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AN EXAMPLE OF NON-HOMEOMORPHIC CONJUGATE VARIETIES

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ABSTRACT. We give examples of smooth quasi-projective varieties over complex numbers, in the context of connected Shimura varieties, which are not homeomorphic to a conjugate of itself by an automorphism of the complex numbers.

1. Introduction

Let X be a quasi-projective variety defined over $\mathbb{C}$. Suppose σ is an automorphism of $\mathbb{C}$. Denote by $X^\sigma := X \times_\sigma \mathbb{C}$, the conjugate of X by the automorphism σ of $\mathbb{C}$, obtained by applying the automorphism σ to the coefficients of the polynomials defining X. It is known that the varieties X and X^σ have the same Betti numbers. In [Se], Serre gave an example such that the topological spaces $X(\mathbb{C})$ and $X^\sigma(\mathbb{C})$ are not homeomorphic.

Recently, Milne and Suh [MS], gave further examples in the context of connected Shimura varieties. Their method is to find a conjugate such that the reductive group underlying the Shimura datum is different, and then apply the super-rigidity results of Margulis.

Our examples are in the same context as that of Milne and Suh, but we work with Shimura's construction of canonical models ([Sh]). Shimura's construction allows us to identify the adelic congruence subgroup defining the conjugate variety as a conjugate by an element of the adjoint group. We then appeal to Mostow rigidity and the failure of strong approximation (or non-triviality of class number) for the adjoint group to get at the desired examples. In our example, the congruent lattices defining the variety and it's conjugate are commensurable. Earlier in [R], we observed using Shimura's construction coupled with the theorems of Labesse and Langlands on the mulitplicity of cusp forms for $SL(1, D)$, that a Galois twist of these spaces attached to $SL(1, D)$ over the reflex field preserves the spectrum of the Laplacian; this provides examples of locally symmetric spaces attached to a quaternion division algebra over a number field which are isospectral but not isometric,

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2. The example

Let F be a totally real number field of degree at least two. Let D be an indefinite quaternion algebra defined over F. We assume that D is split at exactly one real place, say τ_1 of F. This assumption allows us to assume that the reflex field of (F, τ_1) to be F itself. Let V be a vector space of rank $n \geq 2$ over D, equipped with a hermitian inner product with respect to the standard involution on D. We assume that the inner product is definite on the spaces $V \otimes_\tau \mathbb{R}$, for all real embeddings τ of F different from τ_1. In particular, since we have assumed that the degree of F is at least two, the form h is anisotropic. Let G be the group of unitary similitudes of h. We consider G as an algebraic group defined over $\mathbb{Q}$, and let G_d the derived group of G. We let PG denote the projective group attached to G, the group obtained by taking the quotient of G modulo it's centre. Under our assumptions, it follows that

$$G_d(\mathbb{R}) \simeq Sp(2n, \mathbb{R}) \times \text{ a compact group}, \quad n \geq 2.$$

Let K_∞ be a maximal compact subgroup of $G_d(\mathbb{R})$ and let $X = G_d(\mathbb{R})/K_\infty$ be the non-compact symmetric space associated to G. By our assumptions, X is isomorphic to the Siegel upper half space $\mathbb{H}_n$ of dimension n. Denote by $\mathbf{A}$ the adele ring of F, and by $\mathbf{A}_f$ the subring of finite adeles. Let K be a compact open subgroup of $G(\mathbf{A}_f)$, and let $K_d = K \cap G_d(\mathbf{A}_f)$. Denote by

$$\Gamma_K = G(\mathbb{R})K \cap G(\mathbb{Q}) \quad \text{and} \quad \Gamma_{d,K} = G_d(\mathbb{R})K_d \cap G_d(\mathbb{Q}),$$

the corresponding arithmetic lattices in $G(\mathbb{R})$ and $G_d(\mathbb{R})$ respectively. We assume that K is such that $\Gamma_{d,K}$ is torsion-free, and the natural inclusion $\Gamma_{d,K} \subset \Gamma_K$ is an isomorphism modulo the centre of Γ_K.

By a theorem of Baily-Borel, the quotient space $X_K = \Gamma_K \backslash X$ is a connected, smooth, projective variety. The fundamental group $\overline{\Gamma}_K$ of the variety X_K can be identified with the projection of Γ_K to $PG(\mathbb{R})$, and also with the lattice $\Gamma_{d,K}$ contained in $G_d(\mathbb{R})$.

For an element $x \in G(\mathbf{A}_f)$, denote by K^x the conjugate lattice $x^{-1}Kx$, and by $\overline{x}$ its image in $PG(\mathbf{A}_f)$. Further, let $N(\overline{K})$ denote the normalizer of $\overline{K}$ in $PG(\mathbf{A}_f)$, where $\overline{K}$ is the image of K_d in $PG(\mathbf{A}_f)$. The desired example is provided by the following theorem:

Theorem 1. *With notation and assumptions as above, suppose x is an element in $G(\mathbf{A}_f)$ such that $\overline{x}$ does not belong to the set $N(\overline{K})PG(\mathbb{Q})$. Then X_K and X_{K^x} are conjugate by an automorphism σ of $\mathbb{C}$, but the respective fundamental groups $\overline{\Gamma}_K$ and $\overline{\Gamma}_{K^x}$ are not isomorphic. In particular, X_K and X_{K^x} are not homeomorphic.*

Remark. The hypothesis can be seen to hold from two different but related aspects of the arithmetic of algebraic groups. The normalizer $N(\overline{K})$ is a compact open subgroup of $PG(\mathbf{A}_f)$. By the failure of strong approximation for the adjoint group PG (see [PR, Proposition 7.13]), the rational points $PG(\mathbb{Q})$ are not dense in $PG(\mathbf{A}_f)$. Hence, the hypothesis that $\overline{x}$ does not belong to the double coset $N(\overline{K})PG(\mathbb{Q})$ is satisfied provided $N(\overline{K})$ is small enough.

On the other hand, the adjoint group PG is not simply connected, hence has a non-trivial fundamental group. The results of Section 8.2 of [PR], show that the class group of G is non-trivial for suitably chosen congruence lattices in $PG(\mathbf{A}_f)$. This allows us to work with large congruence lattices in $PG(\mathbf{A}_f)$.

Proof. We first show that the varieties X_K and X_{K^x} are conjugate by an automorphism of $\mathbb{C}$. For this, we recall Shimura's theory of canonical models [Sh]. Let $\nu : G \to \mathbf{G}_m$ be the reduced norm. By class field theory, the subgroup $F^*\nu(K)$ of the idele group $\mathbf{A}^*$ defines an abelian extension F_K of F. The reciprocity morphism of class field,

$$\mathrm{rec} : \mathbf{A}^*/F^* \to \mathrm{Gal}(F^{ab}/F),$$

defines an element $\sigma(x) \in \mathrm{Gal}(F^{ab}/F)$ by the prescription

$$\sigma(x) = \mathrm{rec}(\nu(x)^{-1}).$$

As a consequence of the main theorem of canonical models in [Sh, Theorem 2.5, page 159, Section 2.6], the variety X_K has a model defined over the field F_K, and

$$X_K^{\sigma(x)} \simeq X_{K^x}. \tag{2.1}$$

Thus the varieties X_K and X_{K^x} are conjugate.

Suppose on the contrary, that X_K and X_{K^x} have isomorphic fundamental groups. Since these spaces are Eilenberg-Maclane spaces, there exists a homotopy equivalence

$$\phi : X_K \to X_{K^x}.$$

Since the lattices are irreducible in $PG(\mathbb{R})$ and the real rank of PG is at least two, by Mostow rigidity [Mo], the spaces X_K and X_{K^x} are isometric.

Hence, there exists $\overline{g} \in PG(\mathbb{R})$ such that

$$\overline{g}^{-1}\overline{\Gamma}_{K^x}\overline{g} = \overline{\Gamma}_K.$$

Since the lattices $\overline{\Gamma}_K$ and $\overline{\Gamma}_{K^x}$ are arithmetic and commensurable, it follows by a theorem of Borel ([Bo]), that $\overline{g} \in PG(\mathbb{Q})$. Hence there is an element $g \in G(\mathbb{Q})$ satisfying,

$$g^{-1}\Gamma_{d,K^x}g = \Gamma_{d,K}.$$

Consider now $G_d(\mathbb{Q})$ embedded diagonally in $G_d(\mathbf{A}_f)$. By the strong approximation theorem for G_d, the closure of $\Gamma_{d,K}$ in $G_d(\mathbf{A}_f)$ can be identified with K_d. Further, the closure of Γ_{d,K^x} in $G_d(\mathbf{A}_f)$ can be identified with $g^{-1}K_d^x g$, where we now consider $g \in G(\mathbb{Q})$ as diagonally embedded in $G(\mathbf{A}_f)$. Hence, we have

$$g^{-1}K_d^x g = K_d.$$

Projecting to PG, we obtain

$$\overline{g}^{-1}\overline{x}^{-1}\overline{K}\overline{x}\overline{g} = \overline{K},$$

where $\overline{K}$ denotes the image of K_d in $PG(\mathbf{A}_f)$. This implies that $\overline{x} \in N(\overline{K})PG(\mathbb{Q})$, contradicting our choice of $\overline{x}$. □

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