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Absolute continuity under flows generated by SDE with measurable drift coefficient

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Abstract

We consider the Itô SDE with non-degenerate diffusion coefficient and measurable drift coefficient. Under the condition that the gradient of the diffusion coefficient and the divergences of the diffusion and drift coefficients are exponentially integrable with respect to the Gaussian measure, we show that the stochastic flow leaves the reference measure absolutely continuous.

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1 Introduction

Let $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathcal{M}_{d,m}$ be a matrix-valued measurable function and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ a measurable vector field, we denote by σ_t and b_t the functions $\sigma(t,\cdot)$ and $b(t,\cdot)$ respectively. Consider the Itô stochastic differential equation (abbreviated as SDE)

$$\mathrm{d}X_{s,t} = \sigma_t(X_{s,t})\,\mathrm{d}w_t + b_t(X_{s,t})\,\mathrm{d}t, \quad t \geq s, \quad X_{s,s} = x \tag{1.1}$$

where $w_t = (w_t^1, \cdots, w_t^m)^*$ is a standard m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well known that if σ_t and b_t are globally Lipschitz continuous with respect to the spacial variable x (uniformly in t), then the above equation has a unique strong solution which defines a stochastic flow of homeomorphisms on $\mathbb{R}^d$. We want to point out that these homeomorphisms are only Hölder continuous of order strictly less than 1 (unlike the solution of ODE under the Lipschitz condition), hence it is not clear whether the push-forward of the reference measure by the flow is absolutely continuous with respect to itself. When the coefficients are time independent, recently it is proved that if in addition the quantity $\sigma(x)^*x$ grows at most linearly, then the stochastic flow leaves the Lebesgue measure quasi-invariant, see [8] Theorem 1.2. The proof of this result is based on an a priori estimate for the Radon-Nikodym density (see Theorem 2.2 in [8]) and a limit theorem (see [12] Theorem A). An interesting point of the limit theorem lies in the fact that if the SDE (1.1) has the pathwise uniqueness, then the locally uniform convergence of the coefficients implies the convergence of the solutions in a certain sense. The quasi-invariance of Lebesgue measure under the stochastic flow is proved in

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[17] for SDE (1.1) with regular diffusion coefficient but the drift satisfying only a log-Lipschitz condition, which generalizes Lemma 4.3.1 in [15].

In the context of ordinary differential equation (ODE for short)

$$\mathrm{d}X_{s,t} = b_t(X_{s,t})\,\mathrm{d}t, \quad t \geq s, \quad X_{s,s} = x, \tag{1.2}$$

it is known to all that if the vector field b_t does not have the (local) Lipschitz continuity, then the ODE (1.2) may have no uniqueness or may have no solution at all. On the other hand, if b_t has the Sobolev or even BV_{loc} regularity, then the celebrated DiPerna-Lions theory says that the vector field b_t generates a unique flow of measurable maps which leaves the reference measure quasi-invariant, provided that its divergence is bounded or exponentially integrable, see [1, 2, 4, 6]. These results have recently been generalized to the infinite dimensional Wiener space, cf. [3, 7]. In a recent paper, Crippa and de Lellis [5] gave a direct construction of the DiPerna-Lions flow, and this method was generalized in [8, 21] to the case of SDE with Sobolev coefficients.

On the other hand, a remarkable result due to Veretennikov says that if σ_t is bounded Lipschitz continuous and satisfies a non-degeneracy condition, then the SDE (1.1) admits a unique strong solution even though b_t is only bounded measurable, see [19]. This result was generalized in [10] to the case where σ_t is locally Lipschitz continuous, and the drift coefficient b_t is dominated by the sum of a positive constant and an integrable function. The proof is based on a convergence result of the solutions of approximating SDEs to that of the limiting SDE, which follows from the Krylov estimate. Further developments in this direction can be found in [14, 20]. Having the existence of the unique strong solution to (1.1) in mind, it is natural to ask whether the reference measures are quasi-invariant under the action of the stochastic flow? To state the main result of this work, we introduce some notations. γ_d is the standard Gaussian measure on $\mathbb{R}^d$ and for any $p \geq 1$, $\mathbb{D}_1^p(\gamma_d)$ is the first order Sobolev space with respect to γ_d. For a vector field $B \in \mathbb{D}_1^p(\gamma_d)$, $\delta(B)$ denotes the divergence with respect to the Gaussian measure γ_d; for a $d \times m$ matrix $\sigma \in \mathbb{D}_1^p(\gamma_d)$, $\delta(\sigma)$ is a $\mathbb{R}^m$-valued function whose components are the divergences $\delta(\sigma^{\cdot j})$ of the j-th column $\sigma^{\cdot j}$ of σ, $j = 1, \cdots, m$. $\|\sigma\|$ is the Hilbert-Schmidt norm of the matrix. We will prove

Theorem 1.1. *Assume that*

- (i) $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathcal{M}_{d,m}$ *is jointly continuous on* $\mathbb{R}_+ \times \mathbb{R}^d$*, and there is* $c_1 > 0$ *such that for all* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\sigma_t(x)(\sigma_t(x))^* \geq c_1 \mathrm{Id}$*;*
- (ii) *for all* $t \geq 0$, $\sigma_t \in \cap_{p>1} \mathbb{D}_1^p(\gamma_d)$ *and* $\sup_{0 \leq u \leq t} \|\nabla \sigma_u\|_{L^{2(d+1)}(\gamma_d)} < \infty$*;*
- (iii) $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *is measurable and* $\delta(b_t)$ *exists for all* $t \geq 0$*;*
- (iv) *for any* $T > 0$*, there is* $L_T > 0$ *such that* $\|\sigma_t(x)\| \vee |b_t(x)| \leq L_T(1 + |x|)$ *for all* $(t, x) \in [0, T] \times \mathbb{R}^d$*;*
- (v) *for any* $T > 0$*, there is* $\lambda_T > 0$ *such that*

$$\int_0^T \int_{\mathbb{R}^d} \exp\left[\lambda_T\left(|\nabla \sigma_t|^2 + |\delta(\sigma_t)|^2 + |\delta(b_t)|\right)\right] \mathrm{d}\gamma_d \mathrm{d}t < +\infty.$$

Then the Gaussian measure γ_d *is absolutely continuous under the action of the stochastic flow* $X_{s,t}$ *generated by equation* (1.1)*, and the density functions belong to the class* $L \log L$*.*

The main difference of this result from [8] Theorem 1.1, besides the time-dependence of the coefficients, is that we do not require the continuity of the drift coefficient b_t, at the price of the non-degeneracy assumption of the diffusion coefficient. Note that under the above assumptions, SDE (1.1) has a unique strong solution (see Theorem 1.1 in [20]). Here we give a short remark on the linear growth assumption (iv) of the coefficients. In view of the a priori estimate of the Radon-Nikodym density in Theorem 2.1, this condition is natural for the diffusion coefficient σ. If σ is bounded, then we may consider the drift coefficient b which is locally unbounded, more precisely, b is dominated by the sum of a positive constant and a nonnegative function in $L^{d+1}(\mathbb{R}_+ \times \mathbb{R}^d)$, as in [10, 20]. But we need also the exponential integrability of b with respect to the Gaussian measure γ_d, see (2.7), since the Lebesgue integrability of a function does not imply that it is exponentially integrable with respect to γ_d. Here is an example: let $d = 1$ and $f(x) = \mathbf{1}_{(0,1]}(x)\, x^{-1/2}$, then $\int_{\mathbb{R}^1} f\,\mathrm{d}x = 2$ but for any $\varepsilon > 0$, $\int_{\mathbb{R}^1} e^{\varepsilon f}\,\mathrm{d}\gamma_1 = +\infty$.

The paper is organized as follows. In Section 2 we generalize Theorem 1.1 in [8] to the case where the coefficients depend on time. This requires a careful analysis of the dependence on time of several quantities. Then in Section 3 we prove a limit theorem which is a modification of Theorem 2.2 in [10]. Finally we give in Section 4 the proof of the main result. As an application of our main result, we consider the corresponding Fokker-Planck equation and we show that if the initial value is absolutely continuous with respect to the Lebesgue measure, then so is its solution, see Theorem 4.3.

2 The case when b is continuous

In this section, we generalize [8] Theorem 1.1 to the case where the coefficients depend on time. First we prove an a priori estimate for the L^p-norm of the Radon-Nikodym density, which is an extension of Theorem 2.2 in [8]. For the moment, we assume that $\sigma \in C(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m)$ and $b \in C(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ such that for any $T \geq 0$, σ_t and b_t are smooth functions of the spacial variable x with compact support, uniformly for $t \in [0, T]$. Then it is well known that the solution $X_{s,t}$ of (1.1) is a stochastic flow of diffeomorphisms on $\mathbb{R}^d$. Let $K_{s,t} = \frac{\mathrm{d}(X_{s,t})_{\#}\gamma_d}{\mathrm{d}\gamma_d}$ and $\tilde{K}_{s,t} = \frac{\mathrm{d}(X_{s,t}^{-1})_{\#}\gamma_d}{\mathrm{d}\gamma_d}$, then by Lemma 4.3.1 in [15],

$$\tilde{K}_{s,t}(x) = \exp\bigg(- \int_s^t \langle \delta(\sigma_u)(X_{s,u}(x)), \circ\, \mathrm{d}w_u \rangle - \int_s^t \delta(\tilde{b}_u)(X_{s,u}(x))\,\mathrm{d}u \bigg), \tag{2.1}$$

where $\circ\, \mathrm{d}w_u$ denotes the Stratonovich differential and $\tilde{b}_u = b_u - \frac{1}{2}\sum_{j=1}^m \langle \sigma_u^{\cdot j}, \nabla \sigma_u^{\cdot j} \rangle$. Recall that $\sigma_u^{\cdot j}$ is the j-th column of σ_u, $j = 1, \cdots, m$. Though the density $K_{s,t}$ does not have such an explicit expression, it is easy to know that

$$K_{s,t}(x) = \big[\tilde{K}_{s,t}\big(X_{s,t}^{-1}(x)\big)\big]^{-1}. \tag{2.2}$$

Theorem 2.1. *For any $p > 1$,*

$$\begin{aligned} &\|K_{s,t}\|_{L^p(\mathbb{P}\times\gamma_d)} \\ &\leq \bigg[\frac{1}{t-s} \int_s^t \int_{\mathbb{R}^d} \exp\Big(p(t-s)\big[2|\delta(b_u)| + \|\sigma_u\|^2 + \|\nabla \sigma_u\|^2 + 2(p-1)|\delta(\sigma_u)|^2\big] \Big) \mathrm{d}\gamma_d \mathrm{d}u \bigg]^{\frac{p-1}{p(2p-1)}}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 2.2 in [8], by keeping in mind the time-dependence of the coefficients. We first rewrite the density (2.1) using Itô integral:

$$\tilde{K}_{s,t}(x) = \exp\bigg(- \int_s^t \langle \delta(\sigma_u)(X_{s,u}(x)), \mathrm{d}w_u \rangle - \int_s^t \Big[\delta(\tilde{b}_u) + \frac{1}{2}\sum_{j=1}^m \big\langle \sigma_u^{\cdot j}, \nabla \delta(\sigma_u^{\cdot j}) \big\rangle \Big] (X_{s,u}(x))\,\mathrm{d}u \bigg). \tag{2.3}$$

It is easy to show that (see [8] Lemma 2.1)

$$\delta(\tilde b_u)+\frac{1}{2}\sum_{j=1}^m\big\langle\sigma_u^{\cdot j},\nabla\delta(\sigma_u^{\cdot j})\big\rangle=\delta(b_u)+\frac{1}{2}\|\sigma_u\|^2+\frac{1}{2}\sum_{j=1}^m\big\langle\nabla\sigma_u^{\cdot j},(\nabla\sigma_u^{\cdot j})^*\big\rangle.$$

To simplify the notation, denote the right hand side of the above equality by Φ_u. Then $\tilde K_{s,t}(x)$ is expressed as

$$\tilde K_{s,t}(x)=\exp\bigg(-\int_s^t\langle\delta(\sigma_u)(X_{s,u}(x)),\mathrm{d}w_u\rangle-\int_s^t\Phi_u(X_{s,u}(x))\,\mathrm{d}u\bigg).$$

Using relation (2.2), we have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}[K_{s,t}^p(x)]\,\mathrm{d}\gamma_d(x)&=\mathbb{E}\int_{\mathbb{R}^d}\big[\tilde K_{s,t}\big(X_{s,t}^{-1}(x)\big)\big]^{-p}\,\mathrm{d}\gamma_d(x)\\&=\mathbb{E}\int_{\mathbb{R}^d}\big[\tilde K_{s,t}(y)\big]^{-p}\tilde K_{s,t}(y)\,\mathrm{d}\gamma_d(y)\\&=\int_{\mathbb{R}^d}\mathbb{E}\big[\big(\tilde K_{s,t}(x)\big)^{-p+1}\big]\,\mathrm{d}\gamma_d(x).\end{aligned}\tag{2.4}$$

Fixing an arbitrary $r>0$, we get

$$\begin{aligned}\big(\tilde K_{s,t}(x)\big)^{-r}&=\exp\bigg(r\int_s^t\langle\delta(\sigma_u)(X_{s,u}(x)),\mathrm{d}w_u\rangle+r\int_s^t\Phi_u(X_{s,u}(x))\,\mathrm{d}u\bigg)\\&=\exp\bigg(r\int_s^t\langle\delta(\sigma_u)(X_{s,u}(x)),\mathrm{d}w_u\rangle-r^2\int_s^t\big|\delta(\sigma_u)(X_{s,u}(x))\big|^2\,\mathrm{d}u\bigg)\\&\quad\times\exp\bigg(\int_s^t\big(r^2|\delta(\sigma_u)|^2+r\Phi_u\big)(X_{s,u}(x))\,\mathrm{d}u\bigg).\end{aligned}$$

Cauchy-Schwarz's inequality gives

$$\begin{aligned}\mathbb{E}\big[\big(\tilde K_{s,t}(x)\big)^{-r}\big]&\le\bigg[\mathbb{E}\exp\bigg(2r\int_s^t\langle\delta(\sigma_u)(X_{s,u}(x)),\mathrm{d}w_u\rangle-2r^2\int_s^t\big|\delta(\sigma_u)(X_{s,u}(x))\big|^2\,\mathrm{d}u\bigg)\bigg]^{1/2}\\&\quad\times\bigg[\mathbb{E}\exp\bigg(\int_s^t\big(2r^2|\delta(\sigma_u)|^2+2r\Phi_u\big)(X_{s,u}(x))\,\mathrm{d}u\bigg)\bigg]^{1/2}\\&=\bigg[\mathbb{E}\exp\bigg(\int_s^t\big(2r^2|\delta(\sigma_u)|^2+2r\Phi_u\big)(X_{s,u}(x))\,\mathrm{d}u\bigg)\bigg]^{1/2},\end{aligned}\tag{2.5}$$

since by the Novikov condition, the first term on the right hand side is the expectation of a martingale. Let

$$\Phi_u^{(r)}=2r|\delta(b_u)|+r\big(\|\sigma_u\|^2+\|\nabla\sigma_u\|^2+2r|\delta(\sigma_u)|^2\big).$$

Then by (2.5), along with the definition of Φ_u and Cauchy-Schwarz's inequality, we obtain

$$\int_{\mathbb{R}^d}\mathbb{E}\big[\big(\tilde K_{s,t}(x)\big)^{-r}\big]\mathrm{d}\gamma_d(x)\le\bigg[\int_{\mathbb{R}^d}\mathbb{E}\exp\bigg(\int_s^t\Phi_u^{(r)}(X_{s,u}(x))\,\mathrm{d}u\bigg)\,\mathrm{d}\gamma_d(x)\bigg]^{1/2}.\tag{2.6}$$

By Jensen's inequality,

$$\exp\bigg(\int_s^t\Phi_u^{(r)}(X_{s,u}(x))\,\mathrm{d}u\bigg)=\exp\bigg(\int_s^t(t-s)\,\Phi_u^{(r)}(X_{s,u}(x))\,\frac{\mathrm{d}u}{t-s}\bigg)$$

$$\leq \frac{1}{t-s}\int_s^t e^{(t-s)\,\Phi_u^{(r)}(X_{s,u}(x))}\,\mathrm{d}u.$$

Define $I_{s,t} = \sup_{s\leq u\leq t}\int_{\mathbb{R}^d}\mathbb{E}[K_{s,u}^p(x)]\,\mathrm{d}\gamma_d(x)$. Integrating on both sides of the above inequality and by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\exp\bigg(\int_s^t \Phi_u^{(r)}(X_{s,u}(x))\,\mathrm{d}u\bigg)\mathrm{d}\gamma_d(x) &\leq \frac{1}{t-s}\int_s^t \mathbb{E}\int_{\mathbb{R}^d} e^{(t-s)\,\Phi_u^{(r)}(X_{s,u}(x))}\,\mathrm{d}\gamma_d(x)\,\mathrm{d}u\\ &= \frac{1}{t-s}\int_s^t \mathbb{E}\int_{\mathbb{R}^d} e^{(t-s)\,\Phi_u^{(r)}(y)}K_{s,u}(y)\,\mathrm{d}\gamma_d(y)\,\mathrm{d}u\\ &\leq \frac{1}{t-s}\int_s^t \big\|e^{(t-s)\,\Phi_u^{(r)}}\big\|_{L^q(\gamma_d)}\|K_{s,u}\|_{L^p(\mathbb{P}\times\gamma_d)}\,\mathrm{d}u\\ &\leq \bigg(\frac{1}{t-s}\int_s^t \big\|e^{(t-s)\,\Phi_u^{(r)}}\big\|_{L^q(\gamma_d)}\mathrm{d}u\bigg)\,I_{s,t}^{1/p},\end{aligned}$$

where q is the conjugate number of p. Thus it follows from (2.6) and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\big[\big(\tilde K_{s,t}(x)\big)^{-r}\big]\,\mathrm{d}\gamma_d(x) &\leq \bigg(\frac{1}{t-s}\int_s^t \big\|e^{(t-s)\,\Phi_u^{(r)}}\big\|_{L^q(\gamma_d)}\mathrm{d}u\bigg)^{1/2} I_{s,t}^{1/2p}\\ &\leq \bigg(\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{q(t-s)\,\Phi_u^{(r)}}\,\mathrm{d}\gamma_d\mathrm{d}u\bigg)^{1/2q} I_{s,t}^{1/2p}.\end{aligned}$$

Taking $r = p-1$ in the above estimate and by (2.4), we obtain

$$\int_{\mathbb{R}^d}\mathbb{E}[K_{s,t}^p(x)]\,\mathrm{d}\gamma_d(x) \leq \bigg(\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{q(t-s)\,\Phi_u^{(p-1)}}\,\mathrm{d}\gamma_d\mathrm{d}u\bigg)^{1/2q} I_{s,t}^{1/2p}.$$

For any nonnegative measurable function $g:\mathbb{R}_+\to\mathbb{R}_+$, using the power series expansion of the exponential function, it is easy to know that the quantity $\frac{1}{t-s}\int_s^t e^{(t-s)g_u}\mathrm{d}u$ is increasing in t and decreasing in s. Thus we have

$$I_{s,t} \leq \bigg(\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{q(t-s)\,\Phi_u^{(p-1)}}\,\mathrm{d}\gamma_d\mathrm{d}u\bigg)^{1/2q} I_{s,t}^{1/2p}.$$

Solving this inequality for $I_{s,t}$, we get

$$\int_{\mathbb{R}^d}\mathbb{E}[K_{s,t}^p(x)]\,\mathrm{d}\gamma_d(x) \leq I_{s,t} \leq \bigg(\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d}\exp\bigg[\frac{p(t-s)}{p-1}\Phi_u^{(p-1)}\bigg]\,\mathrm{d}\gamma_d\mathrm{d}u\bigg)^{\frac{p-1}{2p-1}}.$$

The desired result follows from the definition of $\Phi_u^{(p-1)}$. □

The rest of this section follows the argument in Section 3 of [8], by taking care of the time-dependence of the coefficients. We assume the following conditions:

(A1) $\sigma:\mathbb{R}_+\times\mathbb{R}^d\to\mathcal{M}_{d,m}$ and $b:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ are jointly continuous and for any $T>0$, there is $L_T>0$ such that $\|\sigma_t(x)\|\vee|b_t(x)|\leq L_T(1+|x|)$ for all $(t,x)\in[0,T]\times\mathbb{R}^d$;

(A2) for any $t\geq 0$, $\sigma_t\in\cap_{p>1}\mathbb{D}_1^p(\gamma_d)$ and $\delta(b_t)$ exists;

(A3) for any $T>0$, there is $\lambda_T>0$, such that

$$\Sigma_T := \int_0^T\int_{\mathbb{R}^d}\exp\big[\lambda_T\big(\|\nabla\sigma_t\|^2+|\delta(\sigma_t)|^2+|\delta(b_t)|\big)\big]\mathrm{d}\gamma_d\mathrm{d}t < +\infty.$$

As we choose the Gaussian measure γ_d as the reference measure, it is natural to regularize functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ using the Ornstein-Uhlenbeck semigroup $(P_\varepsilon)_{\varepsilon>0}$ on $\mathbb{R}^d$:

$$P_\varepsilon f_t(x) = \int_{\mathbb{R}^d} f_t\big(e^{-\varepsilon}x + \sqrt{1-e^{-2\varepsilon}}\, y\big)\mathrm{d}\gamma_d(y).$$

First we have the following simple result (see [8] Lemma 3.1 for the proof).

Lemma 2.2. *Assume that $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ has linear growth with respect to the spacial variable: there is $L_T > 0$ such that $|f_t(x)| \le L_T(1+|x|)$ for all $(t,x) \in [0,T]\times\mathbb{R}^d$, then*

$$\sup_{0\le t\le T}\ \sup_{0<\varepsilon\le 1} |P_\varepsilon f_t(x)| \le L_T(1+M_1)(1+|x|),$$

where $M_1 = \int_{\mathbb{R}^d} |y|\,\mathrm{d}\gamma_d(y)$. If moreover f is jointly continuous, then for any $R>0$,

$$\lim_{\varepsilon\downarrow 0}\ \sup_{0\le t\le T}\ \sup_{x\in B(R)} |P_\varepsilon f_t(x) - f_t(x)| = 0.$$

We introduce a sequence of cut-off functions $\varphi_n \in C_c^\infty(\mathbb{R}^d,[0,1])$ satisfying

$$\varphi_n(x) = 1 \text{ if } |x|\le n, \quad \varphi_n(x) = 0 \text{ if } |x| \ge n+2 \quad \text{and} \quad \|\nabla\varphi_n\|_\infty \le 1.$$

Now define

$$\sigma_t^n = \varphi_n P_{1/n}\sigma_t, \quad b_t^n = \varphi_n P_{1/n} b_t$$

and consider

$$\mathrm{d}X_{s,t}^n = \sigma_t^n(X_{s,t}^n)\,\mathrm{d}w_t + b_t^n(X_{s,t}^n)\,\mathrm{d}t, \quad t\ge s, \quad X_{s,s}^n = x.$$

By the discussions at the beginning of this section, we know that the density function $K_{s,t}^n$ of $(X_{s,t}^n)_\#\gamma_d$ with respect to γ_d exists. We want to find an explicit upper bound for the norms of $K_{s,t}^n$. To this end, applying Theorem 2.1 with $p=2$, we obtain

$$\|K_{s,t}^n\|_{L^2(\mathbb{P}\times\gamma_d)} \le \left[\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} \exp\Big(2(t-s)\big[2|\delta(b_u^n)| + \|\sigma_u^n\|^2 + \|\nabla\sigma_u^n\|^2 + 2|\delta(\sigma_u^n)|^2\big]\Big)\mathrm{d}\gamma_d\mathrm{d}u\right]^{\frac{1}{6}}.$$

By the definitions of σ_t^n and b_t^n, it is easy to show that (see Lemma 3.2 in [8])

$$\begin{aligned}&2|\delta(b_u^n)| + \|\sigma_u^n\|^2 + \|\nabla\sigma_u^n\|^2 + 2|\delta(\sigma_u^n)|^2 \\ &\quad \le P_{1/n}\big(2|b_u| + 2e|\delta(b_u)| + 7\|\sigma_u\|^2 + 2\|\nabla\sigma_u\|^2 + 2e^2|\delta(\sigma_u)|^2\big).\end{aligned}$$

Let

$$\Phi_u^{(1)} = 14\big(|b_u| + \|\sigma_u\|^2\big) \quad \text{and} \quad \Phi_u^{(2)} = 4e^2\big(|\delta(b_u)| + \|\nabla\sigma_u\|^2 + |\delta(\sigma_u)|^2\big),$$

then by Jensen's inequality and the quasi-invariance of γ_d under $P_{1/n}$, we obtain

$$\|K_{s,t}^n\|_{L^2(\mathbb{P}\times\gamma_d)} \le \left[\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{(t-s)\left(\Phi_u^{(1)}+\Phi_u^{(2)}\right)}\mathrm{d}\gamma_d\mathrm{d}u\right]^{\frac{1}{6}}. \tag{2.7}$$

Let $F_{s,t}$ be the quantity in the square bracket on the right hand side of (2.7). By Cauchy's inequality,

$$F_{s,t} \le \left[\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{2(t-s)\Phi_u^{(1)}}\mathrm{d}\gamma_d\mathrm{d}u\right]^{\frac{1}{2}} \cdot \left[\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{2(t-s)\Phi_u^{(2)}}\mathrm{d}\gamma_d\mathrm{d}u\right]^{\frac{1}{2}}. \tag{2.8}$$

By the growth conditions on b and σ, we have for any $u \le T$,

$$\Phi_u^{(1)} \le 14\big[L_T(1+|x|) + L_T^2(1+|x|)^2\big] \le 14L_T(1+L_T)(1+|x|)^2.$$

As a consequence, if $t-s \le 1/112L_T(1+L_T)$, we obtain

$$\begin{aligned}\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{2(t-s)\Phi_u^{(1)}}\mathrm{d}\gamma_d\mathrm{d}u &\le \frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{28(t-s)L_T(1+L_T)(1+|x|)^2}\mathrm{d}\gamma_d\mathrm{d}u \\ &= \int_{\mathbb{R}^d} e^{28(t-s)L_T(1+L_T)(1+|x|)^2}\mathrm{d}\gamma_d \\ &\le \int_{\mathbb{R}^d} e^{(1+|x|)^2/4}\mathrm{d}\gamma_d =: M_2 \end{aligned}\tag{2.9}$$

which is finite. Again noticing that for any nonnegative measurable function $g:\mathbb{R}_+ \to \mathbb{R}_+$, using the power series expansion of the exponential function, the quantity $\frac{1}{t-s}\int_s^t e^{(t-s)g_u}\mathrm{d}u$ is increasing in t and decreasing in s. Hence by assumption (A3), if $t-s \le \lambda_T/8e^2$, then

$$\frac{1}{t-s}\int_s^t\int_{\mathbb{R}^d} e^{2(t-s)\Phi_u^{(2)}}\mathrm{d}\gamma_d\mathrm{d}u \le \frac{8e^2}{\lambda_T}\int_0^T\int_{\mathbb{R}^d} e^{\lambda_T(|\delta(b_u)|+\|\nabla\sigma_u\|^2+|\delta(\sigma_u)|^2)}\mathrm{d}\gamma_d\mathrm{d}u = \frac{8e^2}{\lambda_T}\Sigma_T. \tag{2.10}$$

Set

$$T_0 = \frac{1}{112L_T(1+L_T)} \wedge \frac{\lambda_T}{8e^2},$$

then for all $t-s \le T_0$, we obtain by combining (2.8)–(2.10) that

$$F_{s,t} \le \left(\frac{M_2\Sigma_T}{T_0}\right)^{\frac{1}{2}}.$$

Substituting this estimate into (2.7), we deduce that for all $0 \le s < t \le T$ with $t-s \le T_0$,

$$\sup_{n\ge1}\|K_{s,t}^n\|_{L^2(\mathbb{P}\times\gamma_d)} \le \Lambda_{T_0} := \left(\frac{M_2\Sigma_T}{T_0}\right)^{\frac{1}{12}}. \tag{2.11}$$

Having this explicit estimate in hand, we can now prove

Theorem 2.3. *Under the assumptions* (A1)–(A3)*, there are constants* $C_1, C_2 > 0$ *such that*

$$\sup_{n\ge1}\mathbb{E}\int_{\mathbb{R}^d} K_{s,t}^n|\log K_{s,t}^n|\,\mathrm{d}\gamma_d \le 2\,C_1T^{1/2}\Lambda_{T_0} + C_2T\Lambda_{T_0}^2, \quad \text{for all } 0 \le s < t \le T.$$

Proof. The proof is similar to Theorem 3.3 in [8]. By (2.2) and (2.1), we have

$$K_{s,t}^n(X_{s,t}^n(x)) = \big[\tilde K_{s,t}^n(x)\big]^{-1} = \exp\bigg(\int_s^t\langle\delta(\sigma_u^n)(X_{s,u}^n(x)),\mathrm{d}w_u\rangle + \int_s^t \Phi_u^n(X_{s,u}^n(x))\,\mathrm{d}u\bigg),$$

with

$$\Phi_u^n = \delta(b_u^n) + \frac{1}{2}\|\sigma_u^n\|^2 + \frac{1}{2}\sum_{j=1}^m\big\langle\nabla(\sigma_u^n)^{\cdot j},(\nabla(\sigma_u^n)^{\cdot j})^*\big\rangle,$$

where $(\sigma_u^n)^{\cdot j}$ is the j-th column of σ_u^n. Thus

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d} K_{s,t}^n|\log K_{s,t}^n|\,\mathrm{d}\gamma_d = \mathbb{E}\int_{\mathbb{R}^d}\big|\log K_{s,t}^n(X_{s,t}^n(x))\big|\,\mathrm{d}\gamma_d(x) \\ &\le \mathbb{E}\int_{\mathbb{R}^d}\bigg|\int_s^t\langle\delta(\sigma_u^n)(X_{s,u}^n(x)),\mathrm{d}w_u\rangle\bigg|\,\mathrm{d}\gamma_d(x) + \mathbb{E}\int_{\mathbb{R}^d}\bigg|\int_s^t\Phi_u^n(X_{s,u}^n(x))\,\mathrm{d}u\bigg|\mathrm{d}\gamma_d(x) \\ &=: I_1 + I_2.\end{aligned}\tag{2.12}$$

Using Burkholder's inequality, we get

$$\mathbb{E}\Big|\int_s^t \langle \delta(\sigma_u^n)(X_{s,u}^n(x)), \mathrm{d}w_u\rangle\Big| \le 2\,\mathbb{E}\Big[\Big(\int_s^t \big|\delta(\sigma_u^n)(X_{s,u}^n(x))\big|^2\,\mathrm{d}u\Big)^{1/2}\Big].$$

By Cauchy's inequality,

$$I_1 \le 2\Big[\int_s^t \mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)(X_{s,u}^n(x))\big|\,\mathrm{d}\gamma_d(x)\mathrm{d}u\Big]^{1/2}. \tag{2.13}$$

If $u \in [s, s+T_0]$, then by Cauchy's inequality and (2.11),

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)(X_{s,u}^n(x))\big|^2\,\mathrm{d}\gamma_d(x) &= \mathbb{E}\int_{\mathbb{R}^d} |\delta(\sigma_u^n)(y)|^2 K_{s,u}^n(y)\,\mathrm{d}\gamma_d(y)\\ &\le \|\delta(\sigma_u^n)\|_{L^4(\gamma_d)}^2 \|K_{s,u}^n\|_{L^2(\mathbb{P}\times\gamma_d)}\\ &\le \Lambda_{T_0}\|\delta(\sigma_u^n)\|_{L^4(\gamma_d)}^2.\end{aligned}$$

Now for $u \in]s+T_0, s+2T_0]$, we shall use the flow property:

$$X_{s,u}^n(x,w) = X_{s+T_0,u}^n\big(X_{s,s+T_0}^n(x,w), w\big).$$

Therefore,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)(X_{s,u}^n(x))\big|^2\,\mathrm{d}\gamma_d(x) &= \mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)\big[X_{s+T_0,u}^n\big(X_{s,s+T_0}^n(x)\big)\big]\big|^2\,\mathrm{d}\gamma_d(x)\\ &= \mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)\big(X_{s+T_0,u}^n(y)\big)\big|^2 K_{s,s+T_0}^n(y)\,\mathrm{d}\gamma_d(y)\end{aligned}$$

which is dominated, using Cauchy's inequality, by

$$\begin{aligned}&\Big(\mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)\big(X_{s+T_0,u}^n(y)\big)\big|^4\,\mathrm{d}\gamma_d(y)\Big)^{1/2} \|K_{s,s+T_0}^n\|_{L^2(\mathbb{P}\times\gamma_d)}\\ &\le \Big(\Lambda_{T_0}\|\delta(\sigma_u^n)\|_{L^8(\gamma_d)}^4\Big)^{1/2}\Lambda_{T_0} = \Lambda_{T_0}^{1+2^{-1}}\|\delta(\sigma_u^n)\|_{L^8(\gamma_d)}^2.\end{aligned}$$

Repeating this procedure, we finally obtain, for all $u \in [s,T]$,

$$\mathbb{E}\int_{\mathbb{R}^d} \big|\delta(\sigma_u^n)(X_{s,u}^n(x))\big|^2\,\mathrm{d}\gamma_d(x) \le \Lambda_{T_0}^{1+2^{-1}+\ldots+2^{-N+1}}\|\delta(\sigma_u^n)\|_{L^{2^{N+1}}(\gamma_d)}^2 \le \Lambda_{T_0}^2\|\delta(\sigma_u^n)\|_{L^{2^{N+1}}(\gamma_d)}^2,$$

where $N \in \mathbb{Z}_+$ is the unique integer such that $(N-1)T_0 < T \le NT_0$. This along with (2.13) leads to

$$\begin{aligned}I_1 &\le 2\Big[\int_s^t \Lambda_{T_0}^2\|\delta(\sigma_u^n)\|_{L^{2^{N+1}}(\gamma_d)}^2\mathrm{d}u\Big]^{1/2}\\ &\le 2\Lambda_{T_0}T^{2^{-1}-2^{-N-1}}\Big[\int_0^T\int_{\mathbb{R}^d} |\delta(\sigma_u^n)|^{2^{N+1}}\,\mathrm{d}\gamma_d\mathrm{d}u\Big]^{2^{-N-1}}.\end{aligned}$$

Since $|\delta(\sigma_u^n)| \le P_{1/n}\big(\|\sigma_u\| + e|\delta(\sigma_u)|\big)$, by Jensen's inequality, the invariance of γ_d under the Ornstein-Uhlenbeck group and the assumption on σ, it is easy to know that

$$\|\delta(\sigma_\cdot^n)\|_{L^{2^{N+1}}(\mathcal{L}_T\times\gamma_d)} \le \big\|\,\|\sigma_u\| + e|\delta(\sigma_u)|\,\big\|_{L^{2^{N+1}}(\mathcal{L}_T\times\gamma_d)} =: C_1 \tag{2.14}$$

whose right hand side is finite. Here $\mathcal{L}_T$ means the Lebesgue measure restricted on the interval $[0,T]$. Therefore

$$I_1 \le 2C_1 T^{1/2}\Lambda_{T_0}. \tag{2.15}$$

The same manipulation works for the term I_2 and we get

$$I_2 \le C_2 T \Lambda_{T_0}^2, \tag{2.16}$$

where

$$C_2 = \bigg\| |b_\cdot| + e|\delta(b_\cdot)| + \frac{3}{2}\|\sigma_\cdot\|^2 + \|\nabla\sigma_\cdot\|^2 \bigg\|_{L^{2^N}(\mathcal{L}_T\times\gamma_d)} < \infty. \tag{2.17}$$

Now we draw the conclusion from (2.12), (2.15) and (2.16). □

It follows from Theorem 2.3 that the family $\{K_{s,t}^n\}_{n\ge1}$ is weakly compact in $L^1(\Omega\times\mathbb{R}^d)$. Along a subsequence, $K_{s,t}^n$ converges weakly to some $K_{s,t}\in L^1(\Omega\times\mathbb{R}^d)$ as $n\to\infty$. Let

$$\mathcal{C} = \bigg\{ u\in L^1(\Omega\times\mathbb{R}^d):\ u\ge 0,\ \int_{\mathbb{R}^d}\mathbb{E}(u\log u)\,\mathrm{d}\gamma_d \le 2\,C_1T^{1/2}\Lambda_{T_0} + C_2T\Lambda_{T_0}^2\bigg\}.$$

By the convexity of the function $s\to s\log s$, it is clear that $\mathcal{C}$ is a convex subset of $L^1(\Omega\times\mathbb{R}^d)$. Since the weak closure of $\mathcal{C}$ coincides with the strong one, there exists a sequence of functions $u^{(n)}\in\mathcal{C}$ which converges to $K_{s,t}$ in $L^1(\Omega\times\mathbb{R}^d)$. Along a subsequence, $u^{(n)}$ converges to $K_{s,t}$ almost everywhere. Hence by Fatou's lemma, we get

$$\int_{\mathbb{R}^d}\mathbb{E}(K_{s,t}\log K_{s,t})\,\mathrm{d}\gamma_d \le 2\,C_1T^{1/2}\Lambda_{T_0} + C_2T\Lambda_{T_0}^2. \tag{2.18}$$

Next we have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}(K_{s,t}|\log K_{s,t}|)\,\mathrm{d}\gamma_d &= \bigg(\int_{\{K_{s,t}>1\}} + \int_{\{K_{s,t}\le1\}}\bigg)K_{s,t}|\log K_{s,t}|\,\mathrm{d}(\mathbb{P}\times\gamma_d)\\ &= \int_{\{K_{s,t}>1\}} K_{s,t}\log K_{s,t}\,\mathrm{d}(\mathbb{P}\times\gamma_d) - \int_{\{K_{s,t}\le1\}} K_{s,t}\log K_{s,t}\,\mathrm{d}(\mathbb{P}\times\gamma_d).\end{aligned}$$

Since $x\log x\ge -e^{-1}$ for all $x\in[0,1]$, we obtain from (2.18) that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}(K_{s,t}|\log K_{s,t}|)\,\mathrm{d}\gamma_d &= \int_{\Omega\times\mathbb{R}^d} K_{s,t}\log K_{s,t}\mathrm{d}(\mathbb{P}\times\gamma_d) - 2\int_{\{K_{s,t}\le1\}} K_{s,t}\log K_{s,t}\,\mathrm{d}(\mathbb{P}\times\gamma_d)\\ &\le 2\,C_1T^{1/2}\Lambda_{T_0} + C_2T\Lambda_{T_0}^2 + 2e^{-1}.\end{aligned} \tag{2.19}$$

Finally we can prove the main result of this section.

Theorem 2.4. *Suppose the conditions* (A1)–(A3) *and that SDE* (1.1) *has pathwise uniqueness. Then for any* $T>0$ *and* $0\le s<t\le T$, *almost surely* $(X_{s,t})_\#\gamma_d = K_{s,t}\gamma_d$ *and the estimate* (2.19) *holds.*

Proof. The proof is similar to that of Theorem 3.4 in [8]. □

3 Limit theorem

Now we turn to establish a limit theorem, following the idea of Theorem 2.2 in [10] (see also Theorem 1 on p.87 of [13]). First we need a version of the Krylov estimate.

Lemma 3.1. *Assume that for some $T > 0$,*

(1) *σ and b have linear growth with respect to the spacial variable, uniformly in $t \in [0, T]$;*

(2) *σ is uniformly non-degenerate: there is $c_\sigma > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}^d$, $\sigma_t(x)\sigma_t^*(x) \geq c_\sigma \mathrm{Id}$.*

Let $X_{s,t}(x)$ be a solution to (1.1), then for any Borel function $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ and $\lambda > 0$, we have

$$\mathbb{E}\int_s^T e^{-\lambda t} f(t, X_{s,t}(x))\,\mathrm{d}t \leq N\|f\|_{L^{d+1}(\mathbb{R}_+\times\mathbb{R}^d)},$$

where N is a constant depending only on T, d, c_σ, λ and $x \in \mathbb{R}^d$.

Proof. The proof is similar to that of [10] Corollary 3.2. In our case, the inequality (3.2) on p.769 of [10] becomes

$$\mathbb{E}\int_s^{T\wedge\tau_R} e^{-\lambda t} f(t, X_{s,t}(x))\,\mathrm{d}t \leq C_{d,c_\sigma}(\mathbb{A} + \mathbb{B}^2)^{\frac{d}{2(d+1)}} \left(\int_s^\infty\int_{B(R)} |f(t,y)|^{d+1}\mathrm{d}y\mathrm{d}t\right)^{\frac{1}{d+1}}, \tag{3.1}$$

where τ_R is the first exit time of $X_{s,t}(x)$ from the ball $B(R)$, and by the linear growth of σ_t, b_t, we have

$$\mathbb{A} = \mathbb{E}\int_s^{T\wedge\tau_R} e^{-\lambda t}\cdot\frac{1}{2}\|\sigma_t(X_{s,t}(x))\|^2\mathrm{d}t \leq C_T\int_s^T \mathbb{E}(1 + |X_{s,t}(x)|^2)\,\mathrm{d}t \leq C_T'(1 + |x|^2),$$

and

$$\mathbb{B} = \mathbb{E}\int_s^{T\wedge\tau_R} e^{-\lambda t}|b_t(X_{s,t}(x))|\,\mathrm{d}t \leq C_T\int_s^T \mathbb{E}(1 + |X_{s,t}(x)|)\,\mathrm{d}t \leq C_T'(1 + |x|).$$

Now letting $R \to \infty$ in (3.1) gives the desired estimate. □

The next result, which is a stronger version of Lemma 5.2 in [10], will be used to prove the limit theorem.

Lemma 3.2. *Let η_t and $\{\eta_t^n : n \geq 1\}$ be $\mathcal{M}_{d,m}$-valued stochastic processes, and w, w^n Brownian motions such that the Itô integrals $I_t = \int_0^t \eta_s\,\mathrm{d}w_s$ and $I_t^n = \int_0^t \eta_s^n\,\mathrm{d}w_s^n$ are well defined. Assume that for some $\alpha > 0$,*

$$C_0 := \left(\mathbb{E}\int_0^T \|\eta_s\|^{2+\alpha}\mathrm{d}s\right)\bigvee\left(\sup_{n\geq1}\mathbb{E}\int_0^T \|\eta_s^n\|^{2+\alpha}\mathrm{d}s\right) < \infty,$$

and $\eta_t^n \to \eta_t$ and $w_t^n \to w_t$ in probability for all $t \in [0, T]$. Then

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{0\leq t\leq T}|I_t^n - I_t|^2\right) = 0.$$

Proof. For any $R>0$, define $\psi_R:\mathbb{R}\to\mathbb{R}$ by $\psi_R(x)=\big((-R)\vee x\big)\wedge R$. Then ψ_R is uniformly continuous. For a matrix η, we denote by $\psi_R(\eta)$ the matrix $(\psi_R(\eta^{ij}))$. For all $t\in[0,T]$, since $\eta_t^n\to\eta_t$ in probability, we know that $\psi_R(\eta_t^n)$ converges to $\psi_R(\eta_t)$ in probability. Moreover, they are uniformly bounded, then by Lemma 5.2 in [10],

$$\lim_{n\to\infty}\mathbb{P}\bigg(\sup_{0\le t\le T}\bigg|\int_0^t\psi_R(\eta_s^n)\,\mathrm{d}w_s^n-\int_0^t\psi_R(\eta_s)\,\mathrm{d}w_s\bigg|\ge\varepsilon\bigg)=0$$

for every $\varepsilon>0$. Since ψ_R is bounded, the sequence $\int_0^t\psi_R(\eta_t^n)\,\mathrm{d}w_t^n$ is uniformly bounded in any $L^p(\mathbb{P})$, hence

$$\lim_{n\to\infty}\mathbb{E}\bigg(\sup_{0\le t\le T}\bigg|\int_0^t\psi_R(\eta_s^n)\,\mathrm{d}w_s^n-\int_0^t\psi_R(\eta_s)\,\mathrm{d}w_s\bigg|^2\bigg)=0. \tag{3.2}$$

We have

$$\begin{aligned}|I_t^n-I_t|^2&\le 3\bigg|\int_0^t\eta_s^n\,\mathrm{d}w_s^n-\int_0^t\psi_R(\eta_s^n)\,\mathrm{d}w_s^n\bigg|^2+3\bigg|\int_0^t\psi_R(\eta_s^n)\,\mathrm{d}w_s^n-\int_0^t\psi_R(\eta_s)\,\mathrm{d}w_s\bigg|^2\\&\quad+3\bigg|\int_0^t\psi_R(\eta_s)\,\mathrm{d}w_s-\int_0^t\eta_s\,\mathrm{d}w_s\bigg|^2\\&=:3\big(J_1(t)+J_2(t)+J_3(t)\big).\end{aligned} \tag{3.3}$$

By Burkholder's inequality,

$$\mathbb{E}\Big(\sup_{0\le t\le T}J_1(t)\Big)\le 4\,\mathbb{E}\int_0^T\big\|\eta_s^n-\psi_R(\eta_s^n)\big\|^2\mathrm{d}s.$$

Let $\mathcal{L}_T$ be the Lebesgue measure restricted on the interval $[0,T]$, then by Hölder's inequality,

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}J_1(t)\Big)&\le 4\int_{[0,T]\times\Omega}\mathbf{1}_{\{\|\eta_s^n\|>R\}}\|\eta_s^n\|^2\mathrm{d}(\mathcal{L}_T\otimes\mathbb{P})\\&\le 4\big[(\mathcal{L}_T\otimes\mathbb{P})(\|\eta_s^n\|>R)\big]^{\alpha/(2+\alpha)}\bigg(\int_{[0,T]\times\Omega}\|\eta_s^n\|^{2+\alpha}\mathrm{d}(\mathcal{L}_T\otimes\mathbb{P})\bigg)^{2/(2+\alpha)}\\&\le\frac{4}{R^\alpha}\mathbb{E}\int_0^T\|\eta_s^n\|^{2+\alpha}\mathrm{d}s=\frac{4C_0}{R^\alpha}.\end{aligned}$$

Similarly we have $\mathbb{E}(J_3)\le\frac{4C_0}{R^\alpha}$. These estimates together with (3.3) lead to

$$\mathbb{E}\Big(\sup_{0\le t\le T}|I_t^n-I_t|^2\Big)\le\frac{24C_0}{R^\alpha}+3\,\mathbb{E}\bigg(\sup_{0\le t\le T}\bigg|\int_0^t\psi_R(\eta_s^n)\,\mathrm{d}w_s^n-\int_0^t\psi_R(\eta_s)\,\mathrm{d}w_s\bigg|^2\bigg).$$

By (3.2), first letting $n\to\infty$ and then $R\to\infty$, we get the reuslt. □

Suppose we are given two sequences $\sigma^n:[0,T]\times\mathbb{R}^d\to\mathcal{M}_{d,m}$ and $b^n:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ of measurable functions. Consider the SDE

$$\mathrm{d}X_{s,t}^n=\sigma_t^n(X_{s,t}^n)\,\mathrm{d}w_t+b_t^n(X_{s,t}^n)\,\mathrm{d}t,\quad t\ge s,\quad X_{s,s}^n=x. \tag{3.4}$$

We will prove

Proposition 3.3. *Assume that for some $T>0$,*

(1) σ^n *and* b^n *are jointly continuous on* $[0,T]\times\mathbb{R}^d$ *and there is* $L_T>0$*, such that for all* $(t,x)\in[0,T]\times\mathbb{R}^d$*,*

$$\sup_{n\geq 1}\big(\|\sigma_t^n(x)\|\vee|b_t^n(x)|\big)\leq L_T(1+|x|);$$

(2) $\{\sigma^n:n\geq 1\}$ *are uniformly non-degenerate, i.e. there is* $C>0$ *independent of* n *such that for all* $(t,x)\in[0,T]\times\mathbb{R}^d$*,* $\sigma_t^n(x)(\sigma_t^n(x))^*\geq C\,\mathrm{Id}$*;*

(3) *for all* $n\geq 1$*,* (3.4) *has a unique strong solution* $X_{s,t}^n(x)$*;*

(4) *as* $n\to\infty$*,* $\sigma^n\to\sigma$ *in* $L_{loc}^{2(d+1)}([0,T]\times\mathbb{R}^d)$ *and* $b^n\to b$ *in* $L_{loc}^{d+1}([0,T]\times\mathbb{R}^d)$*.*

Then for any $x\in\mathbb{R}^d$ *and* $T>0$*, the sequence* $(X_{s,\cdot}^n(x),w)$ *is tight in* $C([s,T],\mathbb{R}^{d+m})$*, and there exist a subsequence* $\{n_k:k\geq 1\}$ *and a probability space* $\tilde\Omega$ *on which are defined a sequence* $(\tilde X^k,\tilde w^k)$*, a Brownian motion* $(\tilde w_t,\tilde{\mathcal{F}}_t)$ *and an* $\tilde{\mathcal{F}}_t$*-adapted process* $\tilde X$*, such that*

(a) *for each* $k\geq 1$*,* $(X_{s,\cdot}^{n_k}(x),w)$ *and* $(\tilde X^k,\tilde w^k)$ *have the same finite dimensional distributions;*

(b) *almost surely,* $(\tilde X^k,\tilde w^k)\to(\tilde X,\tilde w)$ *as* $k\to\infty$ *uniformly on any finite time interval;*

(c) $(\tilde X,\tilde w)$ *is a weak solution to SDE* (1.1)*.*

Proof. For simplification of notations, we assume $s=0$ and write X_t^n instead of $X_{0,t}^n$. We follow the idea of the proof of Theorem 2.2 in [10] (see also Theorem 1 on p.87 of [13]). In order to apply the Skorohod theorem (see Theorem 4.2 in Chap. I of [11]), we need to verify that the sequence $\{(X^n(x),w):n\geq 1\}$ satisfy the conditions (4.2) and (4.3) on p.17 of [11]. It is enough to do so for the sequence $\{X^n(x):n\geq 1\}$. For each n, $X_0^n(x)=x$, hence condition (4.2) is satisfied. Next by the uniform growth condition (1) on the coefficients, it is easy to know that there is $C_T>0$ such that

$$\sup_{n\geq 1}\mathbb{E}\bigg(\sup_{s\leq u,v\leq t}|X_u^n(x)-X_v^n(x)|^4\bigg)\leq C_T|s-t|^2,\quad 0\leq s<t\leq T. \tag{3.5}$$

Therefore (4.3) is also verified. Then by Skorohod's theorem, there exist a subsequence $X^{n_k}(x)$ and a probability space $\tilde\Omega$ on which are defined a sequence $(\tilde X^k,\tilde w^k)$ and a process $(\tilde X,\tilde w)$, such that the finite dimensional distributions of $(X^{n_k}(x),w)$ and $(\tilde X^k,\tilde w^k)$ coincide, and almost surely, the limits $\tilde X_t^k\to\tilde X_t$, $\tilde w_t^k\to\tilde w_t$ hold uniformly on any finite interval of time. We have by (3.5),

$$\mathbb{E}\big(|\tilde X_s^k-\tilde X_t^k|^4\big)=\mathbb{E}\big(|X_s^{n_k}(x)-X_t^{n_k}(x)|^4\big)\leq C_T|s-t|^2.$$

Using Fatou's lemma, we obtain

$$\mathbb{E}\big(|\tilde X_s-\tilde X_t|^4\big)\leq C_T|s-t|^2,$$

therefore by Kolmogorov's modification theorem, the processes $\tilde X^k$ and $\tilde X$ are continuous. $\tilde w^k$ and $\tilde w$, being Wiener processes, are also continuous.

Let $\mathcal{F}_t$ be the filtration generated by the original Brownian motion w_t appearing in (3.4). Then the process $(X_s^{n_k},w_s)_{s\leq t}$ are independent on the increments of the Brownian motion w after the time t. By the coincidence of the finite dimensional distributions, the processes $(\tilde X_s^k,\tilde w_s^k)_{s\leq t}$ do not depend on the increments of the Brownian motion $\tilde w^k$ after the time t. This property is preserved in the limiting procedure, that is, $(\tilde X_s,\tilde w_s)_{s\leq t}$ is also independent of the increments of $\tilde w$ after t. As a consequence, $\tilde w_t^k$ (resp. $\tilde w_t$) is a Brownian motion with respect to the filtration $\tilde{\mathcal{F}}_t^k$ (resp. $\tilde{\mathcal{F}}_t$) generated by $\{(\tilde X_s^k,\tilde w_s^k):s\leq t\}$ (resp. $\{(\tilde X_s,\tilde w_s):s\leq t\}$). As the process $\tilde X_t^k$ is continuous and $\tilde{\mathcal{F}}_t^k$-adapted, the stochastic integrals considered below make sense.

It remains to prove the assertion (c). By the continuity of σ^k and b^k, it is easy to show that for all $t \geq 0$,

$$\tilde{X}_t^k = x + \int_0^t \sigma_s^k\big(\tilde{X}_s^k\big)\,\mathrm{d}\tilde{w}_s^k + \int_0^t b_s^k\big(\tilde{X}_s^k\big)\,\mathrm{d}s, \tag{3.6}$$

since the processes $(\tilde{X}^k, \tilde{w}^k)$ and $(X^{n_k}(x), w)$ have the same finite dimensional distributions, and $(X^{n_k}(x), w)$ satisfies the SDE (3.4) (see [13] p.89 for a detailed proof). Now we want to take limit $k \to \infty$ in (3.6). Fix some $T > 0$ and consider $t \leq T$. We first show the convergence of the diffusion part. To this end, we fix some integer $k_0 \geq 1$ and define

$$\begin{aligned} I_1(t) &= \int_0^t \sigma_s^k(\tilde{X}_s^k)\,\mathrm{d}\tilde{w}_s^k - \int_0^t \sigma_s^{k_0}(\tilde{X}_s^k)\,\mathrm{d}\tilde{w}_s^k, \\ I_2(t) &= \int_0^t \sigma_s^{k_0}(\tilde{X}_s^k)\,\mathrm{d}\tilde{w}_s^k - \int_0^t \sigma_s^{k_0}(\tilde{X}_s)\,\mathrm{d}\tilde{w}_s, \\ I_3(t) &= \int_0^t \sigma_s^{k_0}(\tilde{X}_s)\,\mathrm{d}\tilde{w}_s - \int_0^t \sigma_s(\tilde{X}_s)\,\mathrm{d}\tilde{w}_s. \end{aligned}$$

By Burkholder's inequality,

$$\begin{aligned} \mathbb{E}\sup_{t\leq T}|I_1(t)| &\leq 2\,\mathbb{E}\bigg[\bigg(\int_0^T \big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s\bigg)^{1/2}\bigg] \\ &\leq 2\bigg(\mathbb{E}\int_0^T \big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s\bigg)^{1/2}. \end{aligned}$$

Take $\varphi \in C(\mathbb{R}_+ \times \mathbb{R}^d, [0,1])$ such that $\varphi(t,x) \equiv 1$ for $|(t,x)| \leq 1/2$ and $\varphi(t,x) = 0$ for $|(t,x)| \geq 1$; define $\varphi_R(t,x) = \varphi(t/R, x/R)$ for $R > 0$. Then

$$\begin{aligned} \mathbb{E}\sup_{t\leq T}|I_1(t)| &\leq 2\bigg(\mathbb{E}\int_0^T \varphi_R\big(s,\tilde{X}_s^k\big)\big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s\bigg)^{1/2} \\ &\quad + 2\bigg(\mathbb{E}\int_0^T \big[1-\varphi_R\big(s,\tilde{X}_s^k\big)\big]\cdot\big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s\bigg)^{1/2}. \end{aligned} \tag{3.7}$$

We have by Lemma 3.1,

$$\begin{aligned} \mathbb{E}\int_0^T \varphi_R\big(s,\tilde{X}_s^k\big)\big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s &\leq Ne^T\big\|\mathbf{1}_{[0,T]\times B(R)}\|\sigma^k - \sigma^{k_0}\|^2\big\|_{L^{d+1}} \\ &= Ne^T\|\sigma^k - \sigma^{k_0}\|^2_{L^{2(d+1)}_{T,R}}, \end{aligned} \tag{3.8}$$

where N is a constant independent of $k \geq 1$ and $\|\cdot\|_{L^{d+1}_{T,R}}$ is the norm in $L^{d+1}([0,T]\times B(R))$. Since σ^k and b^k have uniform linear growth, the standard moment estimate gives us

$$\sup_{k\geq 1}\mathbb{E}\bigg(\sup_{0\leq t\leq T}\big|\tilde{X}_t^k\big|^p\bigg) \leq C_{p,T}(1+|x|^p)$$

for any $p > 1$. Therefore

$$\mathbb{E}\int_0^T \big\|\sigma_s^k(\tilde{X}_s^k) - \sigma_s^{k_0}(\tilde{X}_s^k)\big\|^4\mathrm{d}s \leq C_T\int_0^T \mathbb{E}\big[(1+|\tilde{X}_s^k|)^4\big]\mathrm{d}s \leq \bar{C}_T(1+|x|^4). \tag{3.9}$$

As a result, by the Cauchy inequality,

$$\begin{aligned}\mathbb{E}\int_0^T \big[1-\varphi_R\big(s,\tilde{X}_s^k\big)\big]\cdot\big\|\sigma_s^k(\tilde{X}_s^k)-\sigma_s^{k_0}(\tilde{X}_s^k)\big\|^2\,\mathrm{d}s \\ \leq \bar{C}_T^{1/2}\big(1+|x|^2\big)\bigg(\mathbb{E}\int_0^T \big[1-\varphi_R\big(s,\tilde{X}_s^k\big)\big]^2\,\mathrm{d}s\bigg)^{1/2}.\end{aligned} \tag{3.10}$$

Combining (3.7), (3.8) and (3.10), we obtain

$$\mathbb{E}\sup_{t\leq T}|I_1(t)|\leq 2N^{1/2}e^{T/2}\|\sigma^k-\sigma^{k_0}\|_{L_{T,R}^{2(d+1)}}+2\bar{C}_T^{1/4}(1+|x|)\bigg(\mathbb{E}\int_0^T\big[1-\varphi_R(s,\tilde{X}_s^k)\big]^2\mathrm{d}s\bigg)^{1/4}.$$

As φ_R is continuous and $1-\varphi_R(t,x)\leq 1$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, by Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\mathbb{E}\sup_{t\leq T}|I_1(t)|\leq 2N^{1/2}e^{T/2}\|\sigma-\sigma^{k_0}\|_{L_{T,R}^{2(d+1)}} \\ +2\bar{C}_T^{1/4}(1+|x|)\bigg(\mathbb{E}\int_0^T\big[1-\varphi_R(s,\tilde{X}_s)\big]^2\mathrm{d}s\bigg)^{1/4}.\end{aligned} \tag{3.11}$$

Notice that Lemma 3.1 holds true also for the process $\tilde{X}_s$. Indeed, we first apply Lemma 3.1 to $\tilde{X}^k$ and continuous functions $f\in L^{d+1}$, then by Fatou's lemma, we obtain the inequality for $\tilde{X}$, since the constant N is independent of k. For general Borel function $f\in L^{d+1}$, a measure theoretic argument gives the desired result. Proceeding as above for the term $I_3(t)$, we get

$$\mathbb{E}\sup_{t\leq T}|I_3(t)|\leq 2N^{1/2}e^{T/2}\|\sigma^{k_0}-\sigma\|_{L_{T,R}^{2(d+1)}}+2\bar{C}_T^{1/4}(1+|x|)\bigg(\mathbb{E}\int_0^T\big[1-\varphi_R(s,\tilde{X}_s)\big]^2\mathrm{d}s\bigg)^{1/4}. \tag{3.12}$$

Now we deal with $I_2(t)$. Since σ^{k_0} is continuous, it is clear that $\sigma_s^{k_0}(\tilde{X}_s^k)$ converges to $\sigma_s^{k_0}(\tilde{X}_s)$ as $k\to\infty$. Similar to (3.9), we have for any $\alpha>2$,

$$\mathbb{E}\int_0^T\big\|\sigma_s^{k_0}(\tilde{X}_s^k)\big\|^\alpha\,\mathrm{d}\tilde{w}_s^k\leq\bar{C}_{\alpha,T}(1+|x|^\alpha),$$

whose right hand side is independent of $k\geq 1$. The same estimate holds for $\mathbb{E}\int_0^T\|\sigma_s^{k_0}(\tilde{X}_s)\|^\alpha\,\mathrm{d}\tilde{w}_s$. Therefore by Lemma 3.2, we have

$$\lim_{k\to\infty}\mathbb{E}\sup_{t\leq T}|I_2(t)|=0. \tag{3.13}$$

Now note that

$$\bigg|\int_0^t\sigma_s^k(\tilde{X}_s^k)\,\mathrm{d}\tilde{w}_s^k-\int_0^t\sigma_s(\tilde{X}_s)\,\mathrm{d}\tilde{w}_s\bigg|\leq\sum_{i=1}^3|I_i(t)|.$$

By (3.11)–(3.13), we have

$$\begin{aligned}\limsup_{k\to\infty}\mathbb{E}\sup_{t\leq T}\bigg|\int_0^t\sigma_s^k(\tilde{X}_s^k)\,\mathrm{d}\tilde{w}_s^k-\int_0^t\sigma_s(\tilde{X}_s)\,\mathrm{d}\tilde{w}_s\bigg| \\ \leq 4N^{1/2}e^{T/2}\|\sigma^{k_0}-\sigma\|_{L_{T,R}^{2(d+1)}}+4\bar{C}_T^{1/4}(1+|x|)\bigg(\mathbb{E}\int_0^T\big[1-\varphi_R(s,\tilde{X}_s)\big]^2\mathrm{d}s\bigg)^{1/4}.\end{aligned}$$

First letting $k_0 \to \infty$ and then $R \to \infty$, we finally obtain

$$\lim_{k\to\infty} \mathbb{E} \sup_{t\le T} \bigg| \int_0^t \sigma_s^k(\tilde X_s^k)\,\mathrm{d}\tilde w_s^k - \int_0^t \sigma_s(\tilde X_s)\,\mathrm{d}\tilde w_s \bigg| = 0.$$

The same method works for the convergence of the drift part, hence we also have

$$\lim_{k\to\infty} \mathbb{E} \sup_{t\le T} \bigg| \int_0^t b_s^k(\tilde X_s^k)\,\mathrm{d}s - \int_0^t b_s(\tilde X_s)\,\mathrm{d}s \bigg| = 0.$$

Thus letting $k \to \infty$ in (3.6) leads to

$$\tilde X_t = x + \int_0^t \sigma_s\big(\tilde X_s\big)\,\mathrm{d}\tilde w_s + \int_0^t b_s\big(\tilde X_s\big)\,\mathrm{d}s, \quad \text{for all } t \le T.$$

That is to say, $(\tilde X, \tilde w)$ is a weak solution to (1.1). □

Now we can prove the main result of this section.

Theorem 3.4. *Assume the conditions of Proposition 3.3 and that SDE* (1.1) *has a unique strong solution $X_{s,t}(x)$. Then*

$$\lim_{n\to\infty} \mathbb{E}\bigg(\sup_{s\le t\le T} |X_{s,t}^n(x) - X_{s,t}(x)| \bigg) = 0.$$

Proof. To simplify the notations, we assume again $s = 0$ and denote the solutions $X_{0,t}^n$, $X_{0,t}$ by X_t^n, X_t. We follow the idea on p.781 of [10]. By the linear growth of σ^n and b^n, the classical moment estimate tells us that every pair of subsequences X^l and X^m is tight in $C([0,T],\mathbb{R}^{2d})$. Hence (X^l, X^m, w) is a tight sequence in $C([0,T],\mathbb{R}^{2d+m})$. By Skorohod's representation theorem, there exist a subsequence (X^{l_k}, X^{m_k}, w) and a probability space $\tilde\Omega$ on which is defined a sequence $(\tilde X^{l_k}, \tilde X^{m_k}, \tilde w^k)$, such that for each $k \ge 1$, (X^{l_k}, X^{m_k}, w) and $(\tilde X^{l_k}, \tilde X^{m_k}, \tilde w^k)$ have the same finite dimensional distributions, and the following convergences hold almost surely: $\tilde X^{l_k} \to \tilde X^{(1)}$ and $\tilde X^{m_k} \to \tilde X^{(2)}$ in $C([0,T],\mathbb{R}^d)$ and $\tilde w^k \to \tilde w$ in $C([0,T],\mathbb{R}^m)$. By assertion (c) of Proposition 3.3, we have almost surely, for all $t \in [0,T]$,

$$\tilde X_t^{(i)} = x + \int_0^t \sigma_s\big(\tilde X_s^{(i)}\big)\mathrm{d}\tilde w_s + \int_0^t b_s\big(\tilde X_s^{(i)}\big)\mathrm{d}s,$$

where $i = 1, 2$. Under the assumptions, the above equation has pathwise uniqueness, hence $\tilde X_t^{(1)} = \tilde X_t^{(2)}$ almost surely for all $t \in [0,T]$. This implies that $\sup_{0\le t\le T} \big|\tilde X_t^{l_k} - \tilde X_t^{m_k}\big|$ converges to 0 in probability. Since (X^{l_k}, X^{m_k}) has the same finite dimensional distributions as $(\tilde X^{l_k}, \tilde X^{m_k})$, we obtain the convergence in probability of $\sup_{0\le t\le T} \big|X_t^{l_k} - X_t^{m_k}\big|$ to 0. By the moment estimate, it is easy to show that the sequence $\sup_{0\le t\le T} \big|X_t^{l_k} - X_t^{m_k}\big|$ is uniformly integrable. Hence

$$\lim_{k\to\infty} \mathbb{E}\bigg(\sup_{0\le t\le T} \big|X_t^{l_k} - X_t^{m_k}\big| \bigg) = 0.$$

As a result, the sequence $\{X^n : n \ge 1\}$ is convergent in $L^1\big(\Omega, C([0,T],\mathbb{R}^d)\big)$ to some $\bar X$. Now similar arguments as before show that $\bar X$ solves the SDE (1.1). By the pathwise uniqueness, we know that almost surely, $\bar X_t$ coincides with X_t for all $t \in [0,T]$. So finally we have proved that X^n converge in $L^1\big(\Omega, C([0,T],\mathbb{R}^d)\big)$ to X. □

4 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1, based on Theorems 2.4 and 3.4. In the following we suppose that σ and b satisfy the conditions in Theorem 1.1. Notice that $b:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ is only measurable, we will regularize it as in Section 2 of [4]. First we extend it to negative time by setting $b_t\equiv 0$ for $t<0$. Let $\chi\in C_c^\infty(\mathbb{R},[0,1])$ such that $\mathrm{supp}(\chi)\subset[-1,1]$ and $\int_{\mathbb{R}}\chi\,\mathrm{d}x=1$. For $n\geq 1$, define the convolution kernel $\chi_n(x)=n\chi(nx)$. Set $b_t^{(n)}(x)=(b_\cdot(x)*\chi_n)(t)$ and

$$b_t^n(x)=\big(P_{1/n}b_t^{(n)}\big)(x).$$

Then b^n is a smooth vector field.

Now we check that σ and b^n satisfy the conditions (A1)–(A3) in Section 2. For all $t\in[0,T]$, we have by the definition of χ_n that

$$\big|b_t^{(n)}(x)\big|\leq\int_{\mathbb{R}}|b_s(x)|\chi_n(t-s)\,\mathrm{d}s\leq L_{T+1}(1+|x|),\quad\text{for all } x\in\mathbb{R}^d.$$

Lemma 2.2 gives us

$$|b_t^n(x)|\leq L_{T+1}(1+M_1)(1+|x|).\tag{4.1}$$

Next for any $t\leq T$, it is easy to know that $\delta(b_t^n)=e^{1/n}P_{1/n}\big[(\delta(b_\cdot)*\chi_n)(t)\big]$. By Cauchy's inequality, for some $c>0$,

$$\begin{aligned}&\int_0^T\!\!\int_{\mathbb{R}^d}\exp\Big(c\big(\|\nabla\sigma_t\|^2+|\delta(\sigma_t)|^2+|\delta(b_t^n)|\big)\Big)\mathrm{d}\gamma_d\mathrm{d}t\\ &\leq\bigg[\int_0^T\!\!\int_{\mathbb{R}^d}\exp\Big(2c\big(\|\nabla\sigma_t\|^2+|\delta(\sigma_t)|^2\big)\Big)\mathrm{d}\gamma_d\mathrm{d}t\bigg]^{\frac12}\cdot\bigg[\int_0^T\!\!\int_{\mathbb{R}^d}\exp\big(2c|\delta(b_t^n)|\big)\mathrm{d}\gamma_d\mathrm{d}t\bigg]^{\frac12}.\end{aligned}\tag{4.2}$$

Using Jensen's inequality twice, we obtain

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{R}^d}\exp\big(2c|\delta(b_t^n)|\big)\mathrm{d}\gamma_d\mathrm{d}t&\leq\int_0^T\!\!\int_{\mathbb{R}^d}\exp\Big(2ce^{1/n}P_{1/n}\big|(\delta(b_\cdot)*\chi_n)(t)\big|\Big)\mathrm{d}\gamma_d\mathrm{d}t\\&\leq\int_0^T\!\!\int_{\mathbb{R}^d}\exp\Big(2ce\big|(\delta(b_\cdot)*\chi_n)(t)\big|\Big)\mathrm{d}\gamma_d\mathrm{d}t\\&\leq\int_0^T\!\!\int_{\mathbb{R}^d}\!\int_{\mathbb{R}}e^{2ce|\delta(b_s)|}\chi_n(t-s)\,\mathrm{d}s\mathrm{d}\gamma_d\mathrm{d}t.\end{aligned}$$

Noticing that $\delta(b_s)\equiv 0$ for $s<0$, we deduce easily by changing the order of integration that

$$\int_0^T\!\!\int_{\mathbb{R}}e^{2ce|\delta(b_s)|}\chi_n(t-s)\,\mathrm{d}s\mathrm{d}t\leq\frac1n+\int_0^{T+n^{-1}}e^{2ce|\delta(b_s)|}\,\mathrm{d}s\leq 1+\int_0^{T+1}e^{2ce|\delta(b_s)|}\,\mathrm{d}s.$$

Thus for all $n\geq 1$,

$$\int_0^T\!\!\int_{\mathbb{R}^d}\exp\big(2c|\delta(b_t^n)|\big)\mathrm{d}\gamma_d\mathrm{d}t\leq 1+\int_0^{T+1}\!\!\int_{\mathbb{R}^d}e^{2ce|\delta(b_s)|}\,\mathrm{d}s\mathrm{d}\gamma_d.\tag{4.3}$$

Therefore, taking $c=\lambda_{T+1}/2e$, we have by (4.2) and (4.3) that

$$\int_0^T\!\!\int_{\mathbb{R}^d}\exp\bigg(\frac{\lambda_{T+1}}{2e}\big(\|\nabla\sigma_t\|^2+|\delta(\sigma_t)|^2+|\delta(b_t^n)|\big)\bigg)\mathrm{d}\gamma_d\mathrm{d}t\leq\Sigma_{T+1}^{1/2}\big(1+\Sigma_{T+1}\big)^{1/2}.\tag{4.4}$$

In view of (4.1) and (4.4), we denote by

$$\tilde{L}_T = L_{T+1}(1+M_1), \quad \tilde{\lambda}_T = \lambda_{T+1}/2e \quad \text{and} \quad \tilde{\Sigma}_T = \Sigma_{T+1}^{1/2}\big(1+\Sigma_{T+1}\big)^{1/2}. \tag{4.5}$$

Then the conditions (A1)–(A3) are satisfied by σ and b^n with the constants $\tilde{L}_T$, $\tilde{\lambda}_T$ and $\tilde{\Sigma}_T$. Note that they are independent of $n \geq 1$.

For any $n \geq 1$, consider the SDE

$$\mathrm{d}X_{s,t}^n = \sigma_t(X_{s,t}^n)\,\mathrm{d}w_t + b_t^n(X_{s,t}^n)\,\mathrm{d}t, \quad t \geq s, \quad X_{s,s}^n = x.$$

Under the conditions of Theorem 1.1, the above SDE has a unique strong solution $X_{s,t}^n$ with infinite lifetime (see Theorem 1.1 in [20]). Set

$$\tilde{T} = \frac{1}{112\tilde{L}_T(1+\tilde{L}_T)} \wedge \frac{\lambda_{T+1}}{16e^3} \quad \text{and} \quad \tilde{\Lambda} = \bigg(\frac{M_2\tilde{\Sigma}_T}{\tilde{T}}\bigg)^{\frac{1}{2}},$$

where M_2 is defined in (2.9). By the above discussions and Theorem 2.4, we have $(X_{s,t}^n)_\#\gamma_d = K_{s,t}^n\gamma_d$ and

$$\int_{\mathbb{R}^d} \mathbb{E}(K_{s,t}^n|\log K_{s,t}^n|)\,\mathrm{d}\gamma_d \leq 2\,\tilde{C}_1 T^{1/2}\tilde{\Lambda} + C_{n,2}T\tilde{\Lambda}^2 + 2e^{-1}, \tag{4.6}$$

where, by (2.14),

$$\tilde{C}_1 = \big\|\, \|\sigma_u\| + e|\delta(\sigma_u)|\,\big\|_{L^{2^{\tilde{N}+1}}(\mathcal{L}_T\times\gamma_d)}$$

with $\tilde{N} = \lceil T/\tilde{T}\rceil$ being the minimum integer that is greater than $T/\tilde{T}$, and by (2.17),

$$C_{n,2} = \bigg\||b_\cdot^n| + e|\delta(b_\cdot^n)| + \frac{3}{2}\|\sigma_\cdot\|^2 + \|\nabla\sigma_\cdot\|^2\bigg\|_{L^{2^{\tilde{N}}}(\mathcal{L}_T\times\gamma_d)}.$$

Since

$$|b_t^n| + e|\delta(b_t^n)| \leq P_{1/n}\big[\big(|b_\cdot| + e^2|\delta(b_\cdot)|\big) * \chi_n\big](t),$$

we have by Jensen's inequality that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\big(|b_t^n| + e|\delta(b_t^n)|\big)^{2^{\tilde{N}}}\mathrm{d}\gamma_d\mathrm{d}t &\leq \int_0^T\int_{\mathbb{R}^d}\bigg(\int_{\mathbb{R}}\big(|b_s| + e^2|\delta(b_s)|\big)\chi_n(t-s)\,\mathrm{d}s\bigg)^{2^{\tilde{N}}}\mathrm{d}\gamma_d\mathrm{d}t\\ &\leq \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}}\big(|b_s| + e^2|\delta(b_s)|\big)^{2^{\tilde{N}}}\chi_n(t-s)\,\mathrm{d}s\mathrm{d}\gamma_d\mathrm{d}t.\end{aligned}$$

Changing the order of integration of the right hand side and noting that $b_s = 0$ for $s < 0$, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\big(|b_t^n| + e|\delta(b_t^n)|\big)^{2^{\tilde{N}}}\mathrm{d}\gamma_d\mathrm{d}t &\leq \int_{\mathbb{R}^d}\int_0^{T+n^{-1}}\big(|b_s| + e^2|\delta(b_s)|\big)^{2^{\tilde{N}}}\mathrm{d}s\mathrm{d}\gamma_d\\ &\leq \int_0^{T+1}\int_{\mathbb{R}^d}\big(|b_s| + e^2|\delta(b_s)|\big)^{2^{\tilde{N}}}\mathrm{d}\gamma_d\mathrm{d}s.\end{aligned}$$

Therefore

$$\begin{aligned}C_{n,2} &\leq \big\||b_\cdot^n| + e|\delta(b_\cdot^n)|\big\|_{L^{2^{\tilde{N}}}(\mathcal{L}_T\times\gamma_d)} + \bigg\|\frac{3}{2}\|\sigma_\cdot\|^2 + \|\nabla\sigma_\cdot\|^2\bigg\|_{L^{2^{\tilde{N}}}(\mathcal{L}_T\times\gamma_d)}\\ &\leq \big\||b_\cdot| + e^2|\delta(b_\cdot)|\big\|_{L^{2^{\tilde{N}}}(\mathcal{L}_{T+1}\times\gamma_d)} + \bigg\|\frac{3}{2}\|\sigma_\cdot\|^2 + \|\nabla\sigma_\cdot\|^2\bigg\|_{L^{2^{\tilde{N}}}(\mathcal{L}_T\times\gamma_d)} =: \tilde{C}_2.\end{aligned}$$

This plus (4.6) gives us that for all $0 \le s < t \le T$,

$$\sup_{n\ge1}\int_{\mathbb{R}^d}\mathbb{E}(K^n_{s,t}|\log K^n_{s,t}|)\,\mathrm{d}\gamma_d \le 2\,\tilde C_1T^{1/2}\tilde\Lambda + \tilde C_2T\tilde\Lambda^2 + 2e^{-1}. \tag{4.7}$$

Now for any fixed $0 \le s < t \le T$, the same argument as that before Theorem 2.3 leads to the existence of some $K_{s,t} \in L^1(\Omega\times\mathbb{R}^d)$, which is a weak limit of a subsequence of $\{K^n_{s,t}\}_{n\ge1}$ and satisfies

$$\int_{\mathbb{R}^d}\mathbb{E}(K_{s,t}|\log K_{s,t}|)\,\mathrm{d}\gamma_d \le 2\,\tilde C_1T^{1/2}\tilde\Lambda + \tilde C_2T\tilde\Lambda^2 + 4e^{-1}. \tag{4.8}$$

Now we are in the position to give

Proof of Theorem 1.1. We follow the idea of the proof of Theorem 3.4 in [8]. To apply the limit result proved in Section 3, we check that σ and b^n satisfy the assumptions of Proposition 3.3. We only have to verify the conditions for b^n. By (4.1), condition (1) in Proposition 3.3 is satisfied. (3) is a consequence of Theorem 1.1 in [20]. Now we check that $b^n \to b$ in $L^{d+1}_{loc}([0,T]\times\mathbb{R}^d)$. It is enough to show that $\lim_{n\to\infty}\|b^n-b\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} = 0$, where $\mathcal{L}_T$ is the Lebesgue measure restricted on $[0,T]$. We have by the triangular inequality,

$$\|b^n-b\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} \le \big\|b^n - P_{1/n}b_\cdot\big\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} + \big\|P_{1/n}b_\cdot - b\big\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)}. \tag{4.9}$$

Jensen's inequality leads to

$$\begin{aligned}\big\|b^n - P_{1/n}b_\cdot\big\|^{d+1}_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} &\le \int_0^T\!\!\int_{\mathbb{R}^d}\big(P_{1/n}|(b_\cdot*\chi_n)(t)-b_t|\big)^{d+1}\mathrm{d}\gamma_d\mathrm{d}t\\ &\le \int_0^T\!\!\int_{\mathbb{R}^d}|(b_\cdot*\chi_n)(t)-b_t|^{d+1}\mathrm{d}\gamma_d\mathrm{d}t.\end{aligned}$$

By the growth condition on b (note that $b_t \equiv 0$ for $t<0$), we deduce easily that for almost every $x\in\mathbb{R}^d$, $b_\cdot(x)*\chi_n \to b_\cdot(x)$ in $L^{d+1}([0,T])$. By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\big\|b^n - P_{1/n}b_\cdot\big\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} = 0. \tag{4.10}$$

Again by the linear growth of b, we have for all $t\in[0,T]$, $\lim_{n\to\infty}\|P_{1/n}b_t - b_t\|_{L^{d+1}(\gamma_d)} = 0$. Using once more Lebesgue's dominated convergence, we obtain

$$\lim_{n\to\infty}\big\|P_{1/n}b_\cdot - b\big\|_{L^{d+1}(\mathcal{L}_T\times\gamma_d)} = 0.$$

This plus (4.9) and (4.10) leads to the desired result. By the above discussion and Theorem 3.4, we have for any $x\in\mathbb{R}^d$,

$$\lim_{n\to\infty}\mathbb{E}\bigg(\sup_{s\le t\le T}|X^n_{s,t}(x) - X_{s,t}(x)|\bigg) = 0. \tag{4.11}$$

Since σ and b have linear growth, the classical moment estimate tells us that $\mathbb{E}|X_{s,t}(x)| \le C(1+|x|)$ and $\sup_{n\ge1}\mathbb{E}|X^n_{s,t}(x)| \le C(1+|x|)$. Now fixing arbitrary $\xi\in L^\infty(\Omega)$ and $\psi\in C_c^\infty(\mathbb{R}^d)$, we have by (4.11) and the dominated convergence theorem,

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}|\xi(\cdot)|\,\big|\psi(X^n_{s,t}(x)) - \psi(X_{s,t}(x))\big|\,\mathrm{d}\gamma_d(x)\\ &\quad\le \|\xi\|_\infty\|\nabla\psi\|_\infty\int_{\mathbb{R}^d}\mathbb{E}\big|X^n_{s,t}(x) - X_{s,t}(x)\big|\mathrm{d}\gamma_d(x) \to 0\end{aligned} \tag{4.12}$$

as n tends to $+\infty$. Therefore

$$\lim_{n\to\infty}\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(X^n_{s,t}(x))\,\mathrm{d}\gamma_d(x)=\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(X_{s,t}(x))\,\mathrm{d}\gamma_d. \tag{4.13}$$

On the other hand, by the above discussion, for each fixed $t\in[0,T]$, up to a subsequence, $K^n_{s,t}$ converges weakly in $L^1(\Omega\times\mathbb{R}^d)$ to some $K_{s,t}$ satisfying (4.8), hence

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi\big(X^n_{s,t}(x)\big)\mathrm{d}\gamma_d(x)&=\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(y)K^n_{s,t}(y)\,\mathrm{d}\gamma_d(y)\\&\to\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(y)K_{s,t}(y)\,\mathrm{d}\gamma_d(y).\end{aligned} \tag{4.14}$$

This together with (4.13) leads to

$$\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(X_{s,t}(x))\,\mathrm{d}\gamma_d(x)=\mathbb{E}\int_{\mathbb{R}^d}\xi\,\psi(y)K_{s,t}(y)\,\mathrm{d}\gamma_d(y).$$

By the arbitrariness of $\xi\in L^\infty(\Omega)$, there exists a full measure subset Ω_ψ of Ω such that

$$\int_{\mathbb{R}^d}\psi(X_{s,t}(x))\,\mathrm{d}\gamma_d(x)=\int_{\mathbb{R}^d}\psi(y)K_{s,t}(y)\,\mathrm{d}\gamma_d(y),\quad\text{for any }\omega\in\Omega_\psi.$$

Now by the separability of $C_c^\infty(\mathbb{R}^d)$, there exists a full subset $\Omega_{s,t}$ such that the above equality holds for any $\psi\in C_c^\infty(\mathbb{R}^d)$. Hence $(X_{s,t})_\#\gamma_d=K_{s,t}\gamma_d$. □

We say that two measures μ, ν on $\mathbb{R}^d$ are equivalent if $\mu\ll\nu$ and $\nu\ll\mu$. We have the following simple result.

Corollary 4.1. *Let μ_0 be a measure on $\mathbb{R}^d$ which is equivalent to γ_d, then $(X_{s,t})_\#\mu_0\ll\mu_0$ for all $0\le s<t\le T$. In particular, the Lebesgue measure is absolutely continuous under the action of the flow $X_{s,t}$.*

Proof. Let $A\subset\mathbb{R}^d$ be such that $\mu_0(A)=0$. Then $\gamma_d(A)=0$, hence by Theorem 1.1, $[(X_{s,t})_\#\gamma_d](A)=0$, or equivalently, the inverse image $(X_{s,t})^{-1}(A)$ is γ_d-negligible. Since μ_0 is also absolutely continuous with respect to γ_d, we deduce that $(X_{s,t})^{-1}(A)$ is μ_0-negligible. That is, $[(X_{s,t})_\#\mu_0](A)=0$. By the arbitrariness of the μ_0-negligible subset A, we conclude the first assertion. □

Remark 4.2. If the inverse flow $(X^{-1}_{s,t})_{s\le t}$ of $(X_{s,t})_{s\le t}$ exists, then there is a simple relation between the density functions. Indeed, let $\mu_0=\rho\gamma_d$ with $\rho(x)>0$ for γ_d-a.e. $x\in\mathbb{R}^d$. Then for any $f\in C_c(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d}f(X_{s,t})\,\mathrm{d}\mu_0=\int_{\mathbb{R}^d}f(X_{s,t})\rho\,\mathrm{d}\gamma_d=\int_{\mathbb{R}^d}f\rho(X^{-1}_{s,t})K_{s,t}\,\mathrm{d}\gamma_d=\int_{\mathbb{R}^d}f\rho(X^{-1}_{s,t})K_{s,t}\rho^{-1}\,\mathrm{d}\mu_0.$$

Therefore $K^{\mu_0}_{s,t}:=\frac{\mathrm{d}[(X_{s,t})_\#\mu_0]}{\mathrm{d}\mu_0}=\rho(X^{-1}_{s,t})K_{s,t}\rho^{-1}$.

Now we apply our result to the Fokker-Planck (or forward Kolmogorov) equation associated to the SDE (1.1), showing that under suitable conditions, the solution of the Fokker-Planck equation consists of absolutely continuous measures with respect to the Lebesgue measure if so is the initial value. Consider

$$\frac{\mathrm{d}\mu_{s,t}}{\mathrm{d}t}+\sum_{i=1}^d\partial_i(b^i_t\mu_{s,t})-\frac{1}{2}\sum_{i,j=1}^d\partial_{ij}(a^{ij}_t\mu_{s,t})=0,\quad t\ge s,\quad\mu_{s,s}=\mu_0, \tag{4.15}$$

where

$$a_t^{ij} = \sum_{k=1}^{m} \sigma_t^{ik} \sigma_t^{jk}, \quad i, j = 1, \cdots, d. \tag{4.16}$$

Define the time dependent second order differential operator

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} a_t^{ij} \partial_{ij} + \sum_{i=1}^{d} b_t^i \partial_i.$$

A measure valued function $\mu_{s,t}$ on $[s, T]$ is called a solution to the Fokker-Planck equation (4.15), if for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \varphi(x)\, \mathrm{d}\mu_{s,t}(x) = \int_{\mathbb{R}^d} L_t \varphi(x)\, \mathrm{d}\mu_{s,t}(x)$$

holds in the distribution sense on $[s, T]$ and $\mu_{s,t}$ is w^*-convergent to μ_0 as $t \downarrow s$. The above equation can simply be written as

$$\frac{\mathrm{d}\mu_{s,t}}{\mathrm{d}t} = L_t^* \mu_{s,t}, \quad t \geq s, \quad \mu_{s,s} = \mu_0, \tag{4.17}$$

where L_t^* is the formal adjoint operator of L_t. If $\mu_{s,t}$ is absolutely continuous with respect to the Lebesgue measure with a density function $u_{s,t}$, then $u_{s,t}$ is also called a solution to (4.15).

By the Itô formula, it is easy to show that the measure defined below

$$\int_{\mathbb{R}^d} \varphi(x)\, \mathrm{d}\mu_{s,t}(x) = \int_{\mathbb{R}^d} \mathbb{E}[\varphi(X_{s,t}(x))]\, \mathrm{d}\mu_0(x), \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d) \tag{4.18}$$

is a solution of (4.15), where $X_{s,t}(x)$ is a weak solution to the SDE (1.1). Under quite general conditions, Figalli studied in [9] the relationship between the well-posedness of the martingale problem of the Itô SDE and the existence and uniqueness of measure valued solutions to the Fokker-Planck equation (see also [18] for extensive investigations in the regular case). Then he proved the existence and uniqueness of solutions to (4.15) under some mild conditions, as a consequence, he obtained the well-posedness of martingale problems for the Itô SDE (1.1). More recently, LeBris and Lions [16] gave a systematical study of the Fokker-Planck type equations with Sobolev coefficients, showing the existence and uniqueness of solutions in suitable spaces.

Besides the existence and uniqueness of solutions to (4.15), we are also interested in the problem that whether the solution $\mu_{s,t}$ has a density with respect to the Lebesgue measure λ. In the smooth case, it is well known that if the differential operator L_t is uniformly elliptic, then we have an affirmative answer even when the initial measure μ_0 is a Dirac mass. The following theorem gives a sufficient condition which guarantees the uniqueness of the equation (4.15) (or equivalently (4.17)), and we also show in a special case that the unique solution has a density with respect to the Lebesgue measure. Denote by $\mathcal{M}_+^f$ the space of measures on $\mathbb{R}^d$ with finite total mass.

Theorem 4.3. *Suppose the conditions of Theorem 1.1. Moreover if σ and b are bounded, then for any $\mu_0 \in \mathcal{M}_+^f$, the Fokker-Planck equation* (4.17) *has a unique finite nonnegative measure valued solution.*

Moreover, if the initial datum μ_0 is equivalent to the Lebesgue measure, then the unique solution $\mu_{s,t}$ to (4.15) *is absolutely continuous with respect to λ.*

Proof. We proceed as in Theorem 3.8 of [17]. Under these conditions, we deduce from Theorem 1.1 in [20] that the Itô SDE (1.1) has a unique strong solution. Therefore the martingale problem for the operator L_t is well posed. Now Lemma 2.3 in [9] gives rise to the first part.

Next we prove the second assertion. Assume $\mu_0 \in \mathcal{M}_+^f$ is equivalent to the Lebesgue measure λ with the density function u_0. Then by Corollary 4.1, μ_0 is absolutely continuous under the action of the stochastic flow $X_{s,t}$ generated by (1.1). Denote by $K_{s,t}^{\mu_0}(x) = \frac{\mathrm{d}[(X_{s,t})_\# \mu_0]}{\mathrm{d}\mu_0}(x)$ the Radon-Nikodym derivative and $k_{s,t}^{\mu_0}(x) = \mathbb{E}(K_{s,t}^{\mu_0}(x))$. Then for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \varphi(X_{s,t}(x))\,\mathrm{d}\mu_0(x) = \int_{\mathbb{R}^d} \varphi(y) K_{s,t}^{\mu_0}(y)\,\mathrm{d}\mu_0(y).$$

Therefore by (4.18),

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\mu_{s,t}(x) = \mathbb{E}\int_{\mathbb{R}^d} \varphi(y) K_{s,t}^{\mu_0}(y)\,\mathrm{d}\mu_0 = \int_{\mathbb{R}^d} \varphi(y) k_{s,t}^{\mu_0}(y)\,\mathrm{d}\mu_0(y),$$

which means that $\frac{\mathrm{d}\mu_{s,t}}{\mathrm{d}\mu_0} = k_{s,t}^{\mu_0}$, and hence the Radon-Nikodym derivative with respect to the Lebesgue measure

$$\frac{\mathrm{d}\mu_{s,t}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mu_{s,t}}{\mathrm{d}\mu_0} \cdot \frac{\mathrm{d}\mu_0}{\mathrm{d}\lambda} = k_{s,t}^{\mu_0} u_0.$$

The proof is complete. □

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