

HERMITE-HADAMARD TYPE INEQUALITIES VIA (α, m) -CONVEXITY

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ABSTRACT. In this paper, we establish some integral inequalities for functions whose second derivatives in absolute value are (α, m) -convex.

1. INTRODUCTION

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

In [2], V.G. Miheşan introduced the class of (α, m) -convex functions as the following:

The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y).$$

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α -starshaped, starshaped, m -convex, convex and α -convex.

Denote by $K_m^\alpha(b)$ the set of all (α, m) -convex functions on $[0, b]$ which $f(0) \leq 0$. For recent results and generalizations concerning m -convex and (α, m) -convex functions see [3], [4] and [7].

In [1], M. Emin Özdemir, Merve Avci and Erhan Set used the following lemma in order to establish some inequalities for m -convex functions.

Lemma 1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° , $a, b \in I$ with $a < b$ and $f'' \in L[a, b]$. Then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta + (1-t)b)dt.$$

In the same paper [1], Özdemir et al. discussed the following new results connecting with m -convex functions:

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Theorem 1. Let $f : I^\circ \rightarrow \mathbb{R}$, where $I^\circ \subset [0, \infty)$ be a twice differentiable function on I° , $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a, b]$. If $|f''|^q$ is m -convex on $[a, b]$ for some fixed $q > 1$ and $m \in (0, 1]$ then the following inequality holds:

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{(b-a)^2}{8} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(\frac{|f''(a)|^q + m |f''(\frac{b}{m})|^q}{2} \right)^{\frac{1}{q}}$$

where $p = \frac{q}{q-1}$.

Corollary 1. With the above assumptions given that $|f''(x)| \leq K$ on $[a, b]$, and $0 < m \leq 1$, we have the inequality

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \leq K \frac{(b-a)^2}{8} \left(\frac{1+m}{2} \right)^{\frac{1}{q}} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}}.$$

In [6], Sarikaya and Aktan obtained the following result concerning Hermite-Hadamard's inequality for functions whose second derivative in absolute value is convex as follows:

Theorem 2. Let $I \subseteq \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \rightarrow \mathbb{R}$ be twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|$ is a convex function on $[a, b]$, then the following inequalities hold:

$$\begin{aligned} & \left| (\lambda - 1) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a) + f(b)}{2} + \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \begin{cases} \frac{(b-a)^2}{12} \left[(\lambda^4 + (1+\lambda)(1-\lambda)^3 + \frac{5\lambda-3}{4}) |f''(a)| \right] & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ + (\lambda^4 + (2-\lambda)\lambda^3 + \frac{1-3\lambda}{4}) |f''(b)|, \\ \frac{(b-a)^2(3\lambda-1)}{48} [|f''(a)| + |f''(b)|], & \text{for } \frac{1}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

In Theorem 2, if we choose $\lambda = 1$ we have

$$(1.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{12} \left[\frac{|f''(a)| + |f''(b)|}{2} \right].$$

The aim of this paper is to establish some inequalities like those given in [1], but now for (α, m) -convex functions. That is, this study is a continuation of [1]. In order to obtain our results, we modified Lemma 1 given in the [1].

2. INEQUALITIES FOR (α, m) -CONVEX FUNCTIONS

Lemma 2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° where $a, b \in I$ with $a < b$ and $m \in (0, 1]$. If $f'' \in L[a, b]$, then the following equality holds:

$$\frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx = \frac{(mb-a)^2}{2} \int_0^1 (t-t^2) f''(ta + m(1-t)b) dt.$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 3. Let $f : I \subset [0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a, b]$ where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is (α, m) -convex on $[a, b]$ for $(\alpha, m) \in [0, 1]^2$, $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \\ & \quad \times \left[|f''(a)|^q \frac{1}{(\alpha+2)(\alpha+3)} + m |f''(b)|^q \left(\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right) \right]^{\frac{1}{q}}. \end{aligned}$$

Proof. Suppose that $q = 1$. From Lemma 2 and using the (α, m) -convexity of $|f''|$ we have

$$\begin{aligned} (2.1) \quad & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \int_0^1 (t-t^2) |f''(ta + m(1-t)b)| dt \\ & \leq \frac{(mb-a)^2}{2} \int_0^1 (t-t^2) [t^\alpha |f(a)| + m(1-t^\alpha) |f(b)|] dt \\ & = \frac{(mb-a)^2}{2} \left[|f''(a)| \frac{1}{(\alpha+2)(\alpha+3)} + m |f''(a)| \left(\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right) \right] \end{aligned}$$

which completes the proof for $q = 1$.

Suppose now that $q > 1$. From Lemma 2 and using the Hölder's integral inequality for $q > 1$, we have

$$\begin{aligned} & \int_0^1 (t-t^2) |f''(ta + m(1-t)b)| dt \\ & = \int_0^1 (t-t^2)^{1-\frac{1}{q}} (t-t^2)^{\frac{1}{q}} |f''(ta + m(1-t)b)| dt \\ (2.2) \quad & \leq \left[\int_0^1 (t-t^2) dt \right]^{1-\frac{1}{q}} \left[\int_0^1 (t-t^2) |f''(ta + m(1-t)b)|^q dt \right]^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since $|f''|^q$ is (α, m) -convex on $[a, b]$ we know that for every $t \in [0, 1]$

$$(2.3) \quad |f''(ta + m(1-t)b)|^q \leq t^\alpha |f''(a)|^q + m(1-t^\alpha) |f''(b)|^q.$$

From (2.1)-(2.3) we have

$$\begin{aligned}
& \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\
& \leq \frac{(mb-a)^2}{2} \left[\int_0^1 (t-t^2) dt \right]^{1-\frac{1}{q}} \left[\int_0^1 (t-t^2) |f''(ta+m(1-t)b)|^q dt \right]^{\frac{1}{q}} \\
& \leq \frac{(mb-a)^2}{2} \left[\int_0^1 (t-t^2) dt \right]^{1-\frac{1}{q}} \\
& \quad \times \left[\int_0^1 (t-t^2) [t^\alpha |f''(a)|^q + m(1-t^\alpha) |f''(b)|^q] dt \right]^{\frac{1}{q}} \\
& = \frac{(mb-a)^2}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{q}} \\
& \quad \times \left[|f''(a)|^q \frac{1}{(\alpha+2)(\alpha+3)} + m |f''(b)|^q \left(\frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)} \right) \right]^{\frac{1}{q}}
\end{aligned}$$

which is the required. □

Remark 1. *If in Theorem 3 we choose $m = \alpha = q = 1$, we obtain*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{24} [|f''(a)| + |f''(b)|]$$

which is the inequality in (1.1).

Theorem 4. *With the assumptions of Theorem 3 the following inequality holds:*

$$\begin{aligned}
& \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\
& \leq \frac{(mb-a)^2}{8} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^q \frac{1}{\alpha+1} + m |f''(b)|^q \left(\frac{\alpha}{\alpha+1} \right) \right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2 and using the Hölder's integral inequality, we obtain

$$\begin{aligned}
& \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\
& \leq \frac{(mb-a)^2}{2} \int_0^1 t(1-t) |f''(ta + m(1-t)b)| dt \\
& \leq \frac{(mb-a)^2}{2} \left(\int_0^1 (t-t^2)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta + m(1-t)b)|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{(mb-a)^2}{2} \left(\int_0^1 (t-t^2)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 [t^\alpha |f''(a)|^q + m(1-t^\alpha) |f''(b)|^q] dt \right)^{\frac{1}{q}} \\
& = \frac{(mb-a)^2}{2} \left(\frac{2^{-1-2p} \sqrt{\pi} \Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^q \frac{1}{\alpha+1} + m |f''(b)|^q \left(\frac{\alpha}{\alpha+1} \right) \right)^{\frac{1}{q}} \\
& = \frac{(mb-a)^2}{2} \frac{(\sqrt{\pi})^{\frac{1}{p}}}{2^{\frac{1}{p}} 2^2} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^q \frac{1}{\alpha+1} + m |f''(b)|^q \left(\frac{\alpha}{\alpha+1} \right) \right)^{\frac{1}{q}}
\end{aligned}$$

and since $\sqrt{\pi} < 2$, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\
& \leq \frac{(mb-a)^2}{8} \left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)} \right)^{\frac{1}{p}} \left(|f''(a)|^q \frac{1}{\alpha+1} + m |f''(b)|^q \left(\frac{\alpha}{\alpha+1} \right) \right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

We note that, the Beta and Gamma function (see [5]),

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0,$$

are used to evaluate the integral

$$\int_0^1 (t-t^2)^p dt = \int_0^1 t^p (1-t)^p dt = \beta(p+1, p+1)$$

where

$$\beta(x, x) = 2^{1-2x} \beta\left(\frac{1}{2}, x\right) \quad \text{and} \quad \beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \text{thus we can obtain that}$$

$$\beta(p+1, p+1) = 2^{1-2(p+1)} \beta\left(\frac{1}{2}, p+1\right) = 2^{1-2(p+1)} \frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)},$$

and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which completes the proof. \square

Remark 2. Suppose that all the assumptions of Theorem 4 are satisfied with $|f''| \leq K$. If we choose $m = \alpha = 1$, we have the inequality in Corollary 1.

Theorem 5. Let $f : I \subset [0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a, b]$ where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is (α, m) -convex on $[a, b]$

for $(\alpha, m) \in [0, 1]^2$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \\ & \quad \times \left(|f''(a)|^q \beta(\alpha+1, q+1) + m |f''(b)|^q \left(\frac{1}{q+1} - \beta(\alpha+1, q+1) \right) \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 2 and using the well known Hölder's integral inequality we have

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \int_0^1 t(1-t) |f''(ta + m(1-t)b)| dt \\ & \leq \frac{(mb-a)^2}{2} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 (1-t)^q |f''(ta + m(1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(mb-a)^2}{2} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 (1-t)^q [t^\alpha |f''(a)|^q + m(1-t^\alpha) |f''(b)|^q] dt \right)^{\frac{1}{q}} \\ & = \frac{(mb-a)^2}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \\ & \quad \times \left(|f''(a)|^q \beta(\alpha+1, q+1) + m |f''(b)|^q \left(\frac{1}{q+1} - \beta(\alpha+1, q+1) \right) \right)^{\frac{1}{q}}. \end{aligned}$$

Since $\frac{1}{2} < \left(\frac{1}{p+1} \right)^{\frac{1}{p}} < 1$, we obtain

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \left(|f''(a)|^q \beta(\alpha+1, q+1) + m |f''(b)|^q \left(\frac{1}{q+1} - \beta(\alpha+1, q+1) \right) \right)^{\frac{1}{q}}. \end{aligned}$$

□

Theorem 6. Let $f : I \subset [0, b^*] \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a, b]$ where $a, b \in I$ with $a < b$, $b^* > 0$. If $|f''|^q$ is (α, m) -convex on $[a, b]$

for $(\alpha, m) \in [0, 1]^2$, and for some fixed $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\left(|f''(a)|^q \beta(\alpha+2, q+1) + m |f''(b)|^q \left(\frac{1}{(q+1)(q+2)} - \beta(\alpha+2, q+1) \right) \right) \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 2 and using the well-known power-mean integral inequality we have

$$\begin{aligned} & \left| \frac{f(a) + f(mb)}{2} - \frac{1}{mb-a} \int_a^{mb} f(x) dx \right| \\ & \leq \frac{(mb-a)^2}{2} \int_0^1 t(1-t) |f''(ta + m(1-t)b)| dt \\ & \leq \frac{(mb-a)^2}{2} \left(\int_0^1 t dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t)^q |f''(ta + m(1-t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(mb-a)^2}{2} \left(\int_0^1 t dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t(1-t)^q [t^\alpha |f''(a)|^q + m(1-t)^\alpha |f''(b)|^q] dt \right)^{\frac{1}{q}} \\ & = \frac{(mb-a)^2}{2} \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\left(|f''(a)|^q \beta(\alpha+2, q+1) + m |f''(b)|^q \left(\frac{1}{(q+1)(q+2)} - \beta(\alpha+2, q+1) \right) \right) \right)^{\frac{1}{q}} \end{aligned}$$

which completes the proof. \square

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