Stability of Strongly Gauduchon Manifolds under Modifications

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Abstract. In our previous works on deformation limits of projective and Moishezon manifolds, we introduced and made crucial use of the notion of *strongly Gauduchon* metrics as a reinforcement of the earlier notion of Gauduchon metrics. Using direct and inverse images of closed positive currents of type (1, 1) and regularisation, we now show that compact complex manifolds carrying *strongly Gauduchon* metrics are stable under modifications. This stability property, known to fail for compact Kähler manifolds, mirrors the modification stability of balanced manifolds proved by Alessandrini and Bassanelli.

1 Introduction

Let X be a compact complex manifold, $\dim_{\mathbb{C}}X = n$. A Hermitian metric on X will be identified throughout with the corresponding positive-definite C^∞ (1, 1)-form ω on X. Recall that a Hermitian metric ω is said to be a *Gauduchon metric* if $\partial\bar\partial\omega^{n-1} = 0$ on X (cf. [Gau77]), a condition that can be reformulated as $\partial\omega^{n-1}$ being $\bar\partial$-closed. Gauduchon proved in [Gau77] that not only do these metrics always exist on any compact complex manifold X, but there is a Gauduchon metric in every conformal class of Hermitian metrics.

In [Pop09] we introduced the notion of a *strongly Gauduchon metric* (also referred to as an *sG metric* in the sequel) by requiring that, for a given Hermitian metric ω, the above-mentioned $(n,\, n-1)$-form $\partial\omega^{n-1}$ be $\bar\partial$-exact on X (rather than merely $\bar\partial$-closed). We showed that such a metric need not exist on an arbitrary X and termed X a *strongly Gauduchon manifold* if it carries a *strongly Gauduchon metric*. We went on to notice that when the $\partial\bar\partial$-lemma holds on X, the notions of Gauduchon and *strongly Gauduchon metrics* coincide, hence every such X is a *strongly Gauduchon manifold*. This is because $d(\partial\omega^{n-1}) = 0$ if ω is a Gauduchon metric ; since the pure-type form $\partial\omega^{n-1}$ is also ∂-exact in an obvious way, it must be $\bar\partial$-exact if the $\partial\bar\partial$-lemma holds on X. We then went on to characterise *strongly Gauduchon manifolds* starting from the following simple observation.

Lemma 1.1 *(Lemma 3.2. in [Pop09]) There exists a strongly Gauduchon metric on a given compact complex manifold X of dimension n if and only* *if there exists a C^∞ $(2n-2)$-form Ω on X such that :*

(a) $\Omega = \overline{\Omega}$ *(i.e. Ω is real) ;*
(b) $d\Omega = 0$ *;*
(c) $\Omega^{n-1,\,n-1} > 0$ *on X (i.e. the component of type $(n-1,\,n-1)$ of Ω is positive-definite).*

Indeed, if γ is a *strongly Gauduchon metric* on X, set $\Omega^{n-1,\,n-1} := \gamma^{n-1}$, take $\Omega^{n,\,n-2}$ to be any smooth $(n,\,n-2)$-form such that $\partial\gamma^{n-1} = -\bar\partial\Omega^{n,\,n-2}$ ($\Omega^{n,\,n-2}$ exists by the sG assumption on γ), set $\Omega^{n-2,\,n} := \overline{\Omega^{n,\,n-2}}$ and $\Omega := \Omega^{n,\,n-2} + \Omega^{n-1,\,n-1} + \Omega^{n-2,\,n}$. Conversely, given an Ω as in the above lemma, we use Michelsohn's procedure for extracting the $(n-1)^{st}$ root of a positive-definite $(n-1,\,n-1)$-form (cf. [Mic83]) and get a unique C^∞ $(1,\,1)$-form γ on X such that $\gamma^{n-1} = \Omega^{n-1,\,n-1}$. That γ is an *sG metric* follows from the properties of Ω.

As a consequence of this observation, we obtained the following intrinsic characterisation of *strongly Gauduchon manifolds*.

Proposition 1.2 *(Proposition 3.3 in [Pop09]) A compact complex manifold X is* **strongly Gauduchon** *if and only if there is no non-zero current T of type $(1,\,1)$ on X such that $T \geq 0$ and T is d-exact on X.*

The object of the present work is to show that the *strongly Gauduchon* property of compact complex manifolds is stable under modifications (i.e. proper, holomorphic, bimeromorphic maps). This provides a sharp contrast to the Kählerness of these manifolds which is only preserved under blowing up (smooth) submanifolds ([Bla58]).

Theorem 1.3 *Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds X and $\widetilde{X}$. Then $\widetilde{X}$ is a* **strongly Gauduchon manifold** *if and only if X is a* **strongly Gauduchon manifold**.

This result parallels the main result of Alessandrini and Bassanelli in [AB95] (see also [AB91] and [AB93]) which asserts that *balanced manifolds* enjoy the same stability property under modifications as above. Recall that *balanced manifolds* were given in [Mic83] an intrinsic characterisation in terms of non-existence of non-trivial positive currents of bidegree $(1,\,1)$ that are components of a boundary. Our criterion listed as Proposition 1.2 above is the analogous intrinsic characterisation of the weaker notion of *strongly Gauduchon manifolds*. The proof of Theorem 1.3 will draw on those of the main results in [AB91], [AB93] and [AB95] with certain arguments handled slightly differently while others are considerably simplified by the fact that d-closed positive $(1,\,1)$-currents always admit unambiguously defined inverse images constructed from their local potentials, unlike the much more delicate-to-handle $\partial\bar\partial$-closed positive (1, 1)-currents that were relevant to the case of *balanced manifolds*. Inverse images for this latter class of currents were painstakingly constructed in [AB93] and a unique choice was shown to enjoy the necessary cohomological properties, rendering the case treated in [AB93] and [AB95] conspicuously more involved than ours.

The motivation for proving stability properties of *strongly Gauduchon manifolds* stems from the relevance of this notion to the study of limits of compact complex manifolds under holomorphic deformations. It has played a key role in proving that the deformation limit of projective (or merely Moishezon) manifolds is Moishezon (cf. [Pop09], resp. [Pop10]) and has similar striking consequences in efforts to tackle analogous but more general conjectures on deformations of *class* $\mathcal{C}$ manifolds. Besides their modification stability exhibited in the present text, *strongly Gauduchon manifolds* are likely to evince stability properties under holomorphic deformations generalising those shown in [Pop09] and [Pop10], a study of which is well underway and will hopefully be the subject of a future paper.

2 Proof of Theorem 1.3

Let $\mu : \widetilde{X} \to X$ be a modification of compact complex manifolds and denote $n = \dim_{\mathbb{C}}\widetilde{X} = \dim_{\mathbb{C}}X$. Let E be the exceptional divisor of μ on $\widetilde{X}$ and let $S \subset X$ be the analytic subset such that the restriction $\mu_{|\widetilde{X}\setminus E} : \widetilde{X} \setminus E \longrightarrow X \setminus S$ is a biholomorphism. Theorem 1.3 falls into two parts.

Theorem 2.1 *If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and X is* **strongly Gauduchon***, then $\widetilde{X}$ is again* **strongly Gauduchon***.*

Proof. We proceed by contradiction. Suppose that $\widetilde{X}$ is not *strongly Gauduchon*. Then, by Proposition 1.2, there exists a current $T \neq 0$ of type (1, 1) on $\widetilde{X}$ such that

$$T \geq 0 \quad \text{and} \quad T \in \operatorname{Im} d \quad \text{on} \quad \widetilde{X}.$$

By compactness of $\widetilde{X}$, the map μ is proper and therefore the direct image under μ of any current on $\widetilde{X}$ is well-defined. Thus $\mu_\star T$ is a well-defined current of type (1, 1) on X. It is clear that

$$\mu_\star T \geq 0 \quad \text{and} \quad \mu_\star T \in \operatorname{Im} d \quad \text{on} \quad X.$$

Indeed, for every C^∞ (1, 1)-form $\omega > 0$ on X, we have

$$\int\limits_X \mu_\star T \wedge \omega^{n-1} = \int\limits_X T \wedge (\mu^\star\omega)^{n-1} \geq 0,$$

a fact that proves the positivity of $\mu_\star T$, while the d-exactness follows from $\mu_\star$ commuting with d. Now we have the following dichotomy.

If $\mu_\star T$ is non-zero, we get a contradiction to the *strongly Gauduchon* assumption on X thanks to Proposition 1.2.

If $\mu_\star T = 0$ on X, we show that $T = 0$ on $\widetilde{X}$, contradicting the choice of T. Indeed, if $\mu_\star T = 0$, the support of T must be contained in the support of E. Since T is a closed positive current of bidegree (1, 1) and the irreducible components E_j of E are all of codimension 1 in $\widetilde{X}$, a classical theorem of support (see e.g. [Dem97, Chapter III, Corollary 2.14]) forces T to have the shape

$$T = \sum_{j\in J} \lambda_j \, [E_j], \quad \text{with coefficients} \quad \lambda_j \geq 0 \quad \text{and some index set } J.$$

Now there are two cases. If all the irreducible components of S are of codimension ≥ 2 in X, then $\text{codim}_X \mu(E_j) \geq 2$ for every $j \in J$. All we have to do is repeat the argument of [AB91, p. 5] that we now recall for the reader's convenience. By [GR70, p. 286], for every $i \geq 0$, there exists a vector subspace $H_i^\star(E) \subset H_i(E)$ and a commutative diagram whose rows are short exact sequences featuring the homology groups H_i of the various spaces involved :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_i^\star(E) & \hookrightarrow & H_i(E) & \xrightarrow{\beta_i} & H_i(S) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_i^\star(E) & \longrightarrow & H_i(\widetilde{X}) & \xrightarrow{\alpha_i} & H_i(X) & \to & 0, \end{array}$$

where $\hookrightarrow$ stands for inclusion. If we denote by $\{\ \}_E$ (respectively $\{\ \}_{\widetilde{X}}$) the homology class of a d-closed current of dimension $2(n-1)$ in the ambient space Supp E (respectively $\widetilde{X}$), we see that

$$\beta_{2(n-1)}\{T\}_E = \sum_j \lambda_j \, \beta_{2(n-1)}\{[E_j]\}_E = 0$$

since the direct images under μ of the currents of integration $[E_j]$ are closed positive (1, 1)-currents on X supported on the analytic subset S with $\text{codim}_X S \geq 2$, hence they must vanish by another classical theorem of support (see e.g. [Dem97, Chapter III, Corollary 2.11]). Thus, from the top exact sequence, we get that $\{T\}_E$ belongs to $H^\star_{2(n-1)}(E)$. The diagram being commutative, the image of $\{T\}_E \in H^\star_{2(n-1)}(E)$ in $H_{2(n-1)}(\widetilde{X})$ under the injective arrow of the bottom exact sequence is $\{T\}_{\widetilde{X}}$. Meanwhile $\{T\}_{\widetilde{X}} = 0$ since T is d-exact on $\widetilde{X}$. We get that $\{T\}_E = 0$. This means that the restriction $T_{|\text{Supp E}}$ is a d-exact current of bidegree (0, 0) on Supp E (since it is of the same bidimension $(n-1,\, n-1)$ as that of T on $\widetilde{X}$). Since the only d-exact current of type (0, 0) is the zero current, we must have $\lambda_j = 0$ for every j. Hence $T = 0$ as a current on $\widetilde{X}$, a contradiction.

If S has irreducible components S_j, $j \in J_0$, of codimension 1 in X, then

$$\mu^{-1}(S_j) = E_j \quad \text{and} \quad \mu^\star([S_j]) = [E_j], \qquad j \in J_0 \subset J.$$

Thus $T = \sum\limits_{j \in J_0} \lambda_j\,[E_j] + \sum\limits_{j \in J \setminus J_0} \lambda_j\,[E_j]$ and $0 = \mu_\star T = \sum\limits_{j \in J_0} \lambda_j\,[S_j]$. Indeed, $\mu_\star[E_j] = [S_j]$ for all $j \in J_0$, while $\mu_\star[E_j] = 0$ for every $j \in J \setminus J_0$ since it is a closed positive (1, 1)-current whose support is contained in an analytic subset of codimension ≥ 2 in X. Hence $\lambda_j = 0$ for all $j \in J_0$. Consequently, $T = \sum\limits_{j \in J \setminus J_0} \lambda_j\,[E_j]$ and we are now in the previous case where we showed that $T = 0$, a contradiction. The proof is complete. $\square$

We now prove the reverse statement.

Theorem 2.2 *If $\mu : \widetilde{X} \to X$ is a modification of compact complex manifolds and $\widetilde{X}$ is* **strongly Gauduchon**, *then X is again* **strongly Gauduchon**.

Proof. We proceed once more by contradiction. Suppose that X is not *strongly Gauduchon*. Then, in view of Proposition 1.2, there exists a current $T \neq 0$ of type (1, 1) on X such that

$$T \geq 0 \quad \text{and} \quad T = dS \quad \text{for some real } 1\text{-current } S \text{ on } \quad X.$$

We shall show that the inverse image current $\mu^\star T$ is a well-defined (1, 1)-current on $\widetilde{X}$ enjoying the same properties as T on X, thus contradicting the *strongly Gauduchon* assumption on $\widetilde{X}$ in view of Proposition 1.2.

Although the inverse image of an arbitrary current is not defined in general, the inverse image of a d-closed positive (1, 1)-current is well-defined under μ by the inverse images of its local $\partial\bar\partial$-potentials (see e.g. [Meo96]). Indeed, following [Meo96], for every open subset $U \subset X$ such that $T_{|U} = i\partial\bar\partial\varphi$ for a psh function φ on U, one defines $(\mu^\star T)_{|\mu^{-1}(U)} := i\partial\bar\partial(\varphi \circ \mu)$. The psh function $\varphi \circ \mu$ is $\not\equiv -\infty$ on every connected component of $\mu^{-1}(U)$ since μ has generically maximal rank and the local pieces $(\mu^\star T)_{|\mu^{-1}(U)}$ glue together into a globally defined d-closed positive (1, 1)-current $\mu^\star T$ on $\widetilde{X}$ that is independent of the choice of open subsets $U \subset X$ and local potentials φ.

It is clear that $\mu^\star T$ is not the zero current on $\widetilde{X}$. Indeed, if we had $\mu^\star T = 0$, the support of T would be contained in S. If all the irreducible components of S were of codimension ≥ 2 in X, a classical theorem of support (see e.g. [Dem97, Chapter III, Corollary 2.11]) would guarantee that the closed positive (1, 1)-current T must be the zero current on X, a contradiction. If S had certain global irreducible components S_j of codimension 1 in X, another theorem of support (cf. [Dem97, Chapter III, Corollary 2.14]) would ensure that T has the shape $T = \sum \lambda_j\,[S_j]$ for some constants $\lambda_j \geq 0$. Then $\mu^\star[S_j]$

would be the current of integration on the inverse-image divisor $\mu^{-1}(S_j) \subset \widetilde{X}$ and $\mu^\star T$ cannot be the zero current unless $\lambda_j = 0$ for all j. However, in this event $T = 0$ on X, a contradiction.

The only thing that has yet to be checked before reaching the desired contradiction is that the non-trivial d-closed positive (1, 1)-current $\mu^\star T$ is d-exact on $\widetilde{X}$. Since the 1-current S cannot be pulled back to $\widetilde{X}$ (the local potential technique is no longer available), we shall use Demailly's regularisation-of-currents theorem [Dem92, Main Theorem 1.1] to get a sequence $(v_j)_{j\in\mathbb{N}}$ of C^∞ (1, 1)-forms on X such that every v_j lies in the same Bott-Chern (hence also De Rham) cohomology class as T with convergence

$$v_j \longrightarrow T \quad \text{weakly as} \quad j \to +\infty, \quad \text{while} \quad v_j \geq -C\omega, \quad j \in \mathbb{N},$$

where ω is any Hermitian metric on X fixed beforehand and $C > 0$ is a constant independent of $j \in \mathbb{N}$.

Since T is d-exact and cohomologous to each v_j, every form v_j is d-exact. Thus, for all $j \in \mathbb{N}$, $v_j = du_j$ for some C^∞ 1-form u_j on X. Unlike S, the C^∞ forms u_j have inverse images under μ and we get

$$\mu^\star v_j = d(\mu^\star u_j) \longrightarrow \mu^\star T \quad \text{weakly as} \quad j \to +\infty, \tag{1}$$

after possibly extracting a subsequence. Indeed, it was shown in [Meo96, Proposition 1] that for every sequence of d-closed positive (1, 1)-currents T_j converging weakly to T, the sequence of inverse-image currents $\mu^\star T_j$ converges weakly to $\mu^\star T$. In our case, the (1, 1)-forms v_j are not necesarily *positive* but only *almost positive* (the negative part being uniformly bounded by $C\omega$). We now spell out the reason why $\mu^\star v_j$ converges weakly to the current $\mu^\star T$ in this slightly more general context. The argument is virtually the same as that of [Meo96].

Pick any C^∞ (1, 1)-form α in the Bott-Chern class of the forms v_j (= the class of T). Then, for every $j \in \mathbb{N}$, we have

$$v_j = \alpha + i\partial\bar\partial\psi_j \geq -C\omega \quad \text{on} \quad X,$$

with C^∞ functions $\psi_j : X \to \mathbb{R}$ that we normalise such that $\int\limits_X \psi_j\,\omega^n = 0$ for every j. This normalisation makes ψ_j unique. Applying the trace w.r.t. ω and using the corresponding Lapacian $\Delta_\omega(\cdot) = \text{Trace}_\omega(i\partial\bar\partial(\cdot))$, we get

$$\Delta_\omega\psi_j = \text{Trace}_\omega(v_j - \alpha), \quad j \in \mathbb{N}.$$

Applying now the Green operator G of Δ_ω and using the normalisation of ψ_j, we get

$$\psi_j = G\,\text{Trace}_\omega(v_j - \alpha), \quad j \in \mathbb{N}.$$

Since G is a compact operator from the Banach space of bounded Borel measures on X to $L^1(X)$ and since the forms v_j converge weakly to T, we infer that some subsequence $(\psi_{j_k})_k$ converges to a limit $\psi \in L^1(X)$ in $L^1(X)$-topology. Thus the weak continuity of $\partial\bar\partial$ gives

$$T = \lim_k (\alpha + i\partial\bar\partial\psi_{j_k}) = \alpha + i\partial\bar\partial\psi \quad \text{ on } \quad X.$$

Now the sequence $(\psi_j)_j$ is uniformly bounded above on X by some constant $C_1 > 0$ thanks to the normalisation imposed on ψ_j and the Green-Riesz representation formula for ψ_j, Δ_ω and G. Hence the sequence $(\psi_j \circ \mu)_j$ is uniformly bounded above on $\widetilde{X}$ by $C_1 > 0$. On the other hand, $\psi_{j_k} \circ \mu$ converges almost everywhere to $\psi \circ \mu$ on $\widetilde{X}$. Since the forms $i\partial\bar\partial(\psi_{j_k} \circ \mu)$ are uniformly bounded below on $\widetilde{X}$ by $-(\mu^\star\alpha + C\mu^\star\omega)$, the almost psh functions $\psi_{j_k} \circ \mu$ can be simultaneously made psh on small open subsets of $\widetilde{X}$ by the addition of a same locally defined smooth psh function. We can thus apply the classical result stating that a sequence of psh functions that are locally uniformly bounded above either converges locally uniformly to $-\infty$ (a case that is ruled out in our present situation), or has a subsequence that converges in L^1_{loc} topology (see e.g. [Hor94, Theorem 3.2.12., p. 149]). We infer that the almost psh functions $\psi_{j_k} \circ \mu$ actually converge in $L^1(\widetilde{X})$-topology (hence also in the weak topology of distributions) and implicitly the forms

$$\mu^\star v_{j_k} = \mu^\star\alpha + i\partial\bar\partial(\psi_{j_k} \circ \mu)$$

converge weakly to the current $\mu^\star T = \mu^\star\alpha + i\partial\bar\partial(\psi \circ \mu)$. Thus the convergence statement (1) is proved.

Since the De Rham class is continuous w.r.t. the weak topology of currents and since each form $\mu^\star v_j = d(\mu^\star u_j)$ has vanishing De Rham class, the limit current $\mu^\star T$ must have vanishing De Rham class. Equivalently, $\mu^\star T$ is d-exact, providing a contradiction to the *strongly Gauduchon* assumption on $\widetilde{X}$ in view of Proposition 1.2. The proof is complete. □

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