SUBGROUPS OF FREE IDEMPOTENT GENERATED SEMIGROUPS: FULL LINEAR MONOIDS

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Abstract. We develop some new topological tools to study maximal subgroups of free idempotent generated semigroups. As an application, we show that the rank 1 component of the free idempotent generated semigroup of the biordered set of a full matrix monoid of size $n\times n, n>2$ over a division ring Q has maximal subgroup isomorphic to the multiplicative subgroup of Q.

1. Introduction

Let S be a semigroup with non-empty set $E=E(S)$ of idempotents, and let $e,f\in E(S)$. It is easy to see that if $e\in Sf\cup fS$, then both ef and fe must also be idempotents of S. Products of idempotents of this form are referred to as *basic products* in E. The set $E=E(S)$ relative to these basic products forms a partial algebra that has been characterized axiomatically as a *regular biordered set* by Nambooripad [13] in the case that S is a (von-Neumann) regular semigroup, and more generally as a *biordered set* by Easdown [4] for an arbitrary semigroup S. The basic products in E may be defined in terms of certain quasi-orders on E that are independent of the specific semigroup S with biordered set E. We refer to [13] and [4] for details of the axiomatic characterization of biordered sets: we will not need these details in the present paper.

Given any (axiomatically characterized) biordered set E, the *free idempotent generated semigroup* on E is the semigroup $IG(E)$ with presentation

$$IG(E)=\langle E : e.f=ef \text{ if } ef \text{ is a basic product } \rangle.$$

Since a product of the form $e.e=e$ is clearly a basic product, it is obvious that $IG(E)$ is an idempotent generated semigroup. A theorem of Easdown [4] shows that the biordered set of idempotents of $IG(E)$ is E, that is, there is a bijection between E and the biordered set of idempotents in $IG(E)$ and these two biordered sets have the same basic products. The semigroup $IG(E)$ is a universal object in the category of idempotent generated semigroups with biordered set E and morphisms that are one to one on idempotents. An analogous result was proved earlier by Nambooripad [13],

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who constructed a free regular idempotent generated semigroup $RIG(E)$ on a regular biordered set E.

Given an idempotent e of any semigroup S, the *maximal subgroup* H_e of S with identity e is the group of units of the submonoid eSe of S. For example, if e is an idempotent $n \times n$ matrix of rank r in the monoid $M_n(Q)$ of $n \times n$ matrices over a division ring Q, then $H_e \cong GL_r(Q)$, the General Linear Group of size r over Q. In [1] it is shown that if E is a regular biordered set and $e \in E$, then the maximal subgroup of $RIG(E)$ with identity e is isomorphic to the maximal subgroup of $IG(E)$ with identity e. A question that has been of interest in the literature is: which groups can arise as maximal subgroups of a free idempotent-generated semigroup $IG(E)$ for some (axiomatically characterized) biordered set E? Early results on this problem (for example [16], [14]) suggested that such groups must be free, and in fact this conjecture explicitly surfaced in the literature in [11], although it had been conjectured since the early 1980's.

In [1], the authors provided the first counterexample to this conjecture, by showing that the free abelian group $\mathbb{Z} \times \mathbb{Z}$ can arise in this context. Subsequently, Gray and Ruskuc [8] have shown that *every* group arises in this context. However, the structure of the maximal subgroups of free idempotent generated semigroups on naturally occurring biordered sets (such as the biordered set of the full linear monoid $M_n(Q)$ over a division ring Q) is far from clear, and the main purpose of the current paper is to provide some new topological tools to study this problem.

We remark that idempotent generated semigroups arise naturally in many parts of mathematics. First of all they are "general". It is well known that every semigroup S embeds into a semigroup generated by a set of idempotents of the same cardinality as S. If S is (finite) countable then S embeds into a (finite) semigroup generated by 3 idempotents. Idempotent generated semigroups play an important part in the theory of reductive algebraic monoids [17, 21]. Putcha's theory of monoids of Lie type shows that one can consider the biordered set of idempotents of such a monoid to be a generalization of a building [18, 19] in the sense of Tits. Thus such objects have a natural geometric structure.

In their paper [1], the authors defined the *Graham-Houghton* 2-complex $GH(E)$ of a regular biordered set E, based on the work of Nambooripad [13], Graham [7] and Houghton [10], and they showed that the maximal subgroups of $IG(E)$ are the fundamental groups of the connected components of $GH(E)$. The 2-cells of $GH(E)$ correspond to the *singular squares* of E defined by Nambooripad [13]. Gray and Ruskuc [8] give a new proof that singular squares give presentations of maximal subgroups of $IG(E)$ for an arbitrary biordered set E.

We outline the main topological idea of this paper. Let S be an idempotent generated regular semigroup with biordered set $E = E(S)$. Then there is a surjective idempotent separating morphism $f : RIG(E) \twoheadrightarrow S$ that is an isomorphism on E. It follows that for every maximal subgroup G of S

there is a unique connected component C of $GH(E)$ and a unique point x of C and a surjective morphism $\pi_1(C, x) \twoheadrightarrow G$. By basic algebraic topology, G acts transitively and fixed point free on each fibre of the connected cover $C(G)$ of C that has fundamental group the kernel of this latter morphism. Thus if $C(G)$ is simply connected, G is isomorphic to a maximal subgroup of $RIG(E)$.

We apply these ideas to show that if Q is a division ring, then the maximal subgroup of $IG(E(M_n(Q)))$ corresponding to an idempotent matrix of rank 1 is Q^*, the multiplicative group of units of Q. The maximal subgroup of $IG(E(M_n(Q)))$ corresponding to an idempotent matrix of rank $n-1$ is a free group, but the structure of the maximal subgroups of $IG(E(M_n(Q)))$ corresponding to idempotent matrices of rank k for $1 < k < n-1$ remains far from clear.

2. Preliminaries

Recall that a semigroup S is called *regular* if $a \in aSa$ for each $a \in S$. In a very influential paper [13], Nambooripad studied the structure of regular semigroups via his theory of *inductive groupoids*. He found an axiomatic characterization of the set of idempotents of a regular semigroup S relative to the basic products in $E(S)$ as a "regular biordered set" and he described the inductive groupoid associated with the free regular idempotent generated set $RIG(E)$ on a regular biordered set E. A presentation for $RIG(E)$ was provided by Pastijn [16]. In [1], the authors showed that if E is a regular biordered set, then the maximal subgroups of $RIG(E)$ are isomorphic to the maximal subgroups of the semigroup $IG(E)$ defined above. We refer the reader to [13] and [1] for details. *We shall assume throughout the remainder of this paper that all biordered sets under consideration are biordered sets of regular semigroups.*

Recall that the Green's relations $\mathcal{R}$ and $\mathcal{L}$ on a semigroup S are defined by $a \mathrel{\mathcal{R}} b$ iff $aS^1 = bS^1$ and $a \mathrel{\mathcal{L}} b$ iff $S^1a = S^1b$. When restricted to $E = E(S)$ these are defined by basic products: $e \mathrel{\mathcal{R}} f$ iff $ef = f$ and $fe = e$, and $e \mathrel{\mathcal{L}} f$ iff $ef = e$ and $fe = f$. Thus, one can consider the $\mathcal{R}$ and $\mathcal{L}$ relations on an arbitrary (axiomatically defined) biordered sets. By the theorems of Nambooripad and Easdown mentioned above, these are exactly the restrictions of the corresponding Green's relations on $RIG(E)$ and $IG(E)$ to E.

We can define Green's relation $\mathcal{D}$ on a biordered set E as the transitive closure of $\mathcal{R} \cup \mathcal{L}$. It follows from the work of Fitz-Gerald [6] and Nambooripad [13] that this is the restriction of Green's relation $\mathcal{D}$ to the idempotents of $RIG(E)$. Thus we identify these relations on E with the classical notions on $RIG(E)$ and $IG(E)$ without further mention. In Nambooripad's language, [13], two idempotents of $RIG(E)$ or $IG(E)$ are $\mathcal{D}$ related if and only if there is an E-path between them. This is just another way of saying that $\mathcal{D}$ is the transitive closure of $\mathcal{R} \cup \mathcal{L}$ restricted to E.

Central to Nambooripad's construction of the inductive groupoid of $RIG(E)$ is the notion of a *singular square* in the (regular) biordered set E. An E*-square* is a sequence (e, f, g, h, e) of elements of E with $e \ \mathcal{R} \ f \ \mathcal{L} \ g \ \mathcal{R} \ h \ \mathcal{L} \ e$. Unless otherwise specified, we will assume that all E-squares are non-degenerate, i.e. the elements e, f, g, h are all distinct. An idempotent $t = t^2 \in E$ *left to right singularizes* the E-square (e, f, g, h, e) if $te = e, th = h, et = f$ and $ht = g$ where all of these products are defined in the biordered set E. Right to left, top to bottom and bottom to top singularization is defined similarly and we call the E-square *singular* if it has a singularizing idempotent of one of these types. All of these products are basic products, so they make sense in any semigroup with biorder isomorphic to E.

The following simple but important fact was first noted by Nambooripad [13]. Recall that the right zero semigroup on a set X is the set X with multiplication $xy = y$ for all $x, y \in X$. A left zero semigroup is the dual notion and a rectangular band is the direct product of some left zero semigroup with some right zero semigroup. Thus a rectangular band is defined on some set of the form $X \times Y$ with multiplication $(x, y)(x\prime, y\prime) = (x, y\prime)$, for all $x, x\prime \in X, y, y\prime \in Y$.

Lemma 2.1. *Let (e, f, g, h, e) be a singular E-square in a semigroup S. Then $efghe = e$: in other words, $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is a rectangular band in any semigroup with biordered set E.*

We remark that the converse of Lemma 2.1 is obviously false. For example if the semigroup S is just a 2×2 rectangular band $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$, then it is not a singular square since there is no idempotent available to singularize it.

Recall from [1] that the *Graham-Houghton* graph of a (regular) biordered set E is the bipartite graph with vertices the disjoint union of the set of $\mathcal{R}$-classes of E and the set of $\mathcal{L}$-classes of E, and with a directed (positively oriented) edge from an $\mathcal{L}$-class L to an $\mathcal{R}$-class R if there is an idempotent $e \in L \cap R$ (and a corresponding inverse edge from R to L in this case). We now add 2-cells to this graph, one for each singular square (e, f, g, h, e). Given this square we sew a 2-cell onto this graph with boundary $ef^{-1}gh^{-1}$. We call the resulting 2-complex the *Graham-Houghton complex* of E and denote it by $GH(E)$.

The following theorem in [1] is based on the work of Nambooripad, and is the principal tool used in [1] to construct maximal subgroups of free idempotent-generated semigroups on biordered sets.

Theorem 2.2. [1] *Let E be a regular biordered set. Then the maximal subgroup of $IG(E)$ based at $e \in E$ is isomorphic to the fundamental group $\pi_1(GH(E), L_e)$ of the Graham-Houghton complex of E based at L_e.*

The following theorem is used crucially in this paper.

Theorem 2.3. *Let E be a regular biordered set. Then for any regular idempotent generated semigroup S with $E \approx E(S)$, there is a one to one correspondence between connected components of $GH(E)$ and $\mathcal{D}$ classes of S. Every maximal subgroup G of S is a quotient of a unique maximal subgroup of $RIG(E)$ that belongs to the unique connected component corresponding to the $\mathcal{D}$-class of G of S.*

Proof Since $E \approx E(S)$, there is a surjective idempotent separating morphism $\varphi : RIG(E) \twoheadrightarrow S$. It is well known that f then induces a bijection between $RIG(E)/\mathcal{K}$ to $S/\mathcal{K}$ for any of Green's relations $\mathcal{K}$. In particular, this is true for $\mathcal{K} = \mathcal{D}$. As mentioned above, two idempotents are $\mathcal{D}$ related in $RIG(E)$ if and only if there is an E-path between them in E. Since positive edges of $GH(E)$ are in one to one correspondence with E, a straightforward induction on the length of a path shows that there is a path between two vertices of $GH(E)$ if and only if there is an E-path in E whose first edge belongs to the first vertex (recall that the vertices of $GH(E)$ are the disjoint union of $\mathcal{R}$ and $\mathcal{L}$ classes of E) and whose last edge belongs to the last vertex. Since idempotent separating morphisms between regular semigroups induce a 1-1 correspondence between $\mathcal{H}$ classes, the second statement of the theorem follows. This completes the proof.

3. A Freeness Criterion

Let S be a regular idempotent generated semigroup and let $E = E(S)$ be its biordered set of idempotents. Let G be a maximal subgroup of S. By Theorem 2.2 and Theorem 2.3 there is a unique connected component $\mathcal{G}$ of $GH(E)$, and a surjective morphism $f : \pi_1(\mathcal{G}) \twoheadrightarrow G$ from the fundamental group of $\mathcal{G}$ to G.

Therefore, there is a unique (up to isomorphism) connected cover $\mathcal{C}(G)$ of $\mathcal{G}$ that has $Ker(f)$ as fundamental group. It is well known, [9] that G acts freely on the fibres of $\mathcal{C}(G)$ and that $\mathcal{G} \approx \mathcal{C}(G)/G$. The construction of $\mathcal{C}(G)$ is a simple exercise in covering theory, but since we need the details, we include them here.

Let $\mathcal{C}$ be a connected 2 complex and let $\varphi : \pi_1(\mathcal{C}) \twoheadrightarrow G$ be a surjective morphism to a group G. Let $\mathcal{T}$ be a spanning tree of $V(\mathcal{C})$. Then $\pi_1(\mathcal{C})$ and G are generated by the positive edges (relative to some orientation) not in $\mathcal{T}$.

For convenience if e is an edge of $\mathcal{C}$ we identify e with its value as a generator in $\pi_1(\mathcal{C})$ as above and $\varphi(e)$ the corresponding value in G. Note that e and $\varphi(e)$ are the identity element if $e \in \mathcal{T}$

Define the cover $\mathcal{C}(G)$ as follows:

Vertices: $G \times V(\mathcal{C})$

Edges: We have an edge from (g, x) to $(h, y), g, h \in G, x, y \in V(\mathcal{C})$ iff there is an edge e from x to y in $\mathcal{C}$ and $g \cdot \varphi(e) = h$ in G.

2-Cells: For every cell of $\mathcal{C}$ sewed on as a loop in $\mathcal{C}$ from some vertex x to itself, we sew a copy of it as a loop from (g,x) to itself following the edges modified by their lifts in the definition of edges above.

Then this is a cover of $\mathcal{C}$ denoted by $\mathcal{C}(G)$ under the projection $G \times V(\mathcal{C}) \to V(\mathcal{C})$. G acts transitively and fixed point free on the left of each fibre by $g(h,x) = (gh,x)$ and the quotient $\mathcal{C}(\mathcal{G})/G \approx \mathcal{C}, \pi_1(\mathcal{C}(G)) \trianglelefteq \pi_1(\mathcal{C})$ and $G \approx \pi_1(\mathcal{C})/\pi_1(\mathcal{C}(G))$

Finally, $\mathcal{C}(G)$ is clearly the universal object in the category of all covers of $\mathcal{C}$ on which G acts transitively on each fibre.

Example 1 If $\mathcal{C}$ is a bouquet of X circles (no 2 cells), then G is an X generated group and $\mathcal{C}(G)$ is just the Cayley graph of G relative to the presentation $f : \pi_1(\mathcal{C}) \to G$.

Example 2 Let E be a regular biordered set and let S be a regular idempotent generated semigroup with $E(S) \approx E$. Let G be a maximal subgroup of S. Then G corresponds by Theorem 2.3 to a unique connected component $\mathcal{C}$ of $GH(E)$ and there is a surjective morphism from the fundamental group corresponding to this component to G.

Putting all this together we get the following criterion for G to be isomorphic to a maximal subgroup of $RIG(E)$ (or equivalently $IG(E)$ [1]). We use the notation in Example 2.

Freeness Criterion If the cover $\mathcal{C}(G)$ of the group G is simply connected, then G is isomorphic to the maximal subgroup of the corresponding component of $RIG(E)$.

4. Matrices over division rings

Throughout this section, Q will be a division ring, $M_n(Q)$ will denote the full linear monoid of $n \times n$ matrices over Q and $GL_n(Q)$ will denote the general linear group, i.e. the group of units of $M_n(Q)$. We will use both lower case and upper case letters to denote matrices. We make use of the covering space methods in the previous section to study the maximal subgroups of the free idempotent generated semigroup on the biordered set of idempotents of this monoid. In particular, we prove the following theorem, which is the main result in this section.

Theorem 4.1. *Let E be the biordered set of $M_n(Q)$, for Q a division ring, and let e be an idempotent matrix of rank 1 in $M_n(Q)$. For $n \geq 3$, the maximal subgroup of $IG(E)$ with identity e is isomorphic to Q^*, the multiplicative group of units of Q.*

For basic facts about matrices over division rings, see the book by Jacobson [12]. There is a great deal of information about full linear monoids, particularly in the case where Q is a field (see for example the books of Putcha [17] and Okniński [15]). Much of the basic structural information about full linear monoids over fields extends to the case where Q is a division ring, but care must be taken to extend some of these results if Q is not commutative. For example, linear combinations of rows will always be considered using left scalar multiplication, and linear combinations of columns using right scalar multiplication; consequently row spaces are left row spaces and column spaces are right column spaces.

It is well known that the set of matrices of a fixed rank $k \leq n$ forms a $\mathcal{J}$-class in the monoid $M_n(Q)$. Here the rank of a matrix $a \in M_n(Q)$ is the (left) row rank of a, which is the same as the (right) column rank of a. In fact, for matrices $a, b \in M_n(Q)$, we have $a \mathcal{J} b$ iff $GL_n(Q)\, a\, GL_n(Q) = GL_n(Q)\, b\, GL_n(Q)$ iff $rank(a) = rank(b)$, and $\mathcal{J}$=$\mathcal{D}$. The maximal subgroup of the $\mathcal{J}$-class of all matrices of rank k is isomorphic to $GL_k(Q)$. The Green's $\mathcal{R}$ and $\mathcal{L}$ relations on $M_n(Q)$ are characterized by

$a \mathcal{R} b$ iff $a\, GL_n(Q) = b\, GL_n(Q)$ iff $Col(a) = Col(b)$, and

$a \mathcal{L} b$ iff $GL_n(Q)\, a = GL_n(Q)\, b$ iff $Row(a) = Row(b)$.

Let D_k be the $\mathcal{D}$-class of $M_n(Q)$ consisting of the rank k matrices, and D_k^0 the corresponding completely 0-simple semigroup. Let $\mathcal{Y}_k$ be the set of all matrices of rank k which are in reduced row echelon form and let $\mathcal{X}_k$ be the set of transposes of these matrices. The structure of D_k^0 is described in the following theorem (see [15]).

Theorem 4.2. *$D_k^0 \cong \mathcal{M}^0(\mathcal{X}_k, GL_k(Q), \mathcal{Y}_k, C_k)$ where the matrix $C_k = (C_k(y, x))$ is defined for $x \in \mathcal{X}_k, y \in \mathcal{Y}_k$ by $C_k(y, x) = yx$ if yx is of rank k and 0 otherwise.*

By Theorem 4.2 and the basic structure of Rees matrix semigroups (see, for example [2]), every matrix a of rank k can be uniquely expressed in the form $a = xhy$ where $x \in \mathcal{X}, y \in \mathcal{Y}$ and h is a block diagonal matrix of the form $\begin{bmatrix} h' & 0 \\ 0 & 0 \end{bmatrix}$ where $h' \in GL_k(Q)$. Since x has $n-k$ columns of zeroes at the right of the matrix and y is the transpose of a matrix of this form we see that we may write the matrix $a = xhy$ of rank k in the form $a = vw^T$ for some $n \times k$ matrices v, w of rank k (choose $v = x'h'$ where x' is obtained from x by deleting the last $n-k$ columns, and w^T is obtained from y by deleting the last $n-k$ rows). Also, we may replace x [resp. y] in the above by matrices of the form $x_1 = xh_1$ [resp. $y_2 = h_2 y$] for any matrices h_1, h_2 in the maximal subgroup of $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ and get isomorphic Rees matrix semigroups. Thus we may replace yx by $w^T v$ as above in the definition of the matrix C_k and obtain an isomorphic Rees matrix semigroup.

If a is a rank-k matrix, expressed as $a = vw^T$ as above, then it is routine to check that a is an idempotent iff $w^T v = I_k$. Two rank-k $n \times n$ matrices $a = v_1 w_1^T, b = v_2 w_2^T$ are $\mathcal{L}$-related iff a and b have the same row space, which in turn is true iff $w_1^T = m w_2^T$ for some non-singular $k \times k$ matrix m. Similarly, a and b are $\mathcal{R}$-related iff they have the same column space, i.e., $v_1 = v_2 m$ for some non-singular $k \times k$ matrix m. An $\mathcal{L}$-class can therefore be identified with the equivalence class $[w^T] = \{mw^T : m \in GL_k(Q)\}$, and an $\mathcal{R}$-class can be identified with $[v] = \{vm : m \in GL_k(Q)\}$ where v, w are $n \times k$ matrices of rank k.

Rectangular bands in the biordered set of idempotents $E = E(M_n(Q))$ may be characterized from the representation of rank k matrices described above.

If $e \in [v_1] \cap [w_1^T], f \in [v_2] \cap [w_1^T], g \in [v_2] \cap [w_2^T], h \in [v_1] \cap [w_2^T]$ forms an E-square, then $e = v_1(w_1^T v_1)^{-1} w_1^T, f = v_2(w_1^T v_2)^{-1} w_1^T, g = v_2(w_2^T v_2)^{-1} w_2^T, h = v_1(w_2^T v_1)^{-1} w_2^T$. Then this E-square is a rectangular band iff $efghe = e$. A calculation of this product shows that this happens iff

$$(*) \ldots (w_1^T v_2)(w_2^T v_2)^{-1}(w_2^T v_1)(w_1^T v_1)^{-1} = I_k.$$

(Note that the identity (*) is independent of the choice of representatives of v_1, v_2, w_1, w_2 in their equivalence classes.) We have the following rather pleasant fact about the semigroup $M_n(Q)$.

Theorem 4.3. *Every non-trivial rectangular band in $M_n(Q)$ (for a division ring Q) is a singular square.*

Proof Given an E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ consisting of $n \times n$ idempotent matrices of rank k with coefficients in the division ring Q, we can, by conjugating e by a change of basis matrix whose columns are a basis for the column space of e followed by a basis for the nullspace of e, assume that $e = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where $I =$ the $k \times k$ identity matrix and the 0's are matrices of 0's of the appropriate size. Then since $ef = f, fe = e$, etc., a routine calculation demonstrates that after the same conjugation we have $f = \begin{bmatrix} I & b \\ 0 & 0 \end{bmatrix}$, $h = \begin{bmatrix} I & 0 \\ a & 0 \end{bmatrix}$, and $g = \begin{bmatrix} I & b \\ a & ab \end{bmatrix}$. Finally, computing $g^2 = g$ we find that $I + ba = I$ (in the upper left corner), so $ba = 0$.

We now show that for this quartet of idempotents there is a matrix η which singularizes the E-square. Conjugating this matrix by the inverse of our change of basis matrix gives a matrix singularizing the original E-square. Writing $\eta = \begin{bmatrix} x & y \\ z & c \end{bmatrix}$ and computing the required (for left-to-right singularization) products $\eta e = e$, $e\eta = f$, etc., together with $\eta^2 = \eta$, we find that η must have the form $\eta = \begin{bmatrix} I & b \\ 0 & c \end{bmatrix}$ with $bc = 0$, $ca = a$, and $c^2 = c$; further, any such matrix will be an idempotent singularizing the E-square. We now proceed to construct the needed matrix c.

Recalling that column spaces always refer to right-linear combinations of columns, note first that the condition $ba = 0$ is equivalent to $b(\mathrm{col}(a)) = \{0\}$, which is equivalent to $\mathrm{col}(a) \subseteq \mathrm{null}(b)$. Similarly, the requirement $bc = 0$ therefore requires $\mathrm{col}(c) \subseteq \mathrm{null}(b)$.

The condition $ca = a$ implies that the columns of a are right-linear combinations of the columns of c, so $\mathrm{col}(a) \subseteq \mathrm{col}(c)$. But conversely, since c is idempotent, $\mathrm{col}(a) \subseteq \mathrm{col}(c)$ implies that $ca = a$. To see this, since every column of a is a right-linear combination of the columns of c we have $a = cs$ for some $n \times n$ matrix s. Then $ca = c(cs) = (c^2)s = cs = a$, as desired.

So our requirements for the matrix c are: c is idempotent ($c^2 = c$) and $\mathrm{col}(a) \subseteq \mathrm{col}(c) \subseteq \mathrm{null}(b)$. Since every $\mathcal{R}$-class of $M_n(Q)$ contains an idempotent we can arrange to have $\mathrm{col}(a) = \mathrm{col}(c)$ (and then $\mathrm{col}(c) = \mathrm{col}(a) \subseteq \mathrm{null}(b)$ is immediate).

This result enables us to complete the description of the Graham-Houghton complex $K = GH(E)$ of the set of idempotents E of the semigroup $M_n(Q)$ of $n \times n$ matrices over the division ring Q. The vertices consist of the equivalence classes $[v]$,$[w^T]$ of sets of k linearly independent column (resp., row) vectors, $0 \le k \le n$; $[v_1] = [v_2]$ iff v_1 and v_2 have the same column space, i.e., $v_2 = v_1 x$ for some non-singular $k \times k$ matrix x; $[w_1^T] = [w_2^T]$ iff they have the same row space, i.e., $w_2^T = x w_1^T$ for $x \in GL_k(Q)$. There is an edge joining $[v]$ and $[w^T]$ iff $w^T v$ is a non-singular $k \times k$ matrix. A non-degenerate 4-cycle $([v_1], [w_1^T], [v_2], [w_2^T])$ (so $[v_1] \neq [v_2], [w_1^T] \neq [w_2^T]$) bounds a 2-cell iff $(w_1^T v_2)(w_2^T v_2)^{-1}(w_2^T v_1)(w_1^T v_1)^{-1} = I_k$. These constitute all of the edges and 2-cells in the complex.

Using this same notation, we can now describe the cover $\widetilde{K}_{n,k}$, defined in section 3 (and referred to as $\mathcal{C}(G)$ there) corresponding to the $\mathcal{D}$-class of the rank-k matrices. $\widetilde{K}_{n,k}$ has vertices the pairs $(g, [w^T])$,$([v], h)$, where $g, h \in GL_k(Q)$; since column spaces are right vector spaces, G acts on v on the right, and so we will write the latter pairs as $([v], h)$. If for each equivalence class we choose a (fixed) representative w_0^T, v_0 then we can identify $(g, [w^T]) = (g, [w_0^T])$ with $g w_0^T$ and $([v], h)$ with $v_0 h^{-1}$. As g, h range over $GL_k(Q)$, this identifies the vertices of $\widetilde{K}_{n,k}$ with the set of all rank-k $k \times n$ and $n \times k$ matrices, respectively. In the notation of section 3, our morphism $\varphi : \pi_1(GH(E)) \to GL_k(Q)$ is $([v], [w^T]) = w_0^T v_0$, where $([v], [w^T])$ is the edge from $[v]$ to $[w^T]$. There is an edge from $g w_0^T$ to $v_0 h^{-1}$ iff $g(w_0^T v_0) = h$, that is, $(g w_0^T)(v_0 h^{-1}) = I_k$, where I_k is the $k \times k$ identity matrix. So the vertices of $\widetilde{K}_{n,k}$ consist of the rank-k $k \times n$ and $n \times k$ matrices, and there is an edge from w^T to v iff $w^T v = I_k$. Finally, there is a 2-cell with boundary any 4-cycle in the 1-skeleton of $\widetilde{K}_{n,k}$.

Proof of Theorem 4.1. We denote by $K_{n,1}$ the subcomplex of K spanned by the rank-1 vertices. By Theorem 2.2 and the Freeness Criterion of section 3, we will be able to prove Theorem 4.1, if we can show that the cover $\widetilde{K}_{n,1}$ is simply connected. By construction, $\widetilde{K}_{n,1}$ has vertex set consisting of all nonzero $n \times 1$ (column) vectors v and all nonzero $1 \times n$ (row) vectors w^T. There is an edge $v \leftrightarrow w^T$ (consisting of a positively oriented edge from w^T to v and its inverse edge from v to w^T) iff $w^Tv = 1$. Finally, each 4-cycle in $\widetilde{K}^{(1)}_{n,1}$ is the boundary of a 2-cell in $\widetilde{K}_{n,1}$.

To show that $\widetilde{K}_{n,1}$ is simply connected, that is, that $\pi_1(\widetilde{K}_{n,1}) = \{1\}$, we need to show that every loop in $\widetilde{K}_{n,1}$ is null-homotopic. More precisely, choosing a maximal tree T in $\widetilde{K}^{(1)}_{n,1}$, $\pi_1(\widetilde{K}_{n,1})$ is generated by loops, one for each edge ϵ not in T. The loops start at the basepoint, run out the tree to one endpoint of ϵ, across ϵ, and then back in the tree to the basepoint. It suffices to show that each of these loops is null-homotopic, and for this it is enough to show that each edge ϵ is homotopic, rel endpoints, to an edge path in T. It is this last statement which we will now prove. We will carry out this verification in steps, in the process building the tree T in steps as well.

The basic shortcut which we will use is the following observation. If $T \subseteq \widetilde{K}^{(1)}_{n,1}$ is a tree and $\epsilon_1, \ldots, \epsilon_n, \epsilon \in K^{(1)}$ are edges with all endpoints lying in T, and if each ϵ_i is homotopic in K, rel endpoints, to an edgepath in T, <u>and</u> ϵ is homotopic in K, rel endpoints, to an edgepath γ in $T \cup \epsilon_1 \cup \cdots \cup \epsilon_n$, then ϵ is also homotopic, rel endpoints, to an edgepath δ in T. This is because we can concatenate a sequence of homotopies, each supported on an edge ϵ_i lying in the edgepath γ, deforming ϵ_i into T, to further homotope ϵ into T. The net effect of this observation is that, in the course of our proof, anytime we have shown that an edge ϵ can be deformed into our tree T, we can act as if ϵ were actually <u>in</u> T and build our further deformations to map into the union of T and ϵ (and all other edges we have shown can deform into T). To reinforce this, we will talk of the edges of our tree as being colored "green", and say that any edge that we can deform into T has turned green. Then to continue to move our proof forward we are required only to show that any further edge can be deformed into the green edges. In this way more and more edges become green; the proof ends when we have shown that every edge can be turned green.

Our approach will be to choose nested collections $\mathcal{V}_i$ of vertices, and then, inductively, extend the tree T_{i-1} from a tree with the previous vertex set $\mathcal{V}_{i-1}$ to a tree whose vertex set is $\mathcal{V}_i$, and show that the edges of the full subcomplex of $\widetilde{K}_{n,1}$ with vertex set $\mathcal{V}_i$ can all be turned green. These edges then can automatically be assumed to be green when moving on to the next vertex set $\mathcal{V}_{i+1}$; a homotopy rel endpoints into the tree T_i is also a homotopy into the tree T_{i+1}. Our approach relies on the fact that the homotopies take place across the 2-cells of $\widetilde{K}_{n,1}$, whose boundaries are the 4-cycles in $\widetilde{K}^{(1)}_{n,1}$. Any time that we can find a 4-cycle three of whose sides have turned green, the square then provides a homotopy of the fourth side into the green edges, enabling us to turn the fourth side green, as well. In what follows, we will signify this by stating that the sequence of edges $v_1 \leftrightarrow v_2 \leftrightarrow v_3 \leftrightarrow v_4$ "yields" the edge $v_1 \leftrightarrow v_4$, meaning that it can now be turned green. Our proof essentially consists of finding a way to list the edges in each of our full subcomplexes so that, for each edge ϵ outside of the tree T_i, there is a square containing ϵ so that the other three edges of the square are each either in the tree T_i or appear earlier in the list, and so, by induction, can be assumed to have turned green. This enables us to turn the edge ϵ green, as well, and continue the induction.

In what follows we denote by $\vec{e}_i$ the vector with 1 in the i-th coordinate and 0 in the remaining coordinates.

We start with the case $n = 3$; our last induction will be on n. Our first vertex set $\mathcal{V}_1$ consists of the nonzero vectors v, w^T all of whose entries are 0 or 1, with at least one 0. We build the tree T_1 by adding the edge from every w^T with first coordinate 1 to $\vec{e}_1$ and then an edge from every v with first coordinate 1 to $\vec{e}_1^T$, and then add, for every vertex v, w^T whose first non-0 coordinate occurs in the i^{th} entry, $i > 1$, the edge from v, w^T to $\vec{e}_1 + \vec{e}_i$ and $\vec{e}_1^T + \vec{e}_i^T$, respectively. (Note that this has, implicitly, already used the hypothesis $n \geq 3$, so that $\vec{e}_1 + \vec{e}_i \in \mathcal{V}_1$.) Since, for every edge in T_1, at the time it is added exactly one of its (non-$\vec{e}_1$) endpoints does not yet lie in the part of T_1 constructed up to that point, their union forms a tree, by induction. The edges already in T_1 are therefore

$(1\ 0\ 0) \leftrightarrow$ each of $(1\ 0\ 0)^T, (1\ 1\ 0)^T, (1\ 0\ 1)^T$,
$(1\ 0\ 0)^T \leftrightarrow$ each of $(1\ 1\ 0), (1\ 0\ 1)$,
$(1\ 1\ 0)^T \leftrightarrow$ each of $(0\ 1\ 0), (0\ 1\ 1)$ and
$(1\ 1\ 0) \leftrightarrow$ each of $(0\ 1\ 0)^T, (0\ 1\ 1)^T$,
$(1\ 0\ 1) \leftrightarrow (0\ 0\ 1)^T$ and $(1\ 0\ 1)^T \leftrightarrow (0\ 0\ 1)$.

By inspection, the remaining edges joining vertices in $\mathcal{V}_1$ are

$(1\ 1\ 0) \leftrightarrow (1\ 0\ 1)^T$ and $(1\ 1\ 0)^T \leftrightarrow (1\ 0\ 1)$,
$(1\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T$ and $(1\ 0\ 1)^T \leftrightarrow (0\ 1\ 1)$,
$(0\ 1\ 0) \leftrightarrow (0\ 1\ 0)^T, (0\ 1\ 1)^T$ and $(0\ 1\ 0)^T \leftrightarrow, (0\ 1\ 1)$,
$(0\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T, (0\ 0\ 1)^T$ and $(0\ 0\ 1)^T \leftrightarrow (0\ 1\ 1)$.

Then the following 4-cycles show how to turn each of these edges, in turn, green:

$$
\begin{array}{lcl}
(1\ 1\ 0) \leftrightarrow (1\ 0\ 0)^T \leftrightarrow (1\ 0\ 0) \leftrightarrow (1\ 0\ 1)^T & \text{yields} & (1\ 1\ 0) \leftrightarrow (1\ 0\ 1)^T \\
(1\ 0\ 1) \leftrightarrow (1\ 0\ 0)^T \leftrightarrow (1\ 0\ 0) \leftrightarrow (1\ 1\ 0)^T & \text{yields} & (1\ 0\ 1) \leftrightarrow (1\ 1\ 0)^T \\
(1\ 0\ 1) \leftrightarrow (1\ 0\ 0)^T \leftrightarrow (1\ 1\ 0) \leftrightarrow (0\ 1\ 1)^T & \text{yields} & (1\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T \\
(0\ 1\ 1) \leftrightarrow (1\ 1\ 0)^T \leftrightarrow (1\ 0\ 0) \leftrightarrow (1\ 0\ 1)^T & \text{yields} & (0\ 1\ 1) \leftrightarrow (1\ 0\ 1)^T \\
(0\ 1\ 0) \leftrightarrow (1\ 1\ 0)^T \leftrightarrow (1\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T & \text{yields} & (0\ 1\ 0) \leftrightarrow (0\ 1\ 1)^T \\
(0\ 1\ 1) \leftrightarrow (1\ 0\ 1)^T \leftrightarrow (1\ 1\ 0) \leftrightarrow (0\ 1\ 0)^T & \text{yields} & (0\ 1\ 1) \leftrightarrow (0\ 1\ 0)^T \\
(0\ 1\ 0) \leftrightarrow (0\ 1\ 1)^T \leftrightarrow (1\ 1\ 0) \leftrightarrow (0\ 1\ 0)^T & \text{yields} & (0\ 1\ 0) \leftrightarrow (0\ 1\ 0)^T \\
(0\ 0\ 1) \leftrightarrow (1\ 0\ 1)^T \leftrightarrow (1\ 1\ 0) \leftrightarrow (0\ 1\ 1)^T & \text{yields} & (0\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T \\
(0\ 0\ 1) \leftrightarrow (0\ 1\ 1)^T \leftrightarrow (1\ 0\ 1) \leftrightarrow (0\ 0\ 1)^T & \text{yields} & (0\ 0\ 1) \leftrightarrow (0\ 0\ 1)^T \\
(0\ 1\ 1) \leftrightarrow (1\ 1\ 0)^T \leftrightarrow (1\ 0\ 1) \leftrightarrow (0\ 0\ 1)^T & \text{yields} & (0\ 1\ 1) \leftrightarrow (0\ 0\ 1)^T
\end{array}
$$

For our next collection $\mathcal{V}_2$ of vertices we add v, w^T all with one entry 0, another entry 1, and the remaining entry $a \neq 0, 1$. We extend T_1 to a tree T_2 with vertex set $\mathcal{V}_2$ by adding the edge from each new vertex to the vertex $\vec{e}_i$ or $\vec{e}_i^T$ (as appropriate), where i is the coordinate with entry equal to 1. In addition to the edges joining vertices in $\mathcal{V}_1$ and those in T_2, the only edges joining vertices in $\mathcal{V}_2$ are of one of the forms

$(0\ 1\ a) \leftrightarrow \left(1\ 0\ a^{-1}\right)^T$
$(0\ 1\ a) \leftrightarrow (b\ 1\ 0)^T$ (with $b \neq 0$)
$(0\ 1\ a) \leftrightarrow (0\ 1-a\ 1)^T$

(together with pairs resulting from simultaneous permutation of the coordinates of each side), since $(0\ 1\ a) \leftrightarrow (x\ y\ z)^T$ requires $y + za = 1$ with at least one of y, z equal to 0 or 1. $y = 0$ implies the first case, $y = 1$ implies $z = 0$ (and vice versa) and implies the second, and $z = 1$ implies the third. For these edges the 4-cycles

$(0\ 1\ a) \leftrightarrow (0\ 1\ 0)^T \leftrightarrow (0\ 1\ 0) \leftrightarrow (b\ 1\ 0)^T$ yields $(0\ 1\ a) \leftrightarrow (b\ 1\ 0)^T$

together with simultaneous permutations, by permuting throughout the 4-cycle. The 4-cycle

$(0\ 1\ a) \leftrightarrow (0\ 1\ 0)^T \leftrightarrow (1\ 1\ 0) \leftrightarrow \left(1\ 0\ a^{-1}\right)^T$ yields $(0\ 1\ a) \leftrightarrow \left(1\ 0\ a^{-1}\right)^T$,

where the edge $(1\ 1\ 0) \leftrightarrow \left(1\ 0\ a^{-1}\right)^T$ is a permutation of $(0\ 1\ 1) \leftrightarrow \left(a^{-1}\ 1\ 0\right)^T$ (cycling to the left), which turned green in the previous step. We again have all simultaneous permutations. Finally, the 4-cycle

$(0\ 1\ a) \leftrightarrow (1\ 1\ 0)^T \leftrightarrow (1\ 0\ 1) \leftrightarrow (0\ 1-a\ 1)^T$ yields $(0\ 1\ a) \leftrightarrow (0\ 1-a\ 1)^T$

together with simultaneous permutations.

This deals with the full subcomplex on the vertices with at least one 0-entry and at least one 1-entry. The next collection $\mathcal{V}_3$ of vertices adds $a\vec{e}_i, a\vec{e}_i^T$ with $a \neq 1$. We extend our tree T_2 to a tree T_3 by adding the edges

$a\vec{e}_i \leftrightarrow \vec{e}_j^T + a^{-1}\vec{e}_i^T$ and $a\vec{e}_i^T \leftrightarrow \vec{e}_j + a^{-1}\vec{e}_i$

where j is the smallest index $\neq i$. The edges still unaccounted for, running between vertices of $\mathcal{V}_3$, are

$a\vec{e}_i \leftrightarrow a^{-1}\vec{e}_i^T$, $a\vec{e}_i \leftrightarrow \vec{e}_k + a^{-1}\vec{e}_i^T$, and $a\vec{e}_i^T \leftrightarrow \vec{e}_k + a^{-1}\vec{e}_i$ (where $k \neq i, j$)

since $a\vec{e}_i \leftrightarrow x\vec{e}_i^T + y\vec{e}_j^T + z\vec{e}_k^T$ and $x\vec{e}_i^T + y\vec{e}_j^T + z\vec{e}_k^T \in \mathcal{V}_3$ requires $x = a^{-1}$ and $\{y, z\} = \{0, 1\}$. The 4-cycles

$a\vec{e}_i \leftrightarrow \vec{e}_j^T + a^{-1}\vec{e}_i^T \leftrightarrow \vec{e}_j + \vec{e}_k \leftrightarrow \vec{e}_k^T + a^{-1}\vec{e}_i^T$,
$a\vec{e}_i^T \leftrightarrow \vec{e}_j + a^{-1}\vec{e}_i \leftrightarrow \vec{e}_j^T + \vec{e}_k^T \leftrightarrow \vec{e}_k + a^{-1}\vec{e}_i$, and
$a\vec{e}_i \leftrightarrow \vec{e}_j^T + a^{-1}\vec{e}_i^T \leftrightarrow \vec{e}_k + a\vec{e}_i \leftrightarrow a^{-1}\vec{e}_i^T$

demonstrate that these edges can be turned green.

For $\mathcal{V}_4$ we add the vertices $a\vec{e}_i + b\vec{e}_j, a\vec{e}_i^T + b\vec{e}_j^T$ with $a, b \neq 0, 1$ and $i < j$. The vertex set $\mathcal{V}_4$ thus consists of all of the vertices with at least one 0-entry. We extend the tree T_3 to a tree T_4 by adding the edges $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow a^{-1}\vec{e}_i$. The edges we need to turn green are of the form $a\vec{e}_i + b\vec{e}_j \leftrightarrow x\vec{e}_i^T + y\vec{e}_j^T + z\vec{e}_k^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow x\vec{e}_i + y\vec{e}_j + z\vec{e}_k$ with

$za + yb + z0 = xa + yb = 1$ (resp. $az + by + 0z = ax + by = 1$; we are working over a division ring!) and at least one of x, y, z equal to 0. This yields the four cases

(two coefficients equal 0): $a\vec{e}_i + b\vec{e}_j \leftrightarrow b^{-1}\vec{e}_j^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow b^{-1}\vec{e}_j$,

($x = 0$): $a\vec{e}_i + b\vec{e}_j \leftrightarrow b^{-1}\vec{e}_j^T + z\vec{e}_k^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow b^{-1}\vec{e}_j + z\vec{e}_k$,

($y = 0$): $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T + z\vec{e}_k^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow a^{-1}\vec{e}_i + z\vec{e}_k$,

($z = 0$): $a\vec{e}_i + b\vec{e}_j \leftrightarrow x\vec{e}_i^T + y\vec{e}_j^T$ and $a\vec{e}_i^T + b\vec{e}_j^T \leftrightarrow x\vec{e}_i + y\vec{e}_j$

These can be turned green by using the 4-cycles

[$y = 0$ and $i < k$]: $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T \leftrightarrow a\vec{e}_i \leftrightarrow a^{-1}\vec{e}_i^T + z\vec{e}_k^T$ (and transposes),

[$y = 0$ and $i > k$, so $k < i < j$]: $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T \leftrightarrow a\vec{e}_i + \vec{e}_j \leftrightarrow z\vec{e}_k^T + \vec{e}_j^T$,

and so $a\vec{e}_i + b\vec{e}_j \leftrightarrow z\vec{e}_k^T + \vec{e}_j^T \leftrightarrow z^{-1}\vec{e}_k \leftrightarrow a^{-1}\vec{e}_i^T + z\vec{e}_k^T$ (together with transposes),

[two coefficients 0]: $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T + \vec{e}_k^T \leftrightarrow b\vec{e}_j + \vec{e}_k \leftrightarrow b^{-1}\vec{e}_j^T$ (and transposes),

[$x = 0$]: $a\vec{e}_i + b\vec{e}_j \leftrightarrow b^{-1}\vec{e}_j^T \leftrightarrow b\vec{e}_j \leftrightarrow b^{-1}\vec{e}_j^T + z\vec{e}_k^T$ (and transposes), and

[$z = 0$]: $a\vec{e}_i + b\vec{e}_j \leftrightarrow a^{-1}\vec{e}_i^T + \vec{e}_k^T \leftrightarrow y^{-1}\vec{e}_j + \vec{e}_k \leftrightarrow x\vec{e}_i^T + y\vec{e}_j^T$ (and transposes).

Finally, $\mathcal{V}_5 = \widetilde{K}_{3,1}^{(0)}$, that is, we add the vertices v, w^T with all entries non-zero. We extend T_4 to a tree T_5 by adding the edges

$(a\ b\ c) \leftrightarrow \left(a^{-1}\ 0\ 0\right)^T$ and $(a\ b\ c)^T \leftrightarrow \left(a^{-1}\ 0\ 0\right)$.

Then the 4-cycles

$(a\ b\ c) \leftrightarrow \left(a^{-1}\ 0\ 0\right)^T \leftrightarrow (a\ b\ 0) \leftrightarrow \left(0\ b^{-1}\ 0\right)^T$

yield $(a\ b\ c) \leftrightarrow \left(0\ b^{-1}\ 0\right)^T$, and

$(a\ b\ c) \leftrightarrow \left(a^{-1}\ 0\ 0\right)^T \leftrightarrow (a\ 0\ c) \leftrightarrow \left(0\ 0\ c^{-1}\right)^T$

yield $(a\ b\ c) \leftrightarrow \left(0\ 0\ c^{-1}\right)^T$ (together with transposes).

Then if $(x\ y\ z)$ has $z = 0$ and $xa + yb = 1$ with $x, y \neq 0$, the 4-cycle

$(a\ b\ c) \leftrightarrow \left(a^{-1}\ 0\ 0\right)^T \leftrightarrow (a\ b\ 0) \leftrightarrow (x\ y\ 0)^T$ yields $(a\ b\ c) \leftrightarrow (x\ y\ 0)^T$ (the transpose relation is similar),

and a similar argument yields the cases where the second or first entry is 0. Finally, if $xa + yb + zc = 1$ with $x, y, z \neq 0$, then the 4-cycle

$$(a\ b\ c) \leftrightarrow \left(a^{-1}\ 0\ 0\right)^T \leftrightarrow \left(a\ b + y^{-1}zc\ 0\right) \leftrightarrow (x\ y\ z)^T$$

yields $(a\ b\ c) \leftrightarrow (x\ y\ z)^T$ (and the transpose relation is, again, similar).

With this, we have constructed a maximal tree $T = T_5$ in $\widetilde{K}_{3,1}^{(1)}$, and have shown how to homotope every edge in $\widetilde{K}_{3,1}^{(1)}$, rel endpoints, to an edge path in T. Consequently, $\widetilde{K}_{3,1}$ is connected and simply connected.

To finish our argument, we show how to extend this result to arbitrary $n \geq 3$. We argue by induction. The base case $n = 3$ is established above. For the inductive step, we assume that we have shown that $\widetilde{K}_{n-1,1}$ is simply-connected. In particular, we have constructed a maximal tree T_{n-1} in $\widetilde{K}_{n-1,1}^{(1)}$

and have shown that each of the edges of $\widetilde{K}^{(1)}_{n-1,1}$ not in T_{n-1} can be deformed, rel endpoints, in $\widetilde{K}_{n-1,1}$, into T_{n-1}.

By appending 0's to the end of every column of the vectors v, w labelling the vertices v, w^T of $\widetilde{K}_{n-1,1}$ (yielding matrices v_+, w_+) and noting that $w_+^T v_+ = 1$ iff $w^T v = 1$, the map $v \mapsto v_+$, $w^T \mapsto w_+^T$ induces an embedding of $\widetilde{K}_{n-1,1}$ into $\widetilde{K}_{n,1}$, and its image is the full subcomplex of $\widetilde{K}_{n,1}$ on the vertex set $\mathcal{V}_6 =$ the image of $\widetilde{K}^{(0)}_{n-1,1}$. The image of the tree T is a tree T_6 which provides the starting point for constructing the needed tree in $\widetilde{K}_{n,1}$.

To build our tree T_7 we add, for the vertices $\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}, \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}^T$ with $a_n \neq 0$ and at least one other entry $a_j \neq 0$ (we may assume j is the smallest such index), the edges $\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \leftrightarrow a_j^{-1}\vec{e}_j^{\,T}$ and $\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}^T \leftrightarrow a_j^{-1}\vec{e}_j$, and then add the edges $a_n\vec{e}_n \leftrightarrow \vec{e}_1^{\,T} + a_n^{-1}\vec{e}_n^{\,T}$ and $a_n\vec{e}_n^{\,T} \leftrightarrow \vec{e}_1 + a_n^{-1}\vec{e}_n$ for each $a_n \neq 0$. This gives a maximal tree in $\widetilde{K}_{n,1}$; by our inductive step we know that every edge in the image of $\widetilde{K}^{(1)}_{n-1,1}$ is homotopic, rel endpoints, to an edge path in T_7, and so can be assumed to be green.

We now work our way through all of the remaining edges of $\widetilde{K}_{n,1}$ in steps, to show that they can all be turned green. If a vertex v, w^T has two or more entries $a_j, a_k \neq 0$, for $j, k < n$ (we may assume j is the smallest such index), then the 4-cycles

$$v \leftrightarrow a_j^{-1}\vec{e}_j^{\,T} \leftrightarrow a_j\vec{e}_j + a_k\vec{e}_k \leftrightarrow a_k^{-1}\vec{e}_k^{\,T} \text{ and } w^T \leftrightarrow a_j^{-1}\vec{e}_j \leftrightarrow a_j\vec{e}_j^{\,T} + a_k\vec{e}_k^{\,T} \leftrightarrow a_k^{-1}\vec{e}_k$$

enable us to make the edges $v \leftrightarrow a_k^{-1}\vec{e}_k^{\,T}$ and $w^T a_k^{-1}\vec{e}_k$ green, and allowing us to base our further arguments off of any non-zero entry of v, w^T (other than the last entry). If $v = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \leftrightarrow \begin{pmatrix} x_1 & \dots & x_{n-1} & 0 \end{pmatrix}^T = w^T$ is an edge, then $x_i a_i \neq 0$ for some i, and the 4-cycle

$$v \leftrightarrow a_i^{-1}\vec{e}_i^{\,T} \leftrightarrow \begin{pmatrix} a_1 & \dots & a_{n-1} & 0 \end{pmatrix} \leftrightarrow w^T$$

enables us to turn the edge $v \leftrightarrow w^T$ green, and a similar argument will allow us to turn the edges $v = \begin{pmatrix} x_1 & \dots & x_{n-1} & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}^T = w^T$ green. With this we can make every edge of $\widetilde{K}_{n,1}$ joining a vertex of $\widetilde{K}_{n,1}$, <u>other than</u> the vertices $a_n\vec{e}_n, a_n\vec{e}_n^{\,T}$, to a vertex in the image of $\widetilde{K}^{(0)}_{n-1,1}$, green. Note, however, that there are no edges between the vertices $a_n\vec{e}_n, a_n\vec{e}_n^{\,T}$ and the vertices in the image of $\widetilde{K}^{(0)}_{n-1,1}$.

For every $i > 1$ the 4-cycle $a_n\vec{e}_n \leftrightarrow \vec{e}_1^{\,T} + a_n^{-1}\vec{e}_n^{\,T} \leftrightarrow \vec{e}_1 + \vec{e}_i \leftrightarrow \vec{e}_i^{\,T} + a_n^{-1}\vec{e}_n^{\,T}$ makes $a_n\vec{e}_n \leftrightarrow \vec{e}_i^{\,T} + a_n^{-1}\vec{e}_n^{\,T}$ green, and a similar argument turns $a_n\vec{e}_n^{\,T} \leftrightarrow \vec{e}_i + a_n^{-1}\vec{e}_n$ green. Then for $i \neq j$, $i, j < n$,

$$\vec{e}_i + a_n\vec{e}_n \leftrightarrow \vec{e}_i^{\,T} + \vec{e}_j^{\,T} \leftrightarrow \vec{e}_j \leftrightarrow \vec{e}_j^{\,T} + a_n^{-1}\vec{e}_n^{\,T}$$

turns $\vec{e}_i + a_n\vec{e}_n \leftrightarrow \vec{e}_j^{\,T} + a_n^{-1}\vec{e}_n^{\,T}$ green. Then (using the fact that $n \geq 3$)

$$a_n\vec{e}_n \leftrightarrow \vec{e}_1^{\,T} + a_n^{-1}\vec{e}_n^{\,T} \leftrightarrow \vec{e}_2 + a_n\vec{e}_n \leftrightarrow a_n^{-1}\vec{e}_n^{\,T}$$

turns $a_n\vec{e}_n \leftrightarrow a_n^{-1}\vec{e}_n^T$ green. If one of $x_1, \dots x_{n-1}$ is non-zero (say x_i, and then choose $j \neq i$, $j < n$), then the 4-cycles

$$a_n\vec{e}_n \leftrightarrow \vec{e}_j^T + a_n^{-1}\vec{e}_n^T \leftrightarrow x_i^{-1}(1-x_j)\vec{e}_i + \vec{e}_j \leftrightarrow \begin{pmatrix} x_1 & \dots & x_{n-1} & a_n^{-1} \end{pmatrix}^T \text{ and}$$
$$a_n\vec{e}_n^T \leftrightarrow \vec{e}_j + a_n^{-1}\vec{e}_n \leftrightarrow x_i^{-1}(1-x_j)\vec{e}_i^T + \vec{e}_j^T \leftrightarrow \begin{pmatrix} x_1 & \dots & x_{n-1} & a_n^{-1} \end{pmatrix}$$

together with the previous step enable us to make every edge with vertex either $a_n\vec{e}_n$ or $a_n\vec{e}_n^T$ green.

All that is left now is to turn the edges $v = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \leftrightarrow \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T = w^T$ with $a_n, x_n \neq 0$ green. By the immediately preceding step we may also assume that $x_i, a_j \neq 0$ for some $i, j < n$. If $i \neq j$ then the 4-cycle

$$v \leftrightarrow a_j^{-1}\vec{e}_j^T \leftrightarrow a_j\vec{e}_j + x_i^{-1}(1-a_jx_j)\vec{e}_i \leftrightarrow w^T$$

turns the edge $v \leftrightarrow w^T$ green. If we cannot find such a pair of distinct indices i, j then our edge must be of the form $a_i\vec{e}_i + a_n\vec{e}_n \leftrightarrow x_i\vec{e}_i^T + x_n\vec{e}_n^T$ with $x_ia_i + x_na_n = 1$ and $a_i, x_i, a_n, x_n \neq 0$. Then (choosing a $j \neq i, n$) the 4-cycle

$$a_i\vec{e}_i + a_n\vec{e}_n \leftrightarrow a_i^{-1}\vec{e}_i^T + \vec{e}_j^T \leftrightarrow x_i^{-1}\vec{e}_i + (1 - a_i^{-1}x_i^{-1})\vec{e}_j \leftrightarrow x_i\vec{e}_i^T + x_n\vec{e}_n^T$$

demonstrates that this final collection of edges can be made green.

Therefore, every edge of $\widetilde{K}_{n,1}^{(1)}$ is homotopic, rel endpoints, to an edge path in T_7, and $\widetilde{K}_{n,1}$ is simply connected. This finishes the inductive step; so for every $n \geq 3$, $\widetilde{K}_{n,1}$ is connected and simply connected, and so is the universal covering space of $K_{n,1}$. This completes the proof of Theorem 4.1.

5. Closing Remarks

Theorem 4.1 provides a natural example of a torsion group that arises as a maximal subgroup of the free idempotent generated semigroup on some (finite) biordered set, answering a question raised in [5]. After the results of this paper were announced, Gray and Ruskuc [8] proved that every group arises as the maximal subgroup of some biordered set overriding this particular example. We remark that if e is an idempotent matrix of rank $n-1$ in $E = E(M_n(Q))$, then the maximal subgroup of $IG(E)$ with identity e must be a free group by Theorem 2.2, since there are no idempotents available to singularize a square consisting of rank $n-1$ idempotent matrices. Thus the maximal subgroup of $IG(E)$ corresponding to an idempotent of rank $n-1$ is not isomorphic to $GL_{n-1}(Q)$.

Based on experimental evidence, we conjecture that the maximal subgroup of $IG(E)$ with identity an idempotent matrix of rank $k < n-1$ is $GL_k(Q)$, at least if $k < n/2$ and $n \geq 3$, but this problem remains open. It is plausible that when $k < n-1$ the subcomplex $\widetilde{K}_{n,k}$ of K spanned by the vertices of rank k is simply connected. The methods in the proof of Theorem 4.1 in the present paper seem difficult to extend. However, there is a lot of structure to the complexes that we have not exploited. The connections to Grassmanians - the vertices of $K_{n,k}$ are (two sets of) the points of the Grassmanian $G_{n,k}$ of k-planes in Q^n, and the vertices of $\widetilde{K}_{n,k}$ are (two sets of) the points of the universal bundle over $G_{n,k}$ - seem worth exploring further, and we expect to consider these ideas in a subsequent paper.

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