Indecomposable representations of the Kronecker quivers

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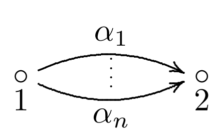
Abstract. Let k be a field and Λ the n-Kronecker algebra, this is the path algebra of the quiver with 2 vertices, a source and a sink, and n arrows from the source to the sink. It is well-known that the dimension vectors of the indecomposable Λ-modules are the positive roots of the corresponding Kac-Moody algebra. Thorsten Weist has shown that for every positive root there are even tree modules with this dimension vector and that for every positive imaginary root there are at least n tree modules. Here, we present a short proof of this result. The considerations used also provide a calculation-free proof that all exceptional modules over the path algebra of a finite quiver are tree modules.

Let k be a field and Q a finite quiver without oriented cycle. Let $\Lambda = kQ$ be the path algebra of Q. The target of the paper is to look for Λ-modules which are tree modules. According to Kac [K], the dimension vectors of the indecomposable Λ-modules are the positive roots of the corresponding Lie algebra: for a real root, there is a unique indecomposable module, for an imaginary root, there are infinitely many provided k is an infinite field. Unfortunately, no effective procedure is known to construct at least one indecomposable module for each positive root. On the other hand, it seems that for each positive root, there exists even a tree module (the definition will be recalled below): that the indecomposable module corresponding to a real root is a tree module, and that for any imaginary root, there are even several different tree modules (see [R3], Problem 9). Thorsten Weist [W] has shown that this is true for all the Kronecker algebras. Here, we present a short proof of his result by determining the dimension vectors of the "cover-thin" Kronecker modules (Proposition 1.1).

The Kronecker algebras are the path algebras of the Kronecker quivers, the n-Kronecker quiver Q with n arrows looks as follows:



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For $n \geq 2$ we obtain in this way representation-infinite algebras, for $n \geq 3$ these algebras are wild. The importance of the Kronecker algebras and their representations is well-known, often they are considered as the basic data in non-commutative geometry.

Let $M = (M_a, M_\alpha)_{a,\alpha}$ be a finite-dimensional representation of a quiver, thus M attaches to each vertex a of the quiver a vector space M_a and to each arrow α a linear map M_α. The sum of the dimension of these vector spaces is called the *total dimension* $\dim M$ of M. In case M is an indecomposable representation with total dimension d, then M is said to be a *tree module* provided that there is a choice of bases for the vector spaces such that the corresponding matrix presentations of the linear maps involve altogether only $d-1$ non-zero entries (so that the "coefficient quiver" is a tree, see [R2]).

The root system for the n-Kronecker algebra is easy to describe: it consists of the vectors $(x, y) \in \mathbb{Z}^2$ with $x^2 + y^2 - nxy \leq 1$. The vectors (x, y) with $x^2 + y^2 - nxy = 1$ are called the real roots, the other roots the imaginary ones. The positive real roots are the dimension vectors of the preprojective and the preinjective modules. Now, the preprojective and the preinjective modules are exceptional modules (a module over a hereditary algebra is said to be exceptional if it is indecomposable and has no self-extension) and exceptional modules over the path algebra of a finite quiver are known to be tree modules [R2]. Thus, in order to show that every positive root is the dimension vector of a tree module, we only have to deal with the imaginary roots.

Theorem. *Let Q be the n-Kronecker quiver. For any positive imaginary root for Q there are at least n tree modules with this dimension vector.*

The proof of the theorem will be given in section 2, it will rely on the use of covering theory. Denote by $\widetilde{Q}$ the universal covering of Q, this is the n-regular tree with bipartite orientation (n-regular means that every vertex has precisely n neighbors, the bipartite orientation is characterized by the property that all vertices are sinks or sources). We denote by $\pi\colon \operatorname{mod} k\widetilde{Q} \to \operatorname{mod} kQ$ the push-down functor.

An indecomposable representation of a quiver is said to be *thin,* provided the non-zero vector spaces used are 1-dimensional. If M is a thin $k\widetilde{Q}$-module, then $\pi(M)$ will be said to be *cover-thin.* Similarly, we say that N is *cover-exceptional* provided there is an exceptional $k\widetilde{Q}$-module M such that $N = \pi(M)$.

1. Cover-thin Kronecker-modules.

1.1. Proposition. *Consider $(x, y) \in \mathbb{N}_0^2$ with $x \leq y$. There exists a cover-thin kQ-module N with dimension vector* $\mathbf{dim}\, N = (x, y)$ *if and only if* $0 < y \leq (n-1)x + 1$. *In this case, there are at least n isomorphism classes of such modules N, unless $(x, y) = (0, 1)$ or $(1, 3)$.*

Proof: Since $k\widetilde{Q}$ is a tree, the thin indecomposable $k\widetilde{Q}$-modules are uniquely determined by the corresponding support, this is just a finite connected subtree of $k\widetilde{Q}$. Take a finite connected subtree T of $k\widetilde{Q}$ with $|T|$ vertices and let $M(T)$ be the $k\widetilde{Q}$-module with support T and $N(T) = \pi(M(T))$ Let (x, y) be the dimension vector of $N(T)$.

If $|T| = 1$, then T consists either of a sink or a source. The condition $x \leq y$ means that $(x, y) = (0, 1)$, that T consists of a sink and that $N(T)$ is the simple projective kQ-module.

Now, let $|T| \geq 2$. Since T is a tree, there is a vertex a in T with a unique neighbor. In case a is a sink, let $b = a$, otherwise denote by b the unique neighbor of a; thus always b is a source. If b is the unique source in T, then $|T| \leq n + 1$ and $\mathbf{dim}\, N(T) = (1, y)$ with $1 \leq y \leq n = (n - 1) + 1$. If $y = n$, then $\pi(M)$ is indecomposable projective and uniquely determined by its dimension vector, otherwise there are at least n isomorphism classes of modules of the form $N(T)$.

Now assume that T contains at least 2 sources. Removing from T the source b we obtain the disjoint union of a connected tree T' with $|T'| \geq 2$ and $t \leq n - 1$ isolated vertices. By induction, we know that $\mathbf{dim}\, \pi(M(T')) = (x', y')$ with $0 < y' \leq (n - 1)x' + 1$ and $(x, y) = (x', y') + (1, t)$. This shows that $y = y' + t \leq (n-1)x' + 1 + (n-1) = (n-1)x + 1$. (Note that only in case $x' = y'$ and $t = 0$, the pair (x, y) will not satisfy the inequality $x \leq y$ we are interested in.) This shows that the dimension vectors (x, y) of the kQ-modules $N(T)$ are as stated.

Conversely, consider (x, y) with $x \leq y$ and $0 < y \leq (n - 1)x + 1$. We try to construct a corresponding T. This is clear for $x \leq 1$ and it is easy to see that for $(x, y) = (0, 1)$ or $(1, 3)$, the corresponding module $N(T)$ is uniquely determined, wheres for $(1, y)$ with $1 \leq y \leq n-1$, there are at least n different isomorphism classes (for $y = 1$ and for $y = n-1$, there are precisely n isomorphism classes.

Thus assume $2 \leq x \leq y \leq (n - 1)x + 1$. Write $y = \sum_{i=1}^{x} y(i)$ with $1 \leq y(i) \leq n - 1$ for $1 \leq i \leq x - 1$ and $1 \leq y(x) \leq n$ (such a decomposition exists, since $x \leq y \leq (n - 1)x + 1$).

Fix some sink s_1 of $\widetilde{Q}$ and take the unique path

$$s_1 \xleftarrow{\alpha_1} t_1 \xrightarrow{\alpha_n} s_2 \xleftarrow{\alpha_1} t_2 \xrightarrow{\alpha_n} \quad \cdots \quad \xleftarrow{\alpha_1} t_{x-1} \xrightarrow{\alpha_n} s_x \xleftarrow{\alpha_1} t_x$$

starting at s_1. For $1 \leq i \leq x$, we add the arrows α_j (and their endpoints) starting at t_i, with $2 \leq j \leq y(i)$. We see that we obtain in this way a subtree T of $\widetilde{Q}$, with x sources and $\sum y(i) = y$ sinks, thus $\mathbf{dim}\, N(T) = (x, y)$.

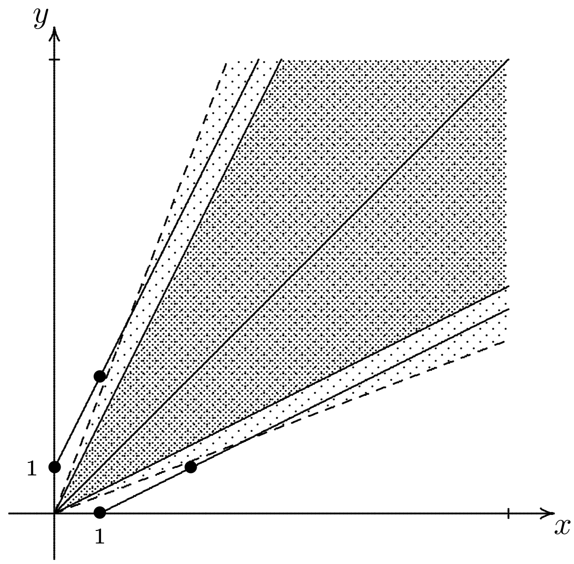
Finally, observe that the module $M(T)$ constructed here for $x \geq 2$ has the property that $\operatorname{Im}(\alpha_1) \cap \operatorname{Im}(\alpha_n) \neq 0$, whereas $\operatorname{Im}(\alpha_i) \cap \operatorname{Im}(\alpha_j) = 0$ for $i < j$ and $(i, j) \neq (1, n)$. Thus, using a permutation of the labels of the arrows, the same construction yields $\binom{n}{2}$ different isomorphism classes and $\binom{n}{2} \geq n$ for $n \geq 3$. It remains to consider the case $n = 2$. Here, $\widetilde{Q}$ is just a line and the positive imaginary roots are of the form (m, m) with $m \geq 1$. Obviously, for every $m \geq 1$, there are precisely two cover-thin kQ-modules. This completes the proof.

Duality provides in a similar way cover-thin kQ-modules with dimension vectors (x, y) where $0 \leq y \leq x$ and $0 < x \leq (n - 1)y + 1$.

It is well-known that the region $\mathcal{F} = \{(x, y) \in \mathbb{N} \mid \frac{1}{n-1}x < y \leq (n - 1)x\}$ is a fundamental domain for the action of the Coxeter transformation on the set of positive imaginary roots. Note that this region is contained in the set of dimension vectors of cover-thin kQ-modules and that the vectors $(0, 1),\ (1, 0),\ (1, 3),\ (3, 1)$ are real roots. Thus we see:

1.2. Corollary. *For every $(x,y) \in \mathcal{F}$, there are at least n isomorphism classes of cover-thin kQ-modules N with* $\mathbf{dim}\, N = (x,y)$.

For the benefit of the reader, we provide an illustration for the case $n = 3$:



The union of the shaded areas is the imaginary cone, the dark part being the fundamental domain $\mathcal{F}$ for the action of the Coxeter transformation on the imaginary cone. The bullets indicate the dimension vectors $(0,1),\ (1,0),\ (1,3),\ (3,1)$, they are outside of the imaginary cone. There are two lines with slope 2 as well as two lines with slope $\frac{1}{2}$: those going through the origin bound the fundamental region $\mathcal{F}$, the parallel ones bound the region of the dimension vectors of cover-thin kQ-modules.

2. Cover-exceptional kQ-modules.

Thin (indecomposable) modules are exceptional. Thus:

2.1. Corollary. *For every positive imaginary root (x,y) there are at least n isomorphism classes of cover-exceptional kQ-modules with* $\mathbf{dim}\, N = (x,y)$.

2.2. Lemma. *Any cover-exceptional module is a tree module.*

Proof: According to [R2], any exceptional module over a hereditary k-algebra is a tree module. But is M is a tree $k\widetilde{Q}$-module, then $\pi(M)$ is a tree kQ-module.

The main theorem is a direct consequence of 2.1 and 2.2.

3. Exceptional modules are tree modules.

The proof of Lemma 2.2 is based on the fact that for Λ a finite-dimensional hereditary k-algebra, any exceptional module is a tree module. On the other hand, one can use the considerations of section 2 in order to provide a proof of this result which avoids any calculations. Indeed, the proof given in [R2] required explicit matrix presentation of the preprojective and preinjective Kronecker modules, and, in this way, was quite technical. Here we show that using induction and the covering theory for the Kronecker algebras, one can avoid the matrix calculations.

Using induction on m, we want to show:

3.1. *Let Λ be the path algebra of a finite quiver and $m > 0$ a natural number. Any exceptional Λ-module of length m is a tree module.*

In the case $m = 1$ nothing has to be shown. Thus let us deal with the induction step, thus let $m > 1$.

First, consider the case where Λ is the n-Kronecker algebra for some $n \geq 1$. In the case $n = 1$, only one module N has to be considered: it has length 2 and obviously is a tree module. Thus, assume that $n \geq 2$. The exceptional Λ-modules are the preprojective modules $P_0, P_1, P_2, \ldots$ and the preinjective modules $Q_0, Q_1.Q_2, \ldots$. These modules are of the form $\pi(M)$ with M an indecomposable representation of $k\widetilde{Q}$ (the corresponding dimension vectors in the case $n = 3$ have been displayed in [FR]). Take such a module M. Of course, we can assume that M is not simple. It is easy to see that there is an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

with $\dim \operatorname{Ext}^1(M'', M') = 1$ such that one of the representations M', M'' is simple and the supports of M', M'' are disjoint (on the level of dimension vectors, we deal with a subtree T of $\widetilde{Q}$ and a vertex a in T having only one neighbor, say b; the coefficients of the dimension vector of M both at a and b are equal to 1. In case a is a sink, $S(a)$ embeds into M say with image M', then $M'' = M/M'$. In case a is a source, there is a surjective map $M \to S(a) = M''$, in this case M' is chosen as its kernel.

Clearly, with M also M' and M'' are exceptional modules. Thus, by induction both are tree modules, and therefore also M is a tree module.

Now assume that we are dealing with an exceptional module M of dimension M such that the support of M has at least three vertices. Schofield induction (see [CB] or also [R1]) asserts that there is an exact sequence

$$0 \to X^a \to M \to Y^b \to 0$$

where X, Y are pairwise orthogonal exceptional modules and the pair (a, b) is the dimension vector of a sincere preprojective or preinjective representation Z of an e-Kronecker module, with $e = \dim \operatorname{Ext}^1(Y, X)$. Since $a > 0, b > 0$, it follows that $\dim X < m$, and $\dim Y < m$. Since the support of M has at least three vertices, we see that not both modules X, Y can be simple, thus also $\dim Z = a + b < m$. By induction, all three modules X, Y, Z are tree modules (here, X, Y are Λ-modules, whereas Z is an e-Kronecker module), but then also M is a tree module, see [R2], section 5. This completes the proof.

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