COMMUTATIVITY DEGREE, ITS GENERALIZATIONS, AND CLASSIFICATION OF FINITE GROUPS

(Abstract)

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Rajat Kanti Nath

Under the supervision of

Ashish Kumar Das

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NORTH-EASTERN HILL UNIVERSITY

SHILLONG – 793022, INDIA

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emails: rajatkantinath@yahoo.com; akdasnehu@gmail.com

Classification of finite groups is a central problem in theory of groups. Even though finite abelian groups have been completely classified, a lot still remains to be done as far as non-abelian groups are concerned. People all over the world have used various types of invariants for classifying finite groups, particularly the non-abelian ones. The commutativity degree of a finite group is one such invariant, and it seems that many interesting results are possible to obtain with the help of this notion and its generalizations.

In recent years there has been a growing interest in the use of the probabilistic methods in the theory of finite groups. These methods have proved useful in the solution of several difficult problems on groups. In some cases the probabilistic nature of the problem is apparent from its formulation, but in other cases the connection to probability seems surprising and can not be anticipated by the nature of the problem.

The roots of the subject matter of this thesis lie in a series of papers by P. Erdös and P. Turán (see [6, 7, 8, 9]) published between 1965 and 1968, and also in the Ph. D thesis of K. S. Joseph [17] submitted in 1969, wherein some problems of statistical group theory and commutativity in non-abelian groups have been considered. In 1973, W. H. Gustafson [15] considered the question – *what is the probability that two group elements commute?* The answer is given by what is known as the commutativity degree of a group. It may be mentioned here that the question, in some sense, was also considered by Erdös and Turán [9].

Formally, the commutativity degree of a finite group G, denoted by $\Pr(G)$, is defined as the ratio

$$\Pr(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G}.$$

In other words, commutativity degree is a kind of measure for abelianness of a group. Note that $\Pr(G) > 0$, and that $\Pr(G) = 1$ if and only if G is abelian. Also, given a finite group G, we have $\Pr(G) \leq \frac{5}{8}$ with equality if and only if $\frac{G}{Z(G)}$ has order 4 (see [2, 15]), where $Z(G)$ denotes the center of G. This gave rise to the problem of determining the numbers in the interval $(0, \frac{5}{8}]$ which can be realized as the commutativity degrees of some finite groups, and also to the problem of classifying all finite groups with a given commutativity degree.

In 1979, D. J. Rusin [23] computed, for a finite group G, the values of $\Pr(G)$ when $G' \subseteq Z(G)$, and also when $G' \cap Z(G)$ is trivial, where G' denotes the commutator subgroup of G. He determined all numbers lying in the interval $(\frac{11}{32}, 1]$ that can be realized as the commutativity degree of some finite groups, and also classified all finite groups whose commutativity degrees lie in the interval $(\frac{11}{32}, 1]$.

In 1995, P. Lescot [18] classified, up to isoclinism, all finite groups whose commutativity degrees are greater than or equal to $\frac{1}{2}$. It may be mentioned here that the concept of isoclinism between any two groups was introduced by P. Hall [16]. A pair $(\phi,\ \psi)$ is said to be an *isoclinism* from a group G to another group H if the following conditions hold:

(a) ϕ is an isomorphism from $G/Z(G)$ to $H/Z(H)$,

(b) ψ is an isomorphism from G' to H', and

(c) the diagram

$$\begin{tikzcd} \frac{G}{Z(G)} \times \frac{G}{Z(G)} \arrow[r, "\phi \times \phi"] \arrow[d, "a_G"'] & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \arrow[d, "a_H"] \\ G' \arrow[r, "\psi"] & H' \end{tikzcd}$$

commutes, that is, $a_H \circ (\phi \times \phi) = \psi \circ a_G$, where a_G and a_H are given respectively by $a_G(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ for all $g_1, g_2 \in G$ and $a_H(h_1Z(H), h_2Z(H)) = [h_1, h_2]$ for all $h_1, h_2 \in H$. Here, given $x, y \in G$, $[x, y]$ stands for the commutator $xyx^{-1}y^{-1}$ of x and y in G.

In 2001, Lescot [19] has also classified, up to isomorphism, all finite groups whose commutativity degrees lie in the interval $[\frac{1}{2}, 1]$.

In 2006, F. Barry, D. MacHale and Á. Ní Shé [1] have shown that if G is a finite group with $|G|$ odd and $\Pr(G) > \frac{11}{75}$, then G is supersolvable. They also proved that if $\Pr(G) > \frac{1}{3}$, then G is supersolvable. It may be mentioned here that a group G is said to be *supersolvable* if there is a series of the form

$$\{1\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = G,$$

where $A_i \trianglelefteq G$ and A_{i+1}/A_i is cyclic for each i with $0 \leq i \leq r-1$.

In the same year 2006, R. M. Guralnick and G. R. Robinson [13] re-established a result of Lescot (see [14]) which says that if G is a finite group with $\Pr(G)$ greater than $\frac{3}{40}$, then either G is solvable, or $G \cong A_5 \times B$, where A_5 is the alternating group of degree 5 and B is some abelian group.

The classical notion of commutativity degree has been generalized in a number of ways. In 2007, A. Erfanian, R. Rezaei and P. Lescot [10] studied the probability $\Pr(H, G)$ that an element of a given subgroup H of a finite group G commutes with an element of G. Note that $\Pr(G, G) = \Pr(G)$. In 2008, M. R. Pournaki and R. Sobhani [22] studied the probability $\Pr_g(G)$ that the commutator of an arbitrarily chosen pair of elements in a finite group G equals a given group element g. They have also extended some of the results obtained by Rusin. It is easy to see that $\Pr_g(G) = \Pr(G)$ if $g = 1$, the identity element of G.

In Chapter 1, we briefly recall a few definitions and well-known results from several relevant topics, which constitute the minimum prerequisites for the subsequent chapters. In this chapter, we also fix certain notations. Given a subgroup K of a group G and an element $x \in G$, we write $C_K(x)$ and $\mathrm{C}\ell_K(x)$ to denote the sets $\{k \in K : kx = xk\}$ and $\{kxk^{-1} \in G : k \in K\}$ respectively; noting that, for $K = G$, these sets coincide respectively with the centralizer and the conjugacy class of x in G. Also, given any two subgroups H and K of a group G, we write $C_K(H) = \{k \in K : hk = kh \text{ for all } h \in H\}$. Note that $C_K(x) = C_K(\langle x \rangle)$, where $\langle x \rangle$ denotes the cyclic subgroup of G generated by $x \in G$.

Further, we write $\mathrm{Irr}(G)$ to denote the set of all irreducible complex characters of G, and $\mathrm{cd}(G)$ to denote the set $\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. If $\chi(1) = |G : Z(G)|^{1/2}$ for some $\chi \in \mathrm{Irr}(G)$, then the group G is said to be of *central type*.

In Chapter 2, which is based on our papers [5] and [20], we determine, for a finite group G, the value of $\Pr(G)$ and the size of $\frac{G}{Z(G)}$ when $|G'| = p^2$ and $|G' \cap Z(G)| = p$, where p is a prime such that $\gcd(p-1, |G|) = 1$. The main result of Section 2.2 is given as follows.

Theorem 2.2.6. *Let G be a finite group and p be a prime such that $\gcd(p-1, |G|) = 1$. If $|G'| = p^2$ and $|G' \cap Z(G)| = p$, then*

(a) $\Pr(G) = \begin{cases} \frac{2p^2-1}{p^4} & \textit{if } C_G(G') \textit{ is abelian} \\ \frac{1}{p^4}\left(\frac{p-1}{p^{2s-1}} + p^2 + p - 1\right) & \textit{otherwise,} \end{cases}$

(b) $|\frac{G}{Z(G)}| = \begin{cases} p^3 & \textit{if } C_G(G') \textit{ is abelian} \\ p^{2s+2} \textit{ or } p^{2s+3} & \textit{otherwise,} \end{cases}$

where $p^{2s} = |C_G(G') : Z(C_G(G'))|$. Moreover,

$$|\frac{G}{G' \cap Z(G)} : Z(\frac{G}{G' \cap Z(G)})| = |\frac{G}{Z(G)} : Z(\frac{G}{Z(G)})| = p^2.$$

This theorem together with few other supplementary results, enable us to classify all finite groups G of odd order with $\Pr(G) \geq \frac{11}{75}$. In the process we also point out a few small but significant lacunae in the work of Rusin [23]. The main result of Section 2.3 is given as follows.

Theorem 2.3.3. *Let G be a finite group. If $|G|$ is odd and $\Pr(G) \geq \frac{11}{75}$, then the possible values of $\Pr(G)$ and the corresponding structures of G', $G' \cap Z(G)$ and $G/Z(G)$ are given as follows:*

$\Pr(G)$	G'	$G' \cap Z(G)$	$G/Z(G)$
1	$\{1\}$	$\{1\}$	$\{1\}$
$\frac{1}{3}(1+\frac{2}{3^{2s}})$	C_3	C_3	$(C_3 \times C_3)^s,\ s \geq 1$
$\frac{1}{5}(1+\frac{4}{5^{2s}})$	C_5	C_5	$(C_5 \times C_5)^s,\ s \geq 1$
$\frac{5}{21}$	C_7	$\{1\}$	$C_7 \rtimes C_3$
$\frac{55}{343}$	C_7	C_7	$C_7 \times C_7$
$\frac{17}{81}$	C_9 *or* $C_3 \times C_3$	C_3	$(C_3 \times C_3) \rtimes C_3$
	$C_3 \times C_3$	$C_3 \times C_3$	C_3^3
$\frac{121}{729}$	$C_3 \times C_3$	$C_3 \times C_3$	C_3^4
$\frac{7}{39}$	C_{13}	$\{1\}$	$C_{13} \rtimes C_3$
$\frac{3}{19}$	C_{19}	$\{1\}$	$C_{19} \rtimes C_3$
$\frac{29}{189}$	C_{21}	C_3	$C_3 \times (C_7 \rtimes C_3)$
$\frac{11}{75}$	$C_5 \times C_5$	$\{1\}$	$(C_5 \times C_5) \rtimes C_3$

In the above table C_n denotes the cyclic group of order n and $\rtimes$ stands for semidirect product.

In [22, Corollary 2.3], M. R. Pournaki and R. Sobhani have proved that, for a finite group G satisfying $|\operatorname{cd}(G)| = 2$, one has

$$\Pr(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'|-1}{|G:Z(G)|}\right)$$

with equality if and only if G is of central type. In Section 2.4, we have improved this result as follows.

Theorem 2.4.1. *If G is a finite group, then*

$$\Pr(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'|-1}{|G:Z(G)|}\right).$$

In particular, $\Pr(G) > \frac{1}{|G'|}$ *if G is non-abelian.*

There are several equivalent conditions that are necessary as well as sufficient for the attainment of the above lower bound for $\Pr(G)$.

Theorem 2.4.3. *For a finite non-abelian group G, the statements given below are equivalent.*

(a) $\Pr(G) = \dfrac{1}{|G'|}\left(1 + \dfrac{|G'|-1}{|G:Z(G)|}\right).$

(b) $\operatorname{cd}(G) = \{1, |G:Z(G)|^{1/2}\}$, *which means that G is of central type with* $|\operatorname{cd}(G)| = 2$.

(c) $|\mathrm{C}\ell_G(x)| = |G'|$ *for all* $x \in G - Z(G)$.

(d) $\mathrm{C}\ell_G(x) = G'x$ *for all* $x \in G - Z(G)$*; in particular, G is a nilpotent group of class* 2.

(e) $C_G(x) \trianglelefteq G$ *and* $G' \cong \frac{G}{C_G(x)}$ *for all* $x \in G - Z(G)$*; in particular, G is a CN-group, that is, the centralizer of every element is normal.*

(f) $G' = \{[y,x] : y \in G\}$ *for all* $x \in G - Z(G)$*; in particular, every element of G' is a commutator.*

Theorem 2.4.1 and Theorem 2.4.3 not only allow us to obtain some characterizations for finite nilpotent groups of class 2 whose commutator subgroups have prime order, but also enable us to re-establish certain facts (essentially due to K. S. Joseph [17]) concerning the smallest prime divisors of the orders of finite groups.

Proposition 2.4.4. *Let G be a finite group and p be the smallest prime divisor of $|G|$.*

(a) *If $p \neq 2$, then $\Pr(G) \neq \frac{1}{p}$.*

(b) *When G is non-abelian, $\Pr(G) > \frac{1}{p}$ if and only if $|G'| = p$ and $G' \subseteq Z(G)$.*

Corrolary 2.4.5. *If G is a finite group with $\Pr(G) = \frac{1}{3}$, then $|G|$ is even.*

Proposition 2.4.7. *Let G be a finite group and p be a prime. Then the following statements are equivalent.*

(a) *$|G'| = p$ and $G' \subseteq Z(G)$.*

(b) *G is of central type with $|\operatorname{cd}(G)| = 2$ and $|G'| = p$.*

(c) *G is a direct product of a p-group P and an abelian group A such that $|P'| = p$ and $\gcd(p, |A|) = 1$.*

(d) *G is isoclinic to an extra-special p-group; consequently, $|G : Z(G)| = p^{2k}$ for some positive integer k.*

In particular, if G is non-abelian and p is the smallest prime divisor of $|G|$, then the above statements are also equivalent to the condition $\Pr(G) > \frac{1}{p}$.

In [19], Lescot deduced that $\Pr(D_{2n}) \to \frac{1}{4}$ and $\Pr(Q_{2^{n+1}}) \to \frac{1}{4}$ as $n \to \infty$, where D_{2n} and $Q_{2^{n+1}}$ denote the dihedral group of order $2n$, $n \geq 1$, and the quaternion group of order 2^{n+1}, $n \geq 2$, respectively. He also enquired whether there are other natural families of finite groups with the same property. In 2007, I. V. Erovenko and B. Sury [11] have shown, in particular, that for every integer $k > 1$ there exists a family $\{G_n\}$ of finite groups such that $\Pr(G_n) \to \frac{1}{k^2}$ as $n \to \infty$. In the last section of Chapter 2 we have considered the question posed by Lescot mentioned above. Moreover, we make the following observation.

Proposition 2.5.1. *For every integer $k > 1$ there exists a family $\{G_n\}$ of finite groups such that $\Pr(G_n) \to \frac{1}{k}$ as $n \to \infty$.*

In the same line, we also have the following result.

Proposition 2.5.2. *For every positive integer n there exists a finite group G such that $\Pr(G) = \frac{1}{n}$.*

In Chapter 3, which is based on our papers [3] and [4], we generalize the following result of F. G. Frobenius [12]:

If G is a finite group and $g \in G$, then the number of solutions of the commutator equation $xyx^{-1}y^{-1} = g$ in G defines a character on G, and is given by

$$\zeta(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|}{\chi(1)} \chi(g).$$

We write $F(x_1, x_2, \ldots, x_n)$ to denote the free group of words on n generators $x_1, x_2, \ldots, x_n$. For $1 \leq i \leq n$, we write '$x_i \in \omega(x_1, x_2, \ldots, x_n)$' to mean that x_i has a non-zero index (that is, x_i^k forms a syllable, with $0 \neq k \in \mathbb{Z}$) in the word $\omega(x_1, x_2, \ldots, x_n) \in F(x_1, x_2, \ldots, x_n)$. We call a word $\omega(x_1, x_2, \ldots, x_n)$ *admissible* if each $x_i \in \omega(x_1, x_2, \ldots, x_n)$ has precisely two non-zero indices, namely, $+1$ and -1. We write $\mathscr{A}(x_1, x_2, \ldots, x_n)$ to denote the set of all admissible words in $F(x_1, x_2, \ldots, x_n)$.

Given a finite group G and an element $g \in G$, let $\zeta_n^\omega(g)$ denote the number of solutions $(g_1, g_2, \ldots, g_n) \in G^n$ of the word equation $\omega(x_1, x_2, \ldots, x_n) = g$, where $G^n = G \times G \times \cdots \times G$ (n times). Thus,

$$\zeta_n^\omega(g) = |\{(g_1, g_2, \ldots, g_n) \in G^n : \omega(g_1, g_2, \ldots, g_n) = g\}|.$$

The main result of Section 3.1 is given as follows.

Theorem 3.1.4. *Let $\omega(x_1, x_2, \ldots, x_n) \in \mathscr{A}(x_1, x_2, \ldots, x_n)$, $n \geq 1$. If G is a finite group, then the map $\zeta_n^\omega : G \longrightarrow \mathbb{C}$ defined by*

$$\zeta_n^\omega(g) = |\{(g_1, g_2, \ldots, g_n) \in G^n : \omega(g_1, g_2, \ldots, g_n) = g\}|, \quad g \in G,$$

is a character of G.

Given any two finite sets X and Y, a function $f : X \longrightarrow Y$ is said to be *almost measure preserving* if there exists a sufficiently small positive real number ϵ such that

$$\left| \frac{|f^{-1}(Y_0)|}{|X|} - \frac{|Y_0|}{|Y|} \right| < \epsilon \quad \text{for all } Y_0 \subseteq Y.$$

In Section 3.2, we consider the following question posed by Aner Shalev [24, Problem 2.10]:

Which words induce almost measure preserving maps on finite simple groups?

More precisely, given an admissible word $\omega(x_1, x_2, \ldots, x_n)$ and the induced word map $\alpha_\omega : G^n \to G$ defined by $\alpha_\omega(g_1, g_2, \ldots, g_n) = \omega(g_1, g_2, \ldots, g_n)$, we proved that

Corollary 3.2.7. *Let G be a finite simple group, and $o(1)$ be a real number depending on G which tends to zero as $|G| \to \infty$.*

(a) *If* $Y \subseteq G$, *then* $\dfrac{|(\alpha_\omega)^{-1}(Y)|}{|G|^n} = \dfrac{|Y|}{|G|} + o(1)$. *This means that the map* α_ω *is almost measure preserving.*

(b) *If* $X \subseteq G^n$, *then* $\dfrac{|\alpha_\omega(X)|}{|G|} \geq \dfrac{|X|}{|G|^n} - o(1)$; *in particular, if* X *is such that* $|X| = (1 - o(1))|G|^n$, *then* $|\alpha_\omega(X)| = (1 - o(1))|G|$. *This means that almost all the elements of* G *can be expressed as* $\omega(g_1, g_2, \ldots, g_n)$ *for some* $g_1, g_2, \ldots, g_n \in G$.

In the last section of Chapter 3, we obtain yet another generalization of Frobenius' result mentioned above. The main results of this section are given as follows.

Theorem 3.3.1. *Let* G *be a finite group,* $H \trianglelefteq G$ *and* $g \in G$. *If* $\tilde{\zeta}(g)$ *denotes the number of elements* $(h_1, g_2) \in H \times G$ *satisfying* $[h_1, g_2] = g$, *then* $\tilde{\zeta}$ *is a class function of* G *and*

$$\tilde{\zeta}(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|H|[\chi_H, \chi_H]}{\chi(1)} \chi(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|H|[\chi_H^G, \chi]}{\chi(1)} \chi(g).$$

Corollary 3.3.2. *Let* G *be a finite group. Then, with notations as above,* $\tilde{\zeta}$ *is a character of* G.

Proposition 3.3.3. *Let* G *be a finite group,* $H \trianglelefteq G$ *and* $g \in G$. *If* $\tilde{\zeta}_{2n}(g)$, $n \geq 1$, *denotes the number of elements* $((h_1, g_1), \ldots, (h_n, g_n)) \in (H \times G)^n$ *satisfying* $[h_1, g_1] \ldots [h_n, g_n] = g$, *then* $\tilde{\zeta}_{2n}$ *is a character of* G *and*

$$\tilde{\zeta}_{2n}(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G|^{n-1}|H|^n[\chi_H, \chi_H]^n}{\chi(1)^{2n-1}} \chi(g).$$

Proposition 3.3.4. *Let H be a subgroup of a finite group G and $g \in G$. Then the number of elements $(g_1, h_2, g_3) \in G \times H \times G$ satisfying $g_1 h_2 g_1^{-1} g_3 h_2^{-1} g_3^{-1} = g$ defines a character of G and is given by*

$$\tilde{\zeta}_3(g) = \sum_{\chi \in \mathrm{Irr}(G)} \frac{|G||H|[\chi_H, \chi_H]}{\chi(1)} \chi(g).$$

In Chapter 4, which is based on our paper [21], we study the probability $\mathrm{Pr}_g^\omega(G)$ that an arbitrarily chosen n-tuple of elements of a given finite group G is mapped to a given group element g under the word map induced by a non-trivial admissible word $\omega(x_1, x_2, \ldots, x_n)$. Formally, we write

$$\mathrm{Pr}_g^\omega(G) = \frac{\zeta_n^\omega(g)}{|G^n|},$$

where $\zeta_n^\omega(g) = |\{(g_1, g_2, \ldots, g_n) \in G^n : \omega(g_1, g_2, \ldots, g_n) = g\}|$. The main results of Section 4.1 are as follows.

Proposition 4.1.1. *Let G be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a non-trivial admissible word. Then*

(a) $\mathrm{Pr}_1^\omega(G) \geq \dfrac{n|G : Z(G)| - n + 1}{|G : Z(G)|^n} \geq \dfrac{1}{|G : Z(G)|^n} \gneq 0,$

(b) $\mathrm{Pr}_1^\omega(G) = 1$ *if and only if* G *is abelian.*

Proposition 4.1.3. *Let G and H be two finite groups and (ϕ, ψ) be an isoclinism from G to H. If $g \in G'$ and $\omega(x_1, x_2, \ldots, x_n)$ is a non-trivial admissible word, then*

$$\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_{\psi(g)}^\omega(H).$$

Proposition 4.1.4. *Let G be a finite group and $\omega(x_1, x_2, \ldots, x_n)$ be a non-trivial admissible word. If $g, h \in G'$ generate the same cyclic subgroup of G, then $\Pr_g^{\omega}(G) = \Pr_h^{\omega}(G)$.*

Proposition 4.1.6. *Let G be a finite group, $g \in G'$ and $\omega(x_1, x_2, \ldots, x_n)$ be a non-trivial admissible word. Then*

(a) $\Pr_g^{\omega}(G) \leq \Pr_1^{\omega}(G) \leq \Pr(G)$,

(b) $\Pr_g^{\omega}(G) = \Pr_1^{\omega}(G)$ *if and only if* $g = 1$.

Let $m_G = \min\{\chi(1) : \chi \in \mathrm{Irr}(G), \chi(1) \neq 1\}$. Considering, in particular, the word $x_1x_2\ldots x_nx_1^{-1}x_2^{-1}\ldots x_n^{-1}$, $n \geq 2$, and writing $\Pr_g^n(G)$ in place of $\Pr_g^{\omega}(G)$, we obtain the following results in the sections 4.2, 4.3 and 4.4.

Proposition 4.2.2. *Let G be a finite non-abelian group, $g \in G'$ and d be an integer such that $2 \leq d \leq m_G$. Then*

(a) $\left|\Pr_g^n(G) - \frac{1}{|G'|}\right| \leq \frac{1}{d^{n-2}}\left(\Pr(G) - \frac{1}{|G'|}\right)$. *In other words,*

$$\frac{1}{d^{n-2}}\left(-\Pr(G) + \frac{d^{n-2}+1}{|G'|}\right) \leq \Pr_g^n(G) \leq \frac{1}{d^{n-2}}\left(\Pr(G) + \frac{d^{n-2}-1}{|G'|}\right).$$

(b) $\left|\Pr_g^n(G) - \frac{1}{|G'|}\right| \leq \frac{1}{d^{n}}\left(1 - \frac{1}{|G'|}\right)$. *In other words,*

$$\frac{1}{d^{n}}\left(-1 + \frac{d^{n}+1}{|G'|}\right) \leq \Pr_g^n(G) \leq \frac{1}{d^{n}}\left(1 + \frac{d^{n}-1}{|G'|}\right).$$

In particular, $\Pr_g^n(G) \leq \frac{2^n+1}{2^{n+1}}$.

Proposition 4.2.3. *If G is a finite non-abelian simple group and $g \in G'$, then*

$$\left|\Pr_g^n(G) - \frac{1}{|G|}\right| \le \frac{1}{3^{n-2}}\left(\frac{1}{12} - \frac{1}{|G|}\right).$$

In other words,

$$\frac{1}{3^{n-2}}\left(\frac{-1}{12} + \frac{3^{n-2}+1}{|G|}\right) \le \Pr_g^n(G) \le \frac{1}{3^{n-2}}\left(\frac{1}{12} + \frac{3^{n-2}-1}{|G|}\right).$$

In particular,

$$\Pr_g^n(G) \le \frac{3^{n-2}+4}{3^{n-1}\times 20}.$$

Corollary 4.3.2. *Let G be a finite non-abelian group with $|\operatorname{cd}(G)| = 2$. Then every element of G' is a generalized commutator of length n for all $n \ge 2$; in particular, every element of G' is a commutator.*

Proposition 4.3.3. *Let G be a finite non-abelian group, $g \in G'$ and d be an integer such that $2 \le d \le m_G$. Then*

(a) $\Pr_g^n(G) = \dfrac{1}{d^{n-2}}\left(\Pr(G) + \dfrac{d^{n-2}-1}{|G'|}\right)$ *if and only if* $g = 1$ *and* $\operatorname{cd}(G) = \{1, d\}$.

(b) $\Pr_g^n(G) = \dfrac{1}{d^{n-2}}\left(-\Pr(G) + \dfrac{d^{n-2}+1}{|G'|}\right)$ *if and only if* $g \ne 1$, $\operatorname{cd}(G) = \{1, d\}$ *and* $|G'| = 2$.

Proposition 4.3.4. *Let G be a finite non-abelian group, $g \in G'$ and d be an integer such that $2 \le d \le m_G$. Then*

(a) $\Pr_g^n(G) = \dfrac{1}{d^{n}}\left(1 + \dfrac{d^{n}-1}{|G'|}\right)$ *if and only if* $g = 1$ *and* $\operatorname{cd}(G) = \{1, d\}$.

(b) $\Pr_g^n(G) = \dfrac{1}{d^n}\left(-1 + \dfrac{d^n+1}{|G'|}\right)$ *if and only if* $g \neq 1$, $\mathrm{cd}(G) = \{1, d\}$ *and* $|G'| = 2$.

Proposition 4.3.7. *Let G be a finite non-abelian group, $g \in G'$ and p be the smallest prime divisor of $|G|$. Then*

$$\Pr_g^n(G) = \frac{p^n + p - 1}{p^{n+1}}$$

if and only if $g = 1$, *and G is isoclinic to*

$$\langle x, y \,:\, x^{p^2} = 1 = y^p,\ y^{-1}xy = x^{p+1}\rangle.$$

In particular, putting $p = 2$, $\Pr_g^n(G) = \dfrac{2^n+1}{2^{n+1}}$ if and only if $g = 1$, and G is isoclinic to D_8, the dihedral group, and hence, to Q_8, the group of quaternions.

Proposition 4.4.1. *Let G be a finite non-abelian group with $|\,\mathrm{cd}(G)| = 2$ and $g \in G'$. Then*

$$\Pr_1^n(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}}\right) \quad \textit{and}$$
$$\Pr_g^n(G) \leq \frac{1}{|G'|}\left(1 - \frac{1}{|G : Z(G)|^{n/2}}\right) \quad \textit{if } g \neq 1.$$

Moreover, in each case, the equality holds if and only if G is of central type.

Corollary 4.4.2. *Let G be a finite non-abelian group and $g \in G'$. If G is of central type with $|\,\mathrm{cd}(G)| = 2$, then*

$$\Pr_1^n(G) \leq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{2^n}\right) \quad \textit{and}$$
$$\Pr_g^n(G) \geq \frac{1}{|G'|}\left(1 - \frac{1}{2^n}\right) \quad \textit{if } g \neq 1.$$

Proposition 4.4.3. *Let G be a finite non-abelian group and $g \in G'$. If $G' \subseteq Z(G)$ and $|G'| = p$, where p is a prime, then*

$$\Pr_g^n(G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{p^{nk}}\right) & \text{if } \ g = 1 \\ \frac{1}{p}\left(1 - \frac{1}{p^{nk}}\right) & \text{if } \ g \neq 1, \end{cases}$$

where $k = \dfrac{1}{2}\log_p |G : Z(G)|$.

Proposition 4.4.5. *Let p be a prime. Let r and s be two positive integers such that $s \mid (p-1)$, and $r^j \equiv 1 \pmod p$ if and only if $s \mid j$. If $G = \langle a, b : a^p = b^s = 1, bab^{-1} = a^r\rangle$ and $g \in G'$, then*

$$\Pr_g^n(G) = \begin{cases} \dfrac{s^n + p - 1}{ps^n} & \text{if } g = 1 \\ \dfrac{s^n - 1}{ps^n} & \text{if } g \neq 1. \end{cases}$$

Proposition 4.4.6. *Let G be a finite non-abelian group and $g \in G'$. If $G' \cap Z(G) = \{1\}$ and $|G'| = p$, where p is a prime, then*

(a) *G is isoclinic to the group $\langle a, b : a^p = b^s = 1, bab^{-1} = a^r\rangle$, where $s \mid (p-1)$, and $r^j \equiv 1 \pmod p$ if and only if $s \mid j$,*

(b) $$\Pr_g^n(G) = \begin{cases} \dfrac{s^n + p - 1}{ps^n} & \text{if } g = 1 \\ \dfrac{s^n - 1}{ps^n} & \text{if } g \neq 1. \end{cases}$$

Proposition 4.4.7. *Let G be a finite non-abelian group and $g \in G'$. If $g \neq 1$, then $\Pr_g^n(G) < \frac{1}{p}$, where p is the smallest prime divisor of $|G|$. In particular, we have $\Pr_g^n(G) < \frac{1}{2}$.*

Proposition 4.4.8. *For each $\varepsilon > 0$ and for each prime p, there exists a finite group G such that*

$$\left|\mathrm{Pr}_g^n(G) - \frac{1}{p}\right| < \varepsilon$$

for all $g \in G'$.

Let G be a finite group and $g \in G'$. Let H and K be two subgroups of G. In Chapter 5, which is based on our paper [4], we study the probability $\mathrm{Pr}_g(H, K)$ that the commutator of a randomly chosen pair of elements (one from H and the other from K) equals g. In other words, we study the ratio

$$\mathrm{Pr}_g(H, K) = \frac{|\{(x, y) \in H \times K : [x, y] = g\}|}{|H||K|},$$

and further extend some of the results obtained in [10] and [22]. Without any loss, we may assume that G is non-abelian. The main results of the sections 5.1 and 5.2 are as follows.

Proposition 5.1.1. *Let G be a finite group and $g \in G'$. If H and K are two subgroups of G, then $\mathrm{Pr}_g(H, K) = \mathrm{Pr}_{g^{-1}}(K, H)$. However, if $g^2 = 1$, or if $g \in H \cup K$ (for example, when H or K is normal in G), we have $\mathrm{Pr}_g(H, K) = \mathrm{Pr}_g(K, H) = \mathrm{Pr}_{g^{-1}}(H, K)$.*

Theorem 5.1.3. *Let G be a finite group and $g \in G'$. If H and K are two subgroups of G, then*

$$\mathrm{Pr}_g(H, K) = \frac{1}{|H||K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} |C_K(x)| = \frac{1}{|H|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathrm{C}\ell_K(x)}} \frac{1}{|\mathrm{C}\ell_K(x)|},$$

where $C_K(x) = \{y \in K : xy = yx\}$ and $\mathrm{C}\ell_K(x) = \{yxy^{-1} : y \in K\}$, the K-conjugacy class of x.

This theorem plays a key role in the study of $\mathrm{Pr}_g(H,K)$. As an immediate consequence, we have the following generalization of the well-known formula $\mathrm{Pr}(G) = \frac{k(G)}{|G|}$.

Corollary 5.1.4. *Let G be a finite group and H, K be two subgroups of G. If $H \trianglelefteq G$, then*

$$\mathrm{Pr}(H,K) = \frac{k_K(H)}{|H|},$$

where $k_K(H)$ is the number of K-conjugacy classes that constitute H.

Proposition 5.1.5. *If H is an abelian normal subgroup of a finite group G with a complement K in G and $g \in G'$, then*

$$\mathrm{Pr}_g(H,G) = \mathrm{Pr}_g(H,K).$$

Corollary 5.1.6. *Let G be a finite group and $g \in G'$. If $H \trianglelefteq G$ with $C_G(x) = H$ for all $x \in H - \{1\}$, then*

$$\mathrm{Pr}_g(H,G) = \mathrm{Pr}_g(H,K),$$

where K is a complement of H in G. In particular,

$$\mathrm{Pr}(H,G) = \frac{1}{|H|} + \frac{|H|-1}{|G|}.$$

Proposition 5.2.1. *Let G be a finite group and $g \in G'$. Let H and K be any two subgroups of G. If $g \neq 1$, then*

(a) $\mathrm{Pr}_g(H,K) \neq 0 \Longrightarrow \mathrm{Pr}_g(H,K) \geq \dfrac{|C_H(K)||C_K(H)|}{|H||K|}$,

(b) $\Pr_g(H,G) \neq 0 \Longrightarrow \Pr_g(H,G) \geq \dfrac{2|H \cap Z(G)||C_G(H)|}{|H||G|}$,

(c) $\Pr_g(G) \neq 0 \Longrightarrow \Pr_g(G) \geq \dfrac{3}{|G : Z(G)|^2}$.

Proposition 5.2.2. *Let G be a finite group and $g \in G'$. If H and K are any two subgroups of G, then*

$$\Pr_g(H,K) \leq \Pr(H,K)$$

with equality if and only if $g = 1$.

Proposition 5.2.3. *Let G be a finite group and $g \in G'$, $g \neq 1$. Let H and K be any two subgroups of G. If p is the smallest prime divisor of $|G|$, then*

$$\Pr_g(H,K) \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.$$

Proposition 5.2.4. *Let H, K_1 and K_2 be subgroups of a finite group G with $K_1 \subseteq K_2$. Then*

$$\Pr(H,K_1) \geq \Pr(H,K_2)$$

with equality if and only if $\mathrm{C}\ell_{K_1}(x) = \mathrm{C}\ell_{K_2}(x)$ *for all $x \in H$.*

Proposition 5.2.5. *Let H, K_1 and K_2 be subgroups of a finite group G with $K_1 \subseteq K_2$. Then*

$$\Pr(H,K_2) \geq \frac{1}{|K_2 : K_1|}\left(\Pr(H,K_1) + \frac{|K_2| - |K_1|}{|H||K_1|}\right)$$

with equality if and only if $C_H(x) = \{1\}$ *for all $x \in K_2 - K_1$.*

Proposition 5.2.6. *Let $H_1 \subseteq H_2$ and $K_1 \subseteq K_2$ be subgroups of a finite group G and $g \in G'$. Then*

$$\Pr_g(H_1,K_1) \leq |H_2 : H_1||K_2 : K_1|\Pr_g(H_2,K_2)$$

with equality if and only if

$$\begin{aligned} & g^{-1}x \notin \mathrm{C}\ell_{K_2}(x) \;\; \textit{for all} \;\; x \in H_2 - H_1, \\ & g^{-1}x \notin \mathrm{C}\ell_{K_2}(x) - \mathrm{C}\ell_{K_1}(x) \;\; \textit{for all} \;\; x \in H_1, \\ \textit{and} \quad & C_{K_1}(x) = C_{K_2}(x) \;\; \textit{for all} \;\; x \in H_1 \;\; \textit{with} \;\; g^{-1}x \in \mathrm{C}\ell_{K_1}(x). \end{aligned}$$

In particular, for $g = 1$, the condition for equality reduces to $H_1 = H_2$, and $K_1 = K_2$.

Corollary 5.2.7. *Let G be a finite group, H be a subgroup of G and $g \in G'$. Then*

$$\Pr_g(H, G) \leq |G : H| \Pr(G)$$

with equality if and only if $g = 1$ and $H = G$.

Theorem 5.2.8. *Let G be a finite group and p be the smallest prime dividing $|G|$. If H and K are any two subgroups of G, then*

$$\begin{aligned} \Pr(H, K) &\geq \frac{|C_H(K)|}{|H|} + \frac{p(|H| - |X_H| - |C_H(K)|) + |X_H|}{|H||K|} \\ \textit{and} \;\; \Pr(H, K) &\leq \frac{(p-1)|C_H(K)| + |H|}{p|H|} - \frac{|X_H|(|K| - p)}{p|H||K|}, \end{aligned}$$

where $X_H = \{x \in H : C_K(x) = 1\}$. Moreover, in each of these bounds, H and K can be interchanged.

Corollary 5.2.9. *Let G be a finite group and p be the smallest prime dividing $|G|$. If H and K are two subgroups of G such that $[H, K] \neq \{1\}$, then*

$$\Pr(H, K) \leq \frac{2p - 1}{p^2}.$$

In particular, $\Pr(H, K) \leq \frac{3}{4}$.

Proposition 5.2.10. *Let G be a finite group and H, K be any two subgroups of G. If $\Pr(H,K) = \frac{2p-1}{p^2}$ for some prime p, then p divides $|G|$. If p happens to be the smallest prime divisor of $|G|$, then*

$$\frac{H}{C_H(K)} \cong C_p \cong \frac{K}{C_K(H)}, \text{ and hence, } H \neq K.$$

In particular, $\frac{H}{C_H(K)} \cong C_2 \cong \frac{K}{C_K(H)}$ if $\Pr(H,K) = \frac{3}{4}$.

In the last section of chapter 5, with H normal in G, we also develop and study a character theoretic formula for $\Pr_g(H,G)$ given by

$$\Pr_g(H,G) = \frac{1}{|G|}\sum_{\chi\in \mathrm{Irr}(G)} \frac{[\chi_H, \chi_H]}{\chi(1)}\chi(g).$$

Proposition 5.3.1. *Let G be a finite group. If H is a normal subgroup of G and $g \in G'$, then*

$$\left|\Pr_g(H,G) - \frac{1}{|G'|}\right| \leq |G:H| \left(\Pr(G) - \frac{1}{|G'|}\right).$$

As an application, we obtain yet another condition under which every element of G' is a commutator.

Proposition 5.3.3. *Let G be a finite group and p be the smallest prime dividing $|G|$. If $|G'| \leq p^2$, then every element of G' is a commutator.*

We conclude the thesis with a discussion, in the last chapter, on some of the possible research problems related to the results obtained in the earlier chapters.

Bibliography

[1] F. Barry, D. MacHale and Á. Ní Shé, *Some supersolvability conditions for finite groups*, Math. Proc. Royal Irish Acad. **106A** (2) (2006), 163–177.

[2] S. M. Belcastro and G. J. Sherman, *Counting centralizers in finite groups*, Math. Magazine **67** (5) (1994), 366–374.

[3] A. K. Das and R. K. Nath, *On solutions of a class of equations in a finite group*, Comm. Algebra **37** (11) (2009), 3904–3911.

[4] A. K. Das and R. K. Nath, *On generalized relative commutativity degree of a finite group*, International Electronic J. Algebra **7** (2010), 140–151.

[5] A. K. Das and R. K. Nath , *A characterization of certain finite groups of odd order*, communicated.

[6] P. Erdös and P. Turán, *On some problems of a statistical group-theory. I*, Z. Wahrschein. Verw. Gebeite **4** (1965), 175–186.

[7] P. Erdös and P. Turán, *On some problems of a statistical group-theory. II*, Acta Math. Acad. Sci. Hungar. **18** (1967), 151–163.

[8] P. Erdös and P. Turán, *On some problems of a statistical group-theory. III*, Acta Math. Acad. Sci. Hungar. **18** (1967), 309–320.

[9] P. Erdös and P. Turán, *On some problems of a statistical group-theory. IV*, Acta. Math. Acad. Sci. Hungar. **19** (1968), 413–435.

[10] A. Erfanian, R. Rezaei and P. Lescot, *On the relative commutativity degree of a subgroup of a finite group*, Comm. Algebra **35** (12) (2007), 4183–4197.

[11] I. V. Erovenko and B. Sury, *Commutativity degree of wreath products of finite abelian groups*, Bull. Aus. Math. Soc. **77** (1) (2008), 31–36.

[12] F. G. Frobenius, *Über Gruppencharaktere*, Gesammelte Abhandlungen Band III, p. 1–37 (J. P. Serre, ed.) Springer-Verlag, Berlin, 1968.

[13] R. M. Guralnick and G. R. Robinson, *On the commuting probability in finite groups*, J. Algebra **300** (2) (2006), 509–528.

[14] R. M. Guralnick and G. R. Robinson, *Addendum to paper "On the commuting probability in finite groups" R. M. Guralnick and G. R. Robinson [J. Algebra 300 (2) (2006), 509–528]*, J. Algebra **319** (2) (2008), 1822.

[15] W. H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly **80** (1973), 1031–1034.

[16] P. Hall, *The classification of prime-power groups*, J. Reine Angew. Math. **182** (1940), 130–141.

[17] K. S. Joseph, *Commutativity in non-abelian groups*, Unpublished Ph. D. thesis, University of California, Los Angeles, 1969.

[18] P. Lescot, *Isoclinism classes and commutativity degrees of finite groups*, J. Algebra **177** (1995), 847–869.

[19] P. Lescot, *Central extensions and commutativity degree*, Comm. Algebra **29** (10) (2001), 4451–4460.

[20] R. K. Nath and A. K. Das, *On a lower bound of commutativity degree*, Rend. Circ. Mat. Palermo **59** (2010), 137–142.

[21] R. K. Nath and A. K. Das, *On generalized commutativity degree of a finite group*, Rocky Mountain J. Math. (to appear).

[22] M. R. Pournaki and R. Sobhani, *Probability that the commutator of two group elements is equal to a given element*, J. Pure Appl. Algebra **212** (2008), 727–734.

[23] D. J. Rusin, *What is the probability that two elements of a finite group commute?*, Pacific J. Math. **82** (1979), 237–247.

[24] A. Shalev, *Commutators, words, conjugacy classes and character methods*, Turk. J. Math. **31** (2007), 131–148.