RELATIVE NODE POLYNOMIALS FOR PLANE CURVES

FLORIAN BLOCK

Abstract. We generalize the recent work of S. Fomin and G. Mikhalkin on polynomial formulas for Severi degrees.

The degree of the Severi variety of plane curves of degree d and δ nodes is given by a polynomial in d, provided δ is fixed and d is large enough. We extend this result to generalized Severi varieties parametrizing plane curves which, in addition, satisfy tangency conditions of given orders with respect to a given line. We show that the degrees of these varieties, appropriately rescaled, are given by a combinatorially defined "relative node polynomial" in the tangency orders, provided the latter are large enough. We describe a method to compute these polynomials for arbitrary δ, and use it to present explicit formulas for $\delta \leq 6$. We also give a threshold for polynomiality, and compute the first few leading terms for any δ.

1. Introduction and Main Results

The *Severi degree* $N^{d,\delta}$ is the degree of the Severi variety of (possibly reducible) nodal plane curves of degree d with δ nodes. Equivalently, $N^{d,\delta}$ is the number of such curves passing through $\frac{(d+3)d}{2} - \delta$ generic points in the complex projective plane $\mathbb{CP}^2$. Severi varieties have received considerable attention since they were introduced by F. Enriques [8] and F. Severi [16] around 1915. Much later, in 1986, J. Harris [12] achieved a celebrated breakthrough by showing their irreducibility.

In 1994, P. Di Francesco and C. Itzykson [7] conjectured that the numbers $N^{d,\delta}$ are given by a polynomial in d, for a fixed number of nodes δ, provided d is large enough. S. Fomin and G. Mikhalkin [9, Theorem 5.1] established this polynomiality in 2009. More precisely, they showed that there exists, for every $\delta \geq 1$, a *node polynomial* $N_\delta(d)$ which satisfies $N^{d,\delta} = N_\delta(d)$ for all $d \geq 2\delta$.

The polynomiality of $N^{d,\delta}$ and the polynomials $N_\delta(d)$ were known in the 19th century for $\delta = 1, 2, 3$. For $\delta = 4, 5, 6$, this was only achieved by I. Vainsencher [19] in 1995. In 2001, S. Kleiman and R. Piene [13] settled the cases $\delta = 7, 8$. In [2], the author computed $N_\delta(d)$ for $\delta \leq 14$ and improved the threshold of S. Fomin and G. Mikhalkin by showing that $N^{d,\delta} = N_\delta(d)$ provided $d \geq \delta$.

Severi degrees can be generalized to incorporate tangency conditions to a fixed line $L \subset \mathbb{CP}^2$. More specifically, the *relative Severi degree* $N^{\delta}_{\alpha,\beta}$ is the number of (possibly reducible) nodal plane curves with δ nodes which have tangency of order i to L at

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α_i *fixed* points (chosen in advance) and tangency of order i to L at β_i *unconstrained* points, for all $i \geq 1$, and which pass through an appropriate number of generic points. Equivalently, $N^{\delta}_{\alpha,\beta}$ is the degree of the *generalized Severi variety* studied in [6, 20]. By Bézout's Theorem, the degree of a curve with tangencies of order (α, β) equals $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$. The number of point conditions (for a potentially finite count) is $\frac{(d+3)d}{2} - \delta - \alpha_1 - \alpha_2 - \cdots$. We recover non-relative Severi degrees by specializing to $\alpha = (0, 0, \dots)$ and $\beta = (d, 0, 0, \dots)$. The numbers $N^{\delta}_{\alpha,\beta}$ are determined by the rather complicated Caporaso-Harris recursion [6].

In this paper, we show that much of the story of (non-relative) node polynomials carries over to relative Severi degrees. Our main result is that, up to a simple combinatorial factor and for fixed $\delta \geq 1$, the relative Severi degrees $N^{\delta}_{\alpha,\beta}$ are given by a multivariate *polynomial* in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$, provided that $\beta_1 + \beta_2 + \dots$ is sufficiently large. (The $\delta = 0$ case is trivial as $N^0_{\alpha,\beta} = 1$ for all α, β.) For a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers with finitely many α_i non-zero we write

$$|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots, \qquad \alpha! \stackrel{\text{def}}{=} \alpha_1! \cdot \alpha_2! \cdot \cdots .$$

Throughout the paper we use the grading $\deg(\alpha_i) = \deg(\beta_i) = 1$ (so that d and $|\beta|$ are homogeneous of degree 1). The following is our main result.

Theorem 1.1. *For every $\delta \geq 1$, there is a combinatorially defined polynomial $N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots)$ of (total) degree 3δ such that, for all $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ with $|\beta| \geq \delta$, the relative Severi degree $N^{\delta}_{\alpha,\beta}$ is given by*

$$N^{\delta}_{\alpha,\beta} = 1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots). \tag{1.1}$$

We call $N_\delta(\alpha; \beta)$ the *relative node polynomial* and use the same notation as in the non-relative case if no confusion can occur. We do not need to specify the number of variables in light of the following stability condition.

Theorem 1.2. *For $\delta \geq 1$ and vectors $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_{m'})$ with $|\beta| \geq \delta$, it holds that*

$$N_\delta(\alpha, 0; \beta) = N_\delta(\alpha; \beta) \quad \text{and} \quad N_\delta(\alpha; \beta, 0) = N_\delta(\alpha; \beta)$$

as polynomials. Therefore, there exists a formal power series $N^{\infty}_{\delta}(\alpha; \beta)$ in infinitely many variables $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ which specializes to all relative node polynomials under $\alpha_{m+1} = \alpha_{m+2} = \cdots = 0$ and $\beta_{m'+1} = \beta_{m'+2} = \cdots = 0$, for various $m, m' \geq 1$.

Using the combinatorial description we provide a method to compute the relative node polynomials for arbitrary δ (see Sections 3 and 4). We utilize it to compute $N_\delta(\alpha; \beta)$ for $\delta \leq 6$. Due to spacial constrains we only tabulate the cases $\delta \leq 3$ in this paper. The polynomials N_0 and N_1 already appeared (implicitly) in [9, Section 4.2].

Theorem 1.3. *The relative node polynomials $N_\delta(\alpha; \beta)$, for $\delta = 0, 1, 2, 3$ (resp., $\delta \leq 6$) are as listed in Appendix A (resp., as provided in the ancillary files accompanying this article).*

The polynomial $N_\delta(\alpha; \beta)$ is of degree 3δ by Theorem 1.1. We compute the terms of $N_\delta(\alpha; \beta)$ of degree $\geq 3\delta - 2$.

Theorem 1.4. *The terms of $N_\delta(\alpha;\beta)$ of (total) degree $\geq 3\delta-2$ are given by*

$$\begin{aligned}N_\delta(\alpha;\beta) = & \frac{3^\delta}{\delta!}\Bigg[d^{2\delta}|\beta|^\delta + \frac{\delta}{3}\Big[-\frac{3}{2}(\delta-1)d^2 - 8d|\beta| + |\beta|\alpha_1 + d\beta_1 + |\beta|\beta_1\Big]d^{2\delta-2}|\beta|^{\delta-1} + \\ & + \frac{\delta}{9}\Big[\frac{3}{8}(\delta-1)(\delta-2)(3\delta-1)d^4 + 12\delta(\delta-1)d^3|\beta| + (11\delta+1)d^2|\beta|^2 + \\ & - \frac{3}{2}\delta(\delta-1)(d^3\beta_1 + d^2|\beta|\alpha_1) - \frac{1}{2}(\delta+5)(3\delta-2)d^2|\beta|\beta_1 - 8(\delta-1)(d|\beta|^2\alpha_1 + d|\beta|^2\beta_1) + \\ & + \frac{1}{2}(\delta-1)(d^2\beta_1^2 + |\beta|^2\alpha_1^2 + |\beta|^2\beta_1^2) + (\delta-1)(d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + |\beta|^2\alpha_1\beta_1)\Big]d^{2\delta-4}|\beta|^{\delta-2} + \cdots\Bigg],\end{aligned}$$

where $d = \sum_{i\geq 1} i(\alpha_i+\beta_i)$.

Theorem 1.4 can be extended to terms of $N_\delta(\alpha,\beta)$ of degree $\geq 3\delta-7$ (see Remark 5.2). We observe that all coefficients of $N_\delta(\alpha;\beta)$ in degree $\geq 3\delta-2$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ. In fact, even more is true. It is conceivable to expect this to hold for arbitrary degrees.

Proposition 1.5. *Every coefficient of $N_\delta(\alpha;\beta)$ in degree $\geq 3\delta-7$ is given, up to a factor of $\frac{3^\delta}{\delta!}$, by a polynomial in δ with rational coefficients.*

Our approach to planar enumerative geometry is combinatorial and inspired by *tropical geometry* which is a general procedure replacing a subvariety of a complex algebraic torus by a piece-wise linear polyhedral complex (see, for example, [10, 15, 17]). By the celebrated Correspondence Theorem of G. Mikhalkin [14, Theorem 1] one can replace the algebraic curve count in $\mathbb{CP}^2$ by an enumeration of certain *tropical curves*. E. Brugallé and G. Mikhalkin [4, 5] introduced a class of decorated graphs, called *(marked) floor diagrams* (see Section 2), which, if counted correctly, are equinumerous to such tropical curves. We use a version of these results which incorporates tangency conditions due to S. Fomin and G. Mikhalkin [9] (see Theorem 2.4). S. Fomin and G. Mikhalkin also introduced a *template decomposition* of floor diagrams which we extend to be suitable for the relative case. This decomposition is crucial in the proofs of all results in this paper, as is the reformulation of algebraic curve counts in terms of floor diagrams.

For some related work see [1], where F. Ardila and the author generalized the polynomiality of Severi degrees to curve counts in (some) toric surfaces including $\mathbb{CP}^1\times\mathbb{CP}^1$ and Hirzebruch surfaces. A main feature is that we show polynomiality not only in the multi-degree of the curves but also "in the surface itself." In [3], A. Gathmann, H. Markwig and the author defined *Psi-floor diagrams* which enumerate plane curves which satisfy point and tangency conditions, and conditions given by *Psi-classes*. We proved a Caporaso-Harris type recursion for Psi-floor diagrams, and show that *relative descendant Gromov-Witten invariants* equal their tropical counterparts.

This paper is organized as follows. In Section 2 we review the definition of floor diagrams and their markings. In Section 3 we introduce a new decomposition of floor diagrams which is compatible with tangency conditions. In Section 4 we prove Theorems 1.1,1.2, 1.3. In Section 5 we show Theorem 1.4 and Proposition 1.5.

2. Floor diagrams and relative markings

Floor diagrams are a class of decorated graphs which, if counted correctly, enumerate plane curves with prescribed properties. They were introduced by E. Brugallé and G. Mikhalkin [4, 5] in the non-relative case and generalized to the relative setting by S. Fomin and G. Mikhalkin [9]. We begin with a review of the relative setup following notation of [9] (where floor diagrams are called "labeled floor diagrams").

Definition 2.1. A *floor diagram* $\mathcal{D}$ on a vertex set $\{1, \dots, d\}$ is a directed graph (possibly with multiple edges) with edge weights $w(e) \in \mathbb{Z}_{>0}$ satisfying:

(1) The edge directions preserve the vertex order, i.e., for each edge $i \to j$ of $\mathcal{D}$ we have $i < j$.
(2) (Divergence Condition) For each vertex j of $\mathcal{D}$:

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} j}} w(e) \leq 1.$$

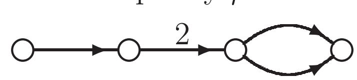
This means that at every vertex of $\mathcal{D}$ the total weight of the outgoing edges is larger by at most 1 than the total weight of the incoming edges.

The *degree* of a floor diagram $\mathcal{D}$ is the number of its vertices. It is *connected* if its underlying graph is. Note that in [9] floor diagrams are required to be connected. If $\mathcal{D}$ is connected its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space). The *cogenus* of a connected floor diagram $\mathcal{D}$ of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If $\mathcal{D}$ is not connected let $d_1, d_2, \dots$ and $\delta_1, \delta_2, \dots$ be the degrees and cogenera, respectively, of its connected components. Then the *cogenus* of $\mathcal{D}$ is $\delta(\mathcal{D}) = \sum_j \delta_j + \sum_{j<j'} d_j d_{j'}$. Via the correspondence between algebraic curves and floor diagrams [5, Theorem 2.5] these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility. Lastly, a marked floor diagram $\mathcal{D}$ has *multiplicity*[1]

$$\mu(\mathcal{D}) \stackrel{\text{def}}{=} \prod_{\text{edges } e} w(e)^2.$$

We draw floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

Example 2.2. An example of a floor diagram of degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\mu = 4$ is drawn below.



To enumerate algebraic curves with floor diagrams we need the notion of markings of such diagrams. Our notation, which is more convenient for our purposes, differs slightly from [9] where S. Fomin and G. Mikhalkin define relative markings relative

[1]If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

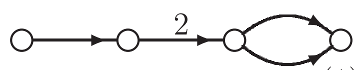
to the partitions $\lambda = \langle 1^{\alpha_1} 2^{\alpha_2} \cdots \rangle$ and $\rho = \langle 1^{\beta_1} 2^{\beta_2} \cdots \rangle$. In the sequel, all sequences are sequences of non-negative integers with finite support.

Definition 2.3. For two sequences α, β we define an (α, β)-*marking* of a floor diagram $\mathcal{D}$ of degree $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ by the following four step process which we illustrate in the case of Example 2.2 for $\alpha = (1, 0, 0, \ldots)$ and $\beta = (1, 1, 0, 0, \ldots)$.

Step 1: Fix a pair of collections of sequences $(\{\alpha^i\}, \{\beta^i\})$, where i runs over the vertices of $\mathcal{D}$, with:

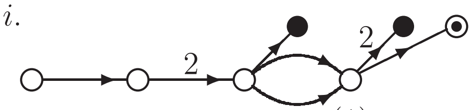
(1) The sums over each collection satisfy $\sum_{i=1}^{d} \alpha^i = \alpha$ and $\sum_{i=1}^{d} \beta^i = \beta$.
(2) For all vertices i of $\mathcal{D}$ we have $\sum_{j \geq 1} j(\alpha_j^i + \beta_j^i) = 1 - \mathrm{div}(i)$.

The second condition says that the "degree of the pair (α^i, β^i)" is compatible with the divergence at vertex i. Each such pair $(\{\alpha^i\}, \{\beta^i\})$ is called *compatible* with $\mathcal{D}$ and (α, β). We omit writing down trailing zeros.



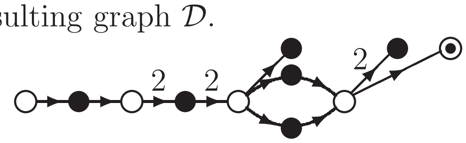
$\alpha^i =$ (1)
$\beta^i =$ (1) $(0,1)$

Step 2: For each vertex i of $\mathcal{D}$ and every $j \geq 1$ create β_j^i new vertices, called β-*vertices* and illustrated as ●, and connected them to i with new edges of weight j directed away from i. For each vertex i of $\mathcal{D}$ and every $j \geq 1$ create α_j^i new vertices, called α-*vertices* and illustrated as ⊙, and connected them to i with new edges of weight j directed away from i.

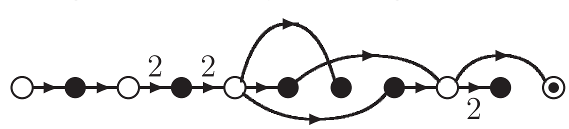


$\alpha^i =$ (1)
$\beta^i =$ (1) $(0,1)$

Step 3: Subdivide each edge of the original floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



Step 4: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original floor diagram $\mathcal{D}$ such that, as in $\mathcal{D}$, each edge is directed from a smaller vertex to a larger vertex. Furthermore, we require that the α-vertices are largest among all vertices, and for every pair of α-vertices $i' > i$, the weight of the i'-adjacent edge is larger than or equal to the weight of the i-adjacent edge.



We call the extended graph $\tilde{\mathcal{D}}$, together with the linear order on its vertices, an (α, β)-*marked floor diagram*, or an (α, β)-*marking* of the floor diagram $\mathcal{D}$.

We need to count (α,β)-marked floor diagrams up to equivalence. Two (α,β)-markings $\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists a weight preserving automorphism of weighted graphs mapping $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$ which fixes the vertices of $\mathcal{D}$. The *number of markings* $\nu_{\alpha,\beta}(\mathcal{D})$ is the number of (α,β)-marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. Furthermore, we write $\mu_\beta(\mathcal{D})$ for the product $1^{\beta_1}2^{\beta_2}\cdots\mu(\mathcal{D})$. The next theorem follows from [9, Theorem 3.18] by a straight-forward extension of the inclusion-exclusion procedure of [9, Section 1] which was used to conclude [9, Corollary 1.9] (the non-relative count of reducible curves via floor diagrams) from [9, Theorem 1.6] (the non-relative count of irreducible curves via floor diagrams).

Theorem 2.4. *For any $\delta \geq 1$, the relative Severi degree $N^\delta_{\alpha,\beta}$ is given by*

$$N^\delta_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D})\nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over all (possibly disconnected) floor diagrams $\mathcal{D}$ of degree $d = \sum_{i\geq 1} i(\alpha_i+\beta_i)$ and cogenus δ.

3. Relative Decomposition of Floor Diagrams

In this section we introduce a new decomposition of floor diagrams compatible with tangency conditions which we use extensively in Sections 4 and 5 to prove all our results stated in Section 1. This decomposition is an extension of ideas of S. Fomin and G. Mikhalkin [9]. We start out by reviewing their key gadget.

Definition 3.1. A *template* Γ is a directed graph (possibly with multiple edges) on vertices $\{0,\ldots,l\}$, where $l \geq 1$, and edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

(1) If $i \to j$ is an edge then $i < j$.
(2) Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")
(3) For each vertex j, $1 \leq j \leq l-1$, there is an edge "covering" it, i.e., there exists an edge $i \to k$ with $i < j < k$.

Every template Γ comes with some numerical data associated with it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of the edge weights is its *multiplicity* $\mu(\Gamma)$. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big].$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \xrightarrow{e} k$ with $i < j \leq k$ and define

$$k_{min}(\Gamma) \stackrel{\text{def}}{=} \max_{1\leq j\leq l}(\varkappa_j - j + 1).$$

This makes $k_{min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \ldots\}$ with left-most vertex k. Figure 1 ([9, Figure 10]) lists all templates Γ with $\delta(\Gamma) \leq 2$.

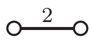
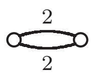
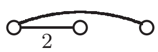
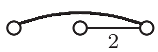
Γ	$\delta(\Gamma)$	$\ell(\Gamma)$	$\mu(\Gamma)$	$\varkappa(\Gamma)$	$k_{\min}(\Gamma)$	$P_\Gamma(k)$
	1	1	4	(2)	2	$k-1$
	1	2	1	(1,1)	1	$2k+1$
	2	1	9	(3)	3	$k-2$
	2	1	16	(4)	4	$\binom{k-2}{2}$
	2	2	1	(2,2)	2	$\binom{2k}{2}$
	2	2	4	(3,1)	3	$2k(k-2)$
	2	2	4	(1,3)	2	$2k(k-1)$
	2	3	1	(1,1,1)	1	$3(k+1)$
	2	3	1	(1,2,1)	1	$k(4k+5)$

FIGURE 1. The templates with $\delta(\Gamma) \le 2$.

Our new decomposition of a floor diagram $\mathcal{D}$ depends on two (infinite) matrices A and B of non-negative integers. We require both to have only finitely many non-zero entries all of which lie above the respective dth row, where d is the degree of $\mathcal{D}$.

The triple $(\mathcal{D}, A, B)$ decomposes as follows. Let $l(A)$ and $l(B)$ be the largest row indices such that A and B have a non-zero entry in this row, respectively. After we remove all "short edges" from $\mathcal{D}$, i.e., all edges of weight 1 between consecutive vertices, the resulting graph is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_r)$, listed left to right. Let k_s be the smallest vertex in $\mathcal{D}$ of each template Γ_s. Record all pairs (Γ_s, k_s) which satisfy $k_s + l(\Gamma_s) \le d - \max(l(A), l(B))$. Record the remaining templates together with all vertices i, for $i \ge \max(l(A), l(B))$ in *one* graph Λ on vertices $0, \ldots, l$ by shifting the vertex labels by $d - l$. See Example 3.4 for an example of this decomposition. Furthermore, by construction, if m is the number of recorded pairs (Γ_s, k_s), we have

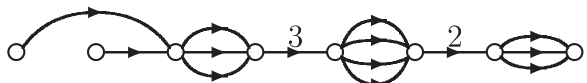
$$\left\{ \begin{array}{rcll} k_i & \ge & k_{\min}(\Gamma_i) & \text{for } 1 \le i \le m, \\ k_{i+1} & \ge & k_i + l(\Gamma_i) & \text{for } 1 \le i \le m-1, \\ k_m + l(\Gamma_m) & \le & d - l(\Lambda). & \end{array} \right. \tag{3.1}$$

Given a floor diagram $\mathcal{D}$, a partitioning of α and β into a compatible pair of collections $(\{\alpha^i, \beta^i\})$ (see Step 1 in Definition 2.3), where i runs over the vertices of $\mathcal{D}$, determines a pair of matrices A, B, if $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are large enough. The ith row vectors a_i and b_i of A and B are given by the sequences α^{d-i} and β^{d-i}, respectively, for $i \ge 1$ (so that a_1 equals the number of α-edges of weight 1 adjacent to the various vertices of Λ, and so on, see Example 3.2). If $d - i \le 0$ set α^{d-i} to be the zero sequence. The sequences α^d and β^d are given by

$$\alpha^d = \alpha - \sum_{i\geq 1} a_i \quad \text{and} \quad \beta^d = \beta - \sum_{i\geq 1} b_i \tag{3.2}$$

if both expression are (component-wise) non-negative.

Example 3.2. For $\alpha = (1,1)$, $\beta = (4,1)$ and the floor diagram $\mathcal{D}$ pictured below, the partitioning

$\mathcal{D} =$ 

$\alpha^i =$ (0, 1) (1)

$\beta^i =$ (1) (1) (2, 1)

determines the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In light of (3.2) we consider, for given tangency conditions α and β, only the triples $(\mathcal{D}, A, B)$ which satisfy

$$\begin{cases} \sum_{i\geq 1} a_i \leq \alpha \text{ (component-wise)}, \\ \sum_{i\geq 1} b_i \leq \beta \text{ (component-wise)}, \end{cases} \tag{3.3}$$

For fixed d, the decomposition

$$\big(\mathcal{D}, A, B\big) \longrightarrow \big(\{(\Gamma_s, k_s)\}, \Lambda, A, B\big). \tag{3.4}$$

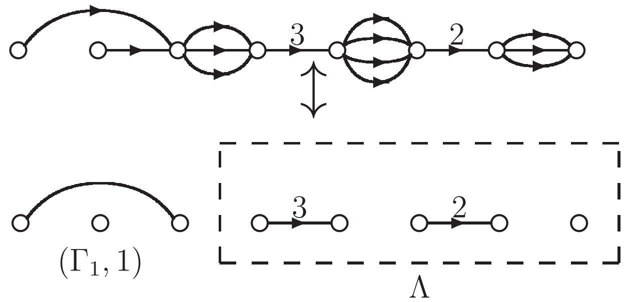
is reversible if the data on the right-hand side satisfies (3.1) and the tuple (Λ, A, B) is an "extended template."

Definition 3.3. A tuple (Λ, A, B) is an *extended template* of *length* $l = l(\Lambda) = l(\Lambda, A, B)$ if Λ is a directed graph (possibly with multiple edges) on vertices $\{0, \ldots, l\}$, where $l \geq 0$, with edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

(1) If $i \to j$ is an edge then $i < j$.
(2) Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")

Moreover, we require A and B to be (infinite) matrices with non-negative integral entries and finite support, and we write $l(A)$ and $l(B)$ for the respective largest row indices of A and B of a non-zero entry. Additionally, we demand that $l(\Lambda) \geq \max(l(A), l(B))$ and that, for each $1 \leq j < l - \max(l(A), l(B))$, there is an edge $i \to k$ of Λ with $i < j < k$.

Example 3.4. An example of a decomposition of a floor diagram $\mathcal{D}$ subject to the matrices A and B of Example 3.2 is pictured below. Once we fix the degree of the floor diagram the decomposition is reversible (here $d = 8$).



The *cogenus* of an extended template (Λ, A, B) is the sum of the cogenera $\delta(\Lambda)$, $\delta(A)$ and $\delta(B)$, where

$$\delta(\Lambda) \overset{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big], \qquad\qquad \delta(A) \overset{\text{def}}{=} \sum_{i,j \geq 1} i \cdot j \cdot a_{i,j},$$

and similarly for B. It is not hard to see that the correspondence (3.4) is cogenus preserving in the sense that (compare with Example 3.4 (cont'd))

$$\delta(\mathcal{D}) = \left(\sum_{i=1}^{m} \delta(\Gamma_i) \right) + \delta(\Lambda) + \delta(A) + \delta(B).$$

Example 3.4 (cont'd). The cogenera of the decomposition are given by

$$\delta(\Gamma_1) + \delta(\Lambda) + \delta(A) + \delta(B) = 1 + 3 + 4 + 6 = 14.$$

This agrees with the cogenus of $\mathcal{D}$ as $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g = \frac{7 \cdot 6}{2} - 7 = 14$.

With an extended template (Λ, A, B) we associate the following numerical data. For $1 \leq j \leq l(\Lambda)$ let $\varkappa_j(\Lambda)$ denote the sum of the weights of edges $i \to k$ of Λ with $i < j \leq k$. Define $d_{min}(\Lambda, A, B)$ to be the smallest positive integer d such that (Λ, A, B) can appear (at the right end) in a floor diagram on $\{1, 2, \ldots, d\}$. We will see later that d_{min} is given by an explicit formula. For a matrix $A = (a_{ij})$ of non-negative integers with finite support define the "weighted lower sum sequence" $\mathrm{wls}(A)$ by

$$\mathrm{wls}(A)_i \overset{\text{def}}{=} \sum_{i' \geq i, j \geq 1} j \cdot a_{i'j}.$$

We now define the number of "markings" of templates and extended templates and relate them to the number of (α, β)-markings of the corresponding floor diagrams. To each template Γ we associate a polynomial as follows. For $k \geq k_{\min}(\Gamma)$ let $\Gamma_{(k)}$ denote the graph obtained from Γ by first adding $k+i-1-\varkappa_i$ short edges connecting $i-1$ to i, for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By [9, Lemma 5.6] the number of linear extensions (up to equivalence, see the paragraph after Definition 2.3) of the vertex poset of the graph $\Gamma_{(k)}$ extending the vertex order of Γ is given by a polynomial $P_\Gamma(k)$ in k, if $k \geq k_{min}(\Gamma)$(see Figure 1).

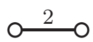
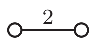
(Λ, A, B)			δ	ℓ	μ	$\varkappa$	$d_{\min}$	$q_{(\Lambda,A,B)}(\alpha;\beta)$ of Lemma 4.1	s
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	0	0	1	()	1	1	0
	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	1	1	(0)	1	1	0
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	1	1	1	(0)	1	$\beta_1(d+\lvert\beta\rvert-1)$	0
	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	4	(2)	4	$(d-3)$	1
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	1	4	(2)	4	$\beta_1(d-3)(d+\lvert\beta\rvert-2)$	1
	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	2	1	(1,1)	3	$2(d-2)$	0
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	2	1	(1,1)	3	$\beta_1(d-2)(2d+2\lvert\beta\rvert-3)$	0
	$\begin{bmatrix}2&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	1	0
	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\beta_1(d+\lvert\beta\rvert-2)$	0
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}2&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\binom{\beta_1}{2}(d^2+2d\lvert\beta\rvert+\lvert\beta\rvert^2-5d-5\lvert\beta\rvert+6)$	0
	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	1	1	(0)	3	1	0
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	2	1	1	(0)	3	$\beta_2(\lvert\beta\rvert-1)(d+\lvert\beta\rvert-2)$	0
	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	2	3	1	(0,0)	3	1	0
	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	2	3	1	(0,0)	3	$\beta_1(\lvert\beta\rvert-1)(2d+\lvert\beta\rvert-3)$	0

FIGURE 2. The extended templates with $\delta(\Lambda, A, B) \leq 2$.

For each pair of sequences (α, β) and each extended template (Λ, A, B) satisfying (3.3) and $d \geq d_{min}$, where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$, we define its "number of markings" as follows. Write $l = l(\Lambda)$ and let $\mathcal{P}(\Lambda, A, B)$ be the poset obtained from Λ by

(1) first creating an additional vertex $l + 1$ $(> l)$,
(2) then adding b_{ij} edges of weight j between $l - i$ and $l + 1$, for all $1 \leq i \leq l$ and $j \geq 1$,
(3) then adding $\beta_j - \sum_{i\geq 1} b_{ij}$ edges of weight j between l and $l + 1$, for $j \geq 1$,
(4) then adding

$$d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} \tag{3.5}$$

("short") edges of weight 1 connecting $i - 1$ and i, for $1 \leq i \leq l$, and finally
(5) subdividing all edges of the resulting graph by introducing a midpoint vertex for each edge.

We denote by $Q_{(\Lambda,A,B)}(\alpha;\beta)$ the number of linear orderings on $\mathcal{P}(\Lambda,A,B)$ (up to equivalence) which extend the linear order on Λ. As $d \geq d_{\min}(\Lambda,A,B)$ if and only if (3.5) is non-negative, for $1 \leq i \leq l$, we have

$$d_{min}(\Lambda,A,B) = \max_{1\leq i\leq l(\Lambda)} (l(\Lambda) - i + 1 + \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}).$$

For sequences $s, t_1, t_2, \ldots$ with $s \geq \sum_i t_i$ (component-wise) we denote by

$$\binom{s}{t_1, t_2, \ldots} \stackrel{\mathrm{def}}{=} \frac{s!}{t_1! t_2! \cdots (s - \sum_i t_i)!}$$

the multinomial coefficient of sequences.

We obtain all (α,β)-markings of the floor diagram $\mathcal{D}$ that come from a compatible pair of sequences $(\{\alpha^i\},\{\beta^i\})$ by independently ordering the α-vertices and the non-α-vertices. Therefore, the number such markings is given (via the correspondence (3.4)) by

$$\Big(\prod_{s=1}^{m} P_{\Gamma_s}(k_s)\Big) \cdot \binom{\alpha}{a_1^T, a_2^T, \ldots} \cdot Q_{(\Lambda,A,B)}(\alpha;\beta), \tag{3.6}$$

where $a_1^T, a_2^T, \ldots$ are the *column* vectors of A. We conclude this section by recasting relative Severi degrees in terms of templates and extended templates.

Proposition 3.5. *For any $\delta \geq 1$, the relative Severi degree $N^{\delta}_{\alpha,\beta}$ is given by*

$$\sum_{\substack{(\Gamma_1,\ldots,\Gamma_m),\\ (\Lambda,A,B)}} \Big(\prod_{s=1}^{m} \mu(\Gamma_s) \sum_{k_1,\ldots k_m} \prod_{s=1}^{m} P_{\Gamma_s}(k_s)\Big) \cdot \Big(\mu(\Lambda) \prod_{i\geq 1} i^{\beta_i} \binom{\alpha}{a_1, a_2, \ldots} Q_{(\Lambda,A,B)}(\alpha;\beta)\Big), \tag{3.7}$$

where the first sum is over all collections $(\Gamma_1,\ldots,\Gamma_m)$ of templates and all extended templates (Λ,A,B) satisfying (3.3), $d \geq d_{min}(\Lambda,A,B)$ and

$$\sum_{i=1}^{m} \delta(\Gamma_i) + \delta(\Lambda) + \delta(A) + \delta(B) = \delta,$$

and the second sum is over all positive integers $k_1,\ldots,k_m$ which satisfy (3.1).

Proof. By Theorem 2.4 the relative Severi degree is given by

$$N^{\delta}_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over all floor diagrams $\mathcal{D}$ of degree $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$ and cogenus δ. The result follows from $\mu_\beta(\mathcal{D}) = \prod_{i\geq 1} i^{\beta_i} \cdot \big(\prod_{s=1}^{m} \mu(\Gamma_s)\big) \cdot \mu(\Lambda)$ and (3.6). □

4. Relative Severi Degrees and Polynomiality

We now turn to the proofs of our main results by first showing a number of technical lemmata. For a graph G, we denote by $\#E(G)$ the number of edges of G. We write $||A||_1 = \sum_{i,j\geq 1} a_{ij}$ for the 1-norm of a (possibly infinite) matrix $A = (a_{ij})$.

Lemma 4.1. *For every extended template* (Λ, A, B) *there is a polynomial* $q_{(\Lambda,A,B)}$ *in* $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ *of degree* $\#E(\Lambda) + ||B||_1 + \delta(B)$ *such that for all* α *and* β *satisfying (3.3) the number* $Q_{(\Lambda,A,B)}(\alpha;\beta)$ *of linear orderings (up to equivalence) of the poset* $\mathcal{P}(\Lambda, A, B)$ *is given by*

$$Q_{(\Lambda,A,B)}(\alpha;\beta) = \frac{(|\beta| - \delta(B))!}{\beta!} \cdot q_{(\Lambda,A,B)}(\alpha;\beta)$$

provided $\sum_{i\geq 1} i(\alpha_i + \beta_i) \geq d_{min}(\Lambda, A, B)$.

Proof. We can choose a linear extension of the order on the vertices of Λ to the poset $\mathcal{P}(\Lambda, A, B)$ in two steps. First, we choose a linear order on the vertices $0, \ldots, l(\Lambda)+1$, the midpoint vertices of the edges of Λ and the midpoint vertices of the edges created in step (2) in the definition of $\mathcal{P}(\Lambda, A, B)$. In a second step, we choose an extension to a linear order on all vertices. Let r_i be the number of vertices between $i-1$ and i after the first extension, for $1 \leq i \leq l(\Lambda)+1$, and let σ_i be the number of equivalent such linear orderings of the interval between $i-1$ and i (σ_i is independent of the particular choice of the linear order). To insert the additional vertices (up to equivalence) between the vertices 0 and $l = l(\Lambda)$ we have

$$\prod_{i=1}^{l} \frac{1}{\sigma_i} \binom{d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} + r_i}{r_i} \tag{4.1}$$

many possibilities where again $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. If $d \geq d_{min}(\Lambda, A, B)$ then expression (4.1) is a polynomial in d of degree $\sum_{i=1}^{l} r_i$, and thus in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$. The number of (equivalent) orderings of the vertices between l and $l+1$ is the multinomial coefficient

$$\binom{|\beta| - ||B||_1 + r_{l+1}}{\beta_1 - |b_1^T|, \beta_2 - |b_2^T|, \ldots}, \tag{4.2}$$

where $|b_j^T|$ denotes the sum of the entries in the jth column of B. As $||B||_1 \leq \delta(B)$, expression (4.2) equals, for all $\beta_1, \beta_2, \cdots \geq 0$,

$$\binom{|\beta|}{\beta_1, \beta_2, \ldots} \frac{(|\beta| - \delta(B))!}{|\beta|!} P(\beta) = \frac{(|\beta| - \delta(B))!}{\beta!} P(\beta) \tag{4.3}$$

for a polynomial P in $\beta_1, \beta_2, \ldots$ of degree $r_{l+1} + \delta(B)$. The product of (4.1) and (4.3) is

$$\frac{(|\beta| - \delta(B))!}{\beta!} P'(\alpha;\beta) \tag{4.4}$$

for a polynomial P' in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ of degree $\#E(\Lambda) + ||B||_1 + \delta(B)$ provided $d \geq d_{min}(\Lambda, A, B)$ where we used that $\sum_{i=1}^{l+1} r_i = \#E(\Lambda) + ||B||_1$. As (4.4) equals the number of linear extensions (up to equivalence) that can be obtained by linearly ordering the vertices in all segments between $i-1$ and i, for $1 \leq i \leq l+1$, the proof is complete. $\square$

Recall that, for an extended template (Λ, A, B), we defined $d_{\min} = d_{\min}(\Lambda, A, B)$ to be the smallest $d \geq 1$ such that $d - l(\Lambda) + i - 1 \geq \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}$ for all $1 \leq i \leq l(\Lambda)$. Let i_0 be the smallest i for which equality is attained (it is easy to see that equality is attained for some i). Define the quantity $s(\Lambda, A, B)$ to be the number of edges of Λ from $i_0 - 1$ to i_0 (of any weight). See Figure 2 for examples.

Lemma 4.2. *For any extended template (Λ, A, B) and any $\alpha, \beta \geq 0$ (componentwise) with*

$$d_{\min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \sum_{i \geq 1} i(\alpha_i + \beta_i) \leq d_{\min}(\Lambda, A, B) - 1$$

we have $q_{(\Lambda, A.B)}(\alpha; \beta) = 0$, where $q_{(\Lambda, A, B)}$ is the polynomial of Lemma 4.1.

Proof. Notice that $d_{\min} - l(\Lambda) + i_0 - 1 = \varkappa_{i_0}(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i_0} + \mathrm{wls}(B)_{l(\Lambda)+1-i_0}$ where $d_{\min} = d_{\min}(\Lambda, A, B)$. Therefore, the number of short edges which are added between $i_0 - 1$ and i_0 in step (3) of the definition of the poset $\mathcal{P}(\Lambda, A, B)$ is $d - d_{\min}$, where as before $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$. Recall that, up to the factor $\frac{(|\beta| - \delta(B))!}{\beta!}$, the polynomial $q_{(\Lambda, A, B)}$ records the number of linear extension of the poset $\mathcal{P}(\Lambda, A, B)$ (up to equivalence). Every such extension is obtained by first linearly ordering the $d - d_{\min}$ midpoints of the short edges between $i_0 - 1$ and i_0 which were added in step (3) together with the $s(\Lambda, A, B)$ midpoints of the edges of Λ between $i_0 - 1$ and i_0, before extending this to a linear order on all the vertices of $\mathcal{P}(\Lambda, A, B)$. Thus, $q_{(\Lambda, A, B)}$ is divisible by the polynomial $(d - d_{\min} + 1) \cdots (d - d_{\min} + s(\Lambda, A, B))$. ☐

The next lemma specifies which extended templates are compatible with a given degree.

Lemma 4.3. *For every extended template (Λ, A, B) we have*

$$d_{min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \delta(\Lambda) + \delta(A) + \delta(B) + 1.$$

Proof. We use the notation from above and write $l = l(\Lambda)$. Notice that

$$d_{\min}(\Lambda, A, B) - l(\Lambda) + i_0 - 1 = \varkappa_{i_0}(\Lambda) + \mathrm{wls}(A)_{l+1-i_0} + \mathrm{wls}(B)_{l+1-i_0}.$$

Therefore, it suffices to show

$$l(\Lambda) \leq \delta(\Lambda) - \varkappa_{i_0}(\Lambda) + s(\Lambda, A, B) + \delta(A) - \mathrm{wls}(A)_{l+1-i_0} + \delta(B) - \mathrm{wls}(B)_{l+1-i_0} + i_0.$$

Let Λ' be the graph obtained from Λ by removing all edges $j \to k$ with either $k < i_0$ or $j \geq i_0$. It is easy to see that $l(\Lambda, A, B) - l(\Lambda', A, B) \leq \delta(\Lambda) - \delta(\Lambda')$. Thus, we can assume without loss of generality that all edges $j \to k$ of Λ satisfy $j < i_0 \leq k$. Therefore, as $\varkappa_{i_0}(\Lambda) = \sum_{\text{edges } e} \mathrm{wt}(e)$, we have

$$\delta(\Lambda) - \varkappa_{i_0} + s(\Lambda, A, B) = \sum_{\text{edges } e} \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big] + s = \sum \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big],$$

where $\mathrm{len}(e)$ is the *length* $k - j$ of an edge $j \xrightarrow{e} k$ and the last sum is over all edges of Λ of length at least 2. It is easy to see that the matrix A satisfies $\delta(A) \geq \mathrm{wls}(A)_i + l(A) - 1$ for all $i \geq 1$, therefore, if $l(A) = l(\Lambda)$, it suffices to show that

$$l(A) \leq \sum \Big[\mathrm{wt}(e)(\mathrm{len}(e) - 1) - 1\Big] + l(A) - 1 + \delta(B) - \mathrm{wls}(B)_{l+1-i_0} + i_0, \tag{4.5}$$

where the sum again runs over all edges of Λ of length at least 2. But (4.5) is clear as all summands in the sum are non-negative and $\delta(B) \geq \mathrm{wls}(B)_{l+1-i_0}$. The same argument also settles the case $l(B) = l(\Lambda)$.

Otherwise, we can assume that $l(\Lambda) > l(A) \geq l(B)$ and that there exists an edge $0 \to i$ of Λ with $l(\Lambda) - l(A) \leq i - 1$. If, additionally, we have $i_0 \leq l(\Lambda) - l(A)$ then $\mathrm{wls}(A)_{l+1-i_0} = 0$ and, using $\delta(B) \geq \mathrm{wls}(B)_{l+1-i_0}$, it suffices to prove that

$$l(A) + i - 1 \leq i - 2 + \delta(A) + 1$$

which is clear as $l(A) \leq \delta(A)$.

Finally, if $i_0 \geq l(\Lambda) - l(A) + 1$, it remains to show that $l(A) + 1 \leq \delta(A) - \mathrm{wls}(A)_{l+1-i_0} + i_0$. We have (by definition of $\delta(A)$ and $\mathrm{wls}(A)_{l+1-i_0}$) that

$$\delta(A) - \mathrm{wls}(A)_{l+1-i_0} + i_0 = \sum (i-1) j a_{ij} + \sum i j a_{ij} + i_0, \tag{4.6}$$

where the first sum runs over $i \geq l + 1 - i_0$, $j \geq 1$ and the second sum runs over $1 \leq i < l + 1 - i_0$, $j \geq 1$. As $i_0 \geq l(\Lambda) - l(A) + 1$ there exists a non-zero entry $a_{i'j'}$ of A with $i' = l(A) \geq l + 1 - i_0$. Therefore, the index set of the first sum of (4.6) is non-empty and the right-hand side of (4.6) is $\geq i' - 1 + i_0 = l(A) + 1$ as $i_0 \geq l(\Lambda) - l(A) + 1 \geq 2$. ☐

We now turn to the proof of the main theorem of this paper.

Proof of Theorem 1.1. We first show that (1.1) holds of all α, β with $d \geq \delta + 1$ where we again write $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$. This implies (1.1) if at least one of $\alpha_1, \alpha_2, \ldots, \beta_2, \beta_3, \ldots$ is non-zero (note that β_1 is omitted), because in that case $|\beta| \geq \delta$ implies $d \geq \delta + 1$.

Notice that we can remove condition (3.3) from formula (3.7) of Proposition 3.5 and still obtain correct relative Severi degrees as $\binom{\alpha}{a_1^T, a_2^T, \ldots} Q_{(\Lambda, A, B)}(\alpha; \beta) = 0$ whenever (3.3) is violated. The first factor of (3.7) equals

$$\sum_{k_m = k_{\min}(\Gamma_m)}^{d - l(\Lambda)} \mu(\Gamma_m) P_{\Gamma_m}(k_m) \sum_{k_{m-1} = k_{\min}(\Gamma_{m-1})}^{k_m - l(\Gamma_{m-1})} \cdots \sum_{k_1 = k_{\min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} \mu(\Gamma_1) P_{\Gamma_1}(k_1) \tag{4.7}$$

and is, therefore, an iterated "discrete integral" of polynomials. By repeated application of [2, Lemma 3.5] (or other means) expression (4.7) is a polynomial in d if $d - l(\Lambda) \geq 2 \sum_{i=1}^m \delta(\Gamma_i)$. Furthermore, as the polynomials $P_{\Gamma_i}(k_i)$ have degrees $\# E(\Gamma_i)$ and each "discrete integration" increases the degree by 1 the polynomial (4.7) is of degree $\sum_{i=1}^m \# E(\Gamma_i) + m$. By a literal application of the argument in Section 4 of [2] one can improve the polynomiality threshold of (4.7) and show that (4.7) is a polynomial in d if $d - l(\Lambda) \geq \sum_{i=1}^m \delta(\Gamma_i) + 1$. Furthermore, we have $l(\Lambda) \leq \delta(\Lambda) + \delta(A) + \delta(B)$. Thus, the first factor of (3.7) is a polynomial in d if $d \geq \delta + 1 = \sum_i \delta(\Gamma_i) + 1 + \delta(\Lambda) + \delta(A) + \delta(B)$.

The multinomial coefficient $\binom{\alpha}{a_1^T, a_2^T, \ldots}$ is a polynomial in $\alpha_1, \alpha_2, \ldots$ for fixed sequences of (column) vectors $a_1^T, a_2^T, \ldots$, if $\alpha_1, \alpha_2, \cdots \geq 0$. Hence, by Lemma 4.1, the second factor of (3.7) is of the form

$$\prod_{i\geq 1} i^{\beta_i} \cdot \frac{(|\beta|-\delta)!}{\beta!} \cdot R_{(\Lambda,A,B)}(\alpha;\beta) \tag{4.8}$$

for a polynomial $R_{(\Lambda,A,B)}(\alpha;\beta)$ in $\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots$ of degree $\#E(\Lambda)+||A||_1+||B||_1+\delta$ provided $d\geq d_{min}(\Lambda,A,B)$, where used that $\delta(B)\leq\delta$. By Lemma 4.2 the second factor of (3.7) equals expression (4.8) for all α,β with $d\geq d_{\min}(\Lambda,A,B)-s(\Lambda,A,B)$. Thus, using Lemma 4.3, if

$$d\geq \delta+1\geq \delta(\Lambda)+\delta(A)+\delta(B)+1\geq d_{\min}(\Lambda,A,B)-s(\Lambda,A,B)$$

the second factor in (3.7) is $\prod_{i\geq 1} i^{\beta_i}\cdot\frac{(|\beta|-\delta)!}{\beta!}$ times a polynomial in $\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots$ of degree $\#E(\Lambda)+||A||_1+||B||_1+\delta$. Hence (1.1) holds if $|\beta|\geq\delta$ and at least one β_i, for $i\geq 2$, or one α_i, for $i\geq 1$, is non-zero. Notice that each summand of (3.7) contributes a polynomial of degree

$$\sum_{i=1}^{m}\#E(\Gamma_i)+m+\#E(\Lambda)+||A||_1+||B||_1+\delta \tag{4.9}$$

to the relative node polynomial $N_\delta(\alpha;\beta)$. It is not hard to see that expression (4.9) is at most 3δ, and that equality is attained by letting $\Gamma_1,\ldots,\Gamma_\delta$ be the unique template on three vertices with cogenus 1 (see Figure 1) and (Λ,A,B) be the unique extended template of cogenus 0 (see Figure 2).

If $\alpha=0$ and $\beta=(d,0,\ldots)$ then $N^\delta_{\alpha,\beta}$ equals the (non-relative) Severi degree $N^{d,\delta}$ which, in turn, is given by the (non-relative) node polynomial $N^{\mathrm{abs}}_\delta(d)$ provided $d\geq\delta$ (see [2, Theorem 1.3]). Therefore, we have $N_\delta(0;d)=N^{\mathrm{abs}}_\delta(d)\cdot d(d-1)\cdots(d-\delta+1)$ as polynomials in d. Applying [2, Theorem 1.3] again completes the proof. □

Remark 4.4. Expression (3.7) gives, in principle, an algorithm to compute the relative node polynomial $N_\delta(\alpha;\beta)$, for any $\delta\geq 1$. In [2, Section 3] we explain how to generate all templates of a given cogenus, and how to compute the first factor in (3.7). The generation of all extended templates of a given cogenus from the templates is straightforward, as is the computation of the second factor in (3.7).

Remark 4.5. The proof of Theorem 1.1 simplifies if we relax the polynomiality threshold. More specifically, without considering the quantity $s(\Lambda,A,B)$ and the rather technical Lemmata 4.2 and 4.3 the argument still implies (1.1) provided $|\beta|\geq 2\delta$ (instead of $|\beta|\geq\delta$).

The immediate conclusion from the proof of Theorem 1.1 is two-fold.

Corollary 4.6. *For $\delta\geq 1$ the relative node polynomial $N_\delta(\alpha,\beta)$ is a polynomial in d, $|\beta|$, $\alpha_1,\ldots,\alpha_\delta$, and $\beta_1,\ldots,\beta_\delta$, where $d=\sum_{i\geq 1} i(\alpha_i+\beta_i)$*

Proof. Every extended template (Λ,A,B) considered in (3.7) satisfies $\delta(A)\leq\delta$ and $\delta(B)\leq\delta$. Therefore, all rows i in A or B are zero for $i>\delta$. □

Proof of Theorem 1.2. By the proof of Lemma 4.1 we have, for every extended template (Λ, A, B),

$$R_{(\Lambda,A,B)}(\alpha, 0; \beta) = R_{(\Lambda,A,B)}(\alpha; \beta) \qquad R_{(\Lambda,A,B)}(\alpha; \beta, 0) = R_{(\Lambda,A,B)}(\alpha; \beta).$$

Hence, by the proof of Theorem 1.1, the result follows. □

Now it is also easy to prove Theorem 1.3.

Proof of Theorem 1.3. Proposition 3.5 gives a combinatorial description of relative Severi degrees. The proof of Lemma 4.1 provides a method to calculate the polynomial $Q_{(\Lambda,A,B)}(\alpha; \beta)$. All terms of expression (3.7) are explicit or can be evaluated using the techniques of [2, Section 3]. This reduces the calculation to a (non-trivial) computer calculation. □

5. Coefficients of Relative Node Polynomials

We now turn towards the computation of the coefficients of the relative node polynomial $N_\delta(\alpha; \beta)$ of large degree for any δ. By Theorem 1.1 the polynomial $N_\delta(\alpha, \beta)$ is of degree 3δ. In the following we propose a method to compute all terms of $N_\delta(\alpha; \beta)$ of degree $\geq 3\delta - t$, for any given $t \geq 0$. This method was used (with $t = 2$) to compute the terms in Theorem 1.4.

The main idea of the algorithm is that, even for general δ, only a small number of summands of (3.7) contribute to the terms of $N_\delta(\alpha; \beta)$ of large degree. A summand of (3.7) is indexed by a collection of templates $\tilde{\Gamma} = \{\Gamma_s\}$ and an extended template (Λ, A, B). To determine whether this summand might contribute to $N_\delta(\alpha; \beta)$ we define the *(degree) defects*

- of the collection of templates $\tilde{\Gamma}$ by

$$\operatorname{def}(\tilde{\Gamma}) \stackrel{\text{def}}{=} \sum_{s=1}^{m} \left[\delta(\Gamma_i)\right] - m, \text{ and}$$

- of the extended template (Λ, A, B) by

$$\operatorname{def}(\Lambda, A, B) \stackrel{\text{def}}{=} \delta(\Lambda) + 2\delta(A) + 2\delta(B) - ||A||_1 - ||B||_1.$$

The following lemma restricts the indexing set of (3.7) to the relevant terms, if only the leading terms of $N_\delta(\alpha; \beta)$ are of interest.

Lemma 5.1. *The summand of (3.7) indexed by $\tilde{\Gamma}$ and (Λ, A, B) is of the form*

$$1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot P(\alpha; \beta),$$

where $P(\alpha; \beta)$ is a polynomial in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ of degree $\leq 3\delta - \operatorname{def}(\tilde{\Gamma}) - \operatorname{def}(\Lambda, A, B)$.

Proof. By [2, Lemma 5.2] the first factor of (3.7) is of degree at most

$$2 \cdot \sum_{s=1}^{m} \delta(\Gamma_s) - \sum_{s=1}^{m} (\delta(\Gamma_s) - 1) = \sum_{s=1}^{m} \delta(\Gamma_s) + m.$$

The multinomial coefficient $\binom{\alpha}{a_1^T, a_2^T, \ldots}$ is a polynomial in α of degree $||A||_1$ if a_j^T are the jth column vector of A. Recall from the proof of Theorem 1.1 that the second factor of (3.7) is

$$\prod_{i\geq 1} i^{\beta_i} \frac{(|\beta|-\delta)!}{\beta!} \text{ times a polynomial in } \alpha, \beta \text{ of degree } \#E(\Lambda) + ||A||_1 + ||B||_1 + \delta.$$

Therefore, the contribution of this summand to the relative node polynomial is at most of degree

$$\begin{aligned} &\sum_{s=1}^{m} \delta(\Gamma_s) + m + \#E(\Lambda) + ||A||_1 + ||B||_1 + \delta \\ &= 3\delta - 2\sum_{s=1}^{m} \delta(\Gamma_s) - 2\delta(\Lambda) - 2\delta(A) - 2\delta(B) + \#E(\Lambda) \\ &= 3\delta - \operatorname{def}(\tilde{\Gamma}) - \operatorname{def}(\Lambda, A, B) - \delta(\Lambda) + \#E(\Lambda). \end{aligned}$$

The result follows as $\delta(\Lambda) \geq \#E(\Lambda)$ □

Therefore, to compute the coefficients of degree $\geq 3\delta - t$ of $N_\delta(\alpha;\beta)$ for some $t \geq 0$, it suffices to consider only summands of (3.7) with $\operatorname{def}(\tilde{\Gamma}) \leq t$ and $\operatorname{def}(\Lambda, A, B) \leq t$.

One can proceed as follows. First, we can compute, for some formal variable $\tilde{\delta}$, the terms of degree $\geq 2\tilde{\delta} - t$ of the first factor of (3.7) to $N_{\tilde{\delta}}(\alpha;\beta)$, that is the terms of degree $\geq 2\tilde{\delta} - t$ of

$$R_{\tilde{\delta}}(d) \stackrel{\text{def}}{=} \sum \prod_{i=1}^{m} \mu(\Gamma_i) \sum_{k_m = k_{min}(\Gamma_m)}^{d - l(\Gamma_m)} P_{\Gamma_m}(k_m) \cdots \sum_{k_1 = k_m in(\Gamma_1)}^{k_2 - l(\Gamma_1)} P_{\Gamma_1}(k_1), \tag{5.1}$$

where the first sum is over all collections of templates $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ with $\delta(\tilde{\Gamma}) = \tilde{\delta}$. (Notice that (5.1) is expression [9, (5.13)] without the "ε-correction" in the sum indexed by k_m.) The leading terms of $R_{\tilde{\delta}}(d)$ can be computed with a slight modification of [2, Algorithm 2] (by replacing, in the notation of [2], C^{end} by C and M^{end} by M). The algorithm relies on the polynomiality of solutions of certain polynomial difference equations, which has been verified for $t \leq 7$, see [2, Section 5] for more details. With a Maple implementation of this algorithm one obtains (with $t = 5$)

$$\begin{aligned} R_{\tilde{\delta}}(d) = \frac{3^{\tilde{\delta}}}{\tilde{\delta}!}\Big[& d^{2\tilde{\delta}} - \frac{8\tilde{\delta}}{3} d^{2\tilde{\delta}-1} + \frac{\tilde{\delta}(11\tilde{\delta}+1)}{3^2} d^{2\tilde{\delta}-2} + \frac{\tilde{\delta}(\tilde{\delta}-1)(496\tilde{\delta}-245)}{6\cdot 3^3} d^{2\tilde{\delta}-3} \\ & - \frac{\tilde{\delta}(\tilde{\delta}-1)(1685\tilde{\delta}^2 - 2773\tilde{\delta} + 1398)}{6\cdot 3^4} d^{2\tilde{\delta}-4} + \\ & - \frac{\tilde{\delta}(\tilde{\delta}-1)(\tilde{\delta}-2)(7352\tilde{\delta}^2 + 11611\tilde{\delta} - 25221)}{30\cdot 3^5} d^{2\tilde{\delta}-5} + \cdots \Big]. \end{aligned}$$

Finally, to compute the coefficients of degree $\geq 3\delta - t$, it remains to compute all extended templates (Λ, A, B) with $\operatorname{def}(\Lambda, A, B) \leq t$ and collect the terms of degree

$\geq 3\delta - t$ of the polynomial

$$R_{\tilde{\delta}}(d - l(\Lambda)) \cdot \mu(\Lambda) \binom{\alpha}{a_1^T, a_2^T, \ldots} \prod_{i=\delta(B)}^{\delta-1} (|\beta| - i) \cdot q_{(\Lambda,A,B)}(\alpha;\beta), \tag{5.2}$$

where, as before, $a_1^T, a_2^T, \ldots$ denote the column vectors of the matrix A, $q_{(\Lambda,A,B)}(\alpha;\beta)$ is the polynomial of Lemma 4.1, and $\tilde{\delta} = \delta - \delta(\Lambda, A, B)$. Notice that, for an indeterminant x and integers $c \geq 0$ and $\delta \geq 1$, we have the expansion

$$\prod_{i=c}^{\delta-1}(x-i) = \sum_{t=0}^{\delta-c} s(\delta - c, \delta - c - t)(x-c)^{\delta-c-t},$$

where $s(n, m)$ is the *Stirling number of the first kind* [18, Section 1.3] for integers $n, m \geq 0$. Furthermore, with $\delta' = \delta - c$ the coefficients $s(\delta', \delta' - t)$ of the sum equal $\delta'(\delta' - 1) \cdots (\delta' - t) \cdot S_t(\delta')$, where S_t is the t-th *Stirling polynomial* [11, (6.45)], for $t \geq 0$, and thus are polynomial in δ'. Therefore, we can compute the leading terms of the product in (5.2) by collecting the leading terms in the sum expansion above.

Proof of Proposition 1.5. Using [2, Algorithm 2] we can compute the terms of the polynomial $R_{\tilde{\Gamma}}(d)$ of degree $\geq 2\tilde{\delta} - 7$ (see [2, Section 5]) and observe that all coefficients are polynomial in $\tilde{\delta}$. By the previous paragraph the coefficients of the expansion of the sum of (5.2) are polynomial in δ. This completes the proof. □

Proof of Theorem 1.4. The method described above is a direct implementation of formula (3.7), which equals the relative Severi degree by the proof of Theorem 1.1. □

Remark 5.2. It is straight-forward to compute the coefficients of $N_\delta(\alpha;\beta)$ of degree $\geq 3\delta - 7$ (and thereby to extend Theorem 1.4). Algorithm 3 of [2] computes the coefficients of the polynomials $R_{\tilde{\delta}}(d)$ of degree $\geq 2\tilde{\delta} - 7$, and thus the desired terms can be collected from (5.2). We expect this method to compute the leading terms of $N_\delta(\alpha, \beta)$ of degree $\geq 3\delta - t$ for arbitrary $t \geq 0$ (see [2, Section 5], especially Conjecture 5.5).

Appendix A. New relative node polynomials

Below we list the relative node polynomials $N_\delta(\alpha;\beta)$ for $\delta \leq 3$. For $\delta \leq 6$ the polynomials $N_\delta(\alpha;\beta)$ are as provided in the ancillary files accompanying this article. All polynomials were obtained by a Maple implementation of the formula (3.7). See Remark 4.4 for more details. For $\delta \leq 1$ this agrees with [9, Corollary 4.5, 4.6]. As before, we write $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$. By Theorem 1.1 the relative Severi degrees $N^\delta_{\alpha,\beta}$ are given by $N^\delta_{\alpha,\beta} = 1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta|-\delta)!}{\beta!} N_\delta(\alpha, \beta)$ provided $|\beta| \geq \delta$.

$$
\begin{aligned}
N_0(\alpha,\beta) &= 1,\\
N_1(\alpha,\beta) &= 3d^2|\beta| - 8d|\beta| + d\beta_1 + |\beta|\alpha_1 + |\beta|\beta_1 + 4|\beta| - \beta_1,\\
N_2(\alpha,\beta) &= \tfrac{9}{2}d^4|\beta|^2 - \tfrac{9}{2}d^4|\beta| - 24d^3|\beta|^2 + 3d^3|\beta|\beta_1 + 3d^2|\beta|^2\alpha_1 + 3d^2|\beta|^2\beta_1 + 24d^3|\beta| - 3d^3\beta_1 + 23d^2|\beta|^2\\
&\quad - 3d^2|\beta|\alpha_1 - 14d^2|\beta|\beta_1 + \tfrac{1}{2}d^2\beta_1^2 - 8d|\beta|^2\alpha_1 - 8d|\beta|^2\beta_1 + d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + \tfrac{1}{2}|\beta|^2\alpha_1^2 + |\beta|^2\alpha_1\beta_1\\
&\quad + \tfrac{1}{2}|\beta|^2\beta_1^2 - 23d^2|\beta| + \tfrac{21}{2}d^2\beta_1 + \tfrac{3}{2}d|\beta|^2 + 8d|\beta|\alpha_1 + 11d|\beta|\beta_1 + d|\beta|\beta_2 - d\alpha_1\beta_1 - \tfrac{5}{2}d\beta_1^2 - \tfrac{1}{2}|\beta|^2\alpha_1\\
&\quad + |\beta|^2\alpha_2 - \tfrac{1}{2}|\beta|^2\beta_1 + |\beta|^2\beta_2 - \tfrac{1}{2}|\beta|\alpha_1^2 - 3|\beta|\alpha_1\beta_1 - \tfrac{5}{2}|\beta|\beta_1^2 - \tfrac{83}{2}d|\beta| - \tfrac{3}{2}d\beta_1 - d\beta_2 - 48|\beta|^2 + \tfrac{1}{2}|\beta|\alpha_1\\
&\quad - |\beta|\alpha_2 + \tfrac{29}{2}|\beta|\beta_1 - 3|\beta|\beta_2 + 2\alpha_1\beta_1 + 3\beta_1^2 + 48|\beta| - 15\beta_1 + 2\beta_2,\\
N_3(\alpha,\beta) &= \tfrac{9}{2}d^6|\beta|^3 - \tfrac{27}{2}d^6|\beta|^2 - 36d^5|\beta|^3 + \tfrac{9}{2}d^5|\beta|^2\beta_1 + \tfrac{9}{2}d^4|\beta|^3\alpha_1 + \tfrac{9}{2}d^4|\beta|^3\beta_1 + 9d^6|\beta| + 108d^5|\beta|^2 - \tfrac{27}{2}d^5|\beta|\beta_1\\
&\quad + 51d^4|\beta|^3 - \tfrac{27}{2}d^4|\beta|^2\alpha_1 - 42d^4|\beta|^2\beta_1 + \tfrac{3}{2}d^4|\beta|\beta_1^2 - 24d^3|\beta|^3\alpha_1 - 24d^3|\beta|^3\beta_1 + 3d^3|\beta|^2\alpha_1\beta_1 + 3d^3|\beta|^2\beta_1^2\\
&\quad + \tfrac{3}{2}d^2|\beta|^3\alpha_1^2 + 3d^2|\beta|^3\alpha_1\beta_1 + \tfrac{3}{2}d^2|\beta|^3\beta_1^2 - 72d^5|\beta| + 9d^5\beta_1 - 153d^4|\beta|^2 + 9d^4|\beta|\alpha_1 + 93d^4|\beta|\beta_1 - 3d^4\beta_1^2\\
&\quad + \tfrac{1243}{6}d^3|\beta|^3 + 72d^3|\beta|^2\alpha_1 + 92d^3|\beta|^2\beta_1 + 3d^3|\beta|^2\beta_2 - 9d^3|\beta|\alpha_1\beta_1 - \tfrac{35}{2}d^3|\beta|\beta_1^2 + \tfrac{1}{6}d^3\beta_1^3 + \tfrac{19}{2}d^2|\beta|^3\alpha_1\\
&\quad + 3d^2|\beta|^3\alpha_2 + \tfrac{19}{2}d^2|\beta|^3\beta_1 + 3d^2|\beta|^3\beta_2 - \tfrac{9}{2}d^2|\beta|^2\alpha_1^2 - 23d^2|\beta|^2\alpha_1\beta_1 - \tfrac{37}{2}d^2|\beta|^2\beta_1^2 + \tfrac{1}{2}d^2|\beta|\alpha_1\beta_1^2\\
&\quad + \tfrac{1}{2}d^2|\beta|\beta_1^3 - 4d|\beta|^3\alpha_1^2 - 8d|\beta|^3\alpha_1\beta_1 - 4d|\beta|^3\beta_1^2 + \tfrac{1}{2}d|\beta|^2\alpha_1^2\beta_1 + d|\beta|^2\alpha_1\beta_1^2 + \tfrac{1}{2}d|\beta|^2\beta_1^3 + \tfrac{1}{6}|\beta|^3\alpha_1^3\\
&\quad + \tfrac{1}{2}|\beta|^3\alpha_1^2\beta_1 + \tfrac{1}{2}|\beta|^3\alpha_1\beta_1^2 + \tfrac{1}{6}|\beta|^3\beta_1^3 + 102d^4|\beta| - 54d^4\beta_1 - \tfrac{1243}{2}d^3|\beta|^2 - 48d^3|\beta|\alpha_1 - \tfrac{199}{2}d^3|\beta|\beta_1\\
&\quad - 9d^3|\beta|\beta_2 + 6d^3\alpha_1\beta_1 + \tfrac{45}{2}d^3\beta_1^2 - 458d^2|\beta|^3 - \tfrac{57}{2}d^2|\beta|^2\alpha_1 - 9d^2|\beta|^2\alpha_2 + 116d^2|\beta|^2\beta_1 - 23d^2|\beta|^2\beta_2\\
&\quad + 3d^2|\beta|\alpha_1^2 + \tfrac{95}{2}d^2|\beta|\alpha_1\beta_1 + \tfrac{105}{2}d^2|\beta|\beta_1^2 + d^2|\beta|\beta_1\beta_2 - d^2\alpha_1\beta_1^2 - 2d^2\beta_1^3 + \tfrac{155}{2}d|\beta|^3\alpha_1 - 8d|\beta|^3\alpha_2\\
&\quad + \tfrac{155}{2}d|\beta|^3\beta_1 - 8d|\beta|^3\beta_2 + 12d|\beta|^2\alpha_1^2 + \tfrac{61}{2}d|\beta|^2\alpha_1\beta_1 + d|\beta|^2\alpha_1\beta_2 + d|\beta|^2\alpha_2\beta_1 + \tfrac{37}{2}d|\beta|^2\beta_1^2 + 2d|\beta|^2\beta_1\beta_2\\
&\quad - \tfrac{3}{2}d|\beta|\alpha_1^2\beta_1 - \tfrac{11}{2}d|\beta|\alpha_1\beta_1^2 - 4d|\beta|\beta_1^3 - \tfrac{5}{2}|\beta|^3\alpha_1^2 + |\beta|^3\alpha_1\alpha_2 - 5|\beta|^3\alpha_1\beta_1 + |\beta|^3\alpha_1\beta_2 + |\beta|^3\alpha_2\beta_1 - \tfrac{5}{2}|\beta|^3\beta_1^2\\
&\quad + |\beta|^3\beta_1\beta_2 - \tfrac{1}{2}|\beta|^2\alpha_1^3 - 3|\beta|^2\alpha_1^2\beta_1 - \tfrac{9}{2}|\beta|^2\alpha_1\beta_1^2 - 2|\beta|^2\beta_1^3 + \tfrac{1243}{3}d^3|\beta| + \tfrac{70}{3}d^3\beta_1 + 6d^3\beta_2 + 1374d^2|\beta|^2\\
&\quad + 19d^2|\beta|\alpha_1 + 6d^2|\beta|\alpha_2 - \tfrac{845}{2}d^2|\beta|\beta_1 + 48d^2|\beta|\beta_2 - 27d^2\alpha_1\beta_1 - 40d^2\beta_1^2 - 2d^2\beta_1\beta_2 - \tfrac{842}{3}d|\beta|^3\\
&\quad - \tfrac{465}{2}d|\beta|^2\alpha_1 + 24d|\beta|^2\alpha_2 - 396d|\beta|^2\beta_1 + 29d|\beta|^2\beta_2 + d|\beta|^2\beta_3 - 8d|\beta|\alpha_1^2 - 33d|\beta|\alpha_1\beta_1 - 3d|\beta|\alpha_1\beta_2\\
&\quad - 3d|\beta|\alpha_2\beta_1 + 2d|\beta|\beta_1^2 - 11d|\beta|\beta_1\beta_2 + d\alpha_1^2\beta_1 + 7d\alpha_1\beta_1^2 + \tfrac{47}{6}d\beta_1^3 - \tfrac{92}{3}|\beta|^3\alpha_1 - 6|\beta|^3\alpha_2 + |\beta|^3\alpha_3\\
&\quad - \tfrac{92}{3}|\beta|^3\beta_1 - 6|\beta|^3\beta_2 + |\beta|^3\beta_3 + \tfrac{15}{2}|\beta|^2\alpha_1^2 - 3|\beta|^2\alpha_1\alpha_2 + \tfrac{87}{2}|\beta|^2\alpha_1\beta_1 - 6|\beta|^2\alpha_1\beta_2 - 6|\beta|^2\alpha_2\beta_1 + 36|\beta|^2\beta_1^2\\
&\quad - 9|\beta|^2\beta_1\beta_2 + \tfrac{1}{3}|\beta|\alpha_1^3 + \tfrac{11}{2}|\beta|\alpha_1^2\beta_1 + 13|\beta|\alpha_1\beta_1^2 + \tfrac{47}{6}|\beta|\beta_1^3 - 916d^2|\beta| + 303d^2\beta_1 - 28d^2\beta_2 + 842d|\beta|^2\\
&\quad + 155d|\beta|\alpha_1 - 16d|\beta|\alpha_2 + \tfrac{1237}{2}d|\beta|\beta_1 - 31d|\beta|\beta_2 - 3d|\beta|\beta_3 + 8d\alpha_1\beta_1 + 2d\alpha_1\beta_2 + 2d\alpha_2\beta_1 - \tfrac{103}{2}d\beta_1^2\\
&\quad + 14d\beta_1\beta_2 + 706|\beta|^3 + 92|\beta|^2\alpha_1 + 18|\beta|^2\alpha_2 - 3|\beta|^2\alpha_3 - 46|\beta|^2\beta_1 + 48|\beta|^2\beta_2 - 6|\beta|^2\beta_3 - 5|\beta|\alpha_1^2\\
&\quad + 2|\beta|\alpha_1\alpha_2 - \tfrac{197}{2}|\beta|\alpha_1\beta_1 + 11|\beta|\alpha_1\beta_2 + 11|\beta|\alpha_2\beta_1 - \tfrac{271}{2}|\beta|\beta_1^2 + 26|\beta|\beta_1\beta_2 - 3\alpha_1^2\beta_1 - 12\alpha_1\beta_1^2 - 10\beta_1^3\\
&\quad - \tfrac{1684}{3}d|\beta| - \tfrac{808}{3}d\beta_1 + 10d\beta_2 + 2d\beta_3 - 2118|\beta|^2 - \tfrac{184}{3}|\beta|\alpha_1 - 12|\beta|\alpha_2 + 2|\beta|\alpha_3 + \tfrac{1184}{3}|\beta|\beta_1 - 102|\beta|\beta_2\\
&\quad + 11|\beta|\beta_3 + 63\alpha_1\beta_1 - 6\alpha_1\beta_2 - 6\alpha_2\beta_1 + 150\beta_1^2 - 24\beta_1\beta_2 + 1412|\beta| - 362\beta_1 + 60\beta_2 - 6\beta_3.
\end{aligned}
$$

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.
E-mail address: blockf@umich.edu