words to Proposition 2.5 is as follows. First, L can treated as empty. Second, the degree of p_j is constant (and maximal) on $[0, \nu_0]$, and is followed by a downslope in the remaining range. The opposite result holds for negative braid words.

Since torus links are so widely studied, it is appropriate to make a statement regarding the degree of their Homflypt coefficient polynomials. This is especially true since there is no similar general expression for all positive braid links. The interested reader should note that V.F.R. Jones displays a formula in Def. 6.1 p. 348 [8] for a two-variable link invariant which Prop. 6.2, p. 348 [8], relates to the Homflypt polynomial by: $X_L(q,\lambda) = P_L(v,z)$, with $v = \sqrt{\lambda}\sqrt{q}$ and $z = \sqrt{q} - 1/\sqrt{q}$. One of several calculations of $X_L(q,\lambda)$ for various knots includes a compact but complex formula for torus knots, Thm. 9.7, p. 359 [8].

Proposition 2.44. *For a torus link, $K(r,n)$, with $r \geq n$, the degree of the Homflypt coefficient polynomial, p_j, is $r(n-1) - j(j+1)$ for any $j \in [0, n-1]$. In case $r = n$, there is $p_{n-1,K(n,n)} = (-1)^{n-1}$.*

It is convenient at this point to give definitions for some thresholds that determine when certain bounds apply.

Definition 2.45. The positive n-braid rank threshold, denoted R_n, has the value n for $n \leq 4$, and $\lfloor n^2/4 \rfloor$ for $n \geq 4$. The positive braid coefficient rank threshold, denoted $R_{j,\,n}$, has the value $\lfloor j^2/4 \rfloor$ for $j \in [0,\, n-1]$.

Proposition 2.46. *For a link, $\widehat{\beta}$, represented by a positive n-braid word β, define $\delta(j,\,\beta) = \max\{\,0\,, 2j\,(R_{j+1,\,n} - \rho(\beta))\,\}$.*

We have $\deg p_j \leq |\beta| + 2\,o(\beta) + \delta(j,\,\beta) - j(j+1)$.

Example 2.47. The upper bound for $\deg p_j$ in Prop. 2.43 is achieved in connected sums of elementary torus links. Prop. 2.44 shows torus links, $K(r,n)$, with $R_n \leq r$ achieve the the upper bound of Prop. 2.46 and the lower bound of Prop. 2.43.

Theorem 2.48. *If β is a nonporous positive n-braid word with $\rho(\beta) \geq 2$, we have:*

(i) all p_j are nonzero for $j \in [0\,, \min(\rho(\beta) - 1, n-1)]$,
(ii) when p_j is nonzero and $R_{j+1,\,n} \leq \rho(\beta)$, then $\deg p_j = |\beta| - j(j+1)$.

The following example shows that for $n \geq 5$ and $j \geq 3$, the requirement $R_{j+1,\,n} \leq \rho(\beta)$ cannot be reduced and ensure $\deg p_j = |\beta| - j(j+1)$. For $n = 4$, the three-link $\beta = (\sigma_3^2\sigma_2^2\sigma_1^2)^2$ has $\deg p_3 = 4 > |\beta| - 12$. The rank two behavior is described in Theorem 2.51. Note that for $j \leq 2$, we have $R_{j+1,\,n} \leq 2 \leq \rho(\beta)$.

Example 2.49. Consider $L(r,n)$ which is obtained by taking α_{n-1}^r for any $n \geq 5$ and replacing the exponent of each $\sigma_{\lfloor n/2 \rfloor}$ by two. There is a node in one choice of skein tree for $\widehat{L(r,n)}$ that corresponds to $K(r, \lfloor (n+1)/2) \rfloor \,\sharp\, T_2 \,\sharp\, K(r, \lfloor n/2) \rfloor$. Observe that $\deg p_{j,\,\widehat{L(r,n)}}$ is at least the degree of p_j for this node, i.e. $r(n-2) - \lfloor j^2/2 \rfloor$. Hence $\deg p_{j,\,\widehat{L(r,n)}}$ exceeds the theorem value, $r\,n - j(j+1)$ when $j \geq 3$ and $2 \leq r < R_{j+1,\,n}$.

Theorem 2.51 depends on counting the number of generators that can be rendered trivial by use of the skein relation and Eq. 3.1. Toward this end, the following extension of Definition 2.26 is provided.

Definition 2.50. When $\beta \in B_n$ has rank two, a generator is said to be a1 (minimal, robust) when its syllable length is one in some (both, neither) row(s). A block of a1 (minimal, robust) generators is called an a1block (stripe, rblock), written $i(l,h)$ when l and h are the minimum and maximum block generator subscripts.

The counting function $d(\beta,a,b)$ for $a,b \in [0,n-1]$ is:

(i) when $a \geq b$ or $b \leq 1$ or σ_{b-1} is robust, then $d(\beta,a,b) = 0$, otherwise,
(ii) when σ_b is robust, then $d(\beta,a,b) = d(\beta,a,b-1)$, otherwise,
(iii) when σ_{b-1} is minimal, then $d(\beta,a,b) = 1 + d(\beta,a,b-1)$, otherwise,
(iv) $d(\beta,a,b) = 1 + d(\beta,a,b-2)$.

The counting function $D(\beta)$ is the sum of the $d(\beta,l,h)$ over all a1blocks $i(l,h)$.

Theorem 2.51. *Assume β is a nonporous, rank two, positive n-braid word. Define $[j]_2 = 0$ for j even, $[j]_2 = 1$ for j odd. Set $J = \lfloor (j-1)/2 \rfloor$. Let k_β be the maximum cardinality among all sets of robust interior generators in β with no adjacent pairs.*

We have $p_j = 0$ exactly when $D(\beta) \geq n-j$. When $p_j \neq 0$ we have:

$$\begin{aligned}
\deg p_j &= |\beta| - 3j + [j]_2, \text{ when } k_\beta \geq J, \\
\deg p_j &= |\beta| - 3j - 2(J-k_\beta) + [j]_2, \text{ when } k_\beta < J, \text{ or equivalently}, \\
\deg p_j &= |\beta| - 4j + 2k_\beta + 2, \text{ when } k_\beta < J.
\end{aligned}$$

Theorem 2.51 shows that the maximal v-span is achieved exactly when all a1blocks are thin.

2.2.8. *Links generated by polarized braid words*

The following theorem unifies the results for alternating and positive braid words to the larger class of polarized or homogeneous braid words.

Theorem 2.52. *For a link, $\widehat{\beta}$, represented by a polarized n-braid word β we have:*

(i) *all leading coefficients of nonzero p_j have the sign $(-1)^{j+x_n}$,*
(ii) *in the range $[0,\nu_n)$, the following are true:*
  (a) $\deg p_j \leq |\beta| - 2(\nu_n - j)$,
  (b) *when $p_j \neq 0$,* $\deg p_j < \deg p_{j+1}$,
(iii) $\deg p_j = |\beta|$, *for $j \in [\nu_n, \nu_n+\nu_0]$, hence $|\beta| = |\widehat{\beta}|$,*
(iv) *in the range $[\nu_n+\nu_0, n-1]$, the following are true:*
  (a) $\deg p_j \leq |\beta| - 2(j - \nu_n - \nu_0)$,
  (b) *when $p_j \neq 0$,* $\deg p_j > \deg p_{j+1}$,
(v) *when $\nu_0 = 0$,* $\deg \nabla_{\widehat{\beta}} = \| \beta \| = \| \widehat{\beta} \|$, *and the leading coefficient of $\nabla_{\widehat{\beta}}$ has sign $(-1)^{\nu_n + x_n}$.*

The relation of polarized braids to Prop. 2.5 is exactly as for alternating braids.

3. Proofs, Lemmas, and Tools

Theorems or equations whose proofs were not described above, and that don't follow by simple induction, are proven in a subsection, "Proof of . . . ". The proof of Eq. 2.1 and Eq. 2.2, is in section 3.7; intervening sections provide the necessary tools.

3.1. *Braid Identities and Properties*

The two relations in the next proposition have implications that are critical for the proofs of all major results that depend on braid length reduction arguments.

Proposition 3.1. *When two braid group generators, σ_i and σ_j, are adjacent, i.e. $|i-j|=1$, and e is nonzero, there are the following equivalent relations:*

$$\sigma_j^e \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i^e \,, \tag{3.1}$$

$$\sigma_j \sigma_i^e = \sigma_i^{-1} \sigma_j^e \sigma_i \sigma_j \,. \tag{3.2}$$

From (3.1) it easily follows that when a three-braid word of rank, $r+1 \geq 3$, has nonzero exponents in the expression, $\sigma_2^{E_1}\sigma_1^{e_1}\cdots\sigma_2^{E_r}\sigma_1^{e_r}\sigma_2\sigma_1$, its closure generates the same link as $\sigma_2^{E_1+e_r}\sigma_1^{e_1}\cdots\sigma_2^{1+E_r}\sigma_1$, a braid word of rank r. Of course, when $r=1$, the braid word, $\sigma_2^{E_1+e_1+1}\sigma_1$ generates the same link as $\sigma_2^{E_1}\sigma_1^{e_1}\sigma_2\sigma_1$.

Next, observe that if a braid word, $\beta \in B_n$, has a trivial generator, σ_κ, then $\widehat{\beta}$ can also be generated by a braid word in B_{n-1}. Indeed, when $\kappa \neq n-1$, first replace β by a braid word of the form $\gamma\sigma_\kappa^{\pm 1}\eta$, in which all the generators of subscript higher than κ appear in γ, and all the generators of subscript lower than κ appear in η. Using the relation, (3.2) repetitively, β may be replaced by a braid word in which σ_κ is replaced by the trivial generator σ_{n-1}, and the subscripts of generators above κ are each reduced by one.

When a braid word, $\beta \in B_n$, has a trivial generator, σ_κ, we may view $\widehat{\beta}$ as the connected sum of $\Psi\widehat{(\kappa, n, \beta)}$ and $\Psi\widehat{(0, \kappa, \beta)}$. When σ_κ is instead a null generator, we may view $\widehat{\beta}$ as the disjoint union of $\Psi\widehat{(\kappa, n, \beta)}$ and $\Psi\widehat{(0, \kappa, \beta)}$.

3.2. *Proof of (2.4) (Conway polynomial formula for 2-braid links)*

Proof. Use the following result relating to Pascal's triangle and induction:

$$\binom{p}{j} = \binom{p-1}{j} + \binom{p-1}{j-1} \,. \tag{3.3}$$

□

3.3. *Properties of the Conway polynomial*

3.3.1. *Properties of the Conway polynomial for n-braid links*

The following formula shows how the Conway polynomial can be calculated using braid words of shorter length. For any braid word, $\beta \in B_n$, and any integer, e:

$$\nabla_{\widehat{\beta\sigma_i^e}} = C_e \, \nabla_{\widehat{\beta\sigma_i^{\pm 1}}} + C_{e\mp 1} \, \nabla_{\widehat{\beta}} \, . \tag{3.4}$$

3.3.2. *Properties of the Conway polynomial for two-braid links*

The Conway polynomial, C_p, is central to the results of this paper, and its properties are numerous and remarkable. Most of its properties are most easily derived from the recursive relation, $C_{p+1} = z\, C_p + C_{p-1}$, rather than by using (2.4).

It is helpful in calculations and proofs to know the following:

$$z^m \, C_p = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \, C_{m+p-2j} \, , \text{ for } m > 0. \tag{3.5}$$

In particular, when $p = 1$, there is an expression for z^m.

The following important identities are critical tools to prove the main results. Assume integers, x, y, p, q, and κ, are chosen with $x + y = p + q$ in the equations below. Eq. 3.8 is generally applied with $\kappa = q$. We have:

$$C_{x+y} = C_x \, C_{y+1} + C_{x-1} \, C_y \, , \tag{3.6}$$

$$C_x \, C_k = \sum_{j=0}^{k-1} (-1)^j \, C_{x+k-1-2j} \, , \text{for } k > 0, \tag{3.7}$$

$$C_x \, C_y - C_p \, C_q = (-1)^\kappa \, \{ C_{x-\kappa} \, C_{y-\kappa} - C_{p-\kappa} \, C_{q-\kappa} \} \, . \tag{3.8}$$

Proposition 3.2. *There are no common roots over the complex numbers for* C_p *and* C_{p+1}*, for any integer* p*, hence* $\gcd(C_p, C_{p+1}) = 1$.

For any integers $a \neq 0$*, and* b*, we have* $C_a | C_{ab}$.

Furthermore, when $\gcd(a, b) = g$*, we have* $\gcd(C_a, C_b) = C_g$.

Proof. The first claim follows from the relation, $C_{p+1} = z\, C_p + C_{p-1}$. Next, observe that $C_{ab} = C_a C_{ab-a+1} + C_{a-1} C_{ab-a}$ and use induction. For the third claim, there are integers, λ, μ so that $\lambda a + \mu b = g$. Hence $C_g = C_{\lambda a + \mu b} = C_{\lambda a+1} C_{\mu b} + C_{\lambda a} C_{\mu b-1}$. This implies C_a and C_b have no common roots in $\mathbb{C}$ other than the roots of C_g. □

The following lemma gives the high order term for the difference of the Conway polynomials for two distinct connected sums of positive torus links, with no null generators, in an interesting special case.

Lemma 3.3. *Suppose two non-increasing sequences of* r *positive integers,* $\{p_i\}_{i=1}^r$ *and* $\{q_i\}_{i=1}^r$*, are given with the same sum,* s*. Denote the multiplicities of the final* *terms as* μ_p *and* μ_q. *Assume* $p_r > q_r$, *or* $p_r = q_r$ *and* $\mu_p < \mu_q$. *If* $p_r > q_r$, *let* $\lambda = \mu_q$, *otherwise* $\lambda = \mu_q - \mu_p$. *We have*

$$\prod_{i=1}^{r} C_{p_i} - \prod_{i=1}^{r} C_{q_i} = (-1)^{q_r} \lambda z^{s-2q_r-r} + o(z^{s-2q_r-r}) \,.$$

Proof. Eq. 3.8 implies the result when $r = 2$. When $p_r > q_r$, apply (3.8), (3.6) and induction to obtain the result. When $p_r = q_r$, apply the result for distinct final terms with r replaced by $r - \mu_p$. □

3.3.3. *Properties of the Conway polynomial for three-braid links*

Proposition 3.4. *When* $\gamma \in B_3$ *and* $a > 0$ *we have:*

$$\begin{aligned}
\nabla_{\widehat{\alpha_2^3\gamma}} &= \nabla_{\widehat{\gamma}} + C_{w(\gamma)+5} - C_{w(\gamma)+1} \,, \\
\nabla_{\widehat{\alpha_2^{3a}\gamma}} &= \nabla_{\widehat{\gamma}} + \sum_{j=1}^{a} C_{w(\gamma)+6j-1} - \sum_{j=1}^{a} C_{w(\gamma)+6j-5} \,, \\
\nabla_{\widehat{\alpha_2^{-3a}\gamma}} &= \nabla_{\widehat{\gamma}} + \sum_{j=1}^{a} C_{w(\gamma)-6j+1} - \sum_{j=1}^{a} C_{w(\gamma)-6j+5} \,.
\end{aligned}$$

If $\gamma = \prod_{k=1}^{r} \sigma_2^{-e_{k,2}} \sigma_1^{e_{k,1}}$, *with* $r, e_{k,2}, e_{k,1} > 0$, *and* $1 < E_i = \sum_{k=1}^{r} e_{k,i}$, *we have*

$$\nabla_{\widehat{\gamma}} = (-1)^{E_2+1}\{C_{E_2+E_1-1} - rC_{E_2+E_1-3} + o(C_{E_2+E_1-3})\} \,, \tag{3.9}$$

$$\nabla_{\widehat{\gamma}} = (-1)^{E_2+1}\{C_{E_2}C_{E_1} - \prod_{k=1}^{2} C_{e_{k,2}}C_{e_{k,1}}\} \,, \text{ when } r = 2.$$

Proof. Induction, plus (3.4) and (3.6) for the first and last equations. Eq. 3.8 is also helpful to prove the order of magnitude result in (3.9). □

3.4. *Properties of the Homflypt polynomial*

The following properties may be found in most introductory texts, in particular, [17]. These formulas relate the Homflypt polynomial for the connected sum, disjoint (or distant) union, and mirror image of a link, to the original links.

$$P_{L_1} \sharp P_{L_2} = P_{L_1} P_{L_2} \,, \tag{3.10}$$

$$P_{L_1} \coprod P_{L_2} = (vz)^{-1}(1 - v^2) P_{L_1} P_{L_2} \,, \tag{3.11}$$

$$P_{\overline{L_1}}(v, z) = P_{L_1}(v^{-1}, -z) = P_{L_1}(-v^{-1}, z) \,. \tag{3.12}$$

3.4.1. *Properties of the Homflypt polynomial for n-braid links*

The Homflypt polynomial shares many properties of the Conway polynomial. The following formulas show how the Homflypt polynomial can be calculated using braids of shorter length. For any braid word, $\beta \in B_n$, and any integer, e:

$$P_{\widehat{\beta\sigma_i^e}} = v^{e-1}\, C_e\, P_{\widehat{\beta\sigma_i}} + v^e\, C_{e-1}\, P_{\widehat{\beta}}\,, \tag{3.13}$$

$$P_{\widehat{\beta\sigma_i^e}} = v^{e+1}\, C_e\, P_{\widehat{\beta\sigma_i^{-1}}} + v^e\, C_{e+1}\, P_{\widehat{\beta}}\,. \tag{3.14}$$

In particular, when σ_i is a clustered generator in β, with exponent, e, there is:

$$P_{\widehat{\beta}} = P_{\widehat{\Psi(i,n,\beta)}} P_{T_e} P_{\widehat{\Psi(0,i,\beta)}}\,. \tag{3.15}$$

3.5. *Proof of Lemma 2.1 (Three-Braid Homflypt formula)*

Proof. The proof is an induction on the rank of β. When the rank is one (3.10) implies (2.3) may be used to compute $P_{\widehat{\beta}}$. Use of (3.6) allows the terms to be rewritten in the form of Lemma 2.1. When the rank is two or more, with $\beta = \gamma\sigma_2^E\sigma_1^e$, a double application of (3.13) to the exponents, E, e, in the last row yields

$$\begin{aligned} P_{\widehat{\beta}} = {} & v^{E+e-2}\, C_E\, C_e\, P_{\widehat{\gamma\sigma_2\sigma_1}} + \\ & v^{E+e-1}\,\{\, C_E\, C_{e-1}\, P_{\widehat{\gamma\sigma_2}} + C_{E-1}\, C_e\, P_{\widehat{\gamma\sigma_1}}\,\} + \\ & v^{E+e}\, C_{E-1}\, C_{e-1}\, P_{\widehat{\gamma}}\,. \end{aligned} \tag{3.16}$$

Now reduce the rank of braid words ending in $\sigma_2\sigma_1$, σ_2 or σ_1 in (3.16) per Section 3.1. The result follows by recombining terms using (3.13) and (3.4). □

3.6. *Proof of Prop. 2.4 (When do 3 braid links have $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$)*

Proof. Suppose first that $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$. Since $w(\beta) = w(\gamma)$ is valid, assume $w(\beta) > w(\gamma)$. Prop. 2.3 shows that $p_{2,\widehat{\beta}} = 0$ and $p_{0,\widehat{\gamma}} = 0$, i.e. $\nabla_{\widehat{\beta}} = C_{w(\beta)-1}$ and $\nabla_{\widehat{\gamma}} = C_{w(\gamma)+1}$. Since $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$, we have $C_{w(\beta)-1} = C_{w(\gamma)+1}$, i.e. $w(\beta) = w(\gamma) + 2$ or $w(\beta) = -w(\gamma)$. These correspond to the second and third outcomes.

Conversely, when $\nabla_{\widehat{\beta}} = \nabla_{\widehat{\gamma}}$, the first condition implies $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$ by Prop. 2.2. Prop. 2.3 shows the remaining two cases also imply $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$. □

Proposition 3.5. *When $\gamma \in B_3$ and $a > 0$ we have:*

$$P_{\widehat{\alpha_2^3\gamma}} = v^6 P_{\widehat{\gamma}} + P_{T_{w(\gamma)+5}} - v^6 P_{T_{w(\gamma)+1}}\,,$$

$$P_{\widehat{\alpha_2^{3a}\gamma}} = v^{6a} P_{\widehat{\gamma}} + \sum_{j=1}^{a} v^{6a-6j} P_{T_{w(\gamma)+6j-1}} - \sum_{j=1}^{a} v^{6+6a-6j} P_{T_{w(\gamma)+6j-5}}\,,$$

$$P_{\widehat{\alpha_2^{-3a}\gamma}} = v^{-6a} P_{\widehat{\gamma}} + \sum_{j=1}^{a} v^{6j-6a} P_{T_{w(\gamma)-6j+1}} - \sum_{j=1}^{a} v^{6j-6a-6} P_{T_{w(\gamma)-6j+5}}\,.$$

Proof. Induction, plus Prop. 3.4 and Prop. 2.3 for the first equation. □

3.7. *Proof of (2.1, 2.2) (Forms for Homflypt polynomial)*

Proof. The expression is valid for B_1 and B_2, so the induction hypothesis is this is so for braid groups with fewer strands than n. Observe that O_n satisfies the result, as does the disjoint union of two links each on fewer strands, or their connected sum. The braid length reduction formulas, (3.13, 3.14), and the standard braid relations, show the result is true whenever a braid word has two or more consecutive occurrences of the same generator. Use of the skein relation shows it suffices to consider the case when β is a positive braid word. There is no loss in generality to assume $\beta = \sigma_{n-1}\gamma\sigma_{n-1}\eta$, in which γ has no occurrences of σ_{n-1} and has positive length, and β has no null or trivial generators.

The following argument shows β may be replaced by a braid word that either falls into a case already shown to be true, or has fewer instances of σ_{n-1}, or the length of γ may be reduced, so that the result follows by induction.

When $\gamma = \sigma_{n-2}$, note that β may be replaced by $\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}\eta$, with fewer instances of σ_{n-1}. When the length of γ is one, but $\gamma \neq \sigma_{n-2}$, then β may be replaced by $\gamma\sigma_{n-1}\sigma_{n-1}\eta$, a case already shown to satisfy the result.

For the remaining case that the length of γ exceeds one, consider the maximal prefix of γ in which successive terms have lower subscript than their predecessor term. If this prefix does not begin with σ_{n-2}, then the first term in the prefix can commute to the left past the initial σ_{n-1}, decreasing the length of γ. Observe that in this prefix, the initial σ_{n-2} must be followed by successive generators in which the subscript decreases by one. Indeed any successive terms in the prefix for which the subscripts differ by two or more allow the later term to commute to the left past σ_{n-1}. This would allow γ to be replaced by a shorter word.

The prefix ends with a generator, the prefix-end, whose subscript is lower than all others in the prefix. The next term after the prefix-end, the neighbor, has a higher subscript, which may be $n-1$. We may assume that the neighbor's subscript is one higher than the subscript for the prefix-end by simply commuting the neighbor to the left until this is true. Thus there arises an earliest substring, $\sigma_k\sigma_{k-1}\sigma_k$; when $k = n-1$ we are done. Application of the braid relation allows $\sigma_k\sigma_{k-1}\sigma_k$ to be replaced by $\sigma_{k-1}\sigma_k\sigma_{k-1}$, and now the new initial σ_{k-1} may commute to the left past the initial σ_{n-1}, leading to a shorter γ, as desired. □

The following results are easy consequences of Equations 2.1 and 2.2.

Proposition 3.6. *The coefficient polynomials, p_j and h_j obey the same skein relation and reduction formulas, (3.4), as the Conway polynomial.*

The coefficient polynomials of a link and its mirror image are related as follows:

$$p_{j,\overline{L}}(z) = (-1)^{n-1}p_{n-1-j,L}(-z) = (-1)^{w+n-1}p_{n-1-j,L}(z)\,.$$

When $\beta \in B_n$ is extended to $\beta\sigma_n \in B_{n+1}$ we have $h_{n,\widehat{\beta\sigma_n}} = 0 = p_{n,\widehat{\beta\sigma_n}}$. For all other subscripts, we have $h_{j,\widehat{\beta\sigma_n}} = h_{j,\widehat{\beta}}$ and $p_{j,\widehat{\beta\sigma_n}} = z\,p_{j,\widehat{\beta}}$.

When $\beta \in B_n$ is extended to $\beta\sigma_n^{-1} \in B_{n+1}$, we have $h_{0,\widehat{\beta\sigma_n}} = 0 = p_{0,\widehat{\beta\sigma_n}}$. For all other subscripts, we have $h_{j,\widehat{\beta\sigma_n^{-1}}} = h_{j-1,\widehat{\beta}}$ and $p_{j,\widehat{\beta\sigma_n^{-1}}} = z\,p_{j-1,\widehat{\beta}}$.

Proposition 3.7. *For the torus links, there are the following relations:*

$$p_{j,K(r,n)} = z^{n-r} p_{j,K(n,r)}\,, \text{for } 0 \le j < r < n\,, \qquad (3.17)$$
$$p_{j,K(r,n)} = 0\,, \text{for } r \le j < n\,.$$

3.8. *Proof of Proposition 2.5 (3-Braid Degree Result)*

Proof. It suffices to prove either $\deg p_0 \le \deg p_1$, or $\deg p_2 \le \deg p_1$, and to consider only the case in which p_0 and p_2 are both nonzero. By Prop. 3.6, the writhe may be assumed to be non-negative. Separate consideration of the cases $\nabla_\beta = 0$, $w = 0$, $w = 1$, $\deg \nabla_\beta > w$, $\deg \nabla_\beta = w$, $\deg \nabla_\beta = w - 2$, and $\deg \nabla_\beta < w - 2$ shows the result is true in all cases. □

3.9. *Proof of Theorem 2.7 (Relations for Laurent coefficients)*

Proof. The proof will be in the reverse order of the statement of the theorem.

First, to show (2.7), (2.8) are independent when $n > 1$, it suffices to find two sets of n polynomials each of which satisfies one relation, but not the other.

In case $w \ne n - 1$, Eq. 3.6 implies $h_0 = C_{w+2-n}$, $h_1 = z\,C_{w+1-n}$, and $h_j = 0$, for $j > 1$ satisfy (2.7), but these don't satisfy (2.8). When $w = n - 1$, the choice $h_1 = C_3$ and $h_j = 0$, for $j \ne 1$ satisfies (2.7), but does not satisfy (2.8).

In case $w \ne n - 3$, observe that $h_1 = C_{w+4-n}$, and $h_j = 0$, for $j \ne 1$ satisfy (2.8), but they don't satisfy (2.7). When $w = n - 3$, the choice $h_0 = 1$, and $h_j = 0$, for $j \ne 0$ satisfies (2.8), but does not satisfy (2.7).

The second point is to show whenever a function satisfies (2.7, 2.8), it also satisfies (2.9). First take the difference of (2.7, 2.8), and divide by z to obtain (2.9) with $\kappa = 1$. Now multiply both sides of this expression by C_κ, multiply both sides of (2.8) by $-C_{\kappa-1}$, and add these two products. Apply (3.6) to the right side of the sum and (3.8) to the left side to obtain (2.9).

The proof of (2.7), (2.8) mirrors that in Section 3.7. For $n = 1$, (2.7, 2.8) are trivial, while for $n = 2$, application of (3.6 and 2.3) yields the result.

For O_n observe that $h_j = (-1)^j \binom{n-1}{j} / z^{n-1}$. The proofs for (2.7) and (2.8) are similar. For (2.7) apply (3.5) to the expression, $z^{n-1}\,C_{w+4-n}$, with $w = 0$, to obtain $z^{n-1}\,C_{4-n} = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j\,C_{3-2j}$. This is equivalent to (2.7).

The technique to verify the claims for disjoint unions and connected sums is similar. Only the proof for the latter is included. To show (2.7) holds for the connected sum of two links, $\widehat{\gamma}$ and $\widehat{\eta}$, with r and s strands, respectively, and $r + s - 1 = n$, observe that $h_{j,\widehat{\gamma}\sharp\widehat{\eta}}$ is merely the sum of all terms of the form $h_{k,\widehat{\gamma}}\, h_{j-k,\widehat{\eta}}$.

Since $C_{2j-3} = C_{2k-1}C_{2(j-k)-1} + C_{2k-2}C_{2(j-k)-2}$, observe that $C_{2j-3}\, h_{j,\widehat{\gamma}\sharp\widehat{\eta}}$ is merely two sums: one with all terms of the form $C_{2k-1}\, C_{2(j-k)-1}\, h_{k,\widehat{\gamma}}\, h_{j-k,\widehat{\gamma}}$ and a second sum with all terms of the form $C_{2k-2}\, C_{2(j-k)-2}\, h_{k,\widehat{\gamma}}\, h_{j-k,\widehat{\gamma}}$. The first sum derives from multiplying the left side of (2.8) for $\widehat{\gamma}$ with the same for $\widehat{\eta}$. The second sum derives from multiplying the left side of (2.9) for $\widehat{\gamma}$ with $\kappa = 1$ with the same for $\widehat{\eta}$.

This implies the left side of (2.7) for $\widehat{\gamma}\sharp\widehat{\eta}$ is merely the sum of $C_{w(\gamma)+2-r}\, C_{w(\eta)+2-s}$ and $(-1)\, C_{w(\gamma)+3-r}\, (-1)C_{w(\eta)+3-s}$. This sum may be rewritten as $C_{w(\gamma)+w(\eta)+4-(r+s-1)}$, as desired. Proof of (2.8) for connected sums is similar.

The braid length reduction formulas, (3.13, 3.14), show whenever a braid word has two or more consecutive occurrences of the same generator, the result is true. The remainder of the proof is exactly as in Section 3.7. □

3.10. *Proof of Props. 2.9, 2.10, 2.12 (Membership in $\Upsilon_{w\mp1}$, Υ_x)*

Proof. Prop. 3.4 and properties of ∇ show that no braid word from Ω_0^* belongs to Υ_{w-1}. The five words in the first two list items are the only members from $\bigcup_{i=1}^5 \Omega_i^*$.

In Ω_6^*, with $d = 0$, Prop. 3.4 implies any member has $r = 1$ and $E_1 = 1$. This yields the sixth word, $\sigma_2^{-e}\sigma_1^1$. In Ω_6^*, with $d = 1$, similar reasoning shows $r = 1$ implies $e_1 = 1$ and yields $\alpha_2^3\sigma_2^{-1}\sigma_1^E$. There are no members in $\Omega_6^* \cap \Upsilon_{w-1}$ when $d = 1$ and $r > 1$, and no members when $d = -1$.

By comparing degrees and using Prop. 3.4, we see that when $d > 1$ there are no members in $\Omega_6^* \cap \Upsilon_{w-1}$ with $r > 1$. Indeed when $r > 1$, observe that $e = \sum_1^r e_k \equiv E = \sum_1^r E_k \bmod 2$, and then note that rC_{e+E-3} cannot be accommodated. A similar argument applies when $d < -1$ and $r > 1$. When $d > 1$ and $r = 1$, we need

$$C_{-e}C_E + \sum_{i=1}^{d-1} C_{-e+E+6i-1} - \sum_{i=1}^{d} C_{-e+E+6i-5} = 0, \text{ which is equivalent to}$$

$$-z\sum_{i=1}^{d-1} C_{-e+E+6i} - C_{-e+1}C_{E+1} = 0.$$

When $E = 1$, we see e is odd, so $e = 3d-2$ and d is odd. This yields the eighth member, but $d = 1$ is the seventh member with $E = 1$. When $E > 1$, there are no members for $e = 1$, hence the degree of $C_{-e+1}C_{E+1}$ is $e + E - 2$. As e, E cannot both be even, apply (3.7) with $x = e-1$ and $k = E+1$ when E is odd, or $x = E+1$ and $k = e-1$ when e is odd, to see there are no members with $e > 1$.

Similarly, when $d < -1$ and $r = 1$, we obtain the final entry in the list, $\alpha_2^{-3d}\sigma_2^{-1}\sigma_1^{3d+1}$ with $d \geq 0$ even. Note $d = 0$ yields the sixth member with $e = 1$.

The proof of Props. 2.10 and 2.12 for Υ_{w+1} and Υ_x are similar. □

3.11. *Proof of Prop. 2.16 (Simple V_L properties for 3-braid links)*

Proof. Begin by ignoring the value for $\mu(\beta)$, and observe that $V_{\widehat{\beta}} = V_{\widehat{\gamma}}$, implies $\mu(\widehat{\beta}) = \mu(\widehat{\gamma})$ by (12.1), p. 368 [8], which states that $V_L(1) = (-2)^{\mu-1}$. Set $a = w(\widehat{\beta})$ and $b = w(\widehat{\gamma})$. Since $w \equiv 1+\mu \bmod 2$ for odd strand number, we have $a \equiv b \bmod 2$.

Comparison of the Jones polynomials, (2.6), for $\widehat{\beta}$ and $\widehat{\gamma}$ yields an expression, (3.18), for $\Delta_{\widehat{\gamma}}(t)$ in terms of $\Delta_{\widehat{\beta}}(t)$. Using $\Delta_L(1/t) = (-1)^{\mu(L)-1}\Delta_L(t)$, p. 6 [21], we obtain an expression, (3.19), for $\Delta_{\widehat{\beta}}(t)$. As $a = b$ is always a solution, a more useful form, (3.20), arises by setting $a = 2B + 2k$ with $b = 2B$, and $G = 1 + t + t^2$, then dividing (3.19) by $t^{a/2} - t^{b/2}$ for $k \neq 0$.

Finally use $\mu(\widehat{\beta}) = 3$ to see that when $a \neq b$, Eq. 3.20 implies $\Delta_{\widehat{\beta}}(1) = 1$, which is disallowed. As $a = b$, Prop. 2.2 shows $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$.

$$t^b\Delta_{\widehat{\gamma}}(t) = (t^{3b/2} - t^{3a/2})t/(1+t+t^2) + (-1)^a(t^{b/2} - t^{a/2}) + t^a\Delta_{\widehat{\beta}}(t). \quad (3.18)$$

$$(t^{2a} - t^{2b})\Delta_{\widehat{\beta}}(t) = \{t^{5a/2} - t^{(2a+3b)/2} + (-1)^a(t^{(2a+b)/2} - t^{(4b-a)/2})\}t/(1+t+t^2) + (-1)^a(t^{3a/2} - t^{(2a+b)/2}) + t^{(2a+3b)/2} - t^{(4b+a)/2}. \quad (3.19)$$

$$t^{a/2}(t^{2k}+1)(t^k+1)\Delta_{\widehat{\beta}}(t) = \{(t^{a+k} + (-1)^a)(t^{2k} + t^k + 1)\}t/G + (t^a + (-1)^a t^{3k}), \text{ for } k \neq 0. \quad (3.20)$$

When $w(\beta) = w(\gamma)\pm 2$, i.e. $k = \pm 1$, substitute in (3.20) to see $\Delta_{\widehat{\beta}}(t) = \Delta_{T_{a\mp 1}}(t)$.

When $\nabla_{\widehat{\beta}} = C_{a-1}$, substitute $\Delta_{\widehat{\beta}}(t) = (t^{a-1} + (-1)^a)/t^{(a-2)/2}(t+1)$ in (3.20) to see that $a = b+2$ is a solution. For $\mu(\beta) = 2$ it is the only solution beyond $a = b$. For $\mu(\beta) = 1$, the only novel solutions are $(a, b) = (2, -2)$ or $(0, 2)$. Substitute $a = b+2$ in (3.18) to see $\Delta_{\widehat{\gamma}}(t) = \Delta_{T_{b+1}}(t) = \Delta_{\widehat{\beta}}(t)$. Hence $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$. Evaluation at $(a, b) = (2, -2)$ or $(0, 2)$ shows that $\Delta_{\widehat{\gamma}}(t) = 1 = \Delta_{\widehat{\beta}}(t)$, so also $P_{\widehat{\beta}} = P_{\widehat{\gamma}}$. A similar analysis applies if $\nabla_{\widehat{\beta}} = C_{a+1}$.

In proving the two claims for knots and two-links with $w(\beta) - w(\gamma) = 2k > 2$, we may assume $a \geq 0$, since $\Delta_{\overline{L}}(t) = \Delta_L(1/t)$ and $V_{\overline{L}}(t) = V_L(1/t)$. Inspection of (3.20) shows that $a = 0$ implies $\Delta_{\widehat{\beta}}(t) = 1$ and $k = 1$, so $a > 0$ and $\max\deg\Delta_{\widehat{\beta}}(t) = a/2-1$. Note that $G \mid (t^{2k}+t^k+1)$ exactly when $k \not\equiv 0 \bmod 3$. So when $k \equiv 0 \bmod 3$, we must have $G \mid (t^{a+k} + (-1)^a)$. This is impossible when a is even and implies $a \equiv 3 \bmod 6$ when a is odd.

Assume now that a is even, so $\widehat{\beta}$ and $\widehat{\gamma}$ are knots. Eq. 3.20 shows that when $k = 2$, a zero is present at $\pm\sqrt{-1}$, but for $k > 2$, the roots of $t^k + 1$ force $a = 2mk$ for some $m > 0$. Similarly, for any $k > 1$, the roots of $t^{2k} + 1$ force $a = k(1+4s)$ for some $s \geq 0$. This implies $k = 2$ and $a \equiv 2 \bmod 8$, as desired. When a is even, the same type of analysis shows $a = k(3+4y)$ with k odd and $k \geq 3$.

To see that $V_{\widehat{\eta}} = 1$ implies $\Delta_{\widehat{\eta}} = 1$, first observe that $\mu(\widehat{\eta}) = 1$ and $V_{\widehat{\sigma_2\sigma^{-1}}} = V_{\widehat{\eta}}$. Assign $\beta = \sigma_2\sigma^{-1}$ and $\gamma = \eta$ in (3.19) to see that $w(\eta) \in \{0, \pm 2\}$. Apply these values to (3.18) to see $\Delta_{\widehat{\eta}} = 1$. □

3.12. *Proof of Prop. 2.17 (V is equivalent to P for 3-braid links)*

Proof. The first half of the proof will deal with knots, and the second half with two-links. Prop. 2.16 shows that three-links satisfy the result.

If $\beta \in \Omega_1^* \cup \Omega_2^*$, Eq. 3.20 and Prop. 2.16 show that $a = 10$ is the only solution, and $\beta = \alpha_2^5$. This is the 10-124 knot, $\Delta_{\widehat{\beta}} = (t^8 - t^7 + t^5 - t^4 + t^3 - t^1 + 1)/t^4$. This implies $b = 6$, so $\gamma \in \Omega_6^*$. Also by (3.18), we have $\Delta_{\widehat{\gamma}} = (t^6 - t^4 + t^3 - t^2 + 1)/t^3$. Now $\gamma = \alpha_2^{3d}\eta$, and since $\Delta_{\widehat{\gamma}}$ is not alternating, we have $d \neq 0$, and $w(\eta) \equiv 0 \bmod 6$. Comparing $\Delta_{\widehat{\gamma}}$ to the expression from Section 2.1, and using $\max\deg \Delta_{\widehat{\eta}} = |\eta|/2 - 1$ leads to $|\eta| = 8$. If $w(\eta) = 0$, we have $d = 1$ and $\Delta_{\widehat{\eta}}$ is not alternating, contrary to the property of alternating three-braid links (Prop. 4.2, p. 13 [16]). This implies $\eta \in \{\sigma_2^{-7}\sigma_2^1, \sigma_2^{-1}\sigma_2^7\}$. Neither of these satisfy the required relation, so $\gamma \notin B_3$.

If $\beta \in \Omega_6^*$ creates a knot, use the formula from Section 2.1 in (3.20) to see:

$$(t^4 + 1)Gt^{a/2}\Delta_{\widehat{\eta}} + t(t^4 + 1)\{-t^{3d} - t^{3d+w(\eta)}\} = t^a + t^6$$

Note that $a = 6d + w(\eta)$, so $3d = (a - w(\eta))/2$, divide by $t^{a/2}$ and we obtain:

$$(t^4 + 1)G\Delta_{\widehat{\eta}} = t(t^4 + 1)\{t^{-w(\eta)/2} + t^{w(\eta)/2}\} + t^{a/2} + t^{6-a/2}$$

The case $a = 2$ is readily discarded, so by comparing maximum degrees, we see $a/2$ is the maximum on the right side, and thus $a = 10 + |\eta|$. The cases that $r = 1$ with either $e_1 = 1$ or $E_1 = 1$ are readily discarded, so we are in a situation where we may apply (3.9) tailored to Δ, to see that there are no solutions. This is the last knot case.

When $\beta \in \Omega_3^*$, we have a two link. The expression for Δ from Section 2.1 may be used in (3.20). The only solution is $a = 3k$, so $\Delta_{\widehat{\beta}}(t) = t(t^{3k} - 1)/(Gt^{3k/2})$. We also have $b = k$ with k odd and $k \geq 3$. Use of (3.18) shows that $\Delta_{\widehat{\gamma}}(t) = t^{k/2} - t^{-k/2}$. This implies $\gamma \in \Omega_6^*$, say $\gamma = \alpha_2^{3d}\eta$, with $d \neq 0$. As in the knot case, this leads to a relation, $(t^k - 1)(t^2 + 1) = t^{k/2}G\Delta_{\widehat{\eta}} + t(-t^{3d+w(\eta)} + t^{3d})$, that cannot be satisfied. No member of $\Omega_4^* \cup \Omega_5^* \cup \Omega_6^*$ generates a two-link that satisfies (3.20). □

3.13. *Proof of Proposition 2.21 (General bounds for p_j and ∇)*

Proof. The proof in Section 3.7 can be easily adapted to prove item *i* and *ii*; item *iii* is a corollary of these. For the last item, take γ to be the braid word obtained by dropping all the trivial generators from β and shift subscripts per Section 3.1□

3.14. *Proof of Theorem 2.22 (Results for arbitrary braid)*

Proof. Application of the following lemma to the braid word and its mirror image, and use of Prop. 3.6, suffice to prove Theorem 2.22. □

Lemma 3.8. *For a link, $\widehat{\beta}$, generated by any n-braid word β we have:*

(i) $\deg p_j \leq |\beta| - 2(\nu_n + \nu_m - j)$, *for* $j \in [0, \nu_n + \nu_m]$,
(ii) $\deg p_j \leq |\beta|$, *for* $j \in [\nu_n + \nu_m, n - 1 - \nu_p]$,
(iii) $\deg p_j \leq |\beta| - 2(j + \nu_p - (n - 1))$, *for* $j \in [n - 1 - \nu_p, n - 1]$,
(iv) $p_j = 0$ *for* $j \notin [\nu_{-1}, n - 1 - \nu_1]$.

Proof. It is instructive to first examine the effect on the coefficient polynomials of a disjoint union of O_μ with the closure of a braid, i.e. how a plateau is created in the degree of p_j. This corresponds to increasing ν_0 by μ.

When two links satisfy the result, so does the connected sum and disjoint union. An induction on $|\beta| + \nu_0(\beta)$ for braids on n strands is sufficient to prove the result by using the type of braid length reduction argument in Section 3.7. □

3.15. *Proof of Proposition 2.24 (∇ as complete rank one inv.)*

Proof. Suppose two positive braid words of rank one, with no null generators, generate links with the same Conway polynomial. Since the closure is merely the connected sum of elementary torus links, it may be assumed there are no common twist exponents in the two words.

When one braid word has fewer generators than the other, it may be replaced by a longer braid word, using positive Markov stabilization, so that the two braid words belong to the same braid group. The degree of the Conway polynomial is the length less the number of generators in this setting. This implies the two braid words, with the same number of generators, have the same length and writhe. Eq. 3.8 and Lemma 3.3 now show the set of twist exponents must coincide. □

3.16. *Proof of Proposition 2.25 (Homflypt/Jones polynomials are each complete rank one invariants)*

Proof. The Jones polynomial for an elementary torus link is given by:

$$V_{T_w}(t) = -t^{(w-1)/2}\{t^{w+1} + (-1)^w(1 + t + t^2)\}/(1 + t)\,.$$

As $V_{O_2}(t) = -(1+t)/\sqrt{t}$, it is seen that -1 is a root for only $V_{O_2}(t)$. When $a \neq 0$ is a root of $V_{T_M}(t)$ and $V_{T_N}(t)$, with $M \neq N$, we have $|a| = 1 = |1 + a + a^2|$, which is only possible when $a = -1, \pm\sqrt{-1}$. As $\pm\sqrt{-1}$ are at most simple roots, and each $V_{T_w}(t)$ has another nonzero root for any $|w| \geq 2$, the Jones polynomial is a complete invariant for links generated by rank one braid words.

To show the same for the Homflypt polynomial, it may be assumed there are no common twist exponents. As a special case, not relevant to the prior result, it may be assumed there are no null generators in either braid word. Indeed, when the number of null generators match, the two braid words can be replaced by another pair from braid groups of lower strand number. A braid word with null generators generates a link whose Conway polynomial is zero, while this is not true when no null generators exist. Thus both braid words must have initially had the same number of null generators, and these can now be dropped from both.

The proof of Proposition 2.24 shows the set of absolute values of the twist exponents in each braid word must be identical. Since the collections are disjoint, the positive twist exponents in one braid word, $\{p_i\}_{i=1}^{\nu_p}$, must appear, with opposite sign, in the second braid word; similarly for the negative exponents, $\{-q_i\}_{i=1}^{\nu_n}$, with both p_i and q_i positive. Furthermore, no twist exponent appears together with its additive inverse in a single braid word. Finally, it may be assumed no trivial generators remain, since they must be paired across the two braid words. In order for the terms of minimal v-degree to match, (2.3) implies:

$$\prod_{i=1}^{\nu_p} C_{p_i+1} \prod_{i=1}^{\nu_n} C_{q_i-1} = \prod_{i=1}^{\nu_n} C_{q_i+1} \prod_{i=1}^{\nu_p} C_{p_i-1}$$

Since the twist values are positive in the prior equation, each $p_i + 1$ matches some $q_j + 1$ or matches some $p_j - 1$. The former is not possible under the stated assumption no twist exponent appears along with its additive inverse in either braid word. But the latter is also impossible since for the maximal positive twist exponent, p_m, $p_m + 1 = p_j - 1$ cannot have a solution.

The statements regarding the subscript, maximal degree, and coefficient of the term of degree two below the degree of h_{ν_n} are consequences of (2.3) and (2.4). □

3.17. *Proof of Lemma 2.27 (Link comp. for rank two n-braid)*

Proof. The number of link components is the number of disjoint cycles, with all strands present, when β is treated as a permutation on the strands. Hence, any even twist exponents may be ignored and it may be assumed β has only twist exponents of zero or one. The essence of the proof then depends on two properties:

$$\mu(L_1 \sharp L_2) = \mu(L_1) + \mu(L_2) - 1 \text{ for connected sums, and}$$
$$\mu(L_1 \coprod L_2) = \mu(L_1) + \mu(L_2) \text{ for disjoint unions.}$$

Any generator with $L(i) = 1$ ($L(i) = 0$) is a trivial (null) generator and so represents a connected sum (disjoint union) of the left and right words consisting of higher subscript and lower subscript generators, respectively. Finally observe that a band generates a two component link, while an even width ooblock generates a knot, and the result follows. □

3.18. *Proof of Prop. 2.29 (h_j properties for rank two n-braid)*

Proof. Equation 3.15 shows it suffices to consider nonporous braid words, and this is assumed below. When β is a single eeblock, Lemma 2.27 implies it has n components, and the following paragraphs show each h_j achieves the minimal degree exactly when the braid word with one fewer generators has this property. As this is true for elementary torus links, the result is true.

As regards conditions that prevent h_{n-1} from achieving the minimal degree, note that application of (3.13) to an even exponent of a band neighbor, and Lemma 2.27, show the first result term cannot contribute a term of minimal degree to $h_{n-1,\widehat{\beta}}$ as it has fewer components than $\widehat{\beta}$. When a band neighbor has an exponent of one, (3.15) shows the second result term contributes zero to $h_{n-1,\widehat{\beta}}$. This observation implies that $h_{j,\widehat{\beta}}$ does not attain the minimal degree for $j \in [n - \phi_{\partial,1}(\beta), n-1]$. A similar argument applies to the statement for h_0.

When β has a band, the prior techniques show any of the exponents for generators whose subscripts have the same parity as the generator with minimal subscript may be replaced by ± 1 by use of (3.13) or (3.14). By the skein relation, an exponent of one may be replaced by minus one and vice versa.

When β has a band with a range of generator subscripts $[x, y]$, for which no $h_{j,\widehat{\Psi(x-1,y+1,\beta)}}$ achieves the minimal degree, the prior discussion shows the same is true for each $h_{j,\widehat{\beta}}$.

The expressions for the minimum and maximum subscripts for which $h_{j,\widehat{\beta}}$ achieves the minimum degree follow from the prior discussion and elementary properties of polynomial algebra. The result on thin bands is a corollary. The result on n-component links follows immediately from the prior results, and Lemma 2.27, which shows any n-component link must either have $\nu_{0,\beta} = n-1$, i.e. β is a single eeblock, or $\nu_{0,\beta} + \phi(\beta) = n-1$, i.e. each generator has either an even-even exponent pair or is in a thin band. □

3.19. *Proof of Proposition 2.33 (Alternating Braid results)*

Proof. The result is easily seen to be true for elementary torus links, O_n, and connected sums and disjoint unions of links generated by alternating braid words for which the result is true. Also, when β, $\beta\sigma_i^1$, $\beta\sigma_j^{-1}$, alternating braid words, satisfy the result, so does $\beta\sigma_i^e$ and $\beta\sigma_j^{-e}$, for any $e > 0$. Thus it suffices to consider braid words for which all syllables have length one, with no trivial or null generators.

There is no loss in generality to assume a subword of the form $\sigma_i\sigma_j^{-1}\sigma_i$ exists in β with $|i-j| = 1$ (see Section 3.7). When the skein relation is applied to the first σ_i, we obtain two terms, with weights vz and v^2. The first is associated with an alternating braid word whose length is one less than $|\beta|$, and thus has the desired properties. The second term braid word has the same length as β and all exponents are ± 1, but σ_i has changed roles to become a mixed generator.

To see that the second term supports the claim, note that $\sigma_i\sigma_j^{-1}\sigma_i$ in β has become $\sigma_i^{-1}\sigma_j^{-1}\sigma_i$ in the second term, which we may replace by $\sigma_j\sigma_i^{-1}\sigma_j^{-1}$. In case this is preceded by $\sigma_i\sigma_j^{-1}$, the second braid word length may be decreased by four to become alternating, which supports the claim. In case $\sigma_j\sigma_i^{-1}\sigma_j^{-1}$ is preceded by σ_j^{-1}, the second braid word length may be decreased by two, with σ_i a mixed generator. Otherwise, the new second term has two mixed generators, σ_i and σ_j, but one fewer positive and one fewer negative generators compared to β.

When σ_i is not a trivial positive generator in the first term, the coefficient polynomials for the second result term have degree at least two less than the degree of the first term contributions on $[0, n-1]$, once the weights are considered. Theorem 2.22 shows this is true for the final case above. When σ_i is a trivial positive generator in the first term, σ_i is now a trivial negative generator in the second term. Hence the second term is a connected sum of two links generated by alternating braid words. Prop. 3.6 shows that the second term subscripts for the coefficient polynomials are shifted to the right and careful inspection shows that the second term makes the necessary contribution to $p_{j,\widehat{\beta}}$ for $j \in [\nu_n, n-1]$ so that the claim is fulfilled. On $[0,\, \nu_n]$ the first term dominates the second and fulfills the claim. □

3.20. *Proof of Proposition 2.34 (Alternating three braid results)*

Proof. When the rank of β is one, β is the connected sum of two elementary torus links, T_a and T_{-b}, in which a and b are both two or more, $|\beta| = a + b$, and

$$\begin{aligned}
&\text{(i)}\;\; p_0 = C_{a+1}C_{-b+1} = (-1)^b(z^{a+b-2} + (a+b-4)z^{a+b-4} + \cdots)\,,\\
&\text{(ii)}\;\; p_1 = -C_{a+1}C_{-b+1} - C_{a-1}C_{-b-1} = (-1)^{b+1}(z^{a+b} + (a+b-2)z^{a+b-2} + \cdots)\,,\\
&\text{(iii)}\;\; p_2 = C_{a-1}C_{-b-1} = (-1)^b(z^{a+b-2} + (a+b-4)z^{a+b-4} + \cdots)\,.
\end{aligned}$$

Thus, the result holds when the rank is one.

When the rank of β is two, β can be assumed to have at least one exponent with absolute value two or more, since the result is true for $|\beta| = 4$. Treating the case $x_n = 2$ separately from $x_n > 2$ leads to a straightforward proof by induction. The result for $x_p = 2$ follows by applying the result for $x_n = 2$ to the mirror image.

When the rank of β is three, the result follows easily from the values of p_j for $|\beta| = 6$ and the prior rank results. For higher ranks, the result will follow if it can be proven when all x_i, y_i in the expression for β are ± 1. But for $r \geq 3$, it follows that $P_{\widehat{(\sigma_2\sigma_1^{-1})^{r+1}}} = vzP_{\sigma_2\sigma_1^{-2}\widehat{(\sigma_2\sigma_1^{-1})^{r-1}}} + v^2P_{\widehat{(\sigma_2\sigma_1^{-1})^{r-3}\sigma_2\sigma_1^{-3}}}$.

Inspection of the first result node braid word, $\sigma_2\sigma_1^{-2}(\sigma_2\sigma_1^{-1})^{r-1}$, shows it has the same value for x_n as the original braid word, β, but has a length one less and a rank one less. When consideration of the weight, vz, for the first node is given, it fulfills all the conditions of induction hypothesis.

Inspection of the second result node braid word, $(\sigma_2\sigma_1^{-1})^{r-3}\sigma_2\sigma_1^{-3}$, shows it has a length four less and rank three less than β, and the value for x_n is one less. Thus the second node has no effect on the results and the proposition is valid. □

3.21. *Proof of Propositions 2.37, 2.38 ($K_{\pm}(2,n)$ formulas)*

Proof. The skein and braid relations imply the following:

$$v^2 P_{K_{\pm}(2,2m+1)} = -vzP_{K_{\pm}(2,2m)} + P_{K_{\pm}(2,2m-1)}\,, \tag{3.21}$$

$$vzP_{K_{\pm}(2,2m)} = (vz)^2 P_{K_{\pm}(2,2m-1)} + v^2 vzP_{K_{\pm}(2,2m-2)}\,. \tag{3.22}$$

Both propositions are easily verified to be true for the minimal values $m = 1$, $m = 2$ respectively. Assuming the result is true for smaller n, first consider $n = 2m + 1$. Inspection of (3.21) shows $a_{j,k,m} = -b_{j-1,k,m} + a_{j,k,m-1}$, and use of (3.3) shows the result is true. For the even strand case, $n = 2m$, inspection of (3.22) shows $b_{j,k,m} = a_{j,k-1,m-1} + b_{j-2,k,m-1}$. For the proof of (2.15, 2.16) the following identities apply:

$$\binom{f_{2m-j-1}(2m-j-1) + f_0(2m-j-1)}{2m-j-1} = 1\,,$$

$$\binom{f_{2m-j-2}(2m-j-1) + f_0(2m-j-2)}{2m-j-2} = 1\,.$$

The result for $n = 2m$ follows as for the odd strand case.

The Conway polynomial formulas follow from (3.21), (3.22), and induction. □

3.22. *Proof of Prop. 2.40 (Rank two alternating braid results)*

Proof. Eq. 3.15 shows it suffices to consider nonporous braid words, and this is assumed below.

That each h_j is of uniform sign and complete follows easily by an induction on the braid length, together with Proposition 2.33, and the observations following Definition 2.20 in Section 2.2.1.

To show the $h_{j,\widehat{\beta}}$ that achieve the minimal degree are those with subscripts in $[\phi_{\partial,-1}(\beta) + \phi_{o,\sharp}(\beta) - \phi_o(\beta),\, n-1-\phi_{\partial,1}(\beta) - \phi_{e,\sharp}(\beta) + \phi_e(\beta)]$, note that the claim is true for the minimal length nonporous rank two alternating braid words in B_n, for $n \geq 3$. The discussion in Section 3.18 to prove Prop. 2.29 shows that when an even-even generator exists, or when all exponents are odd, the result follows, so assume no eeblocks exist and some eoblock exists.

Write σ_i for the generator with maximal or minimal subscript within any eoblock. Eq. 3.13 applied to the even exponent of σ_i implies that when the number of components for the first result term is less than $\mu(\beta)$, it does not affect the outcome of whether $h_{j,\widehat{\beta}}$ achieves the minimal degree. When the odd exponent of σ_i is not ± 1, the range endpoint values are identical for the second result term and for β, as desired. In case the odd exponent of σ_i is one, the induction hypothesis applies to the second result term when it is treated as the connected sum of two links on $n-1$ total strands, so the range of subscripts is $[0, n-2]$. Inspection of the terms shows h_{n-1} is not expected to be part of the range for β, and the range for the second result term coincides with the desired range for β. A similar analysis applies when the odd exponent of σ_i is minus one.

When σ_i is neighbor to a band, the first result term always has one fewer components than β, so it may be assumed no bands exist within β. The other block types for which σ_i can be a neighbor are even width ooblocks, or eoblocks.

When an even width ooblock exists, apply (3.13) to the even exponent of σ_i. When the odd exponent of σ_i is not ± 1, inspection of the expression for the range for the first term shows it is a subset of the range for the second term, and the latter matches the values for β, as desired. When the odd exponent of σ_i is one, the range for the second result term is $[0, n-2]$, while the range for β is $[0, n-1]$. This time the first result term makes the necessary contribution to h_{n-1} to ensure it achieves minimal degree, since a band with odd subscript has been created. A similar analysis applies when the odd exponent of σ_i is minus one.

The final case to consider is when all generators belong to a single eoblock. Apply (3.13) to the even exponent of σ_1. The first result term is a two-link and the second term is a knot with a trivial positive generator. Inspection of the expressions for the two ranges shows their union is $[0, n-1]$, as desired.

To verify the coefficient of $z^{\|\beta\|-2}$ in h_{ν_n} is $(-1)^{\nu_n+x_n}(\|\beta\| + \tau(\beta) - \tau_\sharp(\beta))$ when β is nonporous, observe this is true for $\alpha^2_{n-1,\pm}$. For braid words of greater length, apply the skein relation to any exponent of a generator, σ_i, that has absolute value two or more. So long as σ_i is not trivial in the second result term, the result follows by induction. Otherwise examination of the terms from the second term that come from a connected sum shows the result is true.

When β has only clustered generators, the expression for the coefficient of $z^{\|\beta\|-2}$ in h_{ν_n} is valid. When there are one or more npblocks, the product of their Homflypt polynomials will produce a polynomial for $v^{2\phi_{e,\sharp}(\beta)}$ that has the maximum degree, and the term of degree two lower, ignoring sign, will have a value that is the sum of the normalized lengths plus the number of npblocks less the number of generators belonging to any npblock. When the clustered generators are taken into account, the coefficient for this term in $v^{2\nu_n}$ becomes the value just described plus the value given by Proposition 2.25, i.e. the sum of the normalized lengths for the clustered generators. The sum of these two contributions is the desired coefficient: $(-1)^{\nu_n+x_n}(\|\beta\| + \tau(\beta) - \tau_\sharp(\beta))$.

To show a nonporous β cannot have rank one, note that a rank one braid word with the same Homflypt polynomial, and no trivial generators, cannot have any null generators, and must have the same w, n, ν_n, and length as β. Comparing the coefficient of the term in h_{ν_n} from Proposition 2.25 to that for a rank two alternating braid word shows they could only be the same when $\tau(\beta) = \tau_\sharp(\beta)$, i.e. each npblock is thin. In this case, β is a connected sum of elementary torus links, i.e. has rank one.

In order for a rank two alternating braid word to generate $\widehat{\beta}$, it must have have an npblock that is not thin, no null generators, and the same number of strands, w, and ν_n, as β. Thus it must have the same value for $\tau(*) - \tau_\sharp(*)$, and this is not possible when a non-trivial clustered generator exists. □

3.23. *Proof of Proposition 2.43 (Positive Braid results)*

Proof. The proof in Section 3.7 can be easily adapted to prove the result. □

3.24. *Proof of Proposition 2.44 (Torus Link results)*

Proof. The result is true for $n = 1, 2$, so make the induction hypothesis that it is true when the number of strands is below n. When the rank matches the number of strands ($r = n$), $K(n, n)$ is the closure of α_{n-1}^n. For convenience, define $\omega_k = \sigma_{n-1} \cdots \sigma_k$, a prefix of α_{n-1} ending with σ_k.

Construct a skein calculation tree by converting σ_1 in the first row to σ_1^0 and σ_1^{-1}, according to the skein relation. The first node is equivalent to the braid word, $\omega_2\, \alpha_{n-1}^{n-1}$, while the second node is $\omega_2\, \alpha_{n-1}^{n-2}\, \alpha_{n-2}$.

Continue in this fashion to convert the generator in the first row with the lowest subscript in each node, to σ_1^0 and σ_1^{-1}, forming two new nodes from each parent node. After the entire first row of the original braid word is converted, there are 2^{n-1} end nodes, each associated with a weight in the form $v^{2n-2-a} z^a$, with a as the number of conversions of generators to zero exponent. Thus the end node has a writhe with the value $n(n-1) - (2n-2-a)$. Note that the nodes are all positive words in B_n, and have a rows of α_{n-1} and $n-1-a$ rows of α_{n-2}.

Observe that each node also appears in the skein calculation tree for $K(n-1, n)$, and this is equivalent to the link $K(n, n-1)$. By the induction hypothesis, $\deg p_{j,K(n,n-1)} = n(n-2) - j(j+1)$, for $j < n-1$. By (3.17), $\deg p_{j,K(n-1,n)}$ is one higher, i.e. $(n-1)^2 - j(j+1)$, for $j < n-1$. The only node that makes a contribution to p_{n-1} for $K(n, n)$ is α_{n-2}^{n-1}, whose closure is a disjoint union of O_1 with $K(n-1, n-1)$. By the induction hypothesis, $p_{n-2,K(n-1,n-1)} = (-1)^{n-2}$, hence $p_{n-1,K(n,n)} = (-1)^{n-1}$.

Now the node with the highest power of z in its weight is also the node with the highest writhe, i.e. α_{n-1}^{n-1}, and the weight is $v^{n-1} z^{n-1}$. Hence the dominant term in the skein tree contributes $z^{n-1} p_{j,K(n-1,n)}$, and so the degree of p_j for $K(n, n)$ is $n - 1 + (n-1)^2 - j(j+1)$, i.e. $n(n-1) - j(j+1)$ for $0 \le j < n-1$.

The same technique may now be used to show the result for $r > n$. □

3.25. *Proof of Prop. 2.46 (Pos. Braid Upper Bound for* $\deg p_j$*)*

Proof. The assertion is easily verified for $n = 2$ and O_n for any n, for $j \le 1$ for any β, and for $|\beta| = 1$. When $\beta = \gamma\sigma_i^2\eta$, we have $p_{j,\widehat{\beta}} = z p_{j,\widehat{\gamma\sigma_i\eta}} + p_{j,\widehat{\gamma\eta}}$. When $\rho(\gamma\eta) = \rho(\beta)$, the result is easily verified. When $\rho(\gamma\eta) = \rho(\beta) - 1$, we have

$\delta(j,\,\gamma\eta) = \max\{\,0\,, 2j\,(R_{j+1,\,n} - \rho(\beta)) + 2j\,\}$. By the induction hypothesis we have $\deg p_{j,\,\widehat{\gamma\eta}} \leq (|\beta|-2) + (2\,o(\beta) - 2n+4) + \delta(j,\,\gamma\eta) - j(j+1)$. This may be rewritten as $\deg p_{j,\,\widehat{\gamma\eta}} \leq |\beta| + 2\,o(\beta) + \delta_* - j(j+1)$, with $\delta_* = \max\{\,-2n+2\,, 2j\,(R_{j+1,\,n} - \rho(\beta)) + 2j - 2n + 2\,\}$. Since $\delta_* \leq \delta(j,\,\gamma\sigma_i\eta)$, the result is true.

A braid length reduction argument as in Section 3.7 concludes the proof. □

3.26. *Proof of Theorem 2.48 (Nonporous Positive Braid Words)*

Proof. Observe that there is a node in a skein calculation tree for $\widehat{\beta}$ which represents $K(\rho(\beta), n)$ with all other nodes associated with positive braid words. By Props. 2.43, 2.44, and 3.7, all of the $p_{j,\,\widehat{\beta}}$ are nonzero for $j \in [0\,, \min(\rho(\beta)-1, n-1)]$. The degree value follows from Props. 2.43 and 2.46. □

3.27. *Proof of Theorem 2.51 (Nonporous Rank 2 Positive Braids)*

Proof. The observation that, when $n \geq 3$ and $j \geq 2$, we have $p_j = 0$ exactly when $D(\beta) \geq n-j$ is readily verified for $n = 3$ and $j = 2$. Assume $n \geq 4$ below.

Suppose $\beta = \sigma_{n-1}^a \sigma_{n-2}^b \gamma \sigma_{n-1}^x \sigma_{n-2}^y \eta$, with $\gamma, \eta \in B_{n-2}$. Set $s = x + b - 2$ and $A = a+1$. Write $\sigma_{n-2}^A \gamma \sigma_{n-2}^y \eta$ as ξ_1, and $\sigma_{n-2}^b \gamma \sigma_{n-2}^y \eta$ as ξ_3 . Apply (3.13) to get:

$$P_{\widehat{\beta}} = C_x C_b v^s P_{\widehat{\xi_1}} + C_x C_{b-1} v^{s+1} P_{T_A \sharp T_y} P_{\widehat{\gamma\eta}} + C_{x-1} v^x P_{T_a} P_{\widehat{\xi_3}}\,.$$

In the first result term, $p_{n-1,\,\widehat{\xi_1}} = 0$. Note that $p_{j,\,\widehat{\beta}} = 0$ exactly when this is true for the corresponding coefficient polynomial in each result term. These are:

(i) $z C_x C_b p_{j,\,\widehat{\xi_1}}$,
(ii) $C_x C_{b-1}(C_{A+1} C_{y+1} p_{j,\,*} - p_{1,\,T_A\sharp T_y}\, p_{j-1,\,*} + C_{A-1} C_{y-1} p_{j-2,\,*})$, with $* = \widehat{\gamma\eta}$,
(iii) $C_{x-1}(C_{a+1} p_{j,\,\widehat{\xi_3}} - C_{a-1} p_{j-1,\,\widehat{\xi_3}})$.

The following observations will be used without comment below. When σ_{n-1} is robust, $D(\beta) = D(\xi_3)$. When σ_{n-2} is robust, all braid words, β, ξ_1, $\gamma\eta$, and ξ_3 , have identical values for $D()$, and $k_\beta = 1 + k_{\gamma\eta}$, while k_{ξ_1}, $k_{\xi_3} \in [k_\beta - 1,\, k_\beta]$. When σ_{n-2} is a1, $k_\beta = k_{\xi_1} = k_{\xi_3}$. When σ_{n-1} and σ_{n-2} are both a1, $D(\beta) = 1 + D(\xi_1)$. When the second (third) term is nonzero, σ_{n-2} (σ_{n-1}) must be robust.

When $j = 2$, the second and third terms can only vanish when σ_{n-2} and σ_{n-1} are a1. The first term vanishes exactly when $D(\xi_1) \geq (n-1) - 2$, so $D(\beta) \geq n-2$. When $j = 3$, the second term can only vanish when σ_{n-2} is a1. When σ_{n-1} is a1, the result follows as for $j = 2$. Otherwise, we must have $D(\xi_3) \geq (n-1) - 2$, so $D(\beta) \geq n-3$. As the conditions are also sufficient, the result holds.

The final case is for $j \geq 4$. When σ_{n-2} is a1, similar reasoning as above shows the result holds. When σ_{n-2} is robust, the second term requires $D(\gamma\eta) \geq n-j$, which is also a sufficient condition. Hence $p_{j,\,\widehat{\beta}} = 0$ exactly when $D(\beta) \geq n-j$.

To establish the value for $\deg p_{j,\,\widehat{\beta}}$ when $j \geq 3$ and $D(\beta) < n-j$, begin with $n = j+1$. The first result term is zero, the second is $C_x C_{b-1} C_a C_{y-1} p_{j-2,\widehat{\gamma\eta}}$, the third is $-C_{x-1}C_{a-1}p_{j-1,\widehat{\xi_3}}$, and $D(\beta) = 0$. For $j = 3$ the second result term contributes zero when σ_2 is a1 and a term with degree $|\beta| - 8$ otherwise, while the third result term contributes zero when σ_3 is a1 and otherwise contributes a term with degree $|\beta| - 10$. This proves the $\deg p_{3,\widehat{\beta}}$ claim when $n = 4$ and $D(\beta) = 0$.

Suppose $n > 4$ and $D(\beta) = n - 4$. When both edge generators are robust, all others must be a1, so only the third result term is nonzero and the degree of $p_{3,\widehat{\beta}}$ is $|\beta| - 10$. When an edge generator is a1, it may be assumed to be σ_{n-1} ,[2], [18], in which case the third result term is zero. When σ_{n-2} is also a1, only the first result term is nonzero. Since the non-edge generators of β which are robust are exactly those of ξ_1, the claim is satisfied. In the remaining case when σ_{n-1} is a1 but σ_{n-2} is robust, only the second result term is nonzero and its degree is $|\beta| - 8$, as desired. When $D(\beta) < n - 4$ the first result term contributes a dominant term for $p_{3,\widehat{\beta}}$ with the desired degree. This concludes the proof for $j = 3$. A similar analysis verifies the result for $j = 4$.

Assume now that $j \geq 5$ and $n \geq 6$. Observe that whenever σ_{n-2} is robust, the second result term contributes a term of the expected degree to $p_{j,\widehat{\beta}}$. This is also true of the third result term when σ_{n-1} is robust and σ_{n-2} is a1. When σ_{n-2} is a1 and $p_{j,\widehat{\xi_1}} \neq 0$, the first term contributes a term of the expected degree to $p_{j,\widehat{\beta}}$.

In fact when $p_{j,\widehat{\xi_1}} \neq 0$, and $k_{\xi_1} = k_\beta$ the first term contributes a term of the expected degree to $p_{j,\widehat{\beta}}$. When $p_{j,\widehat{\xi_1}} \neq 0$, and $k_{\xi_1} = k_\beta - 1$, the first term contributes a term of degree two less than expected for $p_{j,\widehat{\beta}}$ when $k_{\xi_1} < J$, i.e. $k_\beta \leq J$, but a term of the expected degree when $k_{\xi_1} \geq J$, i.e. $k_\beta > J$. A more complicated relation applies to a nonzero third result term. When $k_{\xi_3} = k_\beta$, it contributes a term of the expected degree to $p_{j,\widehat{\beta}}$ provided j is even or $k_\beta < J$, otherwise a term of degree two less. If instead, $k_{\xi_3} = k_\beta - 1$ and j is odd, a nonzero third term always contributes a term two less than expected for $p_{j,\widehat{\beta}}$ while for even j, it contributes a term of the expected degree when $k_\beta > J$, but a term of degree two less otherwise. Thus there are no terms of degree higher than expected. □

3.28. *Proof of Theorem 2.52 (Polarized Braid results)*

Proof. Theorem 2.22 provides the upper bounds in the theorem. The proof in Section 3.7 can be adapted to prove the result. The theorem is easily seen to be true for elementary torus links, O_n, and connected sums and disjoint unions of links generated by polarized braids of lower strand number.

When β has an instance of σ_k^2, say $\beta = \sigma_k^2\gamma$, then $P_{\widehat{\beta}} = vzP_{\widehat{\sigma_k\gamma}} + v^2P_{\widehat{\gamma}}$. Since the braid word for each summand satisfies the induction hypothesis, so does the sum. This is true even when σ_k is a null generator for γ. A similar argument applies when β has an instance of σ_k^{-2} The final argument involves the same braid length reduction technique as in Section 3.7. □

4. Conjectures and Open Questions

Here are some results that seem plausible based on examples and related results.

(i) Proposition 2.5 is true for all links,
(ii) the rank of a link generated by a nonporous homogeneous (alternating, positive) braid word, with all syllable lengths two or more, is the rank of the braid word.

The following are a few open questions with no specific expected outcome:

(i) are the $\{h_j\}_0^{n-1}$ solutions to another family of linear relations as in (2.10), independent from those in Thm. 2.7 , or is there a sequence of rational functions in n variables for which they are solutions?
(ii) is there a reasonable way to determine the rank of a braid word, or link, even if only for certain subclasses ?

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