GROUPS WITH POLYNOMIAL GEODESIC GROWTH

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Abstract. We study the *geodesic growth function* for finitely generated groups, and we show that there is a strong relation between virtually cyclic groups and the existence of generating sets for which geodesic growth is polynomial. We show that any group that maps onto the free abelian group of rank 2 has exponential geodesic growth with respect to any finite generating set.

1. Introduction

Growth for finitely generated groups is a concept that has been studied extensively in the last fifty years, with landmarks of modern group theory such as Gromov's theorem characterizing groups with polynomial growth as virtually nilpotent [7], and Grigorchuk's construction of groups with intermediate growth.

Geodesic growth was defined in [6], who mention it was probably introduced in [9], which itself has its roots in [1]. In [5] Epstein *et al* prove that the geodesic growth series (the formal power series associated to a geodesic growth function) is rational for hyperbolic groups, and there are other results about its rationality for other families of groups, see [2, 8, 10]. An example of Cannon, described in [9], of a virtually-$\mathbb{Z}^2$ group having a regular language of geodesics for one generating set, and not another, shows that the geodesic growth series depends on the choice of generators.

The goal of this paper is to study geodesic growth functions.

The article is organised as follows. In Section 2 we define geodesic growth and outline some basic properties. In Sections 3–5 we consider

Date: September 28, 2010.

2000 *Mathematics Subject Classification.* 20F65.

Key words and phrases. Geodesic growth, virtually nilpotent group with virtually cyclic abelianization.

The second author acknowledges support from MEC grant #MTM2008-01550 and is grateful for the hospitality of the University of Queensland. The third author acknowledges support from the University of Queensland New Staff Research Start-Up Fund project number #2008001627. The fourth author acknowledges support by NSF grant DMS-0805932.

which groups admit polynomial geodesic growth with respect to some generating set: we show that virtually cyclic groups have polynomial geodesic growth for at least one finite generating set; that any group which surjects onto $\mathbb{Z}^2$, and all non-cyclic nilpotent groups, have exponential geodesic growth for all finite generating sets; and lastly we exhibit some virtually abelian groups of rank > 1 having polynomial geodesic growth for some finite generating set – such examples by necessity have virtually cyclic abelianizations. In spite of these results we are unable to give a complete classification of groups with polynomial geodesic growth (for some generating set), but we conjecture that if a group has polynomial geodesic growth for some finite generating set then it must be virtually cyclic or virtually abelian with virtually cyclic abelianization.

In the Section 6 we consider whether intermediate geodesic growth is possible.

We note an interesting paper by Mike Shapiro [11], where Shapiro defines a so-called *Pascal function* which is closely related to geodesic growth.

The authors are very grateful to Warren Dicks, Gretchen Ostheimer and Stephan Tillmann for their helpful feedback and insights.

2. Definition and elementary properties

Throughout the paper, let G be a finitely generated group, with finite monoid generating set $X = \{x_1, \ldots, x_m\}$. Note that we allow repetition of generators, but nevertheless use the set terminology (technically speaking G comes with a projection from the free moniod $F(X)$ generated by X and some images under this projection may coincide).

The central concept in this paper is the one of *geodesic growth*.

Definition 1. The *geodesic growth* of a group G with respect to the generating set X is the function $\Gamma_{G,X}(n)$ counting, for each n, the number of geodesics of length at most n starting at 1 in the Cayley graph of the group G with respect to X.

Observe that the geodesic growth is bounded below by the usual growth $\gamma_{G,X}(n)$, which counts the number of elements in the group at distance at most n from 1. Each one of these elements has at least one geodesic, but in general there will be more than one geodesic for each element. (Shapiro considers groups for which there is exactly one geodesic for each element in [11]).

Proposition 2. *Let G be a group generated by a finite monoid set X. The geodesic growth function $\Gamma_{G,X}(n)$ of G with respect to X is larger* *than the word growth function* $\gamma_{G,X}(n)$ *with respect to* X*, in the sense that, for all* n*,*

$$\Gamma_{G,X}(n) \geq \gamma_{G,X}(n).$$

As the next examples will show, geodesic growth is heavily dependent on the generating set.

Example 3. Consider the group $\mathbb{Z}\times C_2$ (here and thereafter C_k denotes the cyclic group of order k). A standard presentation is $\langle t, a | a^2 = 1, at = ta\rangle$. With respect to the generating set $\{t, a\}$ each geodesic of length n is of the form t^n, t^{-n}, $t^i a^{\pm 1} t^{n-i-1}$, or $t^{-i} a^{\pm 1} t^{i+1-n}$, for $i = 0, \dots, n-1$. Thus the number of geodesics of length n is, for $n \geq 2$, equal to $4n+2$, which is linear, and the number of geodesics of length at most n is $O(n^2)$.

Now consider the presentation $\langle t, c | c^2 = t^2, ct = tc\rangle$, obtained from the previous one by the substitution $c = at$. With respect to the generating set $\{t, c\}$ each word $x_1 x_2 \dots x_n$ with $x_i \in \{tt, cc\}$ is a geodesic for the element t^{2n}. The number of such strings with 2 choices for each x_i is $2^n = (\sqrt{2})^{2n}$, showing that the geodesic growth is exponential.

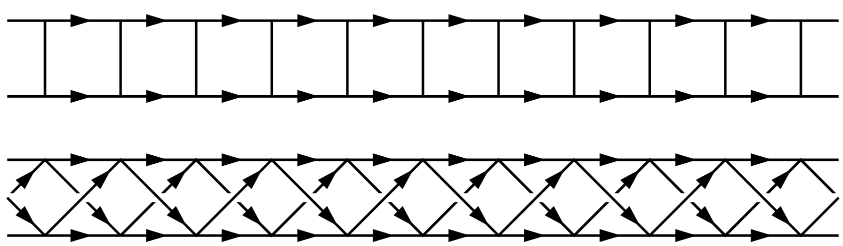

FIGURE 1. Cayley graphs for Example 3.

Note that both generating sets are "minimal" in the sense that one cannot do better than two generators, so both are equally "natural".

Here is an even more extreme example.

Example 4. Consider $\mathbb{Z}$ with the presentation $\langle t \mid \rangle$. For $n = 0$ there is one geodesic of length n, and for $n > 0$ there are exactly two. Now take the presentation $\langle t, s \mid t = s\rangle$. The element t^n has 2^n geodesics, obtained by choosing either s or t for each letter.

By using a similar doubling procedure one can show that every finitely generated group has some generating set for which the geodesic growth is exponential: given generators $a_1, \dots, a_m$, construct a new generating set by adding $b_1, \dots, b_m$, with $b_i = a_i$ in the group, for $i = 1, \dots, m$. This leads us to consider groups which have subexponential (polynomial or intermediate) geodesic growth for *some* finite generating set. Note that, the following proposition is a direct corollary to Proposition 2

Corollary 5. *Every finitely generated group of exponential growth has exponential geodesic growth with respect to any (finite) generating set.*

The lower bound in Proposition 2 allows us also to conclude that if a group has polynomial geodesic growth with respect to some generating set then its word growth must be polynomial. Hence, by Gromov's celebrated theorem [7], we obtain the following.

Corollary 6. *Every group that has polynomial geodesic growth with respect to some (finite) generating set is virtually nilpotent.*

In addition to the obvious lower bound provided by the word growth function given in Proposition 2 there is another lower bound that comes from the good behavior of geodesics under lifts along homomorphisms.

Proposition 7. *Let G be generated by a finite symmetric set X, $\phi : G \to G'$ a surjective homomorphism and $X' = \phi(X)$ the corresponding generating set of G'. The geodesic growth of G with respect to X is bounded below by the geodesic growth of G' with respect to X' in the sense that, for all $n \geq 0$,*

$$\Gamma_{G,X}(n) \geq \Gamma_{G',X'}(n).$$

Proof. For any geodesic in G' we have a natural lift to G, which is a geodesic because if there were a shorter path in G, it would project to a shorter path in G'. Hence, $\Gamma_{G,X}(n) \geq \Gamma_{G',X'}(n)$, for all n. □

3. Geodesic growth for virtually cyclic groups

In this section we prove that the geodesic growth of a virtually cyclic group with respect to any (finite) generating set is either exponential or polynomial. Moreover, we show that both can happen for each virtually cyclic group, depending on the choice of a generating set.

Proposition 8. *Every virtually cyclic group has some finite generating set with respect to which the geodesic growth is polynomial.*

Proof. Let G be virtually cyclic and infinite. Then G has a *normal* subgroup isomorphic to $\mathbb{Z}$.

Call the generator of this subgroup t, the quotient group Q and choose representatives $q_1, \dots, q_m$ for the right cosets of $\langle t \rangle$.

Since $\langle t \rangle$ is normal, conjugating t by a q_i gives an automorphism of $\langle t \rangle$, and hence we have that $q_i t q_i^{-1} = t^{\pm 1}$ for all $i = 1, \dots, m$.

Consider the finite list of all freely reduced words of length 3 in q_i, $i = 1, \dots, m$, of length 3. Then $q_i^{\pm 1} q_j^{\pm 1} q_k^{\pm 1} =_G t^x q_c$, for some integer x

and some c with $0 \leq c \leq m$. Let M be the largest $|x|$ obtained in this way from this finite list.

The set $S = \{ q_1^{\pm 1}, \ldots, q_m^{\pm 1}, t^{\pm 1}, t^{\pm 2}, \ldots, t^{\pm M}\}$ is a symmetric generating set for G.

We will show that there is a uniform upper bound on the number of letters different from $t^{\pm M}$ in any geodesic with respect to S.

Let w be a geodesic with respect to S. Using the fact that $q_i^{\pm 1}t = t^{\pm 1}q_i^{\pm 1}$, for $i = 1, \ldots, m$, we can push all $q_i^{\pm 1}$ letters past all $t^{\pm 1}$ letters to obtain a word of the form $t^k q_{i_1}^{\pm 1} \ldots q_{i_p}^{\pm 1}$. Since w is a geodesic there must be no free cancelation after this. If $p \geq 3$ then we can rewrite $q_{i_1}^{\pm 1} q_j^{\pm 1} q_k^{\pm 1} = t^x q_c$, where $|x| \leq M$ and obtain a shorter representative.

Therefore, every geodesic can have at most two $q_i^{\pm 1}$ letters, i.e., every geodesic is in one of the forms $u\ q_{i_1}^{\pm 1}\ v\ q_{i_2}^{\pm 1}\ w$, or $u\ q_{i_1}^{\pm 1}\ v$ or u, where u, v, w are products of $t^{\pm j}$ letters, for some $j = 1, \ldots, M$.

Consider a word u that is a product of $t^{\pm j}$ letters, for some $j = 1, \ldots, M$. All letters used in u must be of the form t^j or all of them must be of the form t^{-j}. Without loss of generality, assume that all of them are of the form t^j. Assume that the letter t^i appears at least M times, for some $i = 1, \ldots, M-1$. Since $(t^i)^M = (t^M)^i$ and $i < M$, we could rewrite u into a shorter word, a contradiction. Thus, for $i = 1, \ldots, M-1$, each of the letters t^i appears at most $M-1$ times in u.

We conclude that no geodesic can have more than $N = 3(M-1)^2+2$ letters different from $t^{\pm M}$ and that the sign pattern of the letters $t^{\pm M}$ that do appear can possibly change at most twice (if there are two q letters). Therefore, there is a constant k such that, for sufficiently large n, the number of geodesics of length n is no greater than $k\binom{n}{N}$ and the number of geodesics of length at most n is no greater than $kn\binom{n}{N}$, which is polynomial in n. □

Corollary 9. *Let G be a virtually cyclic group generated by a finite symmetric set X. The geodesic growth function $\Gamma_{G,X}$ is either exponential or polynomial.*

Proof. As observed in Section 2 every finitely generated group has some finite generating set for which geodesic growth is exponential. The previous proposition shows that virtually cyclic groups also have at least one generating set for which the geodesic growth is polynomial. The only case we need to rule out is a generating set for which geodesic growth is intermediate.

A virtually cyclic group is hyperbolic, and the language of geodesics is regular for every finite generating set (see [5]). The growth of a regular language is either polynomial or exponential, by [7]. □

4. Geodesic growth for groups that map onto $\mathbb{Z}^2$

The main construction to keep in mind in the abelian case is that in a grid which is p by q, the number of paths to go from a corner to its diagonally opposed one is exactly $\binom{p+q}{p}$. Hence, if we have two generators a and b that commute, then the number of paths from the origin to $a^p b^q$ is exactly that binomial number. This fact is useful to compute the number of geodesics in an abelian group.

For the canonical generating set of $\mathbb{Z}^2$, which is given by two generators a and b which commute, we just see that the element $a^n b^n$ admits $\binom{2n}{n}$ geodesics, by choosing all possible paths in a square grid of side length n. This number is exponential in n. Hence, the geodesic growth for $\mathbb{Z}^2$ is exponential with respect to the canonical generating set.

By replicating this phenomenon, we can extend this result to all generating sets for $\mathbb{Z}^2$.

Proposition 10. *The group $\mathbb{Z}^2$ has exponential geodesic growth for any finite generating set.*

Proof. Let X be a generating set with elements $x_1, \ldots, x_m$. We can consider $\mathbb{Z}^2$ as the lattice of point with integer coordinates in the plane $\mathbb{R}^2$. Each generator x_i can be seen as a vector $(a_i, b_i) \in \mathbb{R}^2$. We keep using the multiplicative notation when we think of x_i as generators of $\mathbb{Z}^2$ and use additive notation when we think of the corresponding vectors.

Consider the convex polygon

$$P = \text{ConvexHull } \{ \; x_i \mid i = 1, \ldots, m \; \}.$$

Note that this polygon is not degenerated to a segment, since X is a generatng set for $\mathbb{Z}^2$. Choose a pair of generators x_r and x_s that are vertices of the polygon P such that the segment $[x_r, x_s]$ is an edge of the polygon. We claim that the path of length $2n$ given by $x_r^n x_s^n$ is a geodesic in $\mathbb{Z}^2$ with respect to X.

Indeed, the vectors that represent the elements of $\mathbb{Z}^2$ of length at most $2n-1$ with respect to X are contained in the polygon $(2n-1)P$, and the vector $nx_r + nx_s$ is on the middle of the edge $[2nx_r, 2nx_s]$ of the polygon $2nP$, which is entirely outside of $(2n-1)P$.

The result follows now by taking the geodesic $x_i^n x_j^n$ and applying the same argument as in the standard generating set. All $\binom{2n}{n}$ permutations of n letters x_i and n letters x_j will yield a geodesic and hence the group has exponential geodesic growth with respect to X. □

Clearly, this result extends to free abelian groups of any rank greater than 2. In fact, by Proposition 7 it extends to any group that maps homomorphically onto $\mathbb{Z}^2$.

Corollary 11. *A finitely generated group G that maps homomorphically onto $\mathbb{Z}^2$ has exponential geodesic growth with respect to any (finite) generating set.*

5. Geodesic growth for nilpotent groups

We show that the groups that have polynomial geodesic growth with respect to some generating set can be fully described within the class of nilpotent groups.

Theorem 12. *A finitely generated, nilpotent group has polynomial geodesic growth with respect to some (finite) generating set if and only if it is virtually cyclic.*

The proof follows easily from the fact that groups that map onto $\mathbb{Z}^2$ always have exponential geodesic growth (Proposition 11) and that virtually cyclic groups sometimes have polynomial geodesic growth (Proposition 8), along with the following well known fact, which we derive from a more general statement on virtually nilpotent groups.

Proposition 13. *A finitely generated, nilpotent group is either virtually cyclic or it maps homomorphically onto $\mathbb{Z}^2$.*

Proof. Let G be a finitely generated, nilpotent group that is not virtually cyclic. By Proposition 14 (see below), G has a homomorphic image H which is virtually free abelian of rank at least 2. Since G is nilpotent, so is H, which implies that H/T is free abelian group of rank at least 2, where T is the torsion subgroup of H. □

Proposition 14. *A finitely generated, virtually nilpotent group is either virtually cyclic or it has a homomorphic image which is virtually free abelian group of finite rank at least 2.*

Proof. Let G be a finitely generated virtually nilpotent group and N be a normal nilpotent subgroup of finite index and with smallest possible nilpotency class c among all such subgroups.

If $c \leq 1$, the group G is virtually abelian and is therefore either virtually cyclic or virtually free abelian of rank at least 2.

Assume that $c \geq 2$. Consider the first three terms of the lower central series $\gamma_1(N) = N$, $\gamma_2(N) = [N, N]$, and $\gamma_3(N) = [\gamma_2(N), N]$. Note that these subgroups, being characteristic in N, are normal in G.

If the abelianization $N/\gamma_2(N)$ has torsion free rank at least 2, then $G/\gamma_2(N)$ is virtually free abelian of rank at least 2 and we are done in this case.

Assume that the abelianization $N/\gamma_2(N)$ is virtually cyclic. We will show that this is impossible. Let H be a subgroup of finite index in N such that $H/\gamma_2(N)$ is cyclic. Let $a\ \gamma_2(N)$ be a generator of $H/\gamma_2(N)$ and $a^m x\ \gamma_3(N)$ and $a^n y\ \gamma_3(N)$ be two arbitrary elements of $H/\gamma_3(N)$, where m and n are integers and x and y are elements of $\gamma_2(N)$. Since the elements of $\gamma_2(N)$ commute with the elements of N (and H) modulo $\gamma_3(M)$ and a^m and a^n commute, we have $a^m x\ \gamma_3(N)\ a^n y\ \gamma_3(N) = a^n y\ \gamma_3(N)\ a^m x\ \gamma_3(N)$, showing that $H/\gamma_3(N)$ is abelian. However, since $\gamma_3(N)$ is nilpotent of class $c-2$ this implies that H is nilpotent of class at most $c-1$. Since H has finite index in G this contradicts the choice of N (H has a subgroup of finite index in G that is normal in G and has nilpotency class at most $c-1$). □

Example 15. It is not possible to extend Theorem 12 to virtually nilpotent groups, since there are virtually abelian groups that are not virtually cyclic but nevertheless have polynomial geodesic growth with respect to some generating sets. Note that such examples must have virtually cyclic abelianization.

Consider the group

$$G = \mathbb{Z}^2 \rtimes C_2 = \langle\ x, y, a \mid [x, y] = 1,\ a^2 = 1,\ x^a = y\ \rangle.$$

and the generating set $S = \{x^{\pm 1}, y^{\pm 1}, X^{\pm 1}, a^{\pm 1}\}$, where $X = x^2$. We will show that the geodesic growth of G with respect to S is polynomial.

This will be accomplished by showing that every geodesic with respect to S contains at most 20 letters that are different from $X^{\pm 1}$.

Let v and w be words over $S' = \{x^{\pm 1}, y^{\pm 1}, X^{\pm 1}\}$. Then $a^{\pm 1} v a^{\pm 1}$ commutes with w and we have $a^{\pm 1} v a^{\pm 1} w a^{\pm 1} = w a^{\pm 1} v a^{\pm 1} a^{\pm 1} = w a^{\pm 1} v$, showing that $a^{\pm 1} v a^{\pm 1} w a^{\pm 1}$ cannot be a geodesic. Thus, no geodesic with respect to S contains more than two letters $a^{\pm 1}$. In other words, all geodesics are in one of the forms $u a^{\pm 1} v a^{\pm 1} w$, or $u a^{\pm 1} v$, or u, for some words u, v, w over S' that represent geodesics in G with respect to S. Consider such a word, say u. It cannot contain both positive and negative powers of x, y or X. Further, since $x^2 = X$, the word u can contain at most one occurrence of $x^{\pm 1}$. Since $y^6 = aX^3a$, the word u can contain at most five occurrences of $y^{\pm 1}$.

Thus any geodesic can have at most 2 occurrences of the letter a, at most 3 occurrences of the letter $x^{\pm 1}$, and at most 15 occurrences of the letter $y^{\pm 1}$, which means that the total number of geodesics of length n is no greater than $k\binom{n}{20}$, for some constant k.

The same trick could be applied to abelian groups of any rank. For example,

$$\mathbb{Z}^3 \rtimes C_3 = \langle x, y, z, a \mid [x, y] = [y, z] = [z, x] = 1,\ a^3 = 1,\ x^a = y,\ y^a = z \rangle$$

If one is concerned about having a minimal generating set, one can use a, t to generate the group, see [3].

We have not been able to repeat the trick to create a virtually nilpotent group with virtually cyclic center, which is not virtually abelian. For example, one could try the Heisenberg group by C_2:

$$G = H \rtimes C_2 = \langle\ x, y, z, a \mid [x, y] = z,\ [x, z] = 1,\ [y, z] = 1\, a^2 = 1,\ x^a = y\ \rangle.$$

Based on our failure to produce such an example we make the following conjecture.

Conjecture 16. *If G is finitely generated and has polynomial geodesic growth with respect to some finite generating set, then G is virtually* abelian *with virtually cyclic abelianization.*

If it turns out to be false, we would replace "abelian" by "nilpotent".

6. Intermediate growth

Groups with intermediate growth have attracted considerable attention since their discovery by Grigorchuk in the 1980s. It seems natural to ask, in the context of this paper, if there exist groups which admit a generating set for which the geodesic growth is intermediate.

If the geodesic growth is intermediate, then the usual growth is either intermediate or polynomial, and in the second case this means the group is virtually nilpotent. If it is virtually cyclic then it is hyperbolic, and in that case, the language of geodesics is a regular language (see [5]). But this contradicts our assumption, because the growth of a regular language is either polynomial or exponential, by [7].

So potential candidates for groups of intermediate geodesic growth either have intermediate usual growth also, or are virtually nilpotent with virtually cyclic abelianization but not virtually cyclic. We believe this second possibility is unlikely.

One of the first examples one might try would be Grigorchuk's group of intermediate growth. With respect to the (standard) generating set of four involutions, the third and forth authors and Gutierrez [4] proved its geodesic growth rate lies between $O(1.41421^n)$ and $O(1.54983^n)$, in particular proving the geodesic growth is not intermediate.

We leave as an open question the following:

Question 17. *Is there a group of intermediate growth (or a virtually nilpotent group that is not virtually cyclic, with virtually cyclic abelianisation) that has intermediate geodesic growth?*

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