BOUNDING AN INDEX BY THE LARGEST CHARACTER DEGREE OF A SOLVABLE GROUP

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Abstract. In this paper, we show that if p is a prime and G is a p-solvable group, then $|G\colon O_p(G)|_p \leq (b(G)^p/p)^{1/(p-1)}$ where $b(G)$ is the largest character degree of G. If p is an odd prime that is not a Mersenne prime or if the nilpotence class of a Sylow p-subgroup of G is at most p, then $|G\colon O_p(G)|_p \leq b(G)$.

1. Introduction

This note was motivated by the preprint [5]. In the preprint [5], Jafari proved the following: Let G be a solvable group and let p be an odd prime. If $b(G) < p^3$, then $|G\colon O_p(G)|_p < p^3$. This result was an improvement on Theorem 12.29 of [3] which stated that if $b(G) < p$, then $|G\colon O_p(G)|_p < p$ and Theorem 12.3 of [3] which stated that if $b(G) < p^{3/2}$, then $|G\colon O_p(G)|_p < p^2$. It also improved the result that Benjamin proved in [1], that if G is p-solvable and $b(G) < p^2$, then $|G\colon O_p(G)|_p < p^2$. Benjamin also proved in her paper that if G is a solvable group and $b(G) < p^\alpha$, then $|G\colon O_p(G)|_p < p^{2\alpha}$, and if $|G|$ is odd, then $|G\colon O_p(G)| \leq p^\alpha$.

Furthermore, in [8], Qian showed that if G is any group and $b(G) < p^2$, then $|G\colon O_p(G)|_p < p^2$. In [9], Qian and Shi showed that if G is any group, then $|G\colon O_p(G)|_p < b(G)^2$ and $|G\colon O_p(G)|_p \leq b(G)$ if G is has an abelian p-Sylow subgroup.

In this note, we will use Jafari's argument to prove the following theorem:

Theorem 1. *Let G be a solvable group and let p be a prime. If $b(G) < p^p$, then $|G\colon O_p|_p < p^p$.*

In addition, we will use Jafari's arguments to prove the following result which is along the lines of the result proved by Qian and Shi when the group has an abelian Sylow p-subgroup.

Theorem 2. *Let G be a solvable group, let p be a prime, and suppose that a Sylow p-subgroup of G has nilpotence class less than p. Then $|G\colon O_p(G)|_p \leq b(G)$.*

Further, Jafari's arguments can also be used to prove a result along the lines of the one that Benjamin proved for groups of odd order.

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Theorem 3. *Let G be a solvable group, and let p be an odd prime that is not a Mersenne prime. Then $|G\colon O_p(G)|_p \leq b(G)$.*

In general, it is not true that $|G\colon O_p(G)|_p \leq b(G)$. We will present examples of solvable groups where $|G\colon O_p(G)|_p$ exceeds $b(G)$. In our examples, $b(G) = p^{p+1}$, G has a Sylow p-subgroup whose nilpotence class is p, and p can be any Mersenne prime. Our examples show that the hypotheses in each of Theorems 1, 2, and 3 are in some sense optimal. However, we are able to obtain a general bound using Jafari's argument. To do this, we need a Theorem of Isaacs regarding the existence of a "large" orbit. The result we obtain is the following:

Theorem 4. *Let G be a solvable group, and let p be either 2 or a Mersenne prime. Then $|G\colon O_p(G)|_p \leq (b(G)^p/p)^{1/(p-1)}$.*

Notice that $(b(G)^p/p)^{1/(p-1)} < b(G)^{p/(p-1)} \leq b(G)^2$, so this bound is better than the bound found by Qian and Shi for general groups and by Benjamin for p-solvable groups. Of course, we are assuming the stronger hypothesis that G is solvable in our result.

In the final section, we prove that these bounds still hold when G is a p-solvable group. At this time, we have not determined whether these same bounds can be proved for any group G or whether the hypothesis that G is p-solvable is necessary to obtain our improved bounds.

2. Solvable Groups

The following is essentially Theorems 4.4 and 4.8 of [6].

Lemma 2.1. *Let P be a p-group, and assume that P acts faithfully and coprimely on an abelian group V. Assume one of the following conditions:*

(1) *p is odd and not a Mersenne prime.*
(2) *$Z_p \wr Z_p$ is not a section of P.*

Then P has a regular orbit on V.

Proof. We work by induction on $|V|$. We may assume that $P > 1$, and since P acts faithfully, this implies that $|V| > 1$. Since the action of P is faithful and coprime on V, we see that the action of P on $V/\Phi(V)$ is faithful and coprime. Using the inductive hypothesis, the result follows if $\Phi(V) > 1$. Thus, we may assume that $\Phi(V) = 1$. Therefore, we can V as a completely reducible P-module, possibly of mixed characteristic.

If V is not irreducible under the action of P, then we have $V = U \oplus W$ where U and W are nontrivial P-submodules of V. Applying the inductive hypothesis, we obtain elements $u \in U$ and $w \in W$ so that $\mathbf{C}_P(u) = \mathbf{C}_P(U)$ and $\mathbf{C}_P(w) = \mathbf{C}_P(W)$. It follows that $\mathbf{C}_P(u + w) = \mathbf{C}_P(u) \cap \mathbf{C}_P(w) = \mathbf{C}_P(U) \cap \mathbf{C}_P(W) = 1$, and we are done. Thus, we may assume that V is irreducible under the action of P. If p is odd and not a Mersenne prime, we can apply Theorem 4.4 of [6] to obtain the result. If $Z_p \wr Z_p$ is not a section of P, then we can apply Theorem 4.8 of [6] to obtain the result. In either case, the lemma is proved. □

Notice that the proof of this next theorem is essentially the proof of Theorem 1 in [5]. Notice that hypothesis (3) gives Theorem 1 and hypothesis gives Theorem 3.

Theorem 2.2. *Let G be a solvable group, and let p be a prime. Assume one of the following conditions:*

1. *p is odd and not a Mersenne prime.*
2. *$Z_p \wr Z_p$ is not a section of G.*
3. *$b(G) < p^p$.*

Then $|G\colon O_p(G)|_p \leq b(G)$.

Proof. Assume the theorem is false and suppose that G is a counterexample of minimal order. Observe that if G is a counterexample, then $G/O_p(G)$ is a counterexample. By the minimality of $|G|$, we deduce that $O_p(G) = 1$. Let P be a Sylow p-subgroup of G, and observe that $|P| = |G|_p = |G\colon O_p(G)|_p$. Let F be the Fitting subgroup of G. Since G is a nontrivial solvable group, it follows that F must be a nontrivial p'-group. We know that $\mathbf{C}_G(F) \leq F$. Observe that $b(PF) \leq b(G)$. If $FP < G$, then since G is a counterexample with $|G|$ minimal, we have that $|P| < b(PF) \leq b(G)$ which is a contradiction. Thus, we may assume that $G = FP$. Since P acts coprimely and faithfully on the nilpotent group F, it follows that P acts coprimely and faithfully on the abelian group $F/\Phi(F)$. Using Brauer's permutation lemma (Theorem 6.32 of [3]), P acts faithfully and coprimely on the abelian group $\mathrm{Irr}(F/\Phi(F))$. Conditions (1) or (2) hold, then we can apply Lemma 2.1. Thus, we assume that condition (3) holds. We can find a subgroup Q in P so that $|P\colon Q| = p$. We know that $b(FQ) \leq b(G) < p^p$, so we may apply the inductive hypothesis to FQ, to obtain the conclusion that $|Q| \leq b(FQ) < p^p$. This implies that $|P| < p^{p+1}$, and so, $Z_p \wr Z_p$ cannot be a section of P (since $|Z_p \wr Z_p| = p^{p+1}$). Thus, we may apply Lemma 2.1 in all cases. By Lemma 2.1, there is a linear character $\lambda \in \mathrm{Irr}(F/\Phi(F))$, so that the stabilizer of λ in P is trivial, and thus, the stabilizer of λ in G is F. This implies that λ^G is irreducible, and so, $|P| = |G\colon F| = \lambda^G(1) \leq b(G)$. This proves the theorem. □

As an immediate corollary, we obtain Theorem 2.

Corollary 2.3. *Let G be a solvable group, and let p be a prime. Assume that a Sylow p-subgroup of G has nilpotence class less than p. Then $|G\colon O_p(G)|_p \leq b(G)$.*

Proof. Notice that if $Z_p \wr Z_p$ is a section of G, then it must be a section of some Sylow p-subgroup of G. This would imply that a Sylow p-subgroup of G has nilpotence class at least p which is a contradiction. Thus, $Z_p \wr Z_p$ is not a section of G, and we can apply Theorem 2.2 to obtain the result. □

Using the following result of Isaacs, we can remove the extra hypotheses on p or on the structure of a Sylow subgroup used in the previous results. Unfortunately, we have to weaken our conclusion in this case, and we will present examples to show that the weaker conclusion is necessary. The following is proved as Theorem A in [4]. While it does not prove that there is a regular orbit, it does prove that there is a "large" orbit.

Theorem 2.4. [4] *Let P be a p-group that acts faithfully and coprimely on a group V. Then there exists an element $v \in V$ so that $|\mathbf{C}_P(v)| \leq (|P|/p)^{1/p}$.*

With this result in hand, we can prove the following which includes Theorem 4.

Theorem 2.5. *Let G be a solvable group and let p be a prime. Then $|G\colon O_p(G)|_p \leq (b(G)^p/p)^{1/(p-1)}$.*

Proof. We work by induction on $|G|$. Using the inductive hypothesis, we may assume that $O_p(G) = 1$. Let P be a Sylow p-subgroup of G, and note that $|P| = |G|_p = |G\colon O_p(G)|_p$. Let F be the Fitting subgroup of G. Since G is solvable, we know that $\mathbf{C}_G(F) \leq F$ and F is a p'-group. Observe that $b(PF) \leq b(G)$. If $PF < G$, then we obtain the result by the inductive hypothesis. Hence, we may assume $G = PF$. Now, P acts faithfully and coprimely on the nilpotent group F. Thus, P will act faithfully and coprimely on $F/\Phi(F)$. Applying Theorem 2.4, we can find a linear character $\lambda \in \mathrm{Irr}(F/\Phi(F))$ so that $|\mathbf{C}_P(\lambda)| \leq (|P|/p)^{1/p}$. Thus, if T is the stabilizer of λ in G, then $T = F\mathbf{C}_P(\lambda)$, and so

$$|G\colon T| = |P\colon \mathbf{C}_P(\lambda)| \geq \frac{|P|}{\left(\frac{|P|}{p}\right)^{1/p}} = \left(|P|^{p-1}p\right)^{1/p}.$$

By Clifford's theorem, we know that it follows that $|G\colon T| \leq b(G)$. This implies that $\left(|P|^{p-1}p\right)^{1/p} \leq b(G)$, and we conclude that $|P| \leq (b(G)^p/p)^{1/(p-1)}$, as desired. □

We now present examples to see that the additional hypothesis is needed in Theorem 2.2 and to see that the bound in Theorem 2.5 is appropriate. All of these examples can be found as Example 4.5 of [6].

(1) Let p be a Mersenne prime, so $p = 2^f - 1 \geq 3$. We take $P = Z_p \wr Z_p$, and we take V to be p copies of $\mathrm{GF}(2^f)$. In [6], they show that P acts faithfully on V, and that P does not have any regular orbits in its action on V. Let G be the semi-direct product of P acting on V. We claim that $b(G) = p^p$. On the other hand, $O_p(G) = 1$, so $|G\colon O_p(G)|_p = |G|_p = |P| = p^{p+1}$ which is larger than $b(G)$. On the other hand, $(b(G)^p/p)^{1/(p-1)} = ((p^p)^p/p)^{1/(p-1)} = (p^{p^2-1})^{1/(p-1)} = p^{p+1}$. Thus, for this group, the bound in Theorem 2.5 is optimal.

(2) In this example, we take $p = 2$. Let q be a Fermat prime, so $q = 2^f + 1 \geq 3$. Let $P = Z_{2^f} \wr Z_2$ and let V be 2 copies of $\mathrm{GF}(q)$. In

[6], they show that P acts faithfully on V, and has no regular orbit. Let G be the semi-direct product of P acting on V. We claim that $b(G) = 2^{2f}$. On the other hand, $O_2(G) = 1$, so $|G\colon O_2(G)|_2 = |G|_2 = |P| = 2^{2f+1}$.

3. p-SOLVABLE GROUPS

This next lemma follows from a corollary of Gluck's regular orbit theorem.

Lemma 3.1 (Corollary 5.7(b) of [6])**.** *Let p be a prime an odd prime, and suppose that P is a p-group that is a permutation group on Ω. Then there exists a set $\Delta \subseteq \Omega$ so that $P_\Delta = 1$.*

We also need the following lemma which is proved in [7].

Lemma 3.2 (Proposition 2.6 of [7])**.** *Let A act faithfully and coprimely on a nonabelian simple group S. Then A has at least 2 regular orbits on $\mathrm{Irr}(S)$.*

Using these two results, we obtain a character that induces irreducibly in the key situation where we have a p-group that is acting on a direct product of copies of a nonabelian simple group.

Lemma 3.3. *Let S be a nonabelian simple group, and let p be a prime that does not divide $|S|$. Suppose $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$. Assume P is a p-group that acts faithfully on V via automorphisms, and assume the action of P transitively permutes the S_i's. If G is the semi-direct product of P acting on V, then there exists $\theta \in \mathrm{Irr}(V)$ so that θ^G is irreducible.*

Proof. Let $\Omega = \{S_1, \ldots, S_n\}$. Let Q be the kernel of the action of P on Ω. Notice that since p does not divide $|S|$, it follows that p is odd. Thus, we can use Lemma 3.1 to obtain a set $\Delta \subseteq \Omega$ so that $\mathbf{C}_P(\Delta) = Q$. Since P acts transitively on Ω, it follows that P acts transitively on the set $\{\mathbf{C}_Q(S_1), \ldots, \mathbf{C}_Q(S_n)\}$. Let $R_i = Q/\mathbf{C}_Q(S_i)$, and this implies that there is a group R so that $R \cong R_i$ for all i. Notice that R acts on S. By Lemma 3.2, we know that R has two regular orbits on $\mathrm{Irr}(S)$. Hence, we can find $\mu, \nu \in \mathrm{Irr}(S)$ in different R-orbits so that $\mathbf{C}_P(\mu) = \mathbf{C}_P(\nu) = 1$. For each i, let μ_i and ν_i be the characters in $\mathrm{Irr}(S)$ that correspond to μ and ν respectively.

We now define $\theta \in \mathrm{Irr}(V)$ as follows. For each i, we take $\theta_i = \mu_i$ if $S_i \in \Delta$ and $\theta_i = \nu_i$ if $S_i \notin \Delta$, and then, $\theta = \theta_1 \times \cdots \times \theta_n$. Observe that $\mathbf{C}_P(\theta) \leq \mathbf{C}_P(\Delta) = Q$, and so,

$$\mathbf{C}_P(\theta) = \mathbf{C}_Q(\theta) = \cap_{i=1}^n \mathbf{C}_Q(\theta_i) = \cap_{i=1}^n \mathbf{C}_Q(S_i) = 1.$$

This implies that the stabilizer of θ in G is V. Hence, θ^G is irreducible. $\square$

We now prove that the results that we proved when G is solvable still hold when G is p-solvable. To simplify the next result we define the following function f. We define $f(G, p) = b(G)$ if p is odd and not a Mersenne prime or if $p = 2$ or p is a Mersenne prime and G has no section isomorphic to

$Z_p \wr Z_p$. We define $f(G,p) = (b(G)^p/p)^{1/(p-1)}$ if either $p = 2$ or p is a Mersenne prime and G has a section isomorphic to $Z_p \wr Z_p$

Theorem 3.4. *Let G be a p-solvable group where p is some prime. Then $|G\colon O_p(G)|_p \leq f(G,p)$.*

Proof. We will work by induction on $|G|$. Suppose H is a section of G. We claim that $f(H,p) \leq f(G,p)$. If p is odd and not a Mersenne prime, then $f(H,p) = b(H) \leq b(G) = f(G,p)$. Suppose for now that $p = 2$ or p is a Mersenne prime. If G does not have a section isomorphic to $Z_p \wr Z_p$, then neither does H, and again, we have $f(H,p) = b(H) \leq b(G) = f(G,p)$. If H has a section isomorphic to $Z_p \wr Z_p$, then so does G, and so, $f(H,p) = (b(H)^p/p)^{1/(p-1)} \leq (b(G)^p/p)^{1/(p-1)} = f(G,p)$. Finally, we must consider the case where G has a section isomorphic to $Z_p \wr Z_p$ and H does not. In this case, since $Z_p \wr Z_p$ is a section of G, it follows that G must have an irreducible character whose degree is at least p, and hence, $p \leq b(G)$. Thus, $b(G)^{1/(p-1)}p \leq b(G)^p$, and $b(G) \leq (b(G)^p/p)^{1/(p-1)}$. We deduce that $f(H,p) = b(H) \leq b(G) \leq (b(G)^p/p)^{1/(p-1)} = f(G,p)$. This proves the claim in all cases.

Using the inductive hypothesis on $G/O_p(G)$, we may assume that $O_p(G) = 1$. Also, using the Frattini argument, one can now show that $O_p(G/\Phi(G)) = 1$, and so, using the inductive hypothesis on $G/\Phi(G)$, we may assume that $\Phi(G) = 1$. If all the minimal normal subgroups of G are solvable, then let F be the Fitting subgroup of G, and it is known that there is a subgroup A so that $G = FA$ and $F \cap A = 1$. It is easy to see that $\mathbf{C}_A(F)$ will be a normal subgroup of G, and since F contains all the minimal normal subgroups of G, we conclude that $\mathbf{C}_A(F) = 1$, and hence, $\mathbf{C}_G(F) \leq F$. So it suffices to show that the result holds in FP, but FP is solvable, and we have seen that the result holds in solvable groups.

Thus, we may assume that G has a nonsolvable minimal normal subgroup V. Since G is p-solvable, we see that p does not divide $|V|$. Notice that this implies that p must be odd. We know that $\mathbf{C}_G(V)$ is normal in G, so $O_p(\mathbf{C}_G(V)) \leq O_p(G) = 1$. Let $H = V\mathbf{C}_G(V)P$. Now, $O_p(H)$ and V are normal in H and of coprime orders, so $O_p(H) \leq \mathbf{C}_G(V)$, and this implies that $O_p(H) = 1$. Thus, we may use the inductive hypothesis to assume that $G = H = V\mathbf{C}_G(V)P$. Observe that $V = S_1 \times \cdots \times S_n$ where $S_i \cong S$ and S is a nonabelian simple group. This implies that $V \cap \mathbf{C}_G(V) = 1$. By Lemma 3.3, we can find $\theta \in \mathrm{Irr}(V)$ so that $\mathbf{C}_P(\theta) = \mathbf{C}_P(V)$. Let $\gamma \in \mathrm{Irr}(\mathbf{C}_G(V))$ so that $\gamma(1) = b(\mathbf{C}_P(V))$. Notice that the stabilizer of $\theta \times \gamma$ is $V\mathbf{C}_G(V)$. This implies that $(\theta\times\gamma)^G$ is irreducible, and hence, $b(G) > |P\colon \mathbf{C}_P(V)|b(\mathbf{C}_G(V))$. By the inductive hypothesis applied in $\mathbf{C}_G(V)$, we have that $|\mathbf{C}_P(V)| \leq f(\mathbf{C}_G(V),p)$. If either p is not a Mersenne prime or $Z_p \wr Z_p$ is not involved in G, then $f(G,p) = b(G)$ and $f(\mathbf{C}_G(V),p) = b(\mathbf{C}_G(V))$, and the result follows. Thus, we may assume that p is a Mersenne prime and $Z_p \wr Z_p$ is involved in G. Then we have that

$$f(G,p) = (b(G)^p/p)^{1/(p-1)} > (|P\colon \mathbf{C}_P(V)|^p b(\mathbf{C}_G(V))^p/p)^{1/(p-1)}.$$

This expression can be rewritten as follows: $|P\colon \mathbf{C}_P(V)|^{p/(p-1)} (b(\mathbf{C}_G(V))^p/p)^{1/(p-1)}$. Using the definition of $f(\mathbf{C}_G(V),p)$, and the fact that $p/(p-1) > 1$, we see that this is larger than $|P\colon \mathbf{C}_P(V)|f(\mathbf{C}_G(V),p)$, and obtain

$$f(G,p) > |P\colon \mathbf{C}_P(V)|f(\mathbf{C}_G(V),p) \geq |P\colon \mathbf{C}_P(V)||\mathbf{C}_P(V)| = |P|.$$

This proves the theorem. □

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