TERNARY SUMS OF SQUARES AND TRIANGULAR NUMBERS

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Abstract. For any integer x, let T_x denote the triangular number $\frac{x(x+1)}{2}$. In this paper we give a complete characterization of all the triples of positive integers (α,β,γ) for which the ternary sums $\alpha x^2+\beta T_y+\gamma T_z$ represent all but finitely many positive integers. This resolves a conjecture of Kane and Sun [7, Conjecture 1.19(i)] and complete the characterization of all almost universal ternary mixed sums of squares and triangular numbers.

1. Introduction

In the Focused Week on Integral Lattices hosted by the Department of Mathematics in University of Florida in February 2010, the first author presented the result in [1] which is a complete characterization of all triples (α,β,γ) of positive integers for which the polynomials $\alpha T_x+\beta T_y+\gamma T_z$ are almost universal, that is, representing all but finitely many positive integers. Here T_x denotes the triangular number $x(x+1)/2$. This resolves a conjecture made by Kane and Sun in [7, Conjecture 1.19(ii)]. In [7] they also study other types of almost universal mixed sums of squares and triangular numbers. In particular, they determine all almost universal ternary sums $\alpha x^2+\beta y^2+\gamma T_z$ [7, Theorem 1.6]. They also formulate a conjecture [7, Conjecture 1.19(i)] about almost universal ternary sums of the form $\alpha x^2+\beta T_y+\gamma T_z$, and an affirmative answer to this conjecture would complete their classification of those almost universal sums. The goal of this paper is to give such a complete characterization via the geometric approach used in [1]. As consequences we resolve Kane and Sun's conjecture and complete the task of characterizing all almost universal ternary mixed sums of squares and triangular numbers.

The basic difference and similarity between our geometric approach and the theta series approach in [7] is briefly explained in [1]. We would like to add a few more comments for the ternary sums $\alpha x^2+\beta T_y+\gamma T_z$ we are considering here. Let $r(n)$ be the number of representations of an integer n by $\alpha x^2+\beta T_y+\gamma T_z$. By the inclusion and exclusion principle $r(n)$ is the $(\beta+\gamma+8n)$-th coefficient of the linear combination of theta series

$$\theta_{f(x,y,z)}-\theta_{f(x,2y,z)}-\theta_{f(x,y,2z)}+\theta_{f(x,2y,2z)},$$

where $\theta_{f(x,y,z)}$ is the theta series of the diagonal quadratic form $f(x,y,z)=8\alpha x^2+\beta y^2+\gamma z^2$. Using this linear combination of theta series, the authors in [7] determine triples (α,β,γ) for which almost all the $r(n)$ are nonzero. On the other hand, our geometric approach is built upon only one ternary $\mathbb{Z}$-lattice (not necessarily diagonal) on the quadratic $\mathbb{Q}$-space associated to the quadratic form $2\alpha x^2+\beta y^2+\gamma z^2$,

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whose representations of integers of the form $\beta+\gamma+8n$ will correspond to the representation of n by $\alpha x^2+\beta T_y+\gamma T_z$. This changes the original problem to the question of determining which ternary quadratic forms represent all sufficiently large integers in an finitely family of positive integers that has some arithmetic interest (in our case, they are the integers congruent to $\beta+\gamma \bmod 8$). Two powerful machinery from the theory of representations of ternary quadratic forms have been proven to be useful in dealing with this kind of questions. The first one is the theorem of Duke and Schulze-Pillot [2] which asserts that a sufficiently large integer is represented by a positive definite ternary quadratic form if that integer is primitively represented by the spinor genus of the quadratic form. The second one is the established theory of primitive spinor exceptions which allows us to determine effectively the set of integers that are primitively represented by a spinor genus. The readers can find all relevant material in [5] and [9].

Our main results will be divided into four theorems which altogether will characterize all triples (α,β,γ) for which $\alpha x^2+\beta T_y+\gamma T_z$ are almost universal. Kane and Sun's conjecture only concerns with the cases in which $\max\{\mathrm{ord}_2(\beta),\mathrm{ord}_2(\gamma)\}$ is either 3 or 4. However, we opt to present the proofs of all cases since it does not take too much extra effort to do so. This also provides a better understanding of the geometric setting we described above.

2. Preliminaries

Henceforth, the language of quadratic spaces and lattices as in [8] will be adopted. We will follow mostly the notations used in [1]. Any unexplained notation and terminology can be found there and in [8]. All the $\mathbb{Z}$-lattices discussed below are positive definite. If K is a $\mathbb{Z}$-lattice and A is a symmetric matrix, we shall write "$K\cong A$" if A is the Gram matrix for K with respect to some basis of K. The discriminant of K, denoted dK, is the determinant of A. An $n\times n$ diagonal matrix with $a_1,\ldots,a_n$ as the diagonal entries is written as $\langle a_1,\ldots,a_n\rangle$. However, we use the notation $[a_1,\ldots,a_n]$ to denote a quadratic space (over any field) with an orthogonal basis whose associated Gram matrix is the diagonal matrix $\langle a_1,\ldots,a_n\rangle$.

The subsequent discussion involves the computation of the spinor norm group of local integral rotations and the relative spinor norm groups of primitive representations of integers by ternary quadratic forms. The formulae for all these computations can be found in [3], [4], [5], and [6]. A correction of some of these formulae can be found in [1, Footnote 1]. The symbol θ always denotes the spinor norm map. If t is an integer represented primitively by $\mathrm{gen}(K)$ and p is a prime, then $\theta^*(K_p,t)$ is the primitive relative spinor norm group of the $\mathbb{Z}_p$-lattice K_p. If E is a quadratic extension of $\mathbb{Q}$, $N_p(E)$ denotes the group of local norms from $E_{\mathfrak{p}}$ to $\mathbb{Q}_p$, where $\mathfrak{p}$ is an extension of p to E.

Let a,b,c be relatively prime positive odd integers, m,r, and s be nonnegative integers such that $r\le s$. Let L be the $\mathbb{Z}$-lattice $\langle 2^{m+1}a,2^rb,2^sc\rangle$ in the orthogonal basis $\{e_1,e_2,e_3\}$. It can easily be shown that an integer n is represented by the ternary sum $2^max^2+2^rbT_y+2^scT_z$ if and only if 2^rb+2^sc+8n is represented by the coset $\omega+2L$ where $\omega=e_2+e_3$.

Lemma 2.1. *If $2^max^2+2^rbT_y+2^scT_z$ is almost universal, then*

(1) *L_p represents all p-adic integers for every odd prime p;*
(2) *$L_p\cong\langle 1,-1,-dL\rangle$ and $\theta(O^+(L_p))\supseteq\mathbb{Z}_p^\times$ for all odd primes p;*
(3) *$\mathbb{Q}_2L$ is anisotropic.*

Proof. See [1, Lemma 2.1]. Note that parts (2) and (3) are consequences of part (1). □

As is explained in [1], Lemma 2.1(1) is equivalent to the following more elementary statement:

> (i) a, b, c are pairwise relatively prime, and (ii) if an odd prime p divides one of $2^m a, 2^r b$ or $2^s c$, then the negative of the product of the other two is a square modulo p.

Let M be the $\mathbb{Z}$-lattice $\mathbb{Z}\omega + 2L$. Relative to the basis $\{2e_1, 2e_2, \omega\}$, M has the following Gram matrix representation:

$$\begin{pmatrix} 2^{m+3}a & 0 & 0 \\ 0 & 2^{r+2}b & 2^{r+1}b \\ 0 & 2^{r+1}b & 2^r b + 2^s c \end{pmatrix}$$

The discriminant of M is $2^{m+r+s+5}abc$. Also of interest is the binary sublattice $P = \mathbb{Z}2e_2 + \mathbb{Z}\omega$ whose Gram matrix is

$$\begin{pmatrix} 2^{r+2}b & 2^{r+1}b \\ 2^{r+1}b & 2^r b + 2^s c \end{pmatrix}$$

and $dP = 2^{r+s+2}bc$.

Lemma 2.2. *Suppose that $2^m ax^2 + 2^r bT_y + 2^s cT_z$ is almost universal. Then*

(1) *$2^r b + 2^s c$ is not divisible by 8 and consequently $r < 2$;*
(2) *r and s cannot both equal 1;*
(3) *$r = 0$ when $m > 0$.*

Proof. (1) Suppose that $2^m ax^2 + 2^r bT_y + 2^s cT_z$ is almost universal. This means that all but finitely many positive integers of the form $2^r b + 2^s c + 8n$ are represented by the coset $\omega + 2L$ and hence by the lattice M. If $2^r b + 2^s c \equiv 0 \bmod 8$, then M_2 represents all 2-adic integers in $8\mathbb{Z}_2$ because $\mathbb{Z}$ is dense in $\mathbb{Z}_2$. But then M_2 would be isotropic and this contradicts Lemma 2.1(3).

(2) Suppose that $r = s = 1$. We know from part (1) that $2b + 2c$ must be congruent to 4 mod 8. Then $u := (b + c)/2$ is a 2-adic unit and

$$M_2^{\frac{1}{4}} \cong \langle 2^{m+1}a, u, 8ubc\rangle.$$

Thus the binary sublattice $\langle 2^{m+1}a, u\rangle$ represents all 2-adic units that are congruent to u mod 2, that is, all 2-adic units. But this is clearly impossible.

(3) This is clear; otherwise $2^m ax^2 + 2^r T_y + 2^s T_z$ only represents even integers. □

Lemma 2.3. *Suppose that $2^m ax^2 + 2^r bT_y + 2^s cT_z$ is almost universal. Then*

(1) *every positive integer of the form $2^r b + 2^s c + 8n$ is represented primitively by $gen(M)$;*
(2) *if t is a primitive spinor exception of M, then $\mathbb{Q}(\sqrt{-tdM})$ is either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$.*

Proof. (1) By Lemma 2.1(2), it suffices to check that M_2 primitively represents all 2-adic integers of the form $2^r b + 2^s c + 8n$. This is clear when $2^r b + 2^s c$ is odd. Suppose that it is even. We first consider the case $r = s = 0$. By Lemma 2.2(1), $b + c \not\equiv 0 \bmod 8$. It can be seen that $P_2^{\frac{1}{2}}$ is a unimodular $\mathbb{Z}_2$-lattice, which is proper when $b + c \equiv 2 \bmod 4$ and improper when $b + c \equiv 4 \bmod 8$. In either case, it is direct to check that P_2 primitively represents all 2-adic integers of the form $b + c + 8n$.

Suppose that $r = 1$. It follows from Lemma 2.2(2) that $s > 1$. In this case,

$$M_2^{\frac{1}{2}} \cong \langle 4a\rangle \perp \langle b + 2^{s-1}c\rangle \perp \langle 2^{s+2}(2b + 2^s c)\rangle.$$

The binary sublattice $\langle 4a, b + 2^{s-1}c\rangle$ represents all units that are congruent to $b + 2^{s-1}c$ mod 4. Hence M_2 primitively represents all integers of the form $2b+2^sc+8n$.

(2) This is the same as the proof of [1, Lemma 2.2(2)]. □

Lemma 2.4. *Suppose that L_p represents all p-adic integers for every odd prime p. If we are not in the exceptional case where $r = s = 0$ and $b + c \equiv 4 \mod 8$, and if $2^r b + 2^s c + 8n$ is not a primitive spinor exception of gen(M) for all $n \geq 0$, then $2^m ax^2 + 2^r bT_y + 2^s cT_z$ is almost universal for every $m \geq 0$.*

Proof. The proof is parallel to that of [1, Lemma 2.3]. We leave the detail to the readers. Note that since we are not in the exceptional case ($r = s = 0$ and $b+c \equiv 4$ mod 8), any representation of $2^r b + 2^s c + 8n$ by M must lie in $\omega + 2L$. □

3. Main Results

We continue to assume that a, b, c are relatively prime positive odd integers. By Lemma 2.2, we only need to address the following four cases:

(i) $m = 0$, $r = 0$, and $s \geq 1$.
(ii) $m = 0$, $r = 1$, and $s > 1$.
(iii) $m > 0$, $r = 0$, and $s > 0$.
(iv) $m \geq 0$, $r = s = 0$.

They will be covered by Theorems 3.1 to 3.4 accordingly. We will provide full detail in the proof of the first theorem. For the proofs of the other three theorems, since the strategy and techniques involved are the same or very similar to the first one, we will just mention the necessary changes that may be less transparent to the readers. In below, the squarefree part of an integer α is denoted by sf(α).

Theorem 3.1. *Suppose that $s \geq 1$. Then $ax^2 + bT_y + 2^s cT_z$ is almost universal if and only if $2ax^2 + by^2 + 2^s cz^2$ represents all p-adic integers over $\mathbb{Z}_p$ for every odd p, and one of the following holds:*

(1) *s is odd, or $s = 2$;*
(2) *sf(abc) is divisible by a prime $p \equiv 5, 7 \mod 8$;*
(3) *$bc \not\equiv 1 \mod 8$;*
(4) *$\frac{sf(abc)-(b+2^s c)}{8}$ is represented by $ax^2 + bT_y + 2^s cT_z$.*

Proof. We assume throughout that $2ax^2 + by^2 + 2^s cz^2$ represents all p-adic integers over $\mathbb{Z}_p$ for every odd p. Note that

$$M_2 \cong \langle b + 2^s c, 8a, (b + 2^s c)2^{s+2}bc\rangle.$$

In below, t is always assumed to be an odd integer primitively represented by gen(M) and E is the quadratic field $\mathbb{Q}(\sqrt{-tdM})$. The strategy of the proof is to show that under (1), (2), or (3), gen(M) does not have any primitive spinor exception of the form $b + 2^s c + 8n$ and hence Lemma 2.4 applies. At the end we show that if (1), (2) and (3) all fail, then $ax^2 + bT_y + 2^s cT_z$ is almost universal if and only if (4) holds.

Suppose that s is odd. Since $\mathrm{ord}_2(-tdM_2) = s + 5$ which is even, E must be $\mathbb{Q}(\sqrt{-1})$. But by [5, Theorem 2(b)(iv)] $\theta^*(M_2, t) \neq N_2(E)$. Therefore, M does not have any odd primitive spinor exceptions. If $s = 2$, M_2 is of *Type E* as defined in

[4, page 531] and so $\theta(O^+(M_2)) = \mathbb{Q}_2^\times$. Combining with Lemma 2.1(2) this shows that gen(M) has only one spinor genus, and hence in this case gen(M) does not have any primitive spinor exception at all.

Now let us assume that s is even and $s \geq 4$. Then E must be $\mathbb{Q}(\sqrt{-2})$. At the primes p where $\left(\frac{-2}{p}\right) = -1$, $\theta(O^+(M_p)) \subseteq N_p(E)$ if and only if $p \nmid \mathrm{sf}(abc)$, as a consequence of [5, Theorem 1] and Lemma 2.1. This means that if sf(abc) is divisible by a prime $p \equiv 5, 7 \mod 8$, then M does not have any odd primitive spinor exceptions.

Suppose that (1) and (2) do not hold, but (3) holds. Let t be a primitive spinor exception of gen(M) of the form $t = b + 2^s c + 8n$. Since $E = \mathbb{Q}(\sqrt{-tdM}) = \mathbb{Q}(\sqrt{-2})$, it follows that $tabc \equiv 1 \mod 8$, and hence $bc \equiv 3 \mod 8$ and $ta \equiv 3 \mod 8$. Over $\mathbb{Z}_2$, $M_2^t \cong \langle 1, 8at, 2^{s+2}bc\rangle$. Let U and W be $\mathbb{Z}_2$-lattices defined by

$$U \cong \langle 1, 24\rangle \text{ and } W \cong 24\langle 1, 2^{s-1}\rangle.$$

Since M_2 is not of *Type E*, $\theta(O^+(M_2) = Q(P(U))Q(P(W))\mathbb{Q}_2^{\times 2}$ by [3, Theorem 2.7] where $P(U)$ is the set of primitive anisotropic vectors in U whose associated symmetries are in $O(U)$ ($P(W)$ is defined similarly). Also, note that $Q(P(U)) = \theta(O^+(U))\mathbb{Q}_2^{\times 2}$ and $Q(P(W)) = 6 \cdot \theta(O^+(\langle 1, 2^{s-1}\rangle))\mathbb{Q}_2^{\times 2}$. It then follows from [3, 1.9] that

$$Q(P(U)) = \{1, 6, -1, -6\}\mathbb{Q}_2^{\times 2} \text{ and } Q(P(W)) \subseteq \{1, 2, 3, 6\}\mathbb{Q}_2^{\times 2}$$

and so $\theta(O^+(M_2))$ is not contained in $N_2(E) = \{1, 2, 3, 6\}\mathbb{Q}_2^{\times 2}$. As a result, none of the positive integers $b + 2^s c + 8n$ is a primitive spinor exception of gen(M).

Let us suppose further that (1), (2), and (3) all fail. Then $bc \equiv 1 \mod 8$ and hence $b \equiv c \equiv 1$ or $3 \mod 8$. So the $\mathbb{Q}_2$ quadratic space underlying M_2 is isometric to either $[2a, 1, 1]$ or $[2a, 3, 3]$ with $a \equiv 1$ or $3 \mod 8$. They are isotropic when $a \equiv 3 \mod 8$ and $a \equiv 1 \mod 8$, respectively, and this is impossible by Lemma 2.1(3). Therefore, we must have $a \equiv c \mod 8$, implying $ac \equiv 1 \mod 8$, and hence $ab \equiv 1 \mod 8$ as well.

Now we claim that sf(abc) is a primitive spinor exception of M. Since $\mathrm{sf}(abc) \equiv b \equiv b + 2^s c \mod 8$, sf$(abc)$ is represented primitively by gen(M). Without causing any confusion, let E denote the field $\mathbb{Q}(\sqrt{-\mathrm{sf}(abc)dM})$, which is just $\mathbb{Q}(\sqrt{-2})$. When p is an odd prime, it follows from [5, Theorem 1] that $\theta(O^+(M_p)) \subseteq N_p(E)$ and $\theta^*(M_p, \mathrm{sf}(abc)) = N_p(E)$. For the prime 2, $M_2 \cong \langle b, 8a, 2^{s+2}c\rangle$ which is not of *Type E*. Hence $\theta(O^+(M_2))$ can be computed as before, and the calculation shows that $\theta(O^+(M_2))$ is exactly equal to $N_2(E)$. By [5, Theorem 2(c)(iii)], we see that $\theta^*(M_2, \mathrm{sf}(abc)) = N_2(E)$. This proves our claim that sf(abc) is a primitive spinor exception of gen(M).

Suppose that sf(abc) is represented by M. If $b + 2^s c + 8n$ is not a primitive spinor exception of gen(M), then $b + 2^s c + 8n$ is represented primitively by spn(M) and hence it is represented by M when n is sufficiently large. Otherwise, $b + 2^s c + 8n$ must be a square multiple of sf(abc) and hence $b + 2^s c + 8n$ is represented by M. So we can conclude that $b + 2^s c + 8n$ is represented by M for almost all n. But any representation of $b + 2^s + 8n$ must be in $\omega + 2L$; hence $ax^2 + bT_y + 2^s T_z$ is almost universal.

Conversely, suppose that sf(abc) is not presented by M. Then there exist, as shown in [9], infinitely many primes p such that $p^2\mathrm{sf}(abc)$ is not represented by M. For each such p, we have $\mathrm{sf}(abc)p^2 \equiv b \equiv b + 2^s c \mod 8$. Therefore, $n :=$

$\frac{\mathrm{sf}(abc)p^2-(b+2^sc)}{8}$ is a positive integer for which $b+2^sc+8n$ is not represented by $\omega+2L$. Therefore, $ax^2+bT_y+2^scT_z$ cannot be almost universal. $\square$

Theorem 3.2. *Suppose $s>1$. Then $ax^2+2bT_y+2^scT_z$ is almost universal if and only if $2ax^2+2by^2+2^scz^2$ represents all p-adic integers over $\mathbb{Z}_p$ for every odd p, and one of the following holds:*

1. *s is even;*
2. *sf(abc) is divisible by a prime $p\equiv 3 \mod 4$;*
3. *$ac\not\equiv 1 \mod 4$;*
4. *$\frac{sf(abc)-(b+2^{s-1}c)}{4}$ is represented by $ax^2+2bT_y+2^scT_z$.*

Proof. The proof is very similar to the one of Theorem 3.1, except that when (3) holds (while both (1) and (2) fail), we can immediately claim that the field E is $\mathbb{Q}(\sqrt{-1})$ and it is never equal to $\mathbb{Q}(\sqrt{-tdM})$ for any odd integer t. $\square$

Theorem 3.3. *Suppose that $m>0$ and $s>0$. Then $2^max^2+bT_y+2^scT_z$ is almost universal if and only if $2^max^2+by^2+2^scz^2$ represents all p-adic integers over $\mathbb{Z}_p$ for every odd p, and one of the following holds:*

1. *m is even and $s=1$ or 2; or, $m=1$ and s is odd;*
2. *sf(abc) is divisible by a prime p for which $\left(\frac{-\delta}{p}\right)=-1$, where $\delta=1$ or 2 when $s+m$ is odd or even accordingly;*
3. *$(b+2^sc)sf(abc)\not\equiv 1 \mod 8$;*
4. *$\frac{sf(abc)-(b+2^sc)}{8}$ is represented by $2^max^2+2bT_y+2^scT_z$.*

Proof. Again we leave the detail to the readers since the strategy of the proof is the same as before. However, there is one difference worth mentioning. As in the proof of Theorem 3.1, we will arrive at the stage where (1), (2), and (3) all fail, and we need to show that $\mathrm{sf}(abc)$ is a primitive spinor exception of $\mathrm{gen}(M)$. There is the case when $s+2=m+3$ and

$$M_2\cong\langle b+2^sc\rangle\perp 2^{s+2}\langle a,(b+2^sc)bc\rangle\cong\langle b\rangle\perp 2^{s+2}\langle a,c\rangle,$$

since $s\geq 3$ when (1) fails. In this case, $E=\mathbb{Q}(\sqrt{-1})$ and $\theta(O^+(M_2))$ is computed using [3, Theorem 3.14(iv)] and [4, 1.2] (see [1, Footnote 1] for corrections). The calculation shows that $\theta(O^+(M_2))=N_2(E)$ since $s+2\geq 4$ and $ac\equiv 1 \bmod 8$. By [5, Theorem 2(b)], we see that $\theta^*(M_2,\mathrm{sf}(abc))=N_2(E)$ no matter what the value of $s+2$ is. When $s+2\neq m+3$, then $\theta(O^+(M_2))$ is computed the same way as before, using [3, 1.9 and Theorem 2.7], and note that $E=\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$ when $s+m$ is odd or even accordingly. $\square$

The following theorem handles the case when $r=s=0$. In this case, if we define $\nu:=\mathrm{ord}_2(b+c)$, then ν is either 0 or 1 by Lemma 2.2.

Theorem 3.4. *Suppose that $m>0$. Then $2^max^2+bT_y+cT_z$ is almost universal if and only if $2^max^2+by^2+cz^2$ represents all p-adic integers over $\mathbb{Z}_p$ for every odd p, and one of the following holds:*

1. *$m\geq 0$ is even and $\nu=2$;*
2. *$m>0$ is odd and $\nu=1$;*
3. *sf(abc) is divisible by a prime $p\equiv 3 \mod 4$;*
4. *$\frac{(b+c)}{2^\nu}sf(abc)\not\equiv 1 \mod 4$;*
5. *$\frac{2^\nu sf(abc)-(b+c)}{8}$ is represented by $ax^2+2bT_y+2^scT_z$.*

Proof. Suppose that $\nu = 2$. In this case, a representation of an integer of the form $b + c + 8n$ by the lattice M is not necessarily from the coset $\omega + 2L$. To amend this, we consider the sublattice $R = \mathbb{Z}[2e_1, b\omega - (b + c)e_2, \omega]$ of M. It is clear that $R_p = M_p$ for every odd prime p, and

$$R_2 \cong \langle 2^{m+3}a, (b + c), (b + c)bc\rangle.$$

Since $b+c \equiv 4 \bmod 8$, $bc \equiv 3 \bmod 8$ and hence the binary $\mathbb{Z}_2$-lattice $\langle(b+c), (b+c)bc\rangle$ represents all elements in $4\mathbb{Z}_2^\times$. This shows that all positive integers of the form $b + c + 8n$ are represented primitively by gen(R). But it is easy to show that the norm ideal of the $\mathbb{Z}$-lattice $R\cap 2L$ is contained in $8\mathbb{Z}$, and hence any representation of an integer $b + c + 8n$ by R must lie insider the subset $R \cap (\omega + 2L)$. Therefore, we can replace M by R in our discussion when $\nu = 2$. The rest of the proof follows the same argument used in the proofs of the previous theorems. □

The following corollary is conjectured by Kane and Sun in [7, Conjecture 1.19(i)]. We state it in an equivalent form that is more in line with our presentation. In below, the odd part of an integer α is denoted by α'.

Corollary 3.5. *When $s \in \{3, 4\}$, if the ternary sum $2^m ax^2 + 2^r bT_y + 2^s cT_z$ is almost universal, then one of the following fails:*

(a) $4 \nmid (2^r b + 2^s c)$ *and* sf$(abc) \equiv (2^r b + 2^s c)' \mod 2^{3-\nu}$ *where* $\nu := ord_2(2^r b + 2^s c)$.
(b) *If* sf$(2^{m+r+s}abc) \equiv (2^r b + 2^s c) \mod 2$ *then* sf(abc) *is only divisible by* $p \equiv 1, 3 \mod 8$. *Otherwise, all divisors of* sf(abc) *are* $p \equiv 1 \mod 4$.
(c) $2^{m+3}ax^2 + 2^r by^2 + 2^s cz^2 = 2^\nu$ sf(abc) *has no integral solutions with y and z odd.*
(d) $\begin{cases} s = 3 \textit{ implies } 4|2^m a \textit{ or } 2|2^r b, \\ s = 4 \textit{ implies } 2|2^m a \textit{ or } 2^m a \equiv 2^r b \mod 8. \end{cases}$

Proof. When $r = 1$, then m is necessarily equal to 0. If, in addition, $s = 4$, then (d) always fails. Let us consider the case $r = 1$ and $s = 3$. Theorem 3.2 applies to this case. We need to show that if any one of conditions (1) to (4) in Theorem 3.2 holds, then one of the statements (a) to (d) must fail. Condition (1) never holds since s is odd. When (2) holds, sf(abc) is divisible by $p \equiv 3 \mod 4$. But sf$(2^4abc) \not\equiv (2b+2^3c) \mod 2$, and this contradicts (b). If (3) holds, we have $ac \not\equiv 1 \mod 4$. This implies $abc \not\equiv b + 2^2c \mod 4$ which means that (a) fails. It is easy to see that (4) holds if and only if (c) fails.

Suppose that $r = 0$ and $s = 3$. When $m = 0$, then $ax^2 + bT_y + 2^3cT_z$ is always almost universal by Theorem 3.1. In this case, however, $4 \nmid a$ and $2 \nmid b$, which implies that (d) always fails. Now, suppose that $m > 0$ and that $2^m ax^2 + bT_y + 2^3cT_z$ is almost universal. We are now covered by Theorem 3.3. If (1) there holds, then since $s = 3$ we must have $m = 1$. However, it is clear that $4 \nmid 2a$ and $2 \nmid b$, and hence (d) fails. Now suppose that (2) holds and so sf(abc) is divisible only by primes p such that $\left(\frac{-\delta}{p}\right) = -1$, where δ is as defined in Theorem 3.3. When $m + s$ is even, these are the primes that are congruent to 5 or 7 $\mod 8$. But at the same time sf$(2^{m+s}abc) \equiv (b + 2^3c) \mod 2$ and therefore (b) fails. When $m + s$ is odd, sf$(2^{m+s}abc) \not\equiv (b + 2^3c) \mod 2$, and (2) implies that sf(abc) is only divisible by primes congruent to 3 $\mod 4$; hence (b) fails again. It is clear that (3) holds if and only if (a) fails, and (4) holds if and only if (c) is not true.

The case $r = 0$ and $s = 4$ can be verified similarly, except that when $m = 0$ we need to make the following adjustment. Suppose that condition (3) of Theorem 3.1 holds, that is, $bc \not\equiv 1 \mod 8$. If (a) fails, then we are done. Therefore, we may assume that (a) holds, and this gives us $\mathrm{sf}(abc) \equiv b \mod 8$, implying $ac \equiv 1 \mod 8$. But then $ab \not\equiv 1 \mod 8$ which makes (d) fail. □

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