Spectra of the Gurtin-Pipkin type equations

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Abstract

We study the spectra of certain integro-differential equations arising in applications. Under some conditions on the kernel of the integral operator, we describe the non-real part of the spectrum.

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1 Introduction

The following integro-differential equations arise in several areas of physics and applied mathematics, namely in heat transfer with finite propagation speed [5], systems with thermal memory [11], viscoelasticity problems [1] and acoustic waves in composite media [4].

(i) Gurtin–Pipkin equations of first order in time (GP1)

$$u_t(x,t)=\int_0^t k(t-s)u_{xx}(x,s)\,ds,\ x\in(0,\pi),\ t>0, \tag{1}$$

(ii) and of second order in time (GP2)

$$u_{tt}(x,t)=au_{xx}-\int_0^t k(t-s)u_{xx}(x,s)\,ds,\ a>0,\ x\in(0,\pi),\ t>0. \tag{2}$$

(iii) Kelvin-Voigt equation (KV)

$$u_{tt}(x,t)=u_{xx}+\epsilon u_{txx}-\int_0^t k(t-s)u_{xx}(x,s)\,ds,\ x\in(0,\pi),\ t>0. \tag{3}$$

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Here

$$k(t) = \int_0^\infty e^{-t\tau} d\mu(\tau)$$

is the Laplace transform of a positive measure $d\mu$. We identify this measure with its distribution function μ, so μ is increasing, continuous from the right, and the integral is interpreted as a Stieltjes integral [14]. We always assume that k is defined and integrable on $(0, \infty)$, that is

$$\int_0^\infty \frac{d\mu(t)}{t} < \infty, \tag{4}$$

and that μ is supported on (d_0, ∞) with some $d_0 > 0$.

Remark 1 *If equation* (1) *can be differentiated with respect to* t*, then we obtain a special case of* (2)*:*

$$u_{tt} = k(0)u_{xx} - \int_0^t \tilde{k}(t-s)u_{xx}ds.$$

with

$$\tilde{k} = -\frac{d}{dt}k. \tag{5}$$

Remark 2 *In the case* $k(t) = \text{const} = \alpha^2$*, equation* (1) *is in fact an integrated wave equation. Indeed, differentiation of* (1) *gives*

$$u_{tt} = \alpha^2 u_{xx}. \tag{6}$$

If $k(t) = \alpha^2 e^{-bt}$*, then differentiation gives a damped wave equation*

$$u_{tt} = \alpha^2 u_{xx} - bu_t. \tag{7}$$

Let the initial conditions be $u(\cdot, 0) = \xi$ for (1), and $u(\cdot, 0) = \xi$, $u_t(\cdot, 0) = \eta$ for (2) and (3).

First we apply Fourier's method: we set $\varphi_n = \sqrt{\frac{2}{\pi}} \sin nx$ and expand the solution and the initial data in a series in φ_n

$$u(x,t) = \sum_1^\infty u_n(t)\varphi_n(x), \ \xi(x) = \sum_1^\infty \xi_n \varphi_n(x), \ \eta(x) = \sum_1^\infty \eta_n \varphi_n(x).$$

The components u_n satisfy ordinary integro-differential equations

(i) GP1

$$\dot{u}_n(t) = -n^2 \int_0^t k(t-s)u_n(s)ds, \ t > 0. \tag{8}$$

(ii) GP2

$$\ddot{u}_n(t) = -an^2 u_n(t) + n^2 \int_0^t k(t-s)u_n(s)ds,\ t > 0. \tag{9}$$

(iii) KV

$$\ddot{u}_n(t) = -n^2 u_n(t) - \epsilon n^2 \dot{u}_n + n^2 \int_0^t k(t-s)u_n(s)ds,\ t > 0. \tag{10}$$

We will denote the Laplace images by the capital letters. Applying the Laplace Transform to (8), (9), and (10), and using the initial conditions, we find

(i) GP1

$$zU_n(z) - \xi_n = -n^2 K(z)U_n(z) \tag{11}$$

or

$$U_n(z) = \frac{\xi_n}{z + n^2 K(z)}. \tag{12}$$

(ii) GP2

$$z^2 U_n(z) - z\xi_n - \eta_n = -an^2 U_n(z) + n^2 K(z)U_n(z)$$

or

$$U_n(z) = \frac{z\xi_n + \eta_n}{z^2 + an^2 - n^2 K(z)}. \tag{13}$$

(iii) KV

$$z^2 U_n(z) - z\xi_n - \eta_n = -n^2 U_n(z) - \epsilon n^2 \left[zU_n - \xi_n\right] + n^2 K(z)U_n(z),$$

or

$$U_n(z) = \frac{\xi_n + z\eta_n - \epsilon n^2 \xi_n}{z^2 + \epsilon z n^2 + n^2 - n^2 K(z)}. \tag{14}$$

Denote the denominators in (12), (13), and (14) by $F_n(z)$, $G_n(z)$, and $H_n(z)$ respectively:

$$F_n(z) = z + n^2 K(z), \quad G_n(z) = z^2 + an^2 - n^2 K(z),$$

and

$$H_n(z) = z^2 + \epsilon z n^2 + n^2 - n^2 K(z).$$

Let F_n^0, G_n^0, and H_n^0 be the sets of zeros of $F_n(z)$, $G_n(z)$, and $H_n(z)$ respectively. Set

$$\Lambda_{GP1} = \bigcup_1^\infty F_n^0,\ \Lambda_{GP2} = \bigcup_1^\infty G_n^0,\ \Lambda_{KV} = \bigcup_1^\infty H_n^0.$$

Definition 3 *The sets Λ_{GP1}, Λ_{GP2}, and Λ_{KV} are the spectra of equations* (1), (2) *and* (3) *respectively.*

Study of the spectra is important for applications. See [4] and the references therein.

Remark 4 *Suppose that in the integro-differential equations (1), (2) and (3) we replace the zero lower limit in the integrals by $-\infty$. Then λ is a point of the spectrum of F_n or G_n or H_n if and only if there is a solution of the form*

$$u_\lambda(x,t) = e^{\lambda t}\varphi_n(x).$$

For such systems a semigroup approach to the equations is possible, (V. V. Vlasov, private communication).

Remark 5 *If* (5) *holds, then $\tilde{K}(z) = k(0) - zK(z)$.*

We do not study here the regularity of solutions and consider the solutions as sequences $\{u_n(t)\}$. Regularity of GP1 is studied in [8] and of GP2 in [12]. In [8], under the assumption that $k(t)$ is twice continuously differentiable it was shown, in particular, that the solution $u(x,t)$ of (1) is a continuous $L^2(0,T)$-valued function. In [12], the conditions on the kernel $k(t)$ and the initial data are found such that there exists a strong solution: for a $\gamma > 0$

$$\int_0^\infty e^{-\gamma t}\left[\|u(\cdot,t)\|^2_{L^2(0,\pi)} + \|u_{xx}(\cdot,t)\|^2_{L^2(0,\pi)} + \|u_{tt}(\cdot,t)\|^2_{L^2(0,\pi)}\right] dt < \infty.$$

We single out the important special case: the case of discrete measure μ with atoms at $b_k > 0$ of mass $a_k > 0$,

$$K(z) = \sum_{k=1}^{\infty} \frac{a_k}{z + b_k}, \quad 0 < b_1 < \ldots \to +\infty, \quad a_k > 0, \tag{15}$$

Discrete measures arise in applications [4], where parameters a_k, b_k are connected with auxiliary boundary value problems arising under averaging.

2 Main Results

In the general case the non-real part of the spectrum is described as follows.

Theorem 6 *(i) For every n, each set F_n^0, G_n^0, or H_n^0 contains at most one point in the upper half-plane, and this point, if exists, belongs to the second quadrant.*

(ii) For n large enough, the set G_n^0 contains a point z_n such that

$$z_n = i\sqrt{a}\, n + o(n). \tag{16}$$

(iii) If

$$A = \int_0^\infty d\mu(t) < \infty, \tag{17}$$

then for n large enough, the set F_n^0 contains a point z_n such that

$$z_n = i\sqrt{A}\, n + o(n). \tag{18}$$

Under the additional assumptions on K, we can find more precise asymptotics of the non-real part of the spectrum.

Theorem 7 *Suppose that*

$$\mu(t) = bt^\rho + O(t^\alpha), \quad t \to \infty, \tag{19}$$

where $0 < \alpha < \rho < 1$. Then:
(i) for n large enough, the set F_n^0 contains a zero z_n of $F_n(z)$ such that

$$z_n = \left(\frac{b\pi\rho}{\sin \pi\rho}\right)^{1/(2-\rho)} e^{i\pi/(2-\rho)} n^{2/(2-\rho)} \left(1 + O(n^{2(\alpha-\rho)/(2-\rho)})\right), \tag{20}$$

(ii) for n large enough, the set G_n^0 contains a zero z_n of $G_n(z)$ such that

$$z_n = i\sqrt{a}n + \frac{b\pi\rho}{2\sin\pi\rho} a^{\rho/2-1} e^{i\pi(\rho/2-1)} n^\rho (1 + o(1)). \tag{21}$$

In the case of a discrete measure with finite number of atoms the next theorem was conjectured by V. V. Vlasov and N. Rautiyan (private communication).

Theorem 8 *Let μ be a measure with compact support, that is*

$$k(t) = \int_{d_0}^d e^{-t\tau} d\mu(\tau), \quad 0 < d_0 < d < \infty. \tag{22}$$

Then the set Λ_{KV} contains finite number of non-real points.

In the case (15) it is not hard to study the real spectrum of the systems, see [7][1] and the discussion below. The simplest versions of theorems 6(iii) and 7 are contained in [4], [6], [7]. In [10], a study of the spectrum shows the lack of controllability of the system.

[1]The assertions of Theorem 1 in [7] are correct only for n large enough.

3 Proof of the Main Results and discussion

Our main tools are the Schwarz Lemma and the Denjoy–Wolff Theorem (see, for example, [9]).

Schwarz's Lemma. *Let f be an analytic function which maps the upper half-plane $\mathbb{C}_+$ into itself. Then the equation $f(z) = z$ has at most one solution w, and if such solution exists then $|f'(w)| < 1$, unless f is an elliptic fractional-linear transformation.*

Denjoy–Wolff Theorem. *Let f be an analytic function which maps $\mathbb{C}_+$ into itself, and suppose that f is not an elliptic fractional-linear transformation. Then there exists a unique point $w \in \mathbb{C}_+ \cup \{\infty\}$ such that the iterates f^{*n} converge to w uniformly on compact subsets of $\mathbb{C}_+$, the angular limit $f(w) = \lim_{z\to w} f(z)$ exists and satisfies $w = f(w)$. Moreover, the angular derivative $f'(w)$ exists and satisfies $|f'(w)| \leq 1$.*

Angular limit means that z is restricted to any angle $\epsilon < \arg(z-w) < \pi-\epsilon$ if $w \in \mathbb{R}$, or $\epsilon < \arg z < \pi - \epsilon$ if $w = \infty$.

Angular derivative is the angular limit

$$f'(w) = \lim_{z\to w}(f(z) - f(w))/(z - w)$$

if $w \in \mathbb{R}$; if $w = \infty$ then it is defined by the angular limit

$$\frac{1}{f'(\infty)} = \lim_{z\to\infty} f(z)/z.$$

The point w in this theorem is called the Denjoy–Wolff point of f. If $w \in \mathbf{R} \cup \{\infty\}$ is such a point that the angular limit $\lim_{z\to w} f(z) = w$ and the angular derivative $|f'(w)| \leq 1$, then w is the Denjoy–Wolff point.

Proof of Theorem 6.

For the equations $F_n(z) = 0$, $G_n(z) = 0$, and $H_n(z) = 0$, we will show that each of them has at most one solution in the upper half-plane. This solution belongs to the second quadrant.

The Laplace transform of $k(t)$ is

$$\begin{aligned} K(z) &= \int_0^\infty e^{-zx} \int_0^\infty e^{-tx} d\mu(t) dx \\ &= \int_0^\infty \int_0^\infty e^{-x(z+t)} dx\, d\mu(t) = \int_0^\infty \frac{d\mu(t)}{z+t}. \end{aligned}$$

This is called the Cauchy transform of the measure $d\mu$. Condition (4) ensures that the integral defining K is absolutely and uniformly convergent on every compact in the z-plane that does not intersect the negative ray. So K is analytic in the plane minus the negative ray.

Moreover,

$$\text{Im } K(z)\text{Im } z < 0, \quad z \in \mathbb{C}\backslash\mathbb{R}_-. \tag{23}$$

We rewrite the equation $F_n(z) = 0$ as $z = f(z) := -n^2 K(z)$, then f maps $\mathbb{C}_+$ to $\mathbb{C}_+$ and by Schwarz's Lemma has at most one fixed point in the upper half-plane. If Re $z \geq 0$, then

$$\text{Re } (z + n^2 K(z)) = \text{Re } z + n^2 \int_0^\infty \frac{t + \text{Re } z}{|t+z|^2} d\mu(t) > 0,$$

so the solution must lie in the second quadrant.

For $G_n(z) = 0$, we first prove that there are no solutions in the first quadrant. Indeed $z^2 + an^2$ maps the first quadrant into $\mathbb{C}_+$, while $n^2 K(z)$ has negative imaginary part in the first quadrant.

To prove that $G_n(z)$ has at most one zero in $\mathbb{C}_+$ we consider a branch ϕ of the square root which maps the lower half-plane onto the second quadrant. Then the equation is equivalent to

$$z = f(z) := n\phi(K(z) - a), \tag{24}$$

because all solutions are in the second quadrant. Function f maps $\mathbb{C}_+$ into itself, and thus by Schwarz's Lemma can have at most one fixed point in $\mathbb{C}_+$.

Similar argument applies to $H_n(z) = 0$. There are no solution in the first quadrant. Indeed, if

$$z^2 + \epsilon z n^2 + n^2 = n^2 K(z)$$

and z is in the first quadrant, then the LHS is in $\mathbb{C}_+$ but the RHS is in $\mathbb{C}_-$. So the equation is equivalent to

$$z = f(z) := n\phi(K(z) - \epsilon z - 1),$$

because all solutions are in the second quadrant. Function F maps $\mathbb{C}_+$ into itself, and thus by Schwarz's Lemma can have at most one fixed point in $\mathbb{C}_+$.

This completes the proof of part (i) of Theorem 6.

To prove part (ii), we first notice that

$$K(z) \to 0 \quad \text{as} \quad z = re^{i\theta}, r \to \infty, \tag{25}$$

uniformly with respect to θ for $|\theta| < \pi - \delta$, for any given $\delta > 0$. This will be expressed by saying that $K \to 0$ as $z \to \infty$ *non-tangentially*. To show this we use the following lemma.

Lemma 9 *If*

$$|\arg z| < \pi - \delta, \tag{26}$$

then

$$|z+t| \asymp |z| + t, \;\; t \geq 0. \tag{27}$$

Proof of the lemma. First, $|z+t| \leq |z| + t$. Second,

$$|z+t| = |z||1 + t/z| \geq |z| \cos\delta,$$

and similarly

$$|z+t| = t|1 + z/t| \geq t\cos\delta.$$

Thus $|z+t| \geq (1/2)(|z| + t)\cos\delta$. This gives (27).

Now

$$|K(z)| \leq C \int_0^\infty \frac{d\mu(t)}{|z| + t},$$

in the sector (26), and we obtain (25).

We rewrite the equation $G_n(z) = 0$ as

$$z_n = in\sqrt{a}\sqrt{1 - K(z_n)/a}.$$

As $K(z) \to 0$ by Lemma 9, we obtain (ii).

Now we prove (iii). Condition (17) permits to obtain asymptotics of K:

$$K(z) = A/z + o(|z|^{-1}), \quad z \to \infty, \tag{28}$$

uniformly with respect to $\arg z$ in $|\arg z| \leq \pi - \epsilon$. Indeed,

$$\begin{aligned} |K(z) - A/z| &= \left| \int_0^\infty \left(\frac{1}{z+t} - \frac{1}{z} \right) d\mu(t) \right| \\ &\leq \frac{1}{|z|} \int_0^\infty \frac{t}{|z|+t} d\mu(t) = o(|z|^{-1}). \end{aligned}$$

Now we rewrite $F_n(z_n) = 0$ as

$$z_n = -n^2 \left(A\frac{1}{z_n} + o(z^{-1}) \right).$$

which gives (iii). This completes the proof of Theorem 6.

To find out whether a solution in $\mathbb{C}_+$ exists for a given n, we consider the case of discrete measure μ. In the case that μ has a finite support,

$$K(z) = \sum_{k=1}^N \frac{a_k}{z + b_k},$$

our arguments are elementary. In this case, $K(z)$ is a real rational function with N poles, all of them on the real line. Then F_n is a rational function of degree $N+1$ because it has an additional pole at infinity. So it must have $N+1$ zeros in the complex plane. On each interval $I_k = (-b_{k+1}, -b_k)$, $1 \le k \le N-1$ there is one zero by the Bolzano-Weierstrass Theorem. The remaining two zeros can lie either one in $\mathbb{C}_+$ and one in $\mathbb{C}_-$ or both on some interval I_k, (which then contains three zeros), or both on the interval $I_0 = (-b_1, 0)$. One can give examples of each possibility.

Examples.

1. Suppose that the measure $d\mu$ has two atoms, that is

$$k(t) = \frac{1}{10}\left(e^{-t} + e^{-2t}\right).$$

Then it easy to check that

$$F_1(z) = K(z) + z = \frac{10}{z+1} + \frac{1}{z+2} + z$$

has 3 real zeros (and no non-real zeros). The additional two zeros are in $(-1, 0)$.

2. Let

$$k(t) = e^{-t} + 200e^{-50t}.$$

It is easy to check that the equation

$$K(x) + x = \frac{1}{x+1} + \frac{200}{x+50} + x$$

has 3 *roots on the interval* $[-50, -1]$.

In the case of a measure with finitely many atoms, the question whether the Denjoy–Wolff point belongs to the real line can be solved in finitely many steps by using the criterion that all roots of a polynomial equation are real, see, for example [3].

In the general case of a discrete measure μ with atoms at b_k, we denote $I_k = (-b_{k+1}, -b_k), k \ge 1$, and $I_0 = (-b_1, 0)$. If there is a solution w in the upper half-plane, it must be the Denjoy–Wolff point of $f(z) = -n^2 K(z)$. If there is no solution in the upper half-plane, then the Denjoy–Wolff point w belongs to some interval I_k, and that $-1 \le f'(w) \le 1$.

So theoretically we can find out whether there is a solution in the upper half-plane, by iterating f, starting from any point in $\mathbb{C}_+$, for example from the point $z_0 = i$. The sequence $z_k = f_n(z_{k-1})$, $k = 0, 1, \ldots$, must converge. If it converges to a point in $\mathbb{C}_+$, then this point is the unique solution in $\mathbb{C}_+$.

Convergence in this case is geometric. If z_k converges to a point on the real line then there is no solution in $\mathbb{C}_+$, but convergence in this case may be extremely slow, if $|f'(w)| = 1$.

Let w be the Denjoy–Wolff point of f. If $w \in \mathbb{C}_+$, then w is the unique solution of $F_n(z) = 0$ in $\mathbb{C}_+$. The case $w = \infty$ is excluded because the angular limit of $f(z)$ as $z \to \infty$ is 0. If $w \in \mathbb{R}$ and $w \in I_k$ for some k, then $f'(w) \in [-1, 1]$ and this I_k is the unique interval of the I_j which contains two additional real zeros of F_n.

We conclude that in the case $w \in \mathbb{C}_+$, each interval $I_k,\ k \geq 1$ contains one solution of $F_n(z) = 0$ while I_0 contains no solutions.

The situation with GP2 and KV are similar: it has a solution in $\mathbb{C}_+$ if and only if the Denjoy–Wolff of the function $f = n\phi(K - a)$ is in $\mathbb{C}_+$, and the Denjoy–Wolff point can in principle be found by iteration.

Now we prove the theorem 7.

Lemma 10 *Under the assumption (19) we have*

$$K(z) = \frac{b\pi\rho}{\sin \pi\rho} z^{\rho-1} + O(z^{\alpha-1}), \quad |z| \to \infty,$$

uniformly with respect to $\arg z$ *in any angle* $|\arg z| < \pi - \delta$. *Here the use the principal branch of* $z^{\rho-1}$ *which is positive on the positive ray.*

Proof. First,

$$\int_0^\infty \frac{dt^\rho}{z+t} = \frac{\pi\rho}{\sin \pi\rho} z^{\rho-1},$$

see, for example, [2, Probl. 28.22] or [13, Probl. 878].

So it is sufficient to prove our Lemma for the case that $\mu(t) = O(t^\alpha)$. We integrate by parts:

$$K(z) = \lim_{R\to\infty} \int_0^R \frac{d\mu(t)}{t+z} = \lim_{R\to\infty} \left(\frac{\mu(R)}{z+R} + \int_0^R \frac{\mu(t)}{(t+z)^2} \right).$$

In a sector $|\arg z| < \pi - \delta$ this gives

$$|K(z)| \leq C \lim_{R\to\infty} \left(\frac{\mu(R)}{|z|+R} + \int_0^R \frac{t^\alpha}{t^2 + |z|^2} dt \right)$$

$$= C \int_0^\infty \frac{t^\alpha}{t^2 + |z|^2} dt = C_1 |z|^{\alpha-1}.$$

This proves the lemma.

Now the proof of Parts (i) and (ii) follows the scheme of the proofs of Theorem 6, Parts (i) and (ii).

Proof of Theorem 8.

In view of Theorem 6, it is enough to prove that all roots of $H_n(z)$ are real for n large enough. We rewrite the equation $H_n(z) = 0$ as

$$n^2 f(z) = z^2,$$

where

$$f(z) = K(z) - \epsilon z - 1.$$

We know from Theorem 6 that all solutions belong to the second quadrant. Function f maps the upper half-plane into the lower half-plane. Let ϕ be the branch of the square root that maps the lower half-plane onto the second quadrant. Now we rewrite our equation as $n\phi(f(z)) = z$. Function f is strictly decreasing on $(-\infty, -d)$ and $f(x) \sim -\epsilon x$ as $x \to -\infty$. So there is a point $r < -d$ such that $f(x) > 0$ on $(-\infty, r]$. We have $n\phi(f(x)) \sim -n\sqrt{-\epsilon x}$ as $x \to -\infty$, It follows that

$$n\phi(f(x)) > x \quad \text{for} \quad x < x_0, \tag{29}$$

where $x_0 < 0$ depends on n, K, ϵ.

Now suppose that n is so large that

$$n\phi(f(r)) < r. \tag{30}$$

This will hold for n large enough because $\phi(f(r)) < 0$ as we established above. Comparison of (29) with (30) shows that there must be a point $r_0 \in (-\infty, r)$ such that

$$n\phi(f(r_0)) = r_0 \quad \text{and} \quad \frac{d}{dx} n\phi(f(x))|_{x=r_0} \in (0, 1].$$

This shows that r_0 is an attracting fixed point of the function $n\phi(f)$, and application of the Denjoy–Wolff theorem completes the proof.

We finish with the following

Question. Can one extend Theorem 8 to arbitrary measure $d\mu$ satisfying (4) ?

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