

ON THE SECOND INNER VARIATION OF THE ALLEN-CAHN FUNCTIONAL AND ITS APPLICATIONS

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ABSTRACT. In this paper, we study the relation between the second inner variations of the Allen-Cahn functional and its Gamma-limit, the area functional. Our result implies that the Allen-Cahn functional only approximates well the area functional up to the first order. However, as an application of our result, we prove, assuming the single-multiplicity property of the limiting energy, that the Morse indices of critical points of the Allen-Cahn functional are bounded from below by the Morse index of the limiting minimal hypersurface.

1. INTRODUCTION AND MAIN RESULTS

Let Ω be an open smooth bounded set in \mathbb{R}^N ($N \geq 2$). Then, for any C^2 , closed hypersurface Γ inside Ω with finite perimeter, we can use the Allen-Cahn functional to approximate the area of Γ . Indeed, for each $\varepsilon > 0$, consider the following Allen-Cahn functional

$$(1.1) \quad E_\varepsilon(u) = \int_\Omega \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon},$$

where $W(u) = \frac{1}{2}(1 - u^2)^2$ is the double-well potential and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a scalar function. This is a typical energy modeling the phase separation phenomena within the van der Waals-Cahn-Hilliard gradient theory of phase transitions [1]. Then, we can find a sequence of scalar functions u^ε such that

(1.2) The zero level sets of u^ε converge to Γ in the Hausdorff distance sense
and

$$(1.3) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) = 2\sigma \mathcal{H}^{N-1}(\Gamma),$$

where $\sigma = \int_{-1}^1 \sqrt{W(s)/2} ds = \frac{2}{3}$ and \mathcal{H}^{N-1} denotes the $(N - 1)$ -dimensional Hausdorff measure. There are many such sequences of u^ε ; the construction of one such sequence follows from the construction part of the general result in the framework of Γ -convergence (see Modica-Mortola [9] and Sternberg [15], for example). Note also that E_ε Γ -converges to the area functional

$$(1.4) \quad E(u) = 2\sigma \mathcal{H}^{N-1}(\Gamma) := E(\Gamma).$$

Here u is a function of bounded variation taking values ± 1 , Γ is the interface separating the phases, i.e, $\Gamma = \partial\{x \in \Omega : u(x) = 1\} \cap \Omega$. Roughly speaking, Γ is the limit of the

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zero level sets of u^ε . Furthermore, we know that (1.2) and (1.3) imply the single-multiplicity property, i.e., in the sense of Radon measures,

$$(1.5) \quad \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \rightharpoonup 2\sigma d\mathcal{H}^{N-1} \llcorner \Gamma.$$

Therefore, from the work of Reshetnyak [11], we can prove that the first inner variation of E_ε at u^ε

$$(1.6) \quad \delta E_\varepsilon(u^\varepsilon, \eta) = \int_\Omega \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) \operatorname{div} \eta - \varepsilon (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \eta),$$

converges to the first inner variation of E at Γ

$$(1.7) \quad \delta E(\Gamma, \eta) = 2\sigma \int_\Gamma (\operatorname{div} \eta - \partial_k \varphi^j n_j \cdot n_k) d\mathcal{H}^{N-1}.$$

In (1.6) and (1.7), $\eta \in (C_c^1(\Omega))^N$ is a vector field and $\vec{n} = (n_1, \dots, n_N)$ denotes the outward unit normal to the region enclosed by Γ ; and (\cdot, \cdot) denotes the standard inner product on R^N .

This convergence result especially imposes the criticality conditions on Γ , typically stationary or minimal condition, if one would like to approximate the area of Γ by u^ε which are critical points of E_ε . The most general result concerning the geometric properties of Γ as the limit of the zero level sets of u^ε with suitable uniform Sobolev bounds on $\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} W'(u^\varepsilon)$ is due to Tonegawa [17]. Regarding L^2 -bounds on $\varepsilon \Delta u^\varepsilon - \varepsilon^{-1} W'(u^\varepsilon)$, one could mention the work of Röger and Schätzle on De Giorgi's conjecture. See [12] and the references therein.

If u^ε (resp. Γ) are critical points of E_ε (resp. E), then a natural question to ask is: what are the relations between the stability of u^ε with respect to E_ε and that of Γ with respect to E ? Assuming the stability of u^ε , by using clever test functions in the stability inequality satisfied by u^ε , Tonegawa [16] proved that Γ must be a stable varifold. Here no assumptions on the regularity of Γ are assumed a priori. Another way to answer the above question is to study the second variations of both functionals E_ε and E . We find that, contrary to the first variation, the second variation of E_ε at u^ε , $\delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta)$, does not in general converges to the second variation of E at Γ , $\delta^2 E(\Gamma, \eta, \zeta)$!

The purpose of this note is two-fold. First, under some regularity assumptions, we provide the precise relation between the limit of the second inner variation of E_ε , assuming the single-multiplicity condition (1.3), and the second inner variation of E . This relation can be of independent interest. Second, we use this relation to estimate from below the Morse indices of the critical points u^ε of E_ε for ε sufficiently small in terms of the Morse index of the critical point Γ of E . Here, again, Γ is the limit of the zero level set of u^ε .

Before stating our main result, we recall some standard definitions. Consider smooth vector fields $\eta, \zeta \in (C_c^1(\Omega))^N$. Then, for t sufficiently small, the map $\Phi_t(x) = x + t\eta(x) + \frac{t^2}{2}\zeta(x)$ is a diffeomorphism of Ω into itself. We think of η and ζ as initial velocity and acceleration vectors when we deform the domain Ω . The second inner variation of E_ε at u^ε with respect to the velocity and acceleration vectors η and ζ is defined by

$$\delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = \left. \frac{d^2}{dt^2} \right|_{t=0} E_\varepsilon(u_t^\varepsilon),$$

where $u_t^\varepsilon(y) = u^\varepsilon(\Phi_t^{-1}(y))$. The second inner variation of E at Γ with respect to the velocity and acceleration vectors η and ζ is defined by

$$\delta^2 E(\Gamma, \eta, \zeta) = \left. \frac{d^2}{dt^2} \right|_{t=0} E(\Gamma_t),$$

where $\Gamma_t = \Phi_t(\Gamma)$.

In this note, we prove the following main result, revealing the exact discrepancy between the second variation of E_ε at u^ε and that of E at Γ .

Theorem 1. *Let Γ be any C^2 , closed hypersurface inside Ω with finite perimeter and let u^ε be any sequence of scalar functions such that the zero level sets of u^ε converge to Γ in the Hausdorff distance sense and*

$$(1.8) \quad \lim_{\varepsilon \rightarrow 0} E_\varepsilon(u^\varepsilon) = 2\sigma \mathcal{H}^{N-1}(\Gamma).$$

Then, for all smooth vector fields $\eta, \zeta \in (C_c^1(\Omega))^N$, we have

$$(1.9) \quad \lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = \delta^2 E(\Gamma, \eta, \zeta) + 2\sigma \int_\Gamma (\vec{n}, \vec{n} \cdot \nabla \eta)^2.$$

Remark 1.1. *The discrepancy term is $2\sigma \int_\Gamma (\vec{n}, \vec{n} \cdot \nabla \eta)^2$. It is new and has a sign. Our result is proved without assuming any criticality conditions on u^ε nor Γ . Thus, on the levels of energy and the first inner variation, the Allen-Cahn functionals approximate well the area functional. This is no longer true for the second inner variation.*

Remark 1.2. *The formula for $\delta^2 E(\Gamma, \eta, \zeta)$ is given by (2.1). Let $\zeta \equiv 0$ and η be a normal vector field defined on Γ , i.e., $\eta = f \vec{n}$ for some function $f \in C_c^1(\Omega)$. Then (1.9) and (2.1) give*

$$(1.10) \quad \lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, f \vec{n}, 0) = \int_\Gamma |\nabla f|^2 - |A|^2 f^2$$

where $|A|$ denotes the length of the second fundamental form A of Γ . The quantity on the right hand side of (1.10) is the one used by Tonegawa in his stability result, [16, Theorem 3]. For the proof of the stability of the interface Γ , this is sufficient. Our formula (1.10) explains that in general, we cannot replace the full gradient $\int_\Gamma |\nabla f|^2$ by the restricted gradient $\int_\Gamma |\nabla^\Gamma f|^2$ in Tonegawa's stability result.

Let us denote by $D^2 E_\varepsilon(u)$ the Hessian of E_ε at u and $Q_\varepsilon(u)$ the associated quadratic function, associated to the bilinear continuous function $B_\varepsilon(u)(\cdot, \cdot)$. If u_t is a variation of u , i.e, $u_0 = u$, then

$$\left. \frac{d^2}{dt^2} \right|_{t=0} E_\varepsilon(u_t) = Q_\varepsilon(u) \left(\left. \frac{d}{dt} \right|_{t=0} u_t \right) = B_\varepsilon(u) \left(\left. \frac{d}{dt} \right|_{t=0} u_t, \left. \frac{d}{dt} \right|_{t=0} u_t \right).$$

Similarly, we can define $D^2 E(u)$, $Q(u)$ and $B(u)$ for E . Now, let $\zeta \equiv 0$ and let η be a normal vector field defined on Γ . Assuming the smoothness of Γ , we can find an extension $\tilde{\eta}$ of η to Ω such that $(\vec{n}, \vec{n} \cdot \nabla \tilde{\eta}) = 0$. In this case, combining (1.9) with Theorem 1.1 in [13], we obtain the following result.

Theorem 2. *Let u^ε be critical points of E_ε , i.e., $\delta E_\varepsilon(u^\varepsilon, \eta) = 0$. Upon extracting a subsequence, $u^\varepsilon \rightarrow u \in BV(\Omega, \{1, -1\})$ in $L^1(\Omega)$. Let Γ be the interface separating the phases of u . Assume that (1.8) holds and Γ is of class C^2 . Denote by n_ε^+ the dimension (possibly infinite) of the space spanned by the eigenvectors of $D^2 E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues, and n^+ the dimension (possibly infinite) of the space spanned by the eigenvectors of $D^2 E(u)$ associated to positive eigenvalues (resp. n_ε^- and n^- for negative eigenvalues). Then, for ε small enough we have*

$$(1.11) \quad n_\varepsilon^+ \geq n^+, \quad n_\varepsilon^- \geq n^-.$$

We note that by writing $\tilde{\eta} = f \vec{n}$, we have

$$Q(u)(\tilde{\eta}) = \int_\Gamma |\nabla^\Gamma f|^2 - |A|^2 f^2$$

where $|A|$ denotes the length of the second fundamental form A of Γ . Therefore, Theorem 2 implies, in particular, that if n_ε^- is the Morse index of u_ε then Γ must be a generalized minimal hypersurface with Morse index n^- satisfying

$$(1.12) \quad \liminf_{\varepsilon \rightarrow 0} n_\varepsilon^- \geq n^-.$$

Thus, if u_ε is stable then Γ is stable, reproving a special case of Tonegawa's stability result when Γ is smooth and (1.8) is satisfied. Regarding stability theory, see Tonegawa [16] for a very general result without assuming (1.8); see also Serfaty [13] for the complex-valued version of u^ε . Regarding regularity theory for stable hypersurface Γ , see a very interesting recent paper by Tonegawa and Wickramasekera [18].

Remark 1.3. *When Γ is a minimal hypersurface satisfying certain nondegeneracy conditions, Pacard and Ritoré [10] constructed critical points u^ε of E_ε whose zero level sets converge to Γ and (1.8) holds. Thus, Theorem 2 provides an estimate for the Morse indices of u^ε constructed in [10] in terms of the Morse index of Γ .*

Remark 1.4. *If $N = 3$, $\Omega = \mathbb{R}^3$ and Γ is a minimal surface, embedded, complete with finite total curvature and is nondegenerate, then del Pino, Kowalczyk and Wei [4] constructed critical points u^ε of E_ε whose zero level sets converge to Γ and $n_\varepsilon^-(u^\varepsilon) = n^-(\Gamma)$ for ε sufficiently small.*

Remark 1.5. *In general, the inequality (1.12) can be strict. Here is one example for $N = 2$ with Γ singular at one point. Let Ω be the unit disc in \mathbb{R}^2 and Γ be the cross $\Gamma = \{(x_1, 0), -1 \leq x_1 \leq 1\} \cup \{(0, x_2), -1 \leq x_2 \leq 1\}$. Using the construction of saddle solutions in Dang, Fife and Peletier [2] (see also Gui [5, Proposition 3.1]), one can construct a sequence of critical points u^ε of E_ε such that the limit of the zero level set of u^ε is the cross Γ . The cross is a stable varifold and thus has Morse index 0. On the other hand, for ε sufficiently small, the critical points u^ε of E_ε are not stable, and thus have Morse index at least 1. The reason that u^ε are not stable is as follows. If otherwise, then by Tonegawa's result [16, Theorem 5], the limit zero level set Γ is a finite number of lines with no intersections or junctions. Therefore, it cannot be the cross, which is a contradiction.*

It would be very interesting to provide estimates similar to (1.11) when the multiplicity one condition (1.8) is dropped. In this regard, we have the following partial result, where we replace (1.8) by the following mild conditions:

- (C1) The limit measure of $\left(\frac{\varepsilon|\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon}\right) dx$ is concentrated on Γ .
(C2) Γ is connected.

Theorem 3. *Let u^ε be critical points of E_ε , i.e., $\delta E_\varepsilon(u^\varepsilon, \eta) = 0$. Upon extracting a subsequence, $u^\varepsilon \rightarrow u \in BV(\Omega, \{1, -1\})$ in $L^1(\Omega)$. Let Γ be the interface separating the phases of u . Assume that (C1) and (C2) are satisfied and Γ is of class C^2 . Denote by n_ε^+ the dimension (possibly infinite) of the space spanned by the eigenvectors of $D^2 E_\varepsilon(u_\varepsilon)$ associated to positive eigenvalues, and n^+ the dimension (possibly infinite) of the space spanned by the eigenvectors of $D^2 E(u)$ associated to positive eigenvalues (resp. n_ε^- and n^- for negative eigenvalues). Then, for ε small enough we have*

$$(1.13) \quad n_\varepsilon^+ \geq n^+, \quad n_\varepsilon^- \geq n^-.$$

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2. PROOF OF THE MAIN RESULTS

This section is entirely devoted to the proof of Theorems 1, 2 and 3.

Proof of Theorem 1. First of all, we have the following formula for the second inner variation of E at Γ (see Simon [14, p. 51], for example)

$$(2.1) \quad \delta^2 E(\Gamma, \eta, \zeta) = 2\sigma \int_\Gamma \left\{ \operatorname{div}^\Gamma \zeta + (\operatorname{div}^\Gamma \eta)^2 + \sum_{i=1}^{N-1} |(D_{\tau_i} \eta)^\perp|^2 - \sum_{i,j=1}^{N-1} (\tau_i \cdot D_{\tau_j} \eta)(\tau_j \cdot D_{\tau_i} \eta) \right\},$$

where $\operatorname{div}^\Gamma \varphi$ denotes the tangential divergence of φ on Γ ; and for each point $x \in \Gamma$, $\{\tau_1(x), \dots, \tau_{N-1}(x)\}$ is any orthonormal basis for the tangent space $T_x(\Gamma)$; for each $\tau \in T_x(\Gamma)$, $D_\tau \eta$ is the directional derivative and the normal part of $D_\tau \eta$ is denoted by

$$(D_{\tau_i} \eta)^\perp = D_{\tau_i} \eta - \sum_{j=1}^{N-1} (\tau_j \cdot D_{\tau_i} \eta) \tau_j.$$

Next, for the second inner variation of E_ε at u^ε , we claim that

$$(2.2) \quad \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = \int_\Omega \left\{ \left(\frac{\varepsilon|\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) (\operatorname{div} \zeta + (\operatorname{div} \eta)^2 - \operatorname{trace}((\nabla \eta)^2)) \right. \\ \left. + \varepsilon |\nabla u^\varepsilon \cdot \nabla \eta|^2 + 2\varepsilon (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot (\nabla \eta)^2) - \varepsilon (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \zeta) \right. \\ \left. - 2\varepsilon (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \eta) \operatorname{div} \eta \right\}.$$

We indicate how to derive this formula. Let $\eta \in (C_c^1(\Omega))^N$ and $\zeta \in (C_c^1(\Omega))^N$ be vector fields and $t \neq 0$ sufficiently small such that the map $\Phi_t(x) = x + t\eta(x) + \frac{t^2}{2}\zeta(x)$ is a diffeomorphism of Ω into itself. Set $u_t^\varepsilon(y) = u^\varepsilon(\Phi_t^{-1}(y))$. We are going to calculate

$$(2.3) \quad \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = \frac{d^2}{dt^2} \Big|_{t=0} E_\varepsilon(u_t^\varepsilon).$$

By change of variables $y = \Phi_t(x)$, we have

$$(2.4) \quad E_\varepsilon(u_t^\varepsilon) = \int_\Omega \left[\frac{\varepsilon |\nabla u^\varepsilon \cdot \nabla \Phi_t^{-1}(\Phi_t(x))|^2}{2} + \frac{(1 - |u^\varepsilon(x)|^2)^2}{2\varepsilon} \right] |\det \nabla \Phi_t(x)| dx.$$

We need to expand the right-hand side of the above formula up to the second power of t . For this purpose, we use the following identity for matrices A and B

$$(2.5) \quad \det(I + tA + \frac{t^2}{2}B) = 1 + t\text{trace}(A) + \frac{t^2}{2}[\text{trace}(B) + (\text{trace}(A))^2 - \text{trace}(A^2)] + O(t^3).$$

Therefore,

$$(2.6) \quad \begin{aligned} \det \nabla \Phi_t(x) &= \det(I + t\nabla \eta(x) + \frac{t^2}{2}\nabla \zeta) \\ &= 1 + t\text{div} \eta + \frac{t^2}{2}[\text{div} \zeta + (\text{div} \eta)^2 - \text{trace}((\nabla \eta)^2)] + O(t^3). \end{aligned}$$

Note that

$$(2.7) \quad \nabla \Phi_t^{-1}(\Phi_t(x)) = [I + t\nabla \eta(x) + \frac{t^2}{2}\nabla \zeta(x)]^{-1} = I - t\nabla \eta - \frac{t^2}{2}\nabla \zeta(x) + t^2(\nabla \eta)^2 + O(t^3).$$

Plugging (2.6) and (2.7) into (2.4), we get (2.2) after some simple calculations. For the sake of completeness, we include here the calculations. First, we note that, for t sufficiently small, $\det \nabla \Phi_t(x) > 0$ and thus $|\det \nabla \Phi_t(x)| = \det \nabla \Phi_t(x)$. Second, from (2.7), we find that

$$\nabla u^\varepsilon \cdot \nabla \Phi_t^{-1}(\Phi_t(x)) = \nabla u^\varepsilon - t\nabla u^\varepsilon \cdot \nabla \eta - \frac{t^2}{2}\nabla u^\varepsilon \cdot \nabla \zeta(x) + t^2\nabla u^\varepsilon \cdot (\nabla \eta)^2 + O(t^3).$$

Hence

$$(2.8) \quad \begin{aligned} \frac{\varepsilon |\nabla u^\varepsilon \cdot \nabla \Phi_t^{-1}(\Phi_t(x))|^2}{2} &= \frac{\varepsilon}{2} \{ |\nabla u^\varepsilon|^2 - 2t(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \eta) + t^2 |\nabla u^\varepsilon \cdot \nabla \eta|^2 \\ &\quad + 2t^2(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot (\nabla \eta)^2) - t^2(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \zeta) + O(t^3) \}. \end{aligned}$$

It follows that

$$(2.9) \quad \begin{aligned} &\left[\frac{\varepsilon |\nabla u^\varepsilon \cdot \nabla \Phi_t^{-1}(\Phi_t(x))|^2}{2} + \frac{(1 - |u^\varepsilon(x)|^2)^2}{2\varepsilon} \right] |\det \nabla \Phi_t(x)| \\ &= \left(\frac{\varepsilon}{2} \{ |\nabla u^\varepsilon|^2 - 2t(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \eta) + t^2 |\nabla u^\varepsilon \cdot \nabla \eta|^2 + 2t^2(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot (\nabla \eta)^2) - t^2(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \zeta) \} \right. \\ &\quad \left. + \frac{(1 - |u^\varepsilon(x)|^2)^2}{2\varepsilon} + O(t^3) \right) \left(1 + t\text{div} \eta + \frac{t^2}{2}[\text{div} \zeta + (\text{div} \eta)^2 - \text{trace}((\nabla \eta)^2)] + O(t^3) \right) \\ &= \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{(1 - |u^\varepsilon(x)|^2)^2}{2\varepsilon} \right) \frac{t^2}{2} [\text{div} \zeta + (\text{div} \eta)^2 - \text{trace}((\nabla \eta)^2)] \\ &\quad + \frac{\varepsilon t^2}{2} \{ |\nabla u^\varepsilon \cdot \nabla \eta|^2 + 2(\nabla u^\varepsilon, \nabla u^\varepsilon \cdot (\nabla \eta)^2) - (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \zeta) \} \\ &\quad - \varepsilon t^2 (\nabla u^\varepsilon, \nabla u^\varepsilon \cdot \nabla \eta) \text{div} \eta \\ &\quad + \text{lower order terms in } t + O(t^3). \end{aligned}$$

Substituting this relation into (2.4), one gets (2.2) as desired.

From (1.8) and the fact that the zero level sets of u^ε converge to Γ in the Hausdorff distance sense, we have the single-multiplicity (1.5). Consequently, from the work of Reshetnyak [11], we can prove that

$$(2.10) \quad \varepsilon \nabla u^\varepsilon \otimes \nabla u^\varepsilon dx \rightharpoonup 2\sigma \vec{n} \otimes \vec{n} \mathcal{H}^{N-1} \llcorner \Gamma.$$

(For a simple proof of this result, see Luckhaus and Modica [8]).

Passing to the limit in (2.2), employing (1.8) and (2.10), we obtain

$$(2.11) \quad \lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = 2\sigma \int_\Gamma \operatorname{div} \zeta + (\operatorname{div} \eta)^2 - \operatorname{trace}((\nabla \eta)^2) \\ + 2\sigma \int_\Gamma \left| \vec{n} \cdot \nabla \eta \right|^2 + 2(\vec{n}, \vec{n} \cdot (\nabla \eta)^2) - (\vec{n}, \vec{n} \cdot \nabla \zeta) - 2(\vec{n}, \vec{n} \cdot \nabla \eta) \operatorname{div} \eta.$$

Note that $\operatorname{div}^\Gamma \eta = \operatorname{div} \eta - (\vec{n}, \vec{n} \cdot \nabla \eta)$. Hence

$$(2.12) \quad \lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = 2\sigma \int_\Omega \operatorname{div}^\Gamma \zeta + (\operatorname{div}^\Gamma \eta)^2 - \operatorname{trace}((\nabla \eta)^2) \\ + 2\sigma \int_\Gamma \left| \vec{n} \cdot \nabla \eta \right|^2 + 2(\vec{n}, \vec{n} \cdot (\nabla \eta)^2) - (\vec{n}, \vec{n} \cdot \nabla \eta)^2.$$

Some calculation using local coordinates completes the proof of (1.9). For the reader's convenience, we include the details. We can choose local coordinates so that $\{\tau_1, \cdot, \tau_{N-1}, \vec{n}\}$ is the orthonormal basis of R^N . Furthermore, $\vec{n} = (0, \dots, 0, 1)$. We calculate successively

- (i) $(\nabla \eta)_{ij} = \frac{\partial \eta^i}{\partial x_j}$,
- (ii) $((\nabla \eta)^2)_{ij} = \sum_k \frac{\partial \eta^i}{\partial x_k} \frac{\partial \eta^k}{\partial x_j}$,
- (iii) $\operatorname{trace}(\nabla \eta)^2 = \sum_i ((\nabla \eta)^2)_{ii} = \sum_{i,k} \frac{\partial \eta^i}{\partial x_k} \frac{\partial \eta^k}{\partial x_i}$,
- (iv) $2(\vec{n}, \vec{n} \cdot (\nabla \eta)^2) = 2 \sum_{i,j} n_i n_j ((\nabla \eta)^2)_{ij} = 2((\nabla \eta)^2)_{NN} = 2 \sum_k \frac{\partial \eta^N}{\partial x_k} \frac{\partial \eta^k}{\partial x_N}$,
- (v) $(\vec{n}, \vec{n} \cdot \nabla \eta)^2 = (\sum_{i,j} n_i n_j \frac{\partial \eta^i}{\partial x_j})^2 = (\frac{\partial \eta^N}{\partial x_N})^2$,
- (vi) $\left| \vec{n} \cdot \nabla \eta \right|^2 = \left| (\frac{\partial \eta^j}{\partial x_i} n_j) \right|^2 = \sum_i \left| \frac{\partial \eta^N}{\partial x_i} \right|^2$,
- (vii) $\left| \vec{n} \cdot \nabla \eta \right|^2 - (\vec{n}, \vec{n} \cdot \nabla \eta)^2 = \sum_{i < N} \left| \frac{\partial \eta^N}{\partial x_i} \right|^2$,
- (viii) $(D_{\tau_i} \eta)^\perp = D_{\tau_i} \eta - \sum_{j=1}^{N-1} (\tau_j \cdot D_{\tau_i} \eta) \tau_j = (\frac{\partial \eta^1}{\partial x_i}, \dots, \frac{\partial \eta^N}{\partial x_i}) - \sum_{j < N} \frac{\partial \eta^j}{\partial x_i} \tau_j = (0, \dots, 0, \frac{\partial \eta^N}{\partial x_i})$,
- (ix) $\sum_{i < N} \left| (D_{\tau_i} \eta)^\perp \right|^2 = \sum_{i < N} \left| \frac{\partial \eta^N}{\partial x_i} \right|^2$,
- (x) $\tau_i \cdot D_{\tau_j} \eta = \frac{\partial \eta^i}{\partial x_j}$,
- (xi) $\sum_{i,j < N} (\tau_i \cdot D_{\tau_j} \eta) (\tau_j \cdot D_{\tau_i} \eta) = \sum_{i,j < N} \frac{\partial \eta^i}{\partial x_j} \frac{\partial \eta^j}{\partial x_i}$,

(xii)

$$\begin{aligned}
& - \text{trace}((\nabla\eta)^2) + \left| \vec{n} \cdot \nabla\eta \right|^2 + 2(\vec{n}, \vec{n} \cdot (\nabla\eta)^2) - (\vec{n}, \vec{n} \cdot \nabla\eta)^2 \\
& = \left| \vec{n} \cdot \nabla\eta \right|^2 - (\vec{n}, \vec{n} \cdot \nabla\eta)^2 + 2(\vec{n}, \vec{n} \cdot (\nabla\eta)^2) - \text{trace}((\nabla\eta)^2) \\
& = \sum_{i=1}^{N-1} |(D_{\tau_i}\eta)^\perp|^2 + \left(2 \sum_k \frac{\partial\eta^N}{\partial x_k} \frac{\partial\eta^k}{\partial x_N} - \sum_{i,k} \frac{\partial\eta^i}{\partial x_k} \frac{\partial\eta^k}{\partial x_i} \right) \\
& = \sum_{i=1}^{N-1} |(D_{\tau_i}\eta)^\perp|^2 + \left(\left| \frac{\partial\eta^N}{\partial x_N} \right|^2 - \sum_{i,j < N} \frac{\partial\eta^i}{\partial x_j} \frac{\partial\eta^j}{\partial x_i} \right) \\
& = \sum_{i=1}^{N-1} |(D_{\tau_i}\eta)^\perp|^2 + \left((\vec{n}, \vec{n} \cdot \nabla\eta)^2 - \sum_{i,j < N} (\tau_i \cdot D_{\tau_j}\eta)(\tau_j \cdot D_{\tau_i}\eta) \right).
\end{aligned}$$

Thus from (2.12), we find that

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) & = 2\sigma \int_\Gamma \text{div}^\Gamma \zeta + (\text{div}^\Gamma \eta)^2 + \sum_{i=1}^{N-1} |(D_{\tau_i}\eta)^\perp|^2 \\
& + \left((\vec{n}, \vec{n} \cdot \nabla\eta)^2 - \sum_{i,j < N} (\tau_i \cdot D_{\tau_j}\eta)(\tau_j \cdot D_{\tau_i}\eta) \right) \\
& = \delta^2 E(\Gamma, \eta, \zeta) + 2\sigma \int_\Gamma (\vec{n}, \vec{n} \cdot \nabla\eta)^2.
\end{aligned}$$

The proof of our theorem is now complete. \square

Proof of Theorem 2. For any vector field V defined on Γ and is normal to Γ , we also denote by V its extension to Ω in such a way that $(\vec{n}, \vec{n} \cdot \nabla V) = 0$. As a consequence, the second term on the right hand side of (1.9) drops. Let V and W be vector fields normal to Γ . Then, let

$$\Phi_{V,t} = x + tV(x), \Phi_{W,t} = x + tW(x)$$

and

$$v_\varepsilon = u^\varepsilon(\Phi_{V,t}^{-1}), w_\varepsilon = u^\varepsilon(\Phi_{W,t}^{-1}).$$

The proof is based on (1.9) together with the following claims.

Claim 2.1. (*polarization*)

$$(2.13) \quad B_\varepsilon(u^\varepsilon)(\partial_t v_\varepsilon(0), \partial_t w_\varepsilon(0)) = B(u)(V, W) + o(1).$$

Claim 2.2. (*injectivity*) *The map $V \mapsto \partial_t v_\varepsilon(0)$ is linear and one-to-one for ε small.*

Now, having these claims, we can complete the proof of Theorem 2, following the arguments in the proof of Theorem 1.1 in [13]. By definition of n^+ , if n^+ is finite, we can find n^+ linearly

independent vector fields V^1, \dots, V^{n^+} which are defined on Γ and normal to Γ such that the quadratic function $Q(u)$ restricted to the space they span is positive, i.e.,

$$(2.14) \quad \min_{\sum_{i=1}^{n^+} a_i^2 = 1} Q(u) \left(\sum_{i=1}^{n^+} a_i V^i \right) > 0.$$

Denote $V_\varepsilon^i = \partial_t v_\varepsilon^i(0) = \frac{d}{dt} \Big|_{t=0} u^\varepsilon \left((x + tV^i(x))^{-1} \right)$. In view of (2.13), we have for all a_i

$$\lim_{\varepsilon \rightarrow 0} Q_\varepsilon(u^\varepsilon) \left(\sum_{i=1}^{n^+} a_i V_\varepsilon^i \right) = Q(u) \left(\sum_{i=1}^{n^+} a_i V^i \right)$$

and the convergence is uniform with respect to (a_i) such that $\sum_{i=1}^{n^+} a_i^2 = 1$. Finally, we deduce from (2.14) that for ε small enough

$$(2.15) \quad \min_{\sum_{i=1}^{n^+} a_i^2 = 1} Q_\varepsilon(u^\varepsilon) \left(\sum_{i=1}^{n^+} a_i V_\varepsilon^i \right) > 0.$$

By Claim 2.2 and the linear independence of V^i , V_ε^i are linearly independent for ε small. Therefore, the V_ε^i span a space of dimension n^+ . This proves that $D^2 E_\varepsilon(u^\varepsilon)$ has at least n^+ positive eigenvalues and thus $n_\varepsilon^+ \geq n^+$. Observe that if $n^+ = +\infty$ then we can apply the previous argument on subspaces of arbitrarily large finite dimension, and find that n_ε^+ is also $+\infty$ for ε small. The same arguments work for n_ε^- and n^- .

We now prove Claim 2.1. Indeed, using (1.9), we see that

$$\frac{d^2}{dt^2} \Big|_{t=0} E_\varepsilon(v_\varepsilon) = Q(u)(V)$$

or, equivalently

$$\lim_{\varepsilon \rightarrow 0} B_\varepsilon(u^\varepsilon)(\partial_t v_\varepsilon(0), \partial_t v_\varepsilon(0)) = B(u)(V, V).$$

Therefore

$$(2.16) \quad B_\varepsilon(u^\varepsilon)(\partial_t v_\varepsilon(0), \partial_t v_\varepsilon(0)) = B(u)(V, V) + o(1).$$

Applying (2.16) to $V + W$ and $V - W$, we get

$$B_\varepsilon(u^\varepsilon)(\partial_t v_\varepsilon(0) + \partial_t w_\varepsilon(0), \partial_t v_\varepsilon(0) + \partial_t w_\varepsilon(0)) = B(u)(V + W, V + W) + o(1)$$

and

$$B_\varepsilon(u^\varepsilon)(\partial_t v_\varepsilon(0) - \partial_t w_\varepsilon(0), \partial_t v_\varepsilon(0) - \partial_t w_\varepsilon(0)) = B(u)(V - W, V - W) + o(1).$$

Subtracting these two relations, we obtain (2.13).

Finally, we prove Claim 2.2. Recall that $\Phi_{V,t}(x) = x + tV(x)$. Note that, for each x , we have

$$x = \Phi_{V,t}(\Phi_{V,t}^{-1}(x)) = \Phi_{V,t}^{-1}(x) + tV(\Phi_{V,t}^{-1}(x)).$$

Hence

$$0 = \frac{d}{dt}(\Phi_{V,t}^{-1}(x)) + t\nabla V(\Phi_{V,t}^{-1}(x)) \cdot \frac{d}{dt}(\Phi_{V,t}^{-1}(x)).$$

Evaluating the above equation at $t = 0$ and noting that $\Phi_{V,0}^{-1}(x) = x$, one obtains

$$\frac{d}{dt} \Big|_{t=0} (\Phi_{V,t}^{-1}(x)) = -V(x).$$

It is now clear that

$$(2.17) \quad V \longmapsto \partial_t v_\varepsilon(0) := \frac{d}{dt} \Big|_{t=0} u^\varepsilon ((x + tV(x))^{-1}) = -\nabla u^\varepsilon \cdot V$$

is a linear map. Let $V = f \vec{n}$ be a normal vector field to Γ . Suppose that $\nabla u^\varepsilon \cdot (f \vec{n}) = 0$ for all ε small. This implies that $\varepsilon \left| \nabla u^\varepsilon \cdot \vec{n} \right|^2 f^2 = 0$. Letting $\varepsilon \rightarrow 0$ and using (2.10), we find that $2\sigma f^2 \equiv 0$ on Γ . Therefore $f = 0$ and $V = 0$. This proves our claim. \square

Proof of Theorem 3. Under the condition (C1) and the fact that u^ε are critical points of E_ε , the work of Hutchinson and Tonegawa [6, Theorem 1] (see also [16]) showed that, in the sense of Radon measures,

$$(2.18) \quad \left(\frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx \rightharpoonup 2m\sigma d\mathcal{H}^{N-1} \llcorner \Gamma,$$

where m is an integer-valued function defined on Γ . Furthermore, we have equipartition of energy, i.e., in the sense of Radon measures

$$(2.19) \quad \left| \frac{\varepsilon |\nabla u^\varepsilon|^2}{2} - \frac{W(u^\varepsilon)}{\varepsilon} \right| dx \rightarrow 0.$$

Because u^ε are critical points of E_ε , Γ is a stationary varifold. Now, it follows from the connectivity of Γ from (C2) and the Constancy Theorem [14, Theorem 41.1] that m must be a constant. From the constancy of m and limiting equipartition of energy (2.19), it can be proved that (see, e.g., [7, Equation (3.5)])

$$(2.20) \quad \varepsilon \nabla u^\varepsilon \otimes \nabla u^\varepsilon dx \rightharpoonup 2m\sigma \vec{n} \otimes \vec{n} \mathcal{H}^{N-1} \llcorner \Gamma.$$

Now, we proceed as in the proof of Theorem 2 which used the result obtained in Theorem 1. First, as in the proof of Theorem 1, we have (2.2). Letting $\varepsilon \rightarrow 0$ in (2.2), using (2.10), and computing as in the proof of Theorem 1, we obtain

$$(2.21) \quad \lim_{\varepsilon \rightarrow 0} \delta^2 E_\varepsilon(u^\varepsilon, \eta, \zeta) = m\delta^2 E(\Gamma, \eta, \zeta) + 2m\sigma \int_\Gamma (\vec{n}, \vec{n} \cdot \nabla \eta)^2$$

for all smooth vector fields $\eta, \zeta \in (C_c^1(\Omega))^N$.

With (2.21), the proof of Theorem 3 can be completed similarly to that of Theorem 2. \square

Remark 2.1. *There exist critical points u^ε of E_ε satisfying (2.18) with any positive integer m for minimal surfaces Γ in bounded domains in \mathbb{R}^2 satisfying certain nondegeneracy conditions. See, e. g., [3].*

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