ON MAXIMAL DISTANCES IN A COMMUTING GRAPH

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Abstract. We study maximal distances in the commuting graphs of matrix algebras defined over algebraically closed fields. In particular, we show that the maximal distance can be attained only between two nonderogatory matrices. We also describe rank-one and semisimple matrices using the distances in the commuting graph.

1. Introduction and preliminaries

One of the options how to study properties in certain non-commutative algebraic domains is a commutator. For example, in algebras the additive commutator, i.e., Lie product $[A, B]_a = AB-BA$ is usually used and with its help some beautiful results were obtained. Let us only mention the famous Kleinecke-Shirokov Theorem [15, 21]. In groups the multiplicative commutator $[A, B]_m = A^{-1}B^{-1}AB$ is used, and it is a central tool in studying solvability of groups and hence in Galois theory of solvability of equations by radicals [11].

Additional information about non-commuting elements is obtained by studying the properties of a commuting graph. For example, if the commuting graphs over two finite semisimple rings are isomorphic, then their noncommutative parts are also isomorphic [3]. Let us remark that commuting graph can also be used in algebraic domains where commutator is not available, e.g., in semigroups or in semirings.

Up until now, one of the prime concerns when studying commuting graphs was calculating its diameter [1, 4, 6, 9, 10, 17, 19]. It turned out that we obtain essentially different results if the matrix algebra $M_n(\mathbb{F})$ is defined over an algebraically closed field $\mathbb{F}$ than if it is defined over non-closed one. While in the former case the diameter is always equal to four, provided $n \geq 3$, in the later case the graph may be disconnected, and if it is connected the diameter is known to be at most six. The hypothesis is that if the commuting graph is connected, its diameter is at most five [4, Conjecture 18]. Note that for $n = 2$ the commuting graph over any field is disconnected [5, Remark 8].

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In the present paper we are interested in commuting graphs of matrix algebras $M_n(\mathbb{F})$ over algebraically closed fields $\mathbb{F}$ with $n \geq 3$. In particular we study the maximal distances between its vertices. It was already proved in [4, Proof of Theorem 3] that an elementary Jordan matrix is always at the maximal distance (i.e., four) from its transpose. In the present paper we show that the maximal distance cannot be achieved when one of the matrices is derogatory. However, if both A and B are non-derogatory we construct an invertible matrix S so that A and $S^{-1}BS$ are at the distance four. We also show that there exist an infinite collection of matrices, pairwise at the maximal distance. Next, we describe rank-one matrices as the ones which are not at the maximal distance from any derogatory matrix. A similar result classifies semisimple (i.e., diagonalizable) matrices. Our paper concludes with a specific example of matrix algebra over algebraically non-closed field, such that the diameter of its commuting graph is greater than four.

Let us briefly recall some standard definitions and notations. Unless explicitly stated otherwise, $\mathbb{F}$ is an algebraically closed field of an arbitrary characteristics. Further, $M_{m,n}(\mathbb{F})$ is the space of $m \times n$ matrices over $\mathbb{F}$ with a standard basis E_{ij}, and $M_n(\mathbb{F}) = M_{n,n}(\mathbb{F})$ is the matrix algebra with identity I. Let $e_1, \ldots, e_n$ be the standard basis of column vectors in $\mathbb{F}^n$ (i.e., of $n \times 1$ matrices). Given an integer $k \geq 2$ denote by $J_k(\mu) = \mu I_k + \sum_{i=1}^{k-1} E_{i(i+1)} \in M_k(\mathbb{F})$ the upper-triangular elementary Jordan cell with μ on its main diagonal, and let $J_1(\mu) = \mu \in \mathbb{F}$. We write shortly $J_k = J_k(0)$. Matrix B is a conjugated matrix of A if $B = S^{-1}AS$ for some invertible matrix S. As usual, A^{tr} is a transpose of $A \in M_n(\mathbb{F})$ and $\operatorname{rk} A$ its rank.

For a matrix algebra $M_n(\mathbb{F})$ over a field $\mathbb{F}$ its commuting graph $\Gamma(M_n(\mathbb{F}))$ is a simple graph (i.e., undirected and loopless), with the vertex set consisting of all non-scalar matrices. Two vertices X, Y form an edge $X - Y$ if the corresponding matrices are different and commute, i.e., if $X \neq Y$ and $XY = YX$. The sequence of successive connected vertices $X_0 - X_1$, $X_1 - X_2$, ..., $X_{k-1} - X_k$ is a path of length k and is denoted by $X_0 - X_1 - \ldots - X_k$. The distance $d(A, B)$ between vertices A and B is the length of the shortest path between them. The diameter of the graph is the maximal distance between any two vertices of the graph.

Given a subset $\Omega \subseteq M_n(\mathbb{F})$, let

$$\mathcal{C}(\Omega) = \{X \in M_n(\mathbb{F});\ \ AX = XA \text{ for every } A \in \Omega\}$$

be its centralizer. If $\Omega = \{A\}$ then we write shortly $\mathcal{C}(A) = \mathcal{C}(\{A\})$. In graph terminology, the set of all non-scalar matrices from the centralizer of A is equal to the set of all vertices X such that $d(A, X) \leq 1$. Note that $\mathbb{F}I \in \mathcal{C}(A)$ for any matrix A and that, by a double centralizer theorem, $\mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$ (see [22, Theorem 2, pp. 106] or [16]). We remark that in different articles a centralizer is also called a commutant and is denoted by $A' = \mathcal{C}(A)$.

A centralizer induces two natural relations on $M_n(\mathbb{F})$. One is the equivalence relation, defined by $A \sim B$ if $\mathcal{C}(A) = \mathcal{C}(B)$. We call any such two matrices equivalent. The other relation is a preorder given by $A \prec B$ if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$. It was already observed that minimal and maximal matrices in this poset are of special importance, see for example [7, 20, 8]. Recall that a matrix A is minimal if $\mathcal{C}(X) \subseteq \mathcal{C}(A)$ implies $\mathcal{C}(X) = \mathcal{C}(A)$. It was shown in [20, Lemma 3.2] that the matrix A is minimal if and only if it is nonderogatory, which means that each of its eigenvalue has geometric multiplicity one, which is further equivalent to the fact that its Jordan canonical form is equal to $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case,

$$\mathcal{C}(J) = \mathbb{F}[J_{n_1}(\lambda_1)] \oplus \cdots \oplus \mathbb{F}[J_{n_k}(\lambda_k)] = \mathbb{F}[J],$$

where $\mathbb{F}[X]$ is an $\mathbb{F}$-algebra generated by X, see [22, Theorem 1, pp. 105] or [12, Theorem 3.2.4.2].

Recall also that a non-scalar matrix A is maximal if $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ implies $\mathcal{C}(A) = \mathcal{C}(X)$ or X is a scalar matrix. It is known (see [7, Lemma 4] and also [20, Lemma 3.1]) that a matrix is maximal if and only if it is equal to $\alpha I + \beta P$ or $\alpha I + \beta N$, where $P^2 = P$ is a non-scalar idempotent, $N \neq 0$ is square-zero (i.e., $N^2 = 0$), and a scalar β is nonzero. It should be noted that the proof of this fact was done only for the field of complex numbers, but can be repeated almost unchanged in an arbitrary algebraically closed field.

2. Results

Throughout this section, with an exception of the last example, $\mathbb{F}$ is an algebraically closed field and $n \geq 3$. We start with three technical lemmas which will be needed in the sequel. First we observe that every matrix commutes with a rank-one matrix.

Lemma 2.1. *For every matrix $A \in M_n(\mathbb{F})$ there exists a rank-one matrix $R \in M_n(\mathbb{F})$ with $d(A, R) \leq 1$.*

Proof. Given any $A \in M_n(\mathbb{F})$, it suffices to show that A commutes with at least one matrix of rank one. Since $\mathbb{F} = \overline{\mathbb{F}}$, the matrix A has at least one eigenvalue λ. So, we may assume without loss of generality that A is singular, otherwise we would consider $A - \lambda I$. Now, let x and y be nonzero vectors in the kernels of A and A^{tr}, respectively. Then, $R = xy^{\mathrm{tr}}$ is a rank-one matrix with $AR = (Ax)y^{\mathrm{tr}} = 0 = x(A^{\mathrm{tr}}y)^{\mathrm{tr}} = RA$. □

Using Lemma 2.1 we can give an alternative proof of the already known fact about the diameter of a commuting graph [4, Corollary 7].

Corollary 2.2. *The distance between any two matrices in the commuting graph is at most four.*

Proof. Let A and B be arbitrary matrices. By Lemma 2.1 there exist rank-one matrices $R_1 = xf^{\mathrm{tr}} \in \mathcal{C}(A)$ and $R_3 = yg^{\mathrm{tr}} \in \mathcal{C}(B)$. Since $n \geq 3$ we can find a nonzero $z \in \mathbb{F}^n$ with $f^{\mathrm{tr}} z = 0 = g^{\mathrm{tr}} z$ and a nonzero $h \in \mathbb{F}^n$ with $h^{\mathrm{tr}} x = 0 = h^{\mathrm{tr}} y$. Then for a rank-one matrix $R_2 = xh^{\mathrm{tr}}$ we obtain $A = R_0 — R_1 — R_2 — R_3 — R_4 = B$. □

Lemma 2.3. *Let $A = J_{k_1} \oplus J_{k_2} \in M_{k_1+k_2}(\mathbb{F})$ be a nilpotent matrix with two Jordan cells of sizes $k_1, k_2 \geq 1$. Then $d(A,R) \leq 2$ for an arbitrary rank-one $R \in M_{k_1+k_2}(\mathbb{F})$.*

Proof. If $k_1 = k_2 = 1$ then A is a zero matrix and the conclusion is then imminent. Otherwise, $k_1 \geq 2$ or $k_2 \geq 2$. Let $k = k_1 + k_2$. It is elementary that the matrix $Z = x_1 E_{1k_1} + x_2 E_{1k} + x_3 E_{(k_1+1)k_1} + x_4 E_{(k_1+1)k}$ commutes with A for any choice of $x_1, x_2, x_3, x_4 \in \mathbb{F}$. Actually, $ZA = AZ = 0$. Moreover, the matrix Z is non-scalar, except when $x_1 = x_2 = x_3 = x_4 = 0$. Therefore, it suffices to show that for an arbitrary rank-one matrix R there exist $x_1, \ldots, x_4 \in \mathbb{F}$, such that at least one of them is nonzero and $ZR = RZ = 0$. To this end, write $R = ab^{\mathrm{tr}}$ for some column vectors $a = (a_1, \ldots, a_k)^{\mathrm{tr}}$ and $b = (b_1, \ldots, b_k)^{\mathrm{tr}}$. Then $ZR = RZ = 0$ is equivalent to $Za = Z^{\mathrm{tr}} b = 0$, hence we must solve a homogeneous system of four linear equations

$$
\begin{aligned}
x_1 a_{k_1} + x_2 a_k &= 0,\\
x_3 a_{k_1} + x_4 a_k &= 0,\\
x_1 b_1 + x_3 b_{k_1+1} &= 0,\\
x_2 b_1 + x_4 b_{k_1+1} &= 0,
\end{aligned}
\tag{1}
$$

x_1, x_2, x_3, x_4 unknown. The corresponding matrix of coefficients is equal to

$$
\begin{bmatrix}
a_{k_1} & a_k & 0 & 0\\
0 & 0 & a_{k_1} & a_k\\
b_1 & 0 & b_{k_1+1} & 0\\
0 & b_1 & 0 & b_{k_1+1}
\end{bmatrix}
$$

and it is easy to check that it is always singular. Therefore the system (1) has a nontrivial solution. This solution defines a non-scalar matrix Z, which commutes with A and R, so $d(A,R) \leq 2$ in $\Gamma(M_{k_1+k_2}(\mathbb{F}))$. □

Lemma 2.4. *Suppose A is not minimal. Then $d(A,R) \leq 2$ for an arbitrary rank-one matrix $R \in M_n(\mathbb{F})$.*

Proof. Using conjugation we might assume A is already in its Jordan form. Since it is not minimal, hence it is derogatory, at least two Jordan cells contain the same eigenvalue. Let $k_1, k_2 \geq 1$ be their sizes. Define also $k = k_1 + k_2$. Moreover, $\mathcal{C}(A) = \mathcal{C}(A - \lambda I)$ so we may also assume that these two Jordan cells are nilpotent and that $A = J_{k_1} \oplus J_{k_2} \oplus \tilde{A}$. It is elementary that

$$\mathcal{C}\big(J_{k_1} \oplus J_{k_2}\big) \oplus (\mathbb{F} I_{n-k}) \subseteq \mathcal{C}(A).$$

Now, let $R = xy^{\mathrm{tr}}$ be an arbitrary rank-one matrix. Decompose $x = x_1 \oplus x_2 \in \mathbb{F}^k \oplus \mathbb{F}^{n-k}$ and $y = y_1 \oplus y_2 \in \mathbb{F}^k \oplus \mathbb{F}^{n-k}$. We claim that there exists a non-scalar matrix $\widehat{Z} \in \mathcal{C}(J_{k_1} \oplus J_{k_2})$ satisfying simultaneously $\widehat{Z} x_1 = \lambda x_1$ as well as $\widehat{Z}^{\mathrm{tr}} y_1 = \lambda y_1$ for some $\lambda \in \mathbb{F}$. In fact, this is trivial when $x_1 = y_1 = 0$. Otherwise we let

$$\widehat{R} = \begin{cases} x_1 y_1^{\mathrm{tr}}; & x_1, y_1 \neq 0 \\ e_1 y_1^{\mathrm{tr}}; & x_1 = 0 \\ x_1 e_1^{\mathrm{tr}}; & y_1 = 0 \end{cases} \in M_k(\mathbb{F}),$$

where $e_1 \in \mathbb{F}^k$ is the first vector of the standard basis. By Lemma 2.3 there exists at least one non-scalar matrix $\widehat{Z} \in M_k(\mathbb{F})$ which commutes with $\widehat{R}$ as well as with $J_{k_1} \oplus J_{k_2} \in M_k(\mathbb{F})$. Therefore, if $x_1, y_1 \neq 0$, then $\widehat{Z} x_1 y_1^{\mathrm{tr}} = x_1 (\widehat{Z}^{\mathrm{tr}} y_1)^{\mathrm{tr}}$ and we obtain $\widehat{Z} x_1 = \lambda x_1$, and $\widehat{Z}^{\mathrm{tr}} y_1 = \lambda y_1$ for some $\lambda \in \mathbb{F}$. If $x_1 = 0$, then similarly as above $\widehat{Z} e_1 = \lambda e_1$, and $\widehat{Z}^{\mathrm{tr}} y_1 = \lambda y_1$. Obviously $\widehat{Z} x_1 = \lambda x_1$. Likewise we argue if $y_1 = 0$.

With the help of $\widehat{Z}$ we define $Z = \widehat{Z} \oplus \lambda I_k \in \mathcal{C}(J_{k_1} \oplus J_{k_2}) \oplus (\mathbb{F} I_{n-k}) \subseteq \mathcal{C}(A)$. Clearly, $Zx = \widehat{Z} x_1 \oplus \lambda x_2 = \lambda x$, and similarly, $Z^{\mathrm{tr}} y = \lambda y$, so Z commutes with $R = xy^{\mathrm{tr}}$ and with A. □

Akbari, Mohammadian, Radjavi, and Raja proved in [4, Lemma 2] that, for matrices of size $n \geq 3$, the diameter of the commuting graph is at most four (see also Corollary 2.2 above) and that $d(J, J^{\mathrm{tr}}) = 4$, thus showing that the diameter of the commuting graph of matrix algebra over algebraically closed fields is equal to four. It is well-known [12, p. 134] that the transpose of a matrix is conjugate to the original, so [4, Lemma 2] implies that the maximal distance from J to some of its conjugates is equal to four. Our next lemma will strengthen their result by considering maximal distances between an arbitrary minimal matrix $A \in M_n(\mathbb{F})$ and matrices from conjugation orbit $\{S^{-1} B S;\ S \text{ invertible}\}$ of another minimal matrix $B \in M_n(\mathbb{F})$. Recall that a minimal matrix is conjugate to $\bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, and where $(n_1, n_2, \ldots, n_k)$ is a partition of n. We will show below that for any two given partitions of n, we can find two minimal matrices with their Jordan forms corresponding to these two partitions, at distance four. One of the matrices is already in its Jordan canonical form, while the other is a matrix, conjugated to its Jordan canonical form by an invertible matrix with all of its minors nonzero. Such invertible matrix is for example a Cauchy matrix $\left[\frac{1}{x_i - y_j}\right]_{ij}$ (see [18]).

Theorem 2.5. *Let S be any matrix with all of its minors nonzero. For any two minimal matrices $A = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) \in M_n(\mathbb{F})$ and $B = \bigoplus_{i=1}^{l} J_{m_i}(\mu_i) \in M_n(\mathbb{F})$, we have $d(A, S^{-1} B S) = 4$.*

Proof. Assume erroneously that A and B, as defined in Lemma, are not at the maximal distance, i.e., $d(A, S^{-1} B S) \leq 3$. Since $\mathcal{C}(A) = \mathcal{C}(\alpha A)$ for all nonzero $\alpha \in \mathbb{F}$, we can lengthen every path by adding vertices which correspond to scalar multiples of matrices. So, there exists a path

$A - X - Y - S^{-1}BS$ of length 3 in $\Gamma(M_n(\mathbb{F}))$. We can assume without loss of generality that X and Y are maximal matrices. Namely, if X is not maximal, then there exists a maximal $X' \succ X$, and since $A, Y \in \mathcal{C}(X) \subseteq \mathcal{C}(X')$, we could consider a path $A - X' - Y - S^{-1}BS$ of length 3. Likewise for Y.

We will show that no two maximal matrices $X \in \mathcal{C}(A)$ and $Y \in \mathcal{C}(S^{-1}BS)$ commute and thus obtain a contradiction to the assumption $d(A, S^{-1}BS) \leq 3$. Since all maximal matrices are equivalent either to a square-zero matrix or to an idempotent matrix we will consider three cases.

First, let us assume that both X and Y are square-zero but nonzero. Since $A = \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i)$ and $B = \bigoplus_{i=1}^{l} J_{m_i}(\mu_i)$ are minimal, so $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for $i \neq j$, we have that

$$X = T_1 \oplus T_2 \oplus \ldots \oplus T_k \quad \text{and} \quad Y = S^{-1}(T_1' \oplus T_2' \oplus \ldots \oplus T_l')S,$$

where all T_i and T_j' are upper triangular Toeplitz matrices. Clearly then $\operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$ for some permutation σ of length n and integer r, $1 \leq r \leq \frac{n}{2}$. Moreover, by the block-Toeplitz structure of $X \neq 0$ there exist indices t and s such that $Xe_t = \alpha e_s \neq 0$. For the sake of simplicity let us denote $T = T_1' \oplus T_2' \oplus \ldots \oplus T_l'$. Now, if $YX = XY$, we would have that $S^{-1}TSXe_t \in \operatorname{Im} X$. This would imply,

$$\alpha TSe_s \in S(\operatorname{Im} X) = \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$$

which is clearly possible if and only if the rank of the $n \times r$ matrix $M = \left[\frac{1}{\alpha}Se_{\sigma(1)}, \frac{1}{\alpha}Se_{\sigma(2)}, \ldots, \frac{1}{\alpha}Se_{\sigma(r)}\right]$ is the same as the rank of the augmented matrix $[M \,|\, TSe_s]$. However, we will show that this is not the case. Since all minors of S are nonzero, its s-es column Se_s has no zero entries and as such cannot be annihilated by a nonzero block-Toeplitz matrix T. Note that T is also square-zero and so it has at least $\frac{n}{2}$ zero rows. Recall that $r \leq \frac{n}{2}$, consequently there exists an $(r+1) \times (r+1)$ submatrix of the augumented matrix, having in the last column exactly r zeros and one nonzero element. By expanding this $(r+1) \times (r+1)$ minor by the last column, we observe that it is equal to a multiple of an $r \times r$ minor of matrix M which is equal to $(\frac{1}{\alpha})^r$ times an $r \times r$ minor of S. By the assumption, every minor of S is nonzero and so $r + 1 = \operatorname{rk}[M \,|\, TSe_s] > \operatorname{rk} M = r$. This implies $TSe_s \notin S(\operatorname{Im} X)$, a contradiction.

Second, suppose a non-scalar idempotent $X \in \mathcal{C}(A)$ commutes with a non-scalar square-zero $Y \in \mathcal{C}(S^{-1}BS)$. Without loss of generality, $r = \operatorname{rk} X \leq \frac{n}{2}$, otherwise take $I - X$ instead of X. So, $X = \sum_{i=1}^{r} E_{\sigma(i)\sigma(i)}$ and $Y = S^{-1}TS$, where σ and $T = T_1' \oplus T_2' \oplus \ldots \oplus T_l'$ are as above. Define $t = \sigma(1)$. Similarly as before, if $YX = XY$ we would have that $S^{-1}TSXe_t \in \operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$, or, equivalently, $TSe_t \in \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$. We proceed as in the first case to obtain a contradiction.

By the symmetry the only case remaining is the case when X and Y are both non-scalar idempotents. Write $X = \sum_{i=1}^{r} E_{\sigma(i)\sigma(i)}$ and $Y = S^{-1}PS$ for $P = \sum_{i=1}^{s} E_{\tau(i)\tau(i)}$. Without loss of generality, $r, s \leq \frac{n}{2}$, since otherwise we would substitute X by $I - X$ or Y by $I - Y$. Again, take $t = \sigma(1)$. If $YX = XY$ then $S^{-1}PSXe_t \in \operatorname{Im} X = \operatorname{Lin}\{e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(r)}\}$, or, equivalently, $PSe_t \in \operatorname{Lin}\{Se_{\sigma(1)}, Se_{\sigma(2)}, \ldots, Se_{\sigma(r)}\}$. Since $\operatorname{rk} P \leq \frac{n}{2}$, it follows that the vector PSe_t has at least $\frac{n}{2}$ zero entries. Note that $\frac{n}{2} \geq r$ and Se_t is the t-th column of S, so it has no zero entries. This gives $PSe_t \neq 0$, a contradiction as in the first case.

This shows $d(A, S^{-1}BS) \geq 4$. But the diameter of commuting graph is equal to four (see [4, Lemma 2]), hence $d(A, S^{-1}BS) = 4$. □

Remark 2.6. *The matrix $S^{-1}BS$ from Theorem 2.5 can be rather complicated. In a special case, when A is nilpotent we can take $A = J_n$ to achieve that $d(J_n, B) = 4$ for any companion matrix B in the lower-triangular form. This can be seen by a slight adaptation of the proof of* [4, Lemma 2]*. For convenience we sketch the main points of the proof. First, it suffices to prove that each maximal $D \in \mathcal{C}(J_n) = \mathbb{F}[J_n]$ satisfies $\mathcal{C}(B) \cap \mathcal{C}(D) = \mathbb{F}I$. We may further assume $D = \sum_{i=r}^{n-1} d_i J_n^i$, $\frac{n}{2} \leq r \leq n-1$, is square-zero and hence write it as a 3×3 block matrix with block at position $(1,3)$ being invertible upper-triangular Toeplitz of size $(n-r) \times (n-r)$, while all the rest blocks are zero. If $Z \in \mathcal{C}(B) \cap \mathcal{C}(D)$ then in particular it commutes with D. By direct computation using block-matrix structure we see that the $(n,1)$ entry of Z is zero. However, $Z \in \mathcal{C}(B)$ and since companion matrices are non-derogatory we have $Z = \sum_{i=0}^{n-1} \lambda_i B^i$. By considering the images of basis vectors we see that $B^i = \begin{bmatrix} 0_{i,(n-i)} & \bigstar_{i,i} \\ I_{n-i} & \bigstar_{(n-i),i} \end{bmatrix}$. Since $(n,1)$ entry of Z is 0 we see that $\lambda_{n-1} = 0$. Proceeding inductively we see that $\lambda_i = 0$ for every $i = (n-1), \ldots, 1$, whence Z is scalar.*

By Theorem 2.5 there exist different types of matrices which are at the maximal distance. Next we show that we can find infinitely many matrices which are in the commuting graph pairwise at the maximal distance. Actually, we find an induced graph which is a tree with an internal vertex and all of its leaves at distance two from the internal vertex.

Theorem 2.7. *There exist an infinite family of matrices $(X_\alpha)_\alpha \in M_n(\mathbb{F})$ and a rank-one matrix Z such that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$ and $d(X_\alpha, Z) = 2$ for all α.*

Proof. We consider three cases separately.

Case $n = 3$. Choose $Z = E_{11}$ and let the infinite family consist of rank one nilpotent matrices

$$R_\alpha = (0, 1, \alpha)^{\text{tr}} (0, \alpha, -1), \quad \alpha \in \mathbb{F}.$$

It is easy to see that each member commutes with E_{11} and that the elements of the family are pairwise at distance two. For each index $\alpha \in \mathbb{F}$ choose a nilpotent X_α such that $X_\alpha^2 = R_\alpha$. Since $n = 3$ all non-scalar matrices, which commute with X_α are equivalent to X_α or to $X_\alpha^2 = R_\alpha$. Therefore, as $d(X_\alpha^2, X_\beta^2) = 2$ for $\alpha \neq \beta$, we see that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$.

Case $n = 4$. Choose $\lambda \in \mathbb{F} \setminus \{0, 1\}$. For nonzero $\alpha \in \mathbb{F}$ consider rank-one nilpotent matrix $N_\alpha = (0, \lambda, \lambda\alpha, \lambda)^{\mathrm{tr}}\,(0, -\alpha, 1, 0)$ and rank-one idempotent $P_\alpha = (0, 1, \alpha, 0)^{\mathrm{tr}}\,(0, 1, 0, -1)$. It is a straightforward calculation that all these matrices are pairwise non-commutative but they all commute with E_{11}, hence

$$d(N_\alpha, N_\beta) = d(P_\alpha, P_\beta) = d(N_\alpha, P_\beta) = 2 \tag{2}$$

for every $\alpha \neq \beta \in \mathbb{F} \setminus \{0\}$. Moreover, there exists a conjugation such that $S_\alpha^{-1} N_\alpha S_\alpha = E_{13}$ and $S_\alpha^{-1} P_\alpha S_\alpha = E_{44}$, for example, take

$$S_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & -1 \\ \alpha\lambda & 0 & 1 & -\alpha \\ \lambda & 0 & 0 & 0 \end{bmatrix}.$$

Then, for each α we can find a minimal matrix $X_\alpha = S_\alpha(J_3 \oplus 1)S_\alpha^{-1}$ with $X_\alpha \prec P_\alpha$ and $X_\alpha \prec N_\alpha$.

We claim that $d(X_\alpha, X_\beta) = 4$. In fact, if a maximal matrix M satisfies $M \succ X_\alpha$, then, up to equivalence, either $M = N_\alpha$, or $M = P_\alpha$. Hence, if $X_\alpha — Y_{\alpha,\beta} — Z_{\alpha,\beta} — X_\beta$ would be a path of length three, connecting X_α and X_β for $\alpha \neq \beta$, then we may assume without loss of generality that $Y_{\alpha,\beta}$ and $Z_{\alpha,\beta}$ are maximal matrices (see the proof of Theorem 2.5). Hence $Y_{\alpha,\beta}$ is equivalent either to N_α or P_α and $Z_{\alpha,\beta}$ is equivalent either to N_β or P_β. This contradicts equation (2), so $d(X_\alpha, X_\beta) \geq 4$ for $\alpha \neq \beta$. Observe that one of the paths from X_α to X_β is $X_\alpha — N_\alpha — E_{11} — N_\beta — X_\beta$.

Case $n \geq 5$. Let $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where λ_i are pairwise distinct. Consider an infinite family of rank one nilpotent matrices R_α indexed by scalars $\alpha \in \mathbb{F}$:

$$R_\alpha = R + \alpha\tilde{R}; \qquad R = x f^{\mathrm{tr}}, \tilde{R} = x g^{\mathrm{tr}},\ x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},\ f = \begin{bmatrix} 2-n \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix},\ g = \begin{bmatrix} 1-n \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix},$$

Note that $S_\alpha = I + R_\alpha$ is invertible with $S_\alpha^{-1} = I - R_\alpha$, and that $S_\alpha^{-1} S_\beta = (I - R_\alpha)(I + R_\beta) = I + (\beta - \alpha)\tilde{R}$. Let us define for every $\alpha \in \mathbb{F}$ the matrix $X_\alpha = S_\alpha A S_\alpha^{-1}$. We will prove that $d(X_\alpha, X_\beta) = 4$ for $\alpha \neq \beta$.

Note first that the distance is invariant for simultaneous conjugation. So, we may replace (X_α, X_β) with $(S_\alpha^{-1} X_\alpha S_\alpha, S_\alpha^{-1} X_\beta S_\alpha) = (A, SAS^{-1})$, where $S = S_\alpha^{-1} S_\beta = I + (\beta - \alpha)\tilde{R}$. Now, to prove $d(A, SAS^{-1}) = 4$ it suffices to show that, given any non-scalar matrices $D_1 \in \mathcal{C}(A)$ and $SD_2S^{-1} \in S\,\mathcal{C}(A)S^{-1}$, they do not commute.

By the choice of minimal A, $\mathcal{C}(A)$ consists of diagonal matrices only, hence D_1 and D_2 are diagonal. Assume erroneously that D_1 and SD_2S^{-1} do commute, i.e., that $D_1(SD_2S^{-1}) = (SD_2S^{-1})D_1$, or equivalently,

$$D_1(I + \tilde{x}g^{\mathrm{tr}})D_2(I - \tilde{x}g^{\mathrm{tr}}) = (I + \tilde{x}g^{\mathrm{tr}})D_2(I - \tilde{x}g^{\mathrm{tr}})D_1,$$

where $\tilde{x} = (\beta - \alpha)x$. Since diagonal matrices commute, we get after expansion and simplification

$$(3)\quad (D_1\tilde{x})(D_2 g)^{\mathrm{tr}} - (D_1 D_2 \tilde{x}) g^{\mathrm{tr}} - (g^{\mathrm{tr}} D_2 \tilde{x}) \cdot (D_1 \tilde{x}) g^{\mathrm{tr}} = \tilde{x}(D_1 D_2 g - (g^{\mathrm{tr}} D_2 \tilde{x}) \cdot D_1 g)^{\mathrm{tr}} - (D_2 \tilde{x})(D_1 g)^{\mathrm{tr}}.$$

Notice that an eigenvector of a non-scalar diagonal matrix has at least one nonzero entry. Hence, $\tilde{x} = (\beta - \alpha)(1, \dots, 1)^{\mathrm{tr}}$ and $g = (1-n, 1, \dots, 1, 1)^{\mathrm{tr}}$ can not be eigenvectors of a non-scalar diagonal matrix. In particular, g and $D_2 g$ are linearly independent and so there exists a vector y such that $g^{\mathrm{tr}} y = 0$ and $(D_2 g)^{\mathrm{tr}} y = 1$. Post-multiplying both sides of equation (3) with y, we now have $D_1 \tilde{x} = \mu \tilde{x} + \nu D_2 \tilde{x}$, $\mu = (D_1 D_2 g - (g^{\mathrm{tr}} D_2 \tilde{x}) D_1 g)^{\mathrm{tr}} y$ and $\nu = -(D_1 g)^{\mathrm{tr}} y$. We infer that $(D_1 - \nu D_2)\tilde{x} = \mu \tilde{x}$, hence $(D_1 - \nu D_2)$ is a scalar matrix because $\tilde{x}$ has all its entries nonzero. Thus, $D_1 = \lambda I + \nu D_2$ for some λ. This simplifies the starting equation $D_1(S D_2 S^{-1}) = (S D_2 S^{-1}) D_1$ into

$$D(SDS^{-1}) = (SDS^{-1})D; \qquad D = D_2,$$

wherefrom also the derived equation (3) simplifies into

$$(4)\quad (D\tilde{x})(Dg)^{\mathrm{tr}} - (D^2 \tilde{x}) g^{\mathrm{tr}} - (g^{\mathrm{tr}} D \tilde{x}) \cdot (D \tilde{x}) g^{\mathrm{tr}} = \tilde{x}(D^2 g - (g^{\mathrm{tr}} D \tilde{x}) \cdot D g)^{\mathrm{tr}} - (D \tilde{x})(D g)^{\mathrm{tr}}.$$

By the similar arguments as above we find a vector z such that $g^{\mathrm{tr}} z = 0$ and $(Dg)^{\mathrm{tr}} z = 1$, and continuing along the lines we see that

$$D\tilde{x} = ((D^2 g)^{\mathrm{tr}} z - (g^{\mathrm{tr}} D \tilde{x})) \cdot \tilde{x} - D\tilde{x}.$$

If $\operatorname{char} \mathbb{F} \neq 2$ then the above equation implies that $\tilde{x}$ is an eigenvector of a diagonal matrix D which is possible only when $D_2 = D$ is scalar, a contradiction.

However, if $\operatorname{char} \mathbb{F} = 2$ then we choose a vector, still named z, such that $g^{\mathrm{tr}} z = 1$ and $(Dg)^{\mathrm{tr}} z = 0$. Similarly as above, this simplifies equation (4) into

$$D^2 \tilde{x} + (g^{\mathrm{tr}} D \tilde{x}) \cdot D \tilde{x} = \mu \tilde{x}; \qquad \mu = (D^2 g)^{\mathrm{tr}} z.$$

Arguing as above, $D^2 + (g^{\mathrm{tr}} D\tilde{x}) D - \mu I = 0$. Thus, $D_2 = D$, being non-scalar diagonal, has exactly two distinct eigenvalues: d_1 and d_2 (with multiplicities k and $n-k$, respectively), because it is annihilated by a quadratic polynomial $p(\lambda) = \lambda^2 + (g^{\mathrm{tr}} D \tilde{x})\lambda - \mu = (\lambda - d_1)(\lambda - d_2)$. With no loss of generality we assume that $d_1 = D_{1,1}$. Then, comparing the coefficients in characteristics 2, gives $d_1 + d_2 = (g^{\mathrm{tr}} D \tilde{x}) = (\beta - \alpha)\big((1-n)d_1 + (k-1)d_1 + (n-k)d_2\big) = (\beta - \alpha)(n-k)(d_1 + d_2)$. Since $\operatorname{char} \mathbb{F} = 2$ and D is not a scalar matrix, we can divide by $d_1 + d_2$ to obtain $(\beta - \alpha)(n-k) = 1$. Observe that in characteristic two, $(\beta - \alpha)(n-k)$ is either equal to 0 or $\beta - \alpha$ and since $(\beta - \alpha)(n-k) = 1$, we have that $\beta - \alpha = (\beta - \alpha)(n-k) = 1$. and thus $\beta = \alpha + 1$. Clearly, we can choose an infinite subset of indices $\mathfrak{A} = \{0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots\} \subset \mathbb{F}$ such that $\alpha - \beta \neq 1$ for $\alpha, \beta \in \mathfrak{A}$. For this subset, $d(X_\alpha, X_\beta) = 4$.

To prove the rest, observe that the rank-one matrix

$$(I + x(f + \alpha g)^{\text{tr}})e_1 e_1^{\text{tr}}(I - x(f + \alpha g)^{\text{tr}}) = S_\alpha E_{11} S_\alpha^{-1}$$

commutes with $X_\alpha = S_\alpha A S_\alpha^{-1}$. Now, since $n \geq 5$ there exists a nonzero vector w with $w^{\text{tr}} e_1 = 0 = w^{\text{tr}} x = w^{\text{tr}} f = w^{\text{tr}} g$. Then, a rank-one matrix $Z = ww^{\text{tr}}$ commutes with $S_\alpha E_{11} S_\alpha^{-1}$ which gives the path

$$X_\alpha - S_\alpha E_{11} S_\alpha^{-1} - Z.$$

Hence, $d(X_\alpha, Z) \leq 2$ for every α. Actually, no shorter path exists, because otherwise, we could join the shorter path for some α with the above path for some other index β to obtain that $d(X_\alpha, X_\beta) \leq 3$, a contradiction. □

We next proceed with the classification of matrices which are equivalent to rank-one matrices. In this classification we will need the following lemma.

Lemma 2.8. *Let $n \geq 4$. Suppose $A \in M_n(\mathbb{F})$ is*
(i) *either a maximal matrix with $2 \leq \operatorname{rk} A \leq n-2$, or*
(ii) *a nilpotent matrix with $A^3 = 0$ and $\operatorname{rk}(A^2) = 1$.*

Then there exists a nonminimal matrix X, such that $d(A, X) \geq 3$.

Proof. (i) As already observed in Preliminaries, a maximal matrix A is either a square-zero matrix or an idempotent, up to equivalence. Let $k = \operatorname{rk} A$.

If A is square-zero, then $2 \leq k \leq \frac{n}{2}$. We define $s_\ell = (1, 1, \ldots, 1)^{\text{tr}} \in \mathbb{F}^\ell$ and $z_{2\ell} = (0, 1, 0, 1, \ldots, 0, 1)^{\text{tr}} \in \mathbb{F}^{2\ell}$. Also, let $N_{2\ell} = \bigoplus_{i=1}^{\ell} J_2^{\text{tr}} \in M_{2\ell}(\mathbb{F})$. Note that $N_{2\ell}^2 = 0$ and $\operatorname{rk} N_{2\ell} = \ell$. It is easy to see that a matrix

$$\begin{bmatrix} N_{2k-2} & 0 & z_{2k-2} \\ 0 & 0_{n-2k+1, n-2k+1} & s_{n-2k+1} \\ 0 & 0 & 0_{1,1} \end{bmatrix} \tag{5}$$

is a square-zero of rank k, hence conjugate to A. So, we can assume without loss of generality that A is already in the form (5).

Next, let us define a matrix $X = J_2 \oplus 0_1 \oplus D$, where D is a diagonal matrix with $n-3$ distinct nonzero diagonal entries. Clearly, X is nonminimal. We will prove that $d(A, X) \geq 3$, i.e., any matrix that commutes with A and X is a scalar matrix.

First, let us assume $k = 2$. It is easy to see that every matrix $B \in \mathcal{C}(X)$ can be decomposed in the following way

$$B = \begin{bmatrix} T & S_1 & 0_{2,1} \\ S_2 & D' & 0_{n-3,1} \\ 0_{1,2} & 0_{1,n-3} & \lambda \end{bmatrix} \tag{6}$$

where $T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in M_2(\mathbb{F})$, $D' = \operatorname{diag}(d_3, d_4, \ldots, d_{n-1}) \in M_{n-3}(\mathbb{F})$, $S_1 \in M_{2,n-3}(\mathbb{F})$ has the only nonzero entry in its upper left corner, $S_2 \in M_{n-3,2}(\mathbb{F})$ has the only nonzero entry in its upper right corner, and $\lambda \in \mathbb{F}$. Note that the blocks in the decomposition of B correspond to the blocks in the decomposition of A. Suppose $B \in \mathcal{C}(X)$ also commutes with $A =$

$\begin{bmatrix} N_2 & 0 & z_2 \\ 0 & 0_{n-3,n-3} & s_{n-3} \\ 0 & 0 & 0_{1,1} \end{bmatrix}$. Then, $N_2S_1 = 0$ and $S_2N_2 = 0$ imply that $S_1 = 0$ and $S_2 = 0$. Moreover, from $D's_{n-3} = \lambda s_{n-3}$ and $Tz_2 = \lambda z_2$ we easily see that $D' = \lambda I_{n-3}$ and $T = \lambda I_2$. Thus, $B = \lambda I$, so $d(A, X) \geq 3$.

Now, let us consider the case $k \geq 3$. Again we decompose every $B \in \mathcal{C}(X)$ to the blocks that correspond to the block decomposition of A:

$$B = \begin{bmatrix} B_1 & 0_{2k-2,n-2k+1} & 0_{2k-2,1} \\ 0_{n-2k+1,2k-2} & D' & 0_{n-2k+1,1} \\ 0_{1,2k-2} & 0_{1,n-2k+1} & \lambda \end{bmatrix}$$

where $B_1 = \begin{bmatrix} a & b & c_1 \\ 0 & a & 0 \\ 0 & c_2 & d_3 \end{bmatrix} \oplus \operatorname{diag}(d_4, d_5, \ldots, d_{2k-2}) \in M_{2k-2}(\mathbb{F})$ and $D' \in M_{n-2k+1}(\mathbb{F})$ is a diagonal matrix. Note that B is the same as in (6) but decomposed in a different way. Suppose B also commutes with A as defined in (5). Similarly as before, we have $D's_{n-2k+1} = \lambda s_{n-2k+1}$ and thus $D' = \lambda I_{n-2k+1}$. Moreover, it is straightforward that from $B_1 z_{2k-2} = \lambda z_{2k-2}$ and $N_{2k-2}B_1 = B_1 N_{2k-2}$ we obtain $B_1 = \lambda I_{2k-2}$. This completes the proof that $d(A, X) \geq 3$.

If A is an idempotent, then $\operatorname{rk}(I - A) = n - \operatorname{rk} A$. Since A and $I - A$ are equivalent, we can thus assume without loss of generality that A is of rank k with $\frac{n}{2} \leq k \leq n-2$. Let W be a $k \times (n-k)$ matrix with the only nonzero elements being

$$W_{1,n-k} = W_{2,n-k-1} = W_{3,n-k-2} = \ldots = W_{n-k,1} = W_{k,1} = W_{k,n-k} = 1\,.$$

Note that, if $k = \frac{n}{2}$, the rows k and $n-k$ coincide. Using an appropriate conjugation we can additionally assume that

$$A = \begin{bmatrix} I_k & W \\ 0_{n-k,k} & 0_{n-k,n-k} \end{bmatrix}.$$

Let us define a nonminimal matrix $X = J_k \oplus 0_1 \oplus I_{n-k-1}$. We will prove that $d(A, X) \geq 3$, i.e., any matrix $B \in \mathcal{C}(A) \cap \mathcal{C}(X)$ is a scalar matrix. It is a straightforward calculation that

$$\mathcal{C}(A) = \left\{ \begin{bmatrix} M & MW - WN \\ 0_{n-k,k} & N \end{bmatrix} ;\ M \in M_k(\mathbb{F}), N \in M_{n-k}(\mathbb{F}) \right\}$$

and that $\mathcal{C}(X)$ consists of all matrices of the form

$$\begin{bmatrix} U & s & 0_{k,n-k-1} \\ v^{\mathrm{tr}} & \lambda & 0_{1,n-k-1} \\ 0_{n-k-1,k} & 0_{n-k-1,1} & Y \end{bmatrix},$$

where U is an upper triangular Toeplitz $k \times k$ matrix, $s = (s_1, 0, \ldots, 0)^{\mathrm{tr}} \in \mathbb{F}^k$, $v = (0, \ldots, 0, v_k)^{\mathrm{tr}} \in \mathbb{F}^k$, $Y = \begin{bmatrix} y_{ij} \end{bmatrix}_{2 \leq i,j \leq n-k} \in M_{n-k-1}(\mathbb{F})$, and $\lambda \in \mathbb{F}$.

Suppose $B = \begin{bmatrix} M & MW-WN \\ 0 & N \end{bmatrix} \in \mathcal{C}(A)\cap\mathcal{C}(X)$. It follows that $M = \sum_{i=1}^{k} m_i J_k^{i-1}$, $N = \lambda \oplus Y$ and $(MW-WN)_{ij} = 0$ except possibly for $i=j=1$.

By equations

$$0 = (MW-WN)_{k,1} = m_1 - \lambda,$$
$$0 = (MW-WN)_{k,n-k} = m_1 - y_{n-k,n-k},$$
$$0 = (MW-WN)_{i,n-k} = m_{k-i+1} \qquad \text{for all } i \text{ with } (k-1) \geq i \geq (n-k)$$

it follows that $m_1 = y_{n-k,n-k} = \lambda$ and $m_2 = m_3 = \ldots = m_{2k-n+1} = 0$. Moreover, by $0 = (MW-WN)_{i,1} = m_{k-i+1}$ for $i = 2,3,\ldots,n-k-1$, it follows that $m_{2k-n+2} = \ldots = m_{k-1} = 0$. Now, equation $0 = (MW-WN)_{1,n-k} = m_k$ completes the proof that $M = \lambda I_k$.

We proceed by $0 = (MW-WN)_{i,n-k-i+1} = m_1 - y_{n-k-i+1,n-k-i+1}$ for $i = 2,\ldots,n-k-1$ and $0 = (MW-WN)_{i,j} = -y_{n-k-i+1,j}$ for $i = 1,2,\ldots,n-k-1$ and $j = 2,3,\ldots,n-k$, such that $i+j \neq n-k+1$. It follows that $N = \lambda I_{n-k}$ and $(MW-WN) = 0$. Thus, $B = \lambda I$ and $d(A,X) \geq 3$.

(ii) Let A be a nilpotent matrix such that $A^3 = 0$ and $\mathrm{rk}(A^2) = 1$. We may assume A is already in its Jordan canonical form, i.e.,

$$A = J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-3-2k}.$$

The centralizer of A is contained in the set of matrices of the form $B = \begin{bmatrix} T & S_1 \\ S_2 & V \end{bmatrix}$, where $T = t_0 I_3 + t_1 J_3 + t_2 J_3^2 \in M_3(\mathbb{F})$, $V \in M_{n-3}(\mathbb{F})$, and where the first column of $S_2 \in M_{n-3,3}(\mathbb{F})$ as well as the last row of $S_1 \in M_{3,n-3}(\mathbb{F})$ contain only zero entries.

Now, let us define the nonminimal matrix $X = 1 \oplus 0 \oplus J_{n-2}$ and take any $B \in \mathcal{C}(A)\cap\mathcal{C}(X)$. Since $B \in \mathcal{C}(X)$, its off-diagonal entries on the first row and the first column are all zero. Comparing with the above form for B we deduce that $T = t_0 I$. Moreover, $B \in \mathcal{C}(X)$ also implies that the bottom-right $(n-2)\times(n-2)$ block of B is upper triangular Toeplitz matrix, which is moreover equal to $t_0' I_{n-2}$ for some $t_0' \in \mathbb{F}$ by the fact that the third row of S_1 vanishes. Actually, $t_0 = t_0'$ because a 3×3 block T overlaps with $(n-2)\times(n-2)$ bottom right block. Further, $B \in \mathcal{C}(X)$ implies that the only possible off-diagonal nonzero entries in the second row and column lie at positions $(2,n)$, and $(3,2)$. Actually, $B_{32} = T_{32} = 0$, while from $B \in \mathcal{C}(A)$ we deduce that if $B_{2n} \neq 0$ then also $B_{1(n-1)} \neq 0$, which would contradict the fact that the first row of B has zero off-diagonal entries. Hence, $B_{2n} = B_{32} = 0$ and so $B = t_0 I$ is a scalar and therefore $d(A,X) \geq 3$. □

Theorem 2.9. *The following statements are equivalent for a non-scalar matrix R.*

(i) *R is equivalent to a matrix of rank one.*
(ii) *$d(R, X) \leq 2$ for every nonminimal matrix X.*

Proof. If $n = 3$, then every nonminimal matrix is equivalent to a rank-one matrix, so we may assume that $n \geq 4$.

To prove that $(i) \Longrightarrow (ii)$, we can assume without loss of generality that $\operatorname{rk} R = 1$. Let X be an arbitrary nonminimal matrix. Then $d(R, X) \leq 2$ by Lemma 2.4.

$\neg(i) \Longrightarrow \neg(ii)$. Suppose R is not equivalent to a rank-one matrix. Note that there exists at least one maximal matrix $M \succ R$. In fact, $M = p(R)$ for some polynomial p. Moreover, we can assume that every maximal $M \succ R$ is either a nonzero square-zero matrix or a non-scalar idempotent. Hence $1 \leq \operatorname{rk} M \leq n - 1$. Note that $\operatorname{rk} M = n - 1$ implies M is an idempotent and therefore it is equivalent to a maximal matrix of rank one. So we can assume that $1 \leq \operatorname{rk} M \leq n - 2$.

If for a maximal $M \succ R$ we have $2 \leq \operatorname{rk} M \leq n - 2$, then by Lemma 2.8 there exists a nonminimal matrix X with $d(M, X) \geq 3$. Hence also $d(R, X) \geq 3$ because $\mathcal{C}(R) \subseteq \mathcal{C}(M)$.

Otherwise, every maximal matrix $M \succ R$ is equivalent to a rank-one matrix. This implies that (i) R is either equivalent to a nilpotent matrix with exactly one Jordan block of dimension 3 and all other cells of dimension at most 2, or (ii) R is equivalent to a matrix whose Jordan structure is equal to $1 \oplus J_2 \oplus 0_{n-3}$, or (iii) R is equivalent to a matrix whose Jordan structure is equal to $1 \oplus J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-3-2k}$. In the first case, Lemma 2.8 assures that there exists a nonminimal X with $d(R, X) \geq 3$. In the case (ii) we have, modulo conjugation, $R = 1 \oplus J_2 \oplus 0_{n-3}$. It is easy to see that $X = J_2 \oplus J_{n-2}$ is nonminimal and $d(R, X) \geq 3$. In case (iii) we have, modulo conjugation, $R \prec R' = 0 \oplus J_3 \oplus \bigoplus_{i=1}^{k} J_2 \oplus 0_{n-3-2k}$. Again, Lemma 2.8 gives a nonminimal matrix X with $d(R', X) \geq 3$, so also $d(R, X) \geq 3$. □

In the previous theorem rank-one matrices are classified with the help of matrices which are not minimal. We next classify minimal matrices as the ones which maximize the distance in a commuting graph.

Theorem 2.10. *The following are equivalent for a matrix $A \in M_n(\mathbb{F})$.*

(i) *A is minimal.*
(ii) *There exists a matrix X such that $d(A, X) = 4$.*

Proof. $(i) \Longrightarrow (ii)$. This follows from Theorem 2.5.

$\neg(i) \Longrightarrow \neg(ii)$. Let A be a non-minimal matrix, and let X be any matrix. By Lemma 2.1, there exists a rank-one matrix R with $d(X, R) \leq 1$. By Theorem 2.9 we have $d(A, R) \leq 2$, so triangle inequality gives $d(A, X) \leq 3$. □

Remark 2.11. *Combining the previous two theorems yields that R is equivalent to rank-one matrix if and only if $d(R, X) \leq 2$ for every matrix X such that $d(X, Z) \leq 3$, for all $Z \in M_n(\mathbb{F})$.*

Semisimple matrices can also be classified using the distance in the commuting graph. Before doing that we need two lemmas.

Lemma 2.12. *Suppose a minimal matrix $B \in M_n(\mathbb{F})$ is semisimple. Then for any $Y - X - B$ there exists a minimal matrix M with $Y - M - X$.*

Proof. Assume with no loss of generality that B is diagonal. Then, every $X \in \mathcal{C}(B)$ is also diagonal. Using simultaneous conjugation on (B, X) we may further assume that $X = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$, with $\lambda_1, \ldots, \lambda_k$ pairwise distinct and $n_1, \ldots, n_k \geq 1$. Now, since Y commutes with X we have that $Y \in \mathcal{C}(X) = M_{n_1}(\mathbb{F}) \oplus \cdots \oplus M_{n_k}(\mathbb{F})$. Consequently, $Y = Y_1 \oplus \cdots \oplus Y_k$ is block-diagonal and we may find an invertible block-diagonal matrix $S = S_1 \oplus \cdots \oplus S_k$ such that $S^{-1}XS = X$ and $S^{-1}YS = \bigoplus_{i=1}^k S_i^{-1} Y_i S_i$ is in Jordan upper-triangular form; say $S^{-1}YS = \bigoplus_{i=1}^s J_{m_i}(\mu_i)$, with $m_i \geq 1$, $s \geq k$. Then we can choose distinct $\nu_1, \ldots, \nu_s \in \mathbb{F}$, such that the matrix $M = S \bigoplus_{i=1}^s J_{m_i}(\nu_i) S^{-1}$ is neither equal to X nor Y. Also, since $\nu_1, \ldots, \nu_s$ are distinct, M is nonderogatory, hence minimal, and it commutes with X and with Y. □

Lemma 2.13. *Suppose a minimal $B \in M_n(\mathbb{F})$ is not semisimple. Then there exist matrices X, Y with $Y - X - B$, but such that no minimal matrix commutes with both X and Y.*

Proof. With no loss of generality assume B is already in its upper-triangular Jordan form, $B = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$ with $\lambda_1, \ldots, \lambda_{n_k}$ distinct and $n_1 \geq 2$. Define $X = E_{1n_1} = J_{n_1}^{n_1 - 1} \oplus 0_{n-n_1} \in \mathcal{C}(B)$ and for an arbitrary $k \in \{1, \ldots, n\} \setminus \{1, n_1\}$ define $Y = E_{1k}$. Clearly, X commutes with Y. Let us show that no minimal A commutes with both X and Y. Assume $A = \bigoplus_{j=1}^s J_{n_j}(\mu_j)$ is already in its Jordan canonical form. Since $X \in \mathcal{C}(A)$ is of rank one, it follows that $X \in \mathbb{F} J_{n_{j_1}}^{n_{j_1} - 1}$ for some j_1, and likewise $Y \in \mathbb{F} J_{n_{j_2}}^{n_{j_2} - 1}$ for some j_2. However, $\operatorname{rk}(X + Y) = 1$ and so $j_1 = j_2$, which gives X and Y must be linearly dependent, a contradiction. □

Theorem 2.14. *Let $A \in M_n(\mathbb{F})$ be a non-scalar matrix. Then the following are equivalent.*

(i) *A is semisimple.*
(ii) *There exists a minimal $B \in \mathcal{C}(A)$ such that for any $Y - X - B$ there exists a minimal matrix M with $Y - M - X$.*

Proof. *(i)* $\Longrightarrow$ *(ii)*. Assume without loss of generality that A is already diagonal. Choose distinct scalars $\mu_1, \ldots, \mu_n$ to form a minimal matrix $B = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ which clearly commutes with A. Then, (ii) follows from Lemma 2.12.

$\neg(i) \Longrightarrow \neg(ii)$. Choose any minimal B which commutes with non-semisimple A (at least one does exist, for example, if $A = S \bigoplus_{i=1}^k J_{n_i}(\lambda_i) S^{-1}$, then for distinct scalars $\lambda_1, \ldots, \lambda_k$ matrices A and $B = S \bigoplus_{i=1}^k J_{n_i}(\lambda_i) S^{-1}$ commute). Since $\mathcal{C}(B) = \mathbb{F}[B]$ it follows that $A \in \mathbb{F}[B]$ which implies that

B itself is not semisimple. It now follows from Lemma 2.13 that there exist X, Y with $Y - X - B$, but no minimal matrix commutes with both of them. So (ii) does not hold. □

Let us conclude with an example of a connected commuting graph over algebraically non-closed field with the diameter strictly larger than 4.

Example 2.15. The commuting graph for $M_9(\mathbb{Z}_2)$ is connected with diameter at least 5.

Note that $\mathbb{Z}_2$ permits only one field extension of degree $n = 9$, and this is the Galois field $GF(2^9)$ which contains $GF(2^3)$ as the only proper intermediate subfield. So, by [2, Theorem 6] the commuting graph of $M_9(\mathbb{Z}_2)$ is connected. To see that its diameter is at least 5, consider an irreducible polynomial $m(\lambda) = \lambda^9 + \lambda^8 + \lambda^4 + \lambda^2 + 1 \in \mathbb{Z}_2[\lambda]$ and let $\widehat{A} = C(m) \in M_9(\mathbb{Z}_2)$ be its companion matrix. Since $\widehat{A}$ has a cyclic vector, $\mathcal{C}(\widehat{A}) = \mathbb{Z}_2[\widehat{A}]$ by a well known Frobenius result on dimension of centralizer (see for example [2, Corollary 1]), and this is a field extension of $\mathbb{Z}_2$ [14, Theorem 4.14, pp. 472] of index $n = 9$. Actually, $\mathcal{C}(\widehat{A})$ is isomorphic to $GF(2^9)$ by the uniqueness of field extensions for finite fields. In the sequel we will identify the two.

Since the field extension $\mathbb{Z}_2 \subset GF(2^9)$ contains only $GF(2^3)$ as a proper intermediate subfield, we see that each $X \in \mathcal{C}(\widehat{A}) \setminus GF(2^3)$ satisfies $\mathbb{Z}_2[X] = \mathbb{Z}_2[\widehat{A}] = \mathcal{C}(\widehat{A})$ and in particular X and $\widehat{A}$ are polynomials in each other so they are equivalent. Moreover, each non-scalar $\widehat{Y} \in GF(2^3)$ satisfies $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$, because no proper intermediate subfields exist between $\mathbb{Z}_2 \subset GF(2^3)$, and in particular, $\mathcal{C}(\widehat{Y}_1) = \mathcal{C}(\widehat{Y}_2)$ for any two non-scalar $\widehat{Y}_1, \widehat{Y}_2 \in GF(2^3) \subset GF(2^9) = \mathcal{C}(\widehat{A})$.

There exists a polynomial p so that $\widehat{Y} = p(\widehat{A}) \in GF(2^3) \setminus \{0, 1\}$. As the field $GF(2^3)$ contains no idempotents other than 0 and 1 we see that the rational canonical form of $\widehat{Y}$ consists only of cells which correspond to powers of the same irreducible polynomials. Likewise, the field contains no nonzero nilpotents, so each cell of $\widehat{Y}$ corresponds to the same irreducible polynomial, raised to power 1. Moreover, $GF(2^3)$ has no subfields other that $\mathbb{Z}_2$, so $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$ and hence the minimal polynomial of $\widehat{Y} \in GF(2^3)$ has degree $[GF(2^3) : \mathbb{Z}_2] = 3$. This polynomial is relatively prime to its derivative, so in a splitting field, $\widehat{Y}$ has three distinct eigenvalues. It easily follows that $\widehat{Y}$ is conjugate to a matrix $C \oplus C \oplus C$, with C being a 3×3 companion matrix of some irreducible polynomial of degree 3. Let S_1 be an invertible matrix such that $\widehat{Y} = S_1^{-1}(C \oplus C \oplus C)S_1$ and define

$$A = S_1 \widehat{A} S_1^{-1}.$$

Clearly then $p(A) = S_1 \widehat{Y} S_1^{-1} = C \oplus C \oplus C$ and it follows that

$$\mathcal{C}(p(A)) = \begin{bmatrix} \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \\ \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \\ \mathbb{Z}_2[C] & \mathbb{Z}_2[C] & \mathbb{Z}_2[C] \end{bmatrix}. \tag{7}$$

Since $\mathbb{Z}_2[\widehat{Y}] = GF(2^3)$ we obtain $\mathbb{Z}_2[C] = GF(2^3)$.

Consider a 3×3 block matrix

$$N = \begin{bmatrix} E_{13} & 0 & 0 \\ 0 & 0 & E_{13} \\ E_{32} & 0 & 0 \end{bmatrix}, \quad E_{13}, E_{13}, E_{32} \in M_3(\mathbb{Z}_2).$$

It is immediate that $N^3 = 0$, so $I + N$ is invertible. Define

$$B = (I + N)A(I + N)^{-1}.$$

We will show that $d(A, B) \geq 5$.

Suppose there exists a path $A — V — Z — W — B$ of length 4. Note that $V \in GF(2^3) \subset \mathcal{C}(A)$. Otherwise, if $V \in \mathcal{C}(A) \setminus GF(2^3)$ then $\mathcal{C}(V) = \mathcal{C}(A)$ and such V has exactly the same neighbours as A. Since $B = (I + N)A(I + N)^{-1}$, it follows $W = (I + N)U(I + N)^{-1}$ for some $U \in GF(2^3) \subset \mathcal{C}(A) = (I + N)^{-1}\mathcal{C}(B)(I + N)$. Recall that any two non-scalar elements in $GF(2^3)$ have the same centralizer. So in particular we might take $U = V = p(A) = C \oplus C \oplus C$ where polynomial p was defined before. For any $Z \in \mathcal{C}(V) \cap \mathcal{C}((I + N)V(I + N)^{-1})$ we have

$$Z = (I + N)\widehat{Z}(I + N)^{-1}, \qquad Z, \widehat{Z} \in \mathcal{C}(V)$$

and hence, by postmultiplying with $(I + N)$ and rearranging,

$$Z - \widehat{Z} = N\widehat{Z} - ZN. \tag{8}$$

Let us write $Z = \big[Z_{ij}\big]_{1 \leq i,j \leq 3}$ and $\widehat{Z} = \big[\widehat{Z}_{ij}\big]_{1 \leq i,j \leq 3}$ as 3×3 block matrices and by (7) we have that $Z_{ij}, \widehat{Z}_{ij} \in \mathbb{Z}_2[C] = GF(2^3) \subseteq M_3(\mathbb{Z}_2)$, hence each of them is either zero or invertible. Then (8) implies

$$\big[Z_{ij} - \widehat{Z}_{ij}\big]_{ij} = \begin{bmatrix} -Z_{11}E_{13} - Z_{13}E_{32} + E_{13}\widehat{Z}_{11} & E_{13}\widehat{Z}_{12} & E_{13}\widehat{Z}_{13} - Z_{12}E_{13} \\ -Z_{21}E_{13} - Z_{23}E_{32} + E_{13}\widehat{Z}_{31} & E_{13}\widehat{Z}_{32} & E_{13}\widehat{Z}_{33} - Z_{22}E_{13} \\ -Z_{31}E_{13} - Z_{33}E_{32} + E_{32}\widehat{Z}_{11} & E_{32}\widehat{Z}_{12} & E_{32}\widehat{Z}_{13} - Z_{32}E_{13} \end{bmatrix}.$$

Observe that each block on the left side belongs to $\mathbb{Z}_2[C] = GF(2^3) \subseteq M_3(\mathbb{Z}_2)$, and so is either zero or invertible. On the other hand, on the right side, each block in the last two columns has rank at most two. We deduce that the last two columns on both sides are zero. In particular, comparing the second columns we see that $\widehat{Z}_{12} = Z_{12} = 0$ and $\widehat{Z}_{32} = 0$, so $Z_{22} = \widehat{Z}_{22}$, and $Z_{32} = \widehat{Z}_{32} = 0$. Putting this in the above equation and simplifying, the last column then gives $\widehat{Z}_{13} = 0$, so $Z_{13} = \widehat{Z}_{13} = 0$, $Z_{23} = \widehat{Z}_{23}$, and $Z_{33} = \widehat{Z}_{33}$. Also, comparing the $(2, 3)$ positions gives

$$0 = Z_{23} - \widehat{Z}_{23} = E_{13}\widehat{Z}_{33} - \widehat{Z}_{22}E_{13} = e_1(\widehat{Z}_{33}^{\mathrm{tr}}e_3)^{\mathrm{tr}} - \widehat{Z}_{22}e_1e_3^{\mathrm{tr}}.$$

Moreover, $\widehat{Z}_{33}^{\mathrm{tr}}e_3 = \lambda e_3$ and $\widehat{Z}_{22}e_1 = \lambda e_1$, $\lambda \in \mathbb{Z}_2$. Since $\widehat{Z}_{33}, \widehat{Z}_{22} \in \mathbb{Z}_2[C]$ and every vector is cyclic for C we see that $\widehat{Z}_{33} = \widehat{Z}_{22} = \lambda I_3$. The matrix equation therefore simplifies to

$$\begin{bmatrix} Z_{11}-\widehat{Z}_{11} & 0 & 0 \\ Z_{21}-\widehat{Z}_{21} & 0 & 0 \\ Z_{31}-\widehat{Z}_{31} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -Z_{11}E_{13}+E_{13}\widehat{Z}_{11} & 0 & 0 \\ -Z_{21}E_{13}-\widehat{Z}_{23}E_{32}+E_{13}\widehat{Z}_{31} & 0 & 0 \\ -Z_{31}E_{13}-\lambda E_{32}+E_{32}\widehat{Z}_{11} & 0 & 0 \end{bmatrix}.$$

Comparing the position $(1,1)$ gives by similar arguments as above that $\widehat{Z}_{11} = Z_{11} = \mu I_3$. Inserting this into the equation we see after rearrangement that the rank of the block at position $(3,1)$ is equal to $\mathrm{rk}((\mu-\lambda)E_{32}-Z_{31}E_{13}) \leq 2$, which forces the two blocks at position $(3,1)$ to be zero, i.e., $Z_{31}-\widehat{Z}_{31} = 0 = (\mu-\lambda)E_{32}-Z_{31}E_{13} = (\mu-\lambda)e_3e_2^{\mathrm{tr}} - Z_{31}e_1e_3^{\mathrm{tr}}$. We immediately get $Z_{31} = \widehat{Z}_{31} = 0 = (\mu-\lambda)$. Therefore, $Z_{11} = Z_{22} = Z_{33} = \lambda I_3$. Finally, comparing the $(2,1)$ positions gives

$$Z_{21}-\widehat{Z}_{21} = -Z_{21}E_{13}-\widehat{Z}_{23}E_{32},$$

and arguing as above, $Z_{21} = \widehat{Z}_{21} = 0$. Hence, Z is scalar. So, $\mathcal{C}(V)\cap\mathcal{C}(W)$ contains only scalars, which gives that $d(A,B)\geq 5$.

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