Pareto and Boltzmann-Gibbs behaviors in a deterministic multi-agent system

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Abstract

A deterministic system of interacting agents is considered as a model for economic dynamics. The dynamics of the system is described by a coupled map lattice with near neighbor interactions. The evolution of each agent results from the competition between two factors: the agent's own tendency to grow and the environmental influence that moderates this growth. Depending on the values of the parameters that control these factors, the system can display Pareto or Boltzmann-Gibbs statistical behaviors in its asymptotic dynamical regime. The regions where these behaviors appear are calculated on the space of parameters of the system. Other statistical properties, such as the mean wealth, the standard deviation, and the Gini coefficient characterizing the degree of equity in the wealth distribution are also calculated on the space of parameters of the system.

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It is currently well established that income or wealth distribution in many western societies presents essentially two phases. This means that the society can be differentiated in two disjoint populations in which the probability distribution of wealth has a different functional form in each of them [1,2,3,4,5]. Analysis of real economic data from U.K. and U.S.A. [6] has shown that one phase possesses an exponential or Boltzmann-Gibbs probability distribution that involves about 95% of individuals, mainly those with low and medium wealths, and that the other phase, consisting of the the 5% of individuals

with highest wealths, shows a power law distribution or Pareto behavior. Several economic models based on diverse probabilistic mechanisms for interaction between agents have been proposed in order to reproduce these types of statistical behavior [7,8,9,10,11]. However, in most cases, both classes of distributions do not appear in a simple model; changes in the interaction rules between agents are required in order to obtain either type of behavior.

Randomness is an essential ingredient in all the former models. Thus, agents behave as a classical gas without notion of locality [5]. The interaction between agents occurs in pairs chosen at random, and these pairs exchange a random quantity of wealth in each transaction; this leads to an asymptotic state where the wealth in the system follows a Boltzmann-Gibbs distribution. The transition from a Boltzmann-Gibbs distribution to a Pareto behavior requires a change of structural properties of the system. A power law distribution can be reached, for instance, by introducing a strong inhomogeneity in the properties of the agents [4,10]. Thus, very different setups are needed in those models in order to simulate the collective behavior of real economic systems. On the other hand, interactions among real economic agents cannot be regarded as fully random. In fact, most economic transactions are driven by some kind of mutual interests or rational forces.

In this article, we study the statistical properties of a recently introduced deterministic, network-based multi-agent dynamical model possessing minimal ingredients [12]. In particular, we show that this simple model is capable of displaying Boltzmann-Gibbs as well as Pareto statistical behaviors in its asymptotic states.

The system consists of N agents placed at the nodes of a network. Each agent, representing an individual, a company, a country or other economic entity, is identified by an index i, with $i = 1, \ldots, N$. The dynamics of each agent is described by a discrete-time map that expresses the competition between its own tendency to grow and an environmental influence that controls this growth. Although the model can be defined on any network of interacting agents, for simplicity we shall consider here a one-dimensional lattice with periodic boundary conditions. The dynamics of the system is described by the coupled map equations [12]

$$
\begin{aligned}
x^i_{t+1} &= r_i\, x^i_t\, \exp(-\mid x^i_t - a_i \Psi^i_t \mid), \\
\Psi^i_t &= \tfrac{1}{2}(x^{i-1}_t + x^{i+1}_t),
\end{aligned}
\tag{1}
$$

where $x^i_t \geq 0$ gives the state of the agent i at discrete time t, and it may denote the *wealth* of this agent; the factor $r_i x^i_t$ expresses the *self-growth capacity* of agent i, characterized by a parameter r_i; Ψ^i_t represents the local field acting at the site i at time t; and a_i measures the coupling of agent i with its
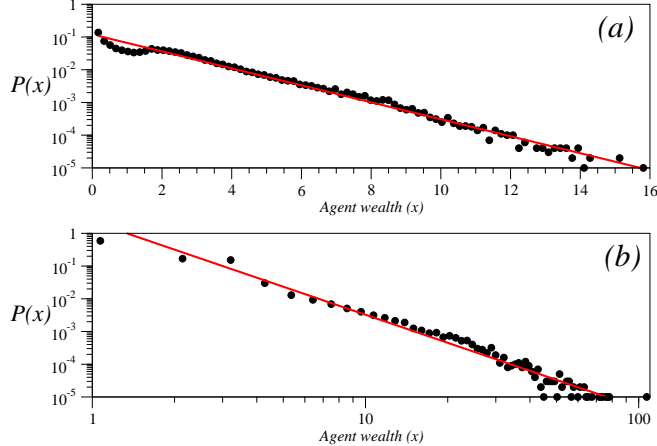
Fig. 1. (a) Semilog plot of the probability distribution $P(x)$ of the states of the agents x_t^i at time $t = 10^4$, for $a = 0.6$ and $r = 4$. (b) Log-log plot of $P(x)$ at time $t = 10^4$, for $a = 0.92$ and $r = 8$.

neighborhood; it can also be interpreted as the *local environmental pressure* exerted on agent i [13]. The negative exponential function acts as a *control factor* that limits this growth with respect to the local field. With the dynamics given by Eqs.(1) the largest possibility of growth for agent i is obtained when $x_t^i \simeq a_i \Psi_t^i$, i.e., when the agent has reached some kind of adaptation to its local environment.

For simplicity, in this paper we focus on a homogeneous system where all agents possess the same growth capacity, $r_i = r$, and are subject to a uniform selection pressure from their environment, $a_i = a$. Thus, the parameter a expresses the homogeneous wish of the agents to reach a wealth level proportional to that of their environment. The value $a = 1$ means a desire of being totally balanced with the neighborhood. The case $a < 1$ could be interpreted as some kind of lack of attitude in the population for improving its relative wealth. When $a > 1$ the agents possess an excess of will (selfishness) for overcoming their local neighbors.

We study the collective behavior of the system described by Eqs.(1) in the space of parameters (a, r). For all the simulations shown, the system size is $N = 10^5$ and the values of the initial conditions are uniformly distributed at random in the interval $x_0^i \in [1, 100]$. Also, a transient of 10^4 iterations is discarded before arriving to the asymptotic regime where all the calculations are carried out. When indicated, time averages are done over the next 100 iterations after the transient, and this result is newly averaged over 100 different realizations of the initial conditions with the same process.

Figure 1 shows the probability distribution of the states of the agents $P(x)$ at time $t = 10^4$ for different values of the parameters a and r. In Fig. 1(a), a semilog plot of $P(x)$ shows that, for the parameters used, the probability
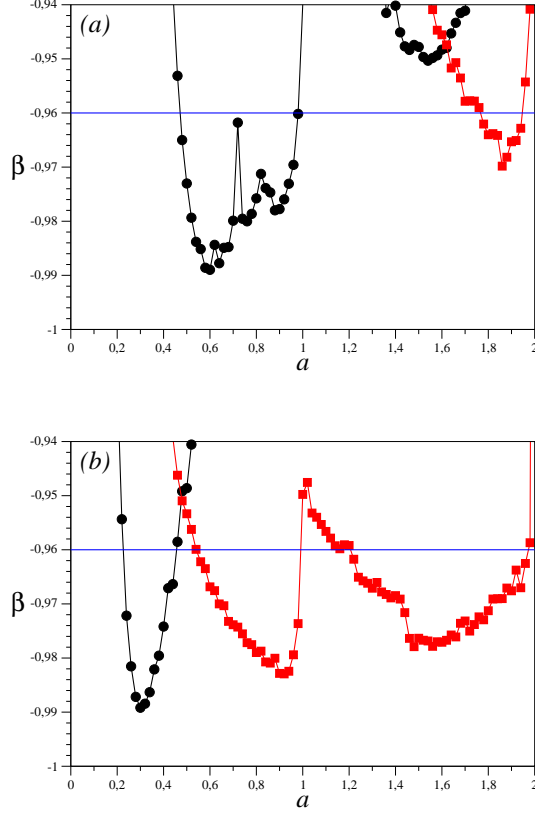
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Fig. 2. Correlation coefficient β of the scaling exponents for the semilog fitting of the Boltzmann-Gibbs (circles) and for the log-log fitting of the Pareto (squares) distributions, as a function of the parameter a for two different values of r. The value $\beta = -0.96$ is indicated by a horizontal line. The values of β shown are the result of averaging the values obtained over 100 iterations for each parameter values, after discarding 10^4 transients. (a) $r = 4$. (b) $r = 8$.

can be well described by a Boltzmann-Gibbs distribution $P(x) \sim e^{-\mu x}$, where $\mu = 0.59$. A thermodynamical simile can be established by defining a kind of 'temperature', $h = 1/\mu = 1.69$, that is related with the mean wealth of the agents in the ensemble. For other values of the parameters, $P(x)$ can display a Pareto-type behavior, as shown in the log-log plot of Fig. 1(b). In this case, $P(x) \sim x^{-\alpha}$, with an exponent $\alpha = 2.84$, a value in agreement with the exponents derived from real economic data [1,14,15].

The exponents μ and α in the distributions shown in Fig. 1 are obtained by linear regression using the least-squares method; this gives a value of the correlation coefficient greater than 0.98 in each case. Boltzmann-Gibbs and Pareto distributions also appear for other values of the parameters (a, r). Figure 2 shows the correlation coefficient β corresponding to the semilog fitting of the Boltzmann-Gibbs as well as the log-log fitting of the Pareto distributions as a function of the parameter a, for two different values of r. We consider that either of these fittings are accurate enough when $|\beta| > 0.96$. The intervals of the parameter a where this condition has been achieved can be identified in

Fig. 2.

Figure 3 shows the regions where the probability distribution $P(x)$ displays Boltzmann-Gibbs and Pareto behaviors in the space of parameters (a, r) for the system Eqs.(1). Both parameters a and r are varied in intervals of size 0.02 and for each pair (a, r) the semilog and the log-log linear regressions described in Fig. 1 are performed after discarding 10^4 transients and averaging over the following 100 iterations, and only those results yielding a correlation coefficient $|\beta| > 0.96$ are shown in Fig. 3.
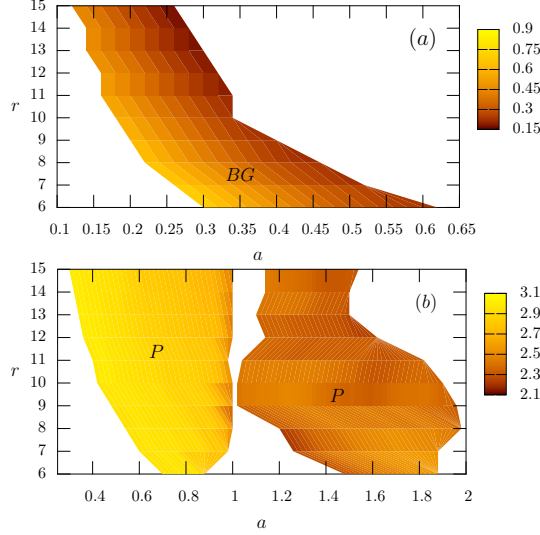

Fig. 3. (a) Regions where Boltzmann-Gibbs behavior $P(x) \sim e^{-\mu x}$ appears on the space of parameters (a, r) are indicated by the label BG. The color code on the right indicates the values of the scaling exponent μ obtained by the procedure explained in Fig. 1(a) and fullfilling the condition $|\beta| > 0.96$. (b) Regions where Pareto behavior $P(x) \sim x^{-\alpha}$ occurs on the plane (a, r) are labeled by P. The color code on the right indicates the values of the exponent α obtained as in Fig. 1(b) and satisfying $|\beta| > 0.96$. In both cases, for each pair of values (a, r) calculations of the scaling exponents are performed after discarding 10^4 transients and averaging over the following 100 iterations.

We note that Boltzmann-Gibbs behavior is found for lower values of the local environmental pressure a. When the value of a increases, the population enters in a competitive regime that provokes the appearance of the Pareto behavior in the system. The scaling exponents obtained for the Pareto behavior observed in Fig. 3(b) are in the range $\alpha \in [2.3, 3.0]$; these values are similar to those found in actual economic data [1,14,15].

The mean field of the system or average wealth per agent at a time t is defined as
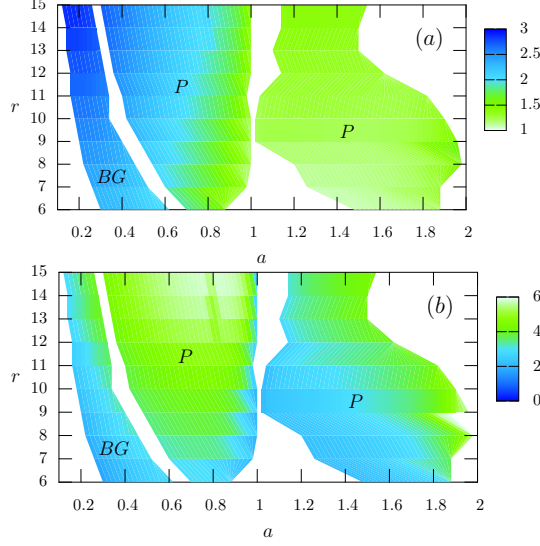
$$H_t = \frac{1}{N} \sum_i^N x_t^i, \tag{2}$$

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Fig. 4. (a) Mean field H_t of the system at $t = 10^4$ for the regions where Boltzmann-Gibbs and Pareto behaviors are found on the plane (a, r), indicated respectively by the labels BG and P. The color code on the right indicates the values taken by H_t. (b) The quantity $\langle \sigma \rangle$ on the space of parameters (a, r). The color code on the right indicates the values of $\langle \sigma \rangle$.

Figure 4(a) shows the asymptotic value of the mean field for the regions where Boltzmann-Gibbs and Pareto behaviors are observed on the space of parameters (a, r). Note that, although the values of the initial states of the agents are randomly distributed on the interval $[1, 100]$, the system evolves to an asymptotic state where H_t takes values on the smaller interval $[0, 3]$. On the other hand, for some values of the parameters a and r, the states of agents in the system at a given time exhibit a large dispersion. Similarly, for those parameters, the values of the state of any agent present large fluctuations over long times. To characterize these fluctuations, we define the instantaneous standard deviation of the mean field as

$$\sigma_t = \left(\frac{1}{N} \sum_{i=1}^{N} \left[x_t^i - H_t \right]^2 \right)^{1/2}. \tag{3}$$

After discarding 10^4 transients, we calculate the mean value of σ_t over 100 iterations, and then average this result over 100 realizations of initial conditions. The resulting average dispersion, denoted by $\langle \sigma \rangle$, is shown in Fig. 4(b) on the plane (a, r) for the same regions indicated in Fig 4(a). Note that for some regions of parameters the quantity $\langle \sigma_t \rangle$ can be much greater than the value of the mean field. For instance, about the line $a = 1.9$, the mean field is small, $H_t \approx 1$, but $\langle \sigma_t \rangle \approx 6$, showing that the fluctuations can be very large in this system. Thus, in spite of its simplicity, the deterministic model given by Eqs. (1) can exhibit great spatiotemporal complexity.
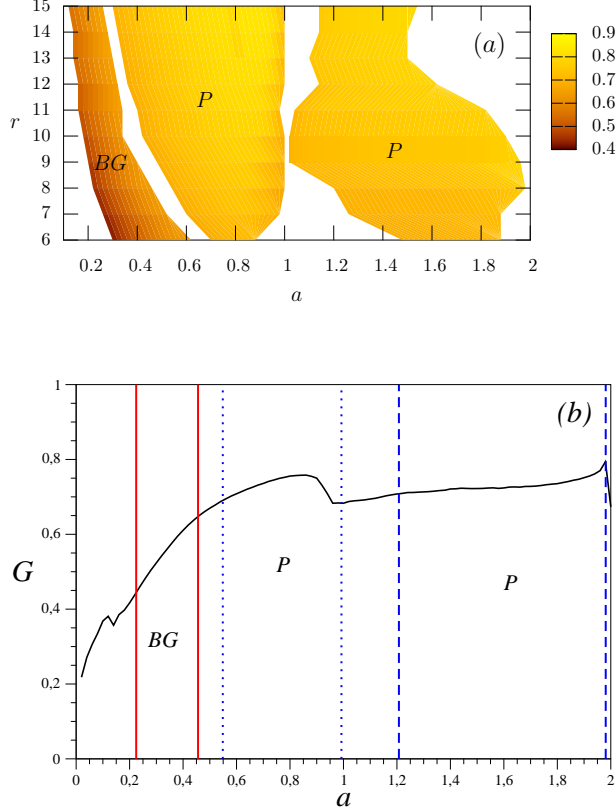
Fig. 5. (a) Gini coefficient at $t = 10^4$ as a function of the parameters a and r. The labels BG and P indicate the regions where Boltzmann-Gibbs and Pareto behaviors are observed on the plane (a, r). The color code on the right indicates the values taken by the Gini coefficient. (b) Gini coefficient vs. a, with fixed $r = 8$, where the labels BG and P correspond to the regions indicated in (a).

The large dispersions observed in Fig. 4(b) reflect the inequity in the wealth distribution among agents in the system. To characterize the degree of inequality in the wealth distribution we use the Gini coefficient defined at a time t as [16]

$$G_t = \frac{1}{2N^2 H_t} \sum_{i,j=1}^{N} |x_t^i - x_t^j|. \tag{4}$$

A perfectly equitable distribution of wealth at time t, where $x_t^i = x_t^j, \forall i, j$, yields a value $G_t = 0$. The opposite situation, where one agent has the total wealth $\sum_{i=1}^{N} x_t^i$, has a value of $G_t = 1$. Figure 5(a) shows the asymptotic value of the Gini coefficient on the plane of parameters (a, r). Note that the Gini coefficient reaches larger values, i.e. $G_t \in [0.6, 0.8]$, in the regions associated to Pareto regimes, while it takes lower values, i.e. $G_t \in [0.4, 0.6]$, in the region corresponding to Boltzmann-Gibbs behavior. This results agree with our qualitative understanding that equity is more favored in the presence of a larger middle economic class in a society, as expressed by a Boltzmann-Gibbs

distribution. A plot of G_t as a function of a for a fixed value $r = 8$ is shown in Fig. 5(b), where the Boltzmann-Gibbs and Pareto regions are also indicated.

In summary, the deterministic model Eqs. (1) shows statistical behaviors described by Boltzmann-Gibbs and Pareto distributions in different regions of its parameters. The appearance of these collective properties does not require the addition of ramdonness or any structural change in the system. Only some appropriate tuning of the parameters of the system is needed to obtain either type of behavior. This property contrasts with most models for economic behavior in the literature which require changes in their dynamical rules in order to yield an exponential or a power law distribution of states. Since it is currently accepted that most western societies consist of two differentiated economic classes characterized by different distribution functions [4], coupled map models such as Eqs. (1) can be useful to study the formation of these two economic populations. Our results support the view that determinism alone can give rise to some relevant collective behaviors observed in economic systems. This basic model can be readily extended to include the considerations of more complex networks of interactions, heterogeneities, and the coevolution of dynamics and the topology of connectivity, among other interesting issues.

Acknowledgments

References

[1] A. Dragulescu and V. M. Yakovenko, Physica A 299 (2001) 213.

[2] A. Chatterjee, S. Yarlagadda, B. K. Chakrabarti (Eds.), Econophysics of Wealth Distributions, Springer Verlag, Milan (2005).

[3] B. K. Chakrabarti, A. Chakraborti, A. Chatterjee (Eds.), Econophysics and Sociophysics, Wiley-VCH, Berlin (2006).

[4] V. M. Yakovenko, preprint arXiv:0709.3662 (2007).

[5]   A. Chatterjee and B. K. Chakrabarti, preprint arXiv:0709.1543v1 (2007).

[6]   A. Dragulescu and V. M. Yakovenko, Eur. Phys. J. B 20 (2001) 585.

[7]   A. Dragulescu and V. M. Yakovenko, Eur. Phys. J. B 17 (2000) 723 .

[8]   R. López-Ruiz, J. Sañudo, and X. Calbet, arXiv:0707.4081 (2007).

[9]   A. Chakraborti and B.K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000)

[10] A. Chatterjee, B. K. Chakrabarti, and S. S. Manna, Physica A 335 (2004) 155.

[11] J. Angle, Physica A 367 (2006) 388 .

[12] J. R. Sánchez, J. González-Estévez, R. López-Ruiz, and M. G. Cosenza, Eur. Phys. J. ST 143 (2007) 241.

[13] M. Ausloos, P. Clippe, and A. Pekalski, Physica A 324 (2003) 330.

[14] M. Levy and S. Solomon, Physica A 242 (1997) 90.

[15] W. Souma, Fractals 9 (2001) 463 .

[16] M. Rodríguez-Achach, and R. Huerta-Quintanilla, Physica A 361 (2006) 309.